

L^p -Liouville Theorems for Invariant Partial Differential Operators in \mathbb{R}^n

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Abstract

We prove some L^p -Liouville theorems for hypoelliptic second order Partial Differential Operators left translation invariant with respect to a Lie group composition law in \mathbb{R}^n . Results for both solutions and subsolutions are given.

1 Introduction

In this paper we are concerned with L^p -Liouville properties for solutions and subsolutions to the equation

$$\mathcal{L}u = 0 \quad \text{in } \mathbb{R}^n, \quad (1.1)$$

where \mathcal{L} is a linear hypoelliptic second order Partial Differential Operator left translation invariant with respect to a Lie group in \mathbb{R}^n . More precisely, the operator \mathcal{L} in (1.1) is of the kind

$$\mathcal{L} := \operatorname{div}(A\nabla) + \langle b, \nabla \rangle,$$

where $A = (a_{ij})_{i,j=1,\dots,n}$ is a $n \times n$ symmetric matrix with real entries a_{ij} in $C^\infty(\mathbb{R}^n, \mathbb{R})$, $b = (b_1, \dots, b_n)$ is a vector valued function with real components b_j in $C^\infty(\mathbb{R}^n, \mathbb{R})$, and, as usual, div , ∇ $\langle \cdot, \cdot \rangle$ denote Euclidean divergence, gradient and inner product in \mathbb{R}^n . We will assume, without further comments,

$$\langle A(x)\xi, \xi \rangle \geq 0 \quad \forall x, \xi \in \mathbb{R}^n \quad \text{and} \quad \operatorname{trace} A(0) > 0.$$

Our crucial assumptions on \mathcal{L} are the following ones.

- (H1) \mathcal{L} is hypoelliptic, that is if u is a distribution in a open set $\Omega \subseteq \mathbb{R}^n$ and $\mathcal{L}u$ is smooth in Ω , then u is smooth in Ω .
- (H2) There exists a Lie group $\mathbb{G} = (\mathbb{R}^n, \circ)$ such that \mathcal{L} is left translation invariant on \mathbb{G} .

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For simplicity reasons we assume the *Lebesgue measure* in \mathbb{R}^n both *left* and *right invariant* on \mathbb{G} . Throughout the paper we will denote by L^p the Lebesgue space $L^p(\mathbb{R}^n, \mathbb{R})$.

We recall that \mathbb{G} is said to be a *homogeneous Lie group* if the following property holds: there exists a n -tuple of real numbers $\sigma = (\sigma_1, \dots, \sigma_n)$, with $1 \leq \sigma_1 \leq \dots \leq \sigma_n$, such that the dilation

$$\delta_\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad \delta_\lambda(x_1, \dots, x_n) = (\lambda^{\sigma_1} x_1, \dots, \lambda^{\sigma_n} x_n) \quad (1.2)$$

is an automorphisms of \mathbb{G} , for every $\lambda > 0$. The real number

$$Q = \sigma_1 + \dots + \sigma_n$$

is called the *homogeneous dimension* of \mathbb{G} w.r.t. $(\delta_\lambda)_{\lambda>0}$.

If $\mathbb{G} = (\mathbb{R}^n, \circ, \delta_\lambda)$ is homogeneous then the Lebesgue measure in \mathbb{R}^n is right and left translation invariant on \mathbb{G} (see e.g. [BLU07]).

Aim of this paper is to prove the following theorems.

Theorem 1.1 *Let $1 \leq p < \infty$ and let $u \in L^p$ be a smooth solution to*

$$\mathcal{L}u = 0 \quad \text{in } \mathbb{R}^n. \quad (1.3)$$

Then $u \equiv 0$.

Nonnegative solutions to the equation (1.3) satisfy also an L^p -Liouville property for $0 < p < 1$. Indeed:

Theorem 1.2 *Let $0 < p < 1$ and let $u \geq 0$, $u^p \in L^1$, be a smooth solution to*

$$\mathcal{L}u = 0 \quad \text{in } \mathbb{R}^n.$$

Then $u \equiv 0$.

Theorem 1.1 extends to the subsolutions as follows.

Theorem 1.3 *Let $u \in L^1_{\text{loc}}$ be a solution to*

$$\mathcal{L}u \geq 0 \quad \text{in } \mathbb{R}^n, \text{ in the weak sense of distributions.}$$

If $u \in L^p$ for a suitable $p \in [1, \infty[$, then $u \leq 0$ a.e. in \mathbb{R}^n .

When \mathbb{G} is a homogeneous group, Theorem 1.3 takes the following sharp form.

Theorem 1.4 *Let \mathbb{G} be a homogeneous Lie group with homogeneous dimension $Q \geq 3$. Assume \mathcal{L} is homogeneous of degree two w.r.t. the dilations in \mathbb{G} . Let $u \in L^1_{\text{loc}}$ be a solution to*

$$\mathcal{L}u \geq 0 \quad \text{in } \mathbb{R}^n, \text{ in the weak sense of distributions.}$$

If $u \in L^p$ for a suitable $p \in [1, 1 + \frac{2}{Q-2}]$, then

$$u \equiv 0 \quad \text{a.e. in } \mathbb{R}^n.$$

Moreover, for every $p > 1 + \frac{2}{Q-2}$, there exists $u \in L^p$, $u \leq 0$, $u \not\equiv 0$, such that

$$\mathcal{L}u \geq 0 \quad \text{in } \mathbb{R}^n, \text{ in the weak sense of distributions.}$$

Our proofs of the previous theorems are based on some devices that allow to obtain, as well, Liouville theorems for semilinear equations as the following one. We stress that this theorem does not requires the homogeneity of \mathbb{G} and \mathcal{L} .

Theorem 1.5 *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a C^1 -increasing function such that $f^{-1}(\{0\}) = 0$. Define*

$$F : \mathbb{R} \rightarrow \mathbb{R}, \quad F(t) = \int_0^t f(s) \, ds. \quad (1.4)$$

Let $u \in C^2(\mathbb{R}^n, \mathbb{R})$ be a classical solution to

$$\mathcal{L}u = f(u) \quad \text{in } \mathbb{R}^n. \quad (1.5)$$

If $F(u) \in L^1(\mathbb{R}^n)$ then $u \equiv 0$.

If in Theorem 1.5 we choose $f(t) = \lambda t$ or $f(t) = |t|^{p-1}t$, we obtain, respectively, the following corollaries.

Corollary 1.6 *Let $u \in C^2(\mathbb{R}^n, \mathbb{R}) \cap L^2(\mathbb{R}^n)$ be such that*

$$\mathcal{L}u = \lambda u \quad \text{in } \mathbb{R}^n, \quad \text{with } \lambda \geq 0.$$

Then $u \equiv 0$.

Corollary 1.7 *Let $1 \leq p < \infty$ and let $u \in L^{p+1}$ be a solution to*

$$\mathcal{L}u = |u|^{p-1}u \quad \text{in } \mathbb{R}^n.$$

Then $u \equiv 0$.

Remark 1.8 Theorem 1.1 does not hold, in general, if we assume $u \in L^\infty$ instead of $u \in L^p$ with $p < \infty$. Indeed, consider the Kolmogorov-type operator in $\mathbb{R}^3 = \mathbb{R}_x^2 \times \mathbb{R}_t$

$$\mathcal{L} = \partial_{x_1}^2 + \left(x_1 - \frac{1}{2}x_2\right) \partial_{x_1} + \left(\frac{1}{2}x_1 - x_2\right) \partial_{x_2} - \partial_t.$$

This operator satisfies (H1) and (H2), however, by a Priola and Zabczyk' s Theorem, it has a bounded solution in \mathbb{R}^3 which is not constant (see Remark 8.1 for details).

Remark 1.9 When \mathbb{G} is a homogeneous group and \mathcal{L} is homogeneous w.r.t. the dilations of \mathbb{G} , Theorem 1.1 and Theorem 1.2 follow from a general Liouville-type theorem of Geller [Gel83, Theorem 2]. We want to stress that Geller' s Theorem also implies L^∞ -Liouville property for \mathcal{L} (if \mathbb{G} and \mathcal{L} are homogeneous).

Remark 1.10 If the operator \mathcal{L} is homogeneous w.r.t. a group of dilations $(\delta_\lambda)_{\lambda>0}$ as in (1.2), then Theorem 1.1 and Theorem 1.2 hold only assuming hypothesis (H1).

This follows from a result of Xuebo who extended Geller's Theorem to homogeneous hypoelliptic operators, not necessarily left invariant on a Lie group (see [Luo97, Theorem 1]).

Remark 1.11 We want to explicitly remark that Geller's and Xuebo's Theorems do not apply to subsolutions.

Remark 1.12 We say that \mathcal{L} satisfies the *one-side Liouville property* if

$$\mathcal{L}u = 0 \text{ in } \mathbb{R}^n, \ u \geq 0 \implies u = \text{const. in } \mathbb{R}^n.$$

This property does not hold, in general, even for left translation invariant and homogenous operators. This is the case, e.g., of the classical heat operator in $\mathbb{R}^n = \mathbb{R}^{N+1} = \mathbb{R}^n_x \times \mathbb{R}_t$

$$\mathcal{H} := \Delta - \partial_t, \quad \Delta = \sum_{j=1}^n \partial_{x_j}^2,$$

which is hypoelliptic, invariant w.r.t. the euclidean translations and homogenous of degree two w.r.t. the dilations

$$\delta_\lambda : \mathbb{R}^{N+1} \longrightarrow \mathbb{R}^{N+1}, \quad \delta_\lambda(x, t) = (\lambda x, \lambda^2 t), \quad \lambda > 0.$$

The function $u(x, t) = \exp(x_1 + \dots + x_N + Nt)$ is a non-constant strictly positive solution to $\mathcal{H}u = 0$ in \mathbb{R}^{N+1} . Several classes of homogenous operators satisfying the one-side Liouville property have been presented in [KL05, KL06, KL07].

Remark 1.13 L^1 -Liouville Theorems for sub-Laplacians in suitable half spaces of stratified Lie groups in \mathbb{R}^n have been proved by Uguzzoni [Ugu99] and Kogoj [Kog15].

Remark 1.14 When $\mathcal{L} = \Delta$ is the classical Laplacian in \mathbb{R}^n , Theorem 1.5 is contained in [CDM08a, Theorem 4.5].

Remark 1.15 For some kind of L^p -Liouville Theorems for sub-Laplacians on stratified Lie groups we directly refer to the monograph [BLU07], Chapter 5, Section 5.8.

Remark 1.16 Liouville-type Theorems based on suitable representation formulae for both solutions and subsolutions to some classes of higher order systems are contained in [CDM08b].

The remaining part of the present paper is organized as follows. In Section 2 we introduce a representation formula which will play a crucial role in the proof of our Liouville-type Theorems. Some properties of the integral operators involved in the representation formula are proved in Section 3. Section 4 is devoted to the proofs of Theorems 1.1 and 1.2, while Theorem 1.3 and 1.4 are proved in Section 5 and 6, respectively. Section 7 contains the short proof of Theorem 1.5. Finally, in Section 8 we show some explicit examples of operators to which our results apply.

2 A representation formula

The assumptions $A(0) \geq 0$ and $(\text{tr } A(0)) > 0$ imply the existence of a index $j \in \{1, \dots, n\}$ such that $a_{jj}(0) > 0$. For simplicity of notation we assume

$$a_{11}(0) > 0.$$

For $0 < \varepsilon < R$, let us define

$$V = V_{\varepsilon, R} := D(Re_1, R + \varepsilon) \cap D(-Re_1, R + \varepsilon),$$

where $e_1 = (1, 0, \dots, 0)$ and $D(\alpha, r)$ denotes the Euclidean ball with center α and radius r . If R is sufficiently big and ε sufficiently smooth, V satisfied the so called non-characteristic exterior ball condition at any point of its boundary. As a consequence, for every $x_0 \in \partial V$ there exists a function $h(x_0, \cdot) \in C^2(\overline{V}, \mathbb{R})$ satisfying

$$\begin{aligned} h(x_0, \cdot) &> 0 \text{ in } \overline{V} \setminus x_0 \quad \text{and} \quad h(x_0, x_0) = 0, \\ \mathcal{L}_\varepsilon h(x_0, \cdot) &\leq -1 \text{ in } \overline{V} \text{ for every } \varepsilon \in [0, 1], \end{aligned}$$

where $\mathcal{L}_\varepsilon = \mathcal{L} + \varepsilon \Delta$ and Δ is the Laplace operator in \mathbb{R}^n .

The proof of this statement follows from very standard arguments (see e.g. [BLU07], pages 383, 384 and 387).

The existence of barrier functions $h(x_0, \cdot)$ implies the following Picone-type estimate: there exists a constant $C > 0$, independent of $\varepsilon \in [0, 1]$, such that

$$\sup_{\overline{V}} |u| \leq \sup_{\partial V} |u| + C \sup_{\overline{V}} |\mathcal{L}_\varepsilon u|, \quad \forall u \in C^2(\overline{V}, \mathbb{R}) \quad (2.1)$$

and for every $\varepsilon \in [0, 1]$.

We also have the following Picone Maximum Principle:
if $u \in C^2(V, \mathbb{R}) \cap C(\overline{V}, \mathbb{R})$ satisfies

$$\mathcal{L}_\varepsilon u \geq 0 \text{ in } V \quad \text{and} \quad u|_{\partial V} \leq 0,$$

then $u \leq 0$ in V (see, e.g. [Lan10]).

The hypoellipticity of \mathcal{L} , the estimate (2.1) and the boundary barrier functions $h(x_0, \cdot)$ allow to prove the solvability of the boundary value problem

$$\begin{cases} \mathcal{L}u = -f & \text{in } V \\ u|_{\partial V} = \varphi \end{cases} \quad (2.2)$$

with an *elliptic regularization procedure*.

Proposition 2.1 *For every $f \in C^\infty(\overline{V}, \mathbb{R})$ and for every $\varphi \in C(\partial V, \mathbb{R})$ the boundary value problem (2.2) has a unique solution $u \in C^\infty(V, \mathbb{R}) \cap C(\overline{V}, \mathbb{R})$. This solution satisfies the estimate*

$$\sup_V |u| \leq \sup_{\partial V} |\varphi| + C \sup_V |f|,$$

where $C > 0$ does not depend on u, φ and f . Moreover, if $f \geq 0$ and $\varphi \geq 0$, then $u \geq 0$.

Proof. It follows, along standard lines, as in [Bon69, Theorem 5.2] and [BLU07, pages 383–387]. \square

We denote by $G(f)$ the solution of (2.2) with $\varphi = 0$ and by $H(\varphi)$ the solution of (2.2) with $f = 0$. Then, letting $C_0(\overline{V}, \mathbb{R}) = \{u \in C(\overline{V}, \mathbb{R}) \mid u = 0 \text{ on } \partial V\}$, the operators

$$G : C^\infty(\overline{V}, \mathbb{R}) \longrightarrow C_0(\overline{V}, \mathbb{R})$$

and

$$H : C(\partial V, \mathbb{R}) \longrightarrow C^\infty(V, \mathbb{R}) \cap C(\overline{V}, \mathbb{R})$$

are linear, nonnegative and satisfy

$$\begin{aligned} \sup_{\overline{V}} |G(f)| &\leq C \sup_{\overline{V}} |f| \quad \forall f \in C^\infty(\overline{V}, \mathbb{R}), \\ \sup_{\overline{V}} |H(\varphi)| &\leq \sup_{\partial V} |\varphi|. \end{aligned}$$

Then, G can be continued to a linear, nonnegative and continuous operator, still denoted by G ,

$$G : C(\overline{V}, \mathbb{R}) \longrightarrow C_0(\overline{V}, \mathbb{R}).$$

Let us now consider the functionals

$$C(\overline{V}, \mathbb{R}) \ni f \longmapsto G(f)(0) \in \mathbb{R}$$

and

$$C(\partial V, \mathbb{R}) \ni \varphi \longmapsto H(\varphi)(0) \in \mathbb{R}.$$

They are linear, nonnegative and continuous. Then there exist two nonnegative Radon measures ν and μ , respectively on \overline{V} and ∂V such that

$$G(f)(0) = \int_{\overline{V}} f \, d\nu \quad \text{for every } f \in C(\overline{V}, \mathbb{R})$$

and

$$H(\varphi)(0) = \int_{\partial V} \varphi \, d\mu \quad \text{for every } \varphi \in C(\partial V, \mathbb{R}).$$

Then, the following proposition holds

Proposition 2.2 *For every $u \in C^2(\overline{V}, \mathbb{R})$ we have*

$$u(0) = \int_{\partial V} u \, d\mu - \int_{\overline{V}} \mathcal{L}u \, d\nu. \quad (2.3)$$

Proof. We first assume $u \in C^\infty(\overline{V}, \mathbb{R})$. Let us put $f = \mathcal{L}u$. By the very definition of G , the function $v := G(f)$ satisfies: $v \in C^\infty(V, \mathbb{R}) \cap C(\overline{V}, \mathbb{R})$ and

$$\begin{cases} \mathcal{L}u = -f = -\mathcal{L}u & \text{in } V, \\ v|_{\partial V} = 0. \end{cases}$$

Then, $\mathcal{L}(v + u) = 0$ and $(v + u)|_{\partial V} = u|_{\partial V}$, so that

$$v + u = H(u|_{\partial V}),$$

i.e.,

$$u = H(u|_{\partial V}) - G(\mathcal{L}u).$$

Hence

$$\begin{aligned} u(0) &= H(u|_{\partial V})(0) - G(\mathcal{L}u)(0) \\ &= \int_{\partial V} u d\mu - \int_{\overline{V}} \mathcal{L}u \, d\nu. \end{aligned}$$

Then (2.3) holds true if $u \in C^\infty(\overline{V}, \mathbb{R})$. On the other hand, if $u \in C^2(\overline{V}, \mathbb{R})$, there exists a sequence (u_n) , with $u_n \in C^\infty(\overline{V}, \mathbb{R})$, such that

$$\begin{aligned} u_n &\longrightarrow u \quad \text{uniformly on } \partial V \\ \text{and} \\ \mathcal{L}u_n &\longrightarrow \mathcal{L}u \quad \text{uniformly on } \overline{V}. \end{aligned}$$

On the other hand, for what we have already proved,

$$u_n(0) = \int_{\partial V} u_n d\mu - \int_{\overline{V}} \mathcal{L}u_n \, d\nu \quad \forall n \in \mathbb{N}.$$

Letting n go to infinity we obtain (2.3). \square

We are now ready to state and prove the main result of this section.

Theorem 2.3 *For every $v \in C^2(\mathbb{R}^n, \mathbb{R})$ we have*

$$v(x) = M(v)(x) - N(\mathcal{L}v)(x) \quad \forall x \in \mathbb{R}^n,$$

where,

$$M(u)(x) = \int_{\partial V} u(x \circ y) \, d\mu(y), \tag{2.4}$$

and

$$N(f)(x) = \int_{\overline{V}} f(x \circ y) \, d\nu(y). \tag{2.5}$$

Here \circ denotes the composition law of \mathbb{G} .

Proof. Let $x \in \mathbb{R}^n$ be fixed and consider the function

$$u : \mathbb{R}^n \longrightarrow \mathbb{R}, \quad u(y) = v(x \circ y).$$

Obviously $u \in C^2(\mathbb{R}^n, \mathbb{R})$ and, since \mathcal{L} is left translation invariant,

$$\mathcal{L}u(y) = (\mathcal{L}v)(x \circ y) \text{ for every } y \in \mathbb{R}^n.$$

Then, by Proposition 2.2,

$$\begin{aligned} v(x) = u(0) &= \int_{\partial V} u(y) \, d\mu(y) - \int_{\overline{V}} \mathcal{L}u(y) \, d\nu(y) \\ &= \int_{\partial V} v(x \circ y) \, d\mu(y) - \int_{\overline{V}} (\mathcal{L}v)(x \circ y) \, d\nu(y) \\ &= M(v)(x) - N(\mathcal{L}(v))(x). \end{aligned}$$

□

3 Some properties of the operators M and N

In this Section we prove some properties of the operators M and N defined in (2.4) and (2.5), respectively.

We start with the following lemma.

Lemma 3.1 *Let v be a continuous and L^1 -function in \mathbb{R}^n . Then $M(v) \in L^1(\mathbb{R}^n)$ and*

$$\int_{\mathbb{R}^n} v(x) \, dx = \int_{\mathbb{R}^n} M(v)(x) \, dx. \quad (3.1)$$

Proof. It follows from Fubini Theorem and the invariance of the Lebesgue measure on \mathbb{G} . Indeed:

$$\begin{aligned} \int_{\mathbb{R}^n} M(v)(x) \, dx &= \int_{\mathbb{R}^n} \left(\int_{\partial V} v(x \circ y) \, d\mu(y) \right) dx \\ &= \int_{\partial V} \left(\int_{\mathbb{R}^n} v(x \circ y) \, dx \right) d\mu(y) \\ &= \left(\int_{\mathbb{R}^n} v(x) \, dx \right) \left(\int_{\partial V} d\mu(y) \right). \end{aligned} \quad (3.2)$$

On the other hand, by Proposition 2.2 applied to the function $u \equiv 1$, we have

$$1 = \int_{\partial V} d\mu.$$

Using this information in (3.2), we obtain (3.1). □

Regarding the operator N we have:

Lemma 3.2 *Let $f \in C(\mathbb{R}^n, \mathbb{R})$. Then the following statements hold.*

- (i) $N(f) \geq 0$ if $f \geq 0$;
- (ii) if $f \geq 0$ and $N(f) \equiv 0$, then $f \equiv 0$;
- (iii) $N(f) \in C(\mathbb{R}^n, \mathbb{R})$.

Proof.

(i) It is obvious.

(ii) Let $f \geq 0$ and $N(f) \equiv 0$ in \mathbb{R}^n . Assume, by contradiction, $f \not\equiv 0$. Then, there exists $x_0 \in \mathbb{R}^n$ such that $f(x_0) > 0$. Since f is continuous, there exists an open set $\Omega \ni x_0$ such that $f(x) > 0$ for every $x \in \Omega$. It follows

$$f(z \circ y) > 0 \quad \forall z, y \in \mathbb{R}^n : z \circ y \in \Omega.$$

As a consequence, for every $z \in \mathbb{R}^n$ we have

$$0 = N(f)(z) = \int_{\overline{V}} f(z \circ y) \, d\nu(y) \geq \int_{\overline{V} \cap (z^{-1} \circ \Omega)} f(z \circ y) \, d\nu(y),$$

so that, since $f(z \circ y) > 0$ for every $y \in z^{-1} \circ \Omega$, we get

$$\nu(\overline{V} \cap (z^{-1} \circ \Omega)) = 0 \quad \forall z \in \mathbb{R}^n. \quad (3.3)$$

Since

$$\bigcup_{z \in \mathbb{R}^n} (z^{-1} \circ \Omega) = \mathbb{R}^n,$$

from (3.3) we obtain

$$\nu(\overline{V}) = 0.$$

As a consequence, by Proposition 2.2,

$$u(0) = \int_{\partial V} u \, d\mu \quad \forall u \in C^2(\overline{V}, \mathbb{R}).$$

In particular:

$$u(0) = 0 \quad \forall u \in C_0^\infty(V, \mathbb{R}),$$

which is absurd. This completes the proof of (ii).

(iii) Since f is continuous and \overline{V} is compact, for every $z_0 \in \mathbb{R}^n$ we have:

$$\sup_{y \in \overline{V}} |f(z \circ y) - f(z_0 \circ y)| \longrightarrow 0 \text{ as } z \longrightarrow z_0.$$

Then

$$\begin{aligned} N(f)(z_0) &= \int_{\overline{V}} f(z_0 \circ y) \, d\nu(y) = \int_{\overline{V}} \lim_{z \rightarrow z_0} f(z \circ y) \, d\nu(y) \\ &= \lim_{z \rightarrow z_0} \int_{\overline{V}} f(z \circ y) \, d\nu(y) \\ &= \lim_{z \rightarrow z_0} N(f)(z). \end{aligned}$$

This proves the continuity of $N(f)$.

□

4 Proof of Theorems 1.1 and 1.2

We start with the following elementary lemma.

Lemma 4.1 *Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be a C^2 -function and let $u \in C^2(\Omega, \mathbb{R})$, $\Omega \subseteq \mathbb{R}^n$, open. Then*

$$v := F(u)$$

is a real C^2 -function in Ω such that

$$\mathcal{L}v = F'(u)\mathcal{L}u + F''(u)|\nabla_A u|^2,$$

where

$$|\nabla_A u|^2 := \langle A\nabla u, \nabla u \rangle.$$

Proof. We show the elementary computations for reader convenience. We have:

$$\begin{aligned} \mathcal{L}v &= \operatorname{div}(A\nabla(F(u)) + \langle b, \nabla(F(u)) \rangle) \\ &= \operatorname{div}(F'(u)A\nabla u) + \langle b, \nabla u \rangle F'(u) \\ &= F'(u)(\operatorname{div}(A\nabla u) + \langle b, \nabla u \rangle) + F''(u)\langle A\nabla u, \nabla u \rangle \\ &= F'(u)\mathcal{L}u + F''(u)|\nabla_A u|^2. \end{aligned}$$

□

To prove our theorems we need another lemma.

Lemma 4.2 *Let $\Omega \subseteq \mathbb{R}^n$ be open and connected and let $v \in C^2(\Omega, \mathbb{R})$ be such that*

$$|\nabla_A u|^2 = 0 \quad \text{and} \quad \langle b, \nabla u \rangle = 0 \quad \text{in } \Omega. \quad (4.1)$$

Then $u = \text{const.}$ in Ω .

Proof. Let us denote by X_1, \dots, X_n the vector fields constructed with the columns of the matrix A , i.e.,

$$X_j = \sum_{k=1}^n a_{k,j} \partial_{x_k}, \quad j = 1, \dots, n.$$

Let us also put

$$X_0 = \sum_{k=1}^n b_k \partial_{x_k}.$$

Then, assumption (4.1) can be written as follows

$$X_j u = 0 \quad \text{in } \Omega \quad \text{for every } j = 1, \dots, n.$$

As a consequence,

$$Yu = 0 \quad \text{in } \Omega \quad \forall Y \in \operatorname{Lie}\{X_0, X_1, \dots, X_n\}. \quad (4.2)$$

On the other hand, since \mathcal{L} is hypoelliptic,

$$\text{rank Lie}\{X_0, X_1, \dots, X_n\}(x) = N \quad \forall x \in \Omega^*,$$

where Ω^* is an open subset of Ω such that $\overline{\Omega^*} \subseteq \Omega$. Then, for every $x \in \Omega^*$ and for every $i \in \{1, \dots, n\}$ there exist $Y_1, \dots, Y_n \in \text{Lie}\{X_0, X_1, \dots, X_n\}$ and real constants $c_i^{(1)}(x), \dots, c_i^{(n)}(x)$ such that

$$\partial_{x_i} = \sum_{j=1}^n c_i^{(j)} Y_j.$$

Thus, from (4.2), we obtain

$$\partial_{x_i} u(x) = 0 \quad \forall x \in \Omega^*, \forall i = 1, \dots, n.$$

Since $u \in C^1(\Omega, \mathbb{R})$ and Ω^* is dense in Ω , this implies

$$\nabla u \equiv 0 \text{ in } \Omega.$$

Then u is constant in Ω . □

A key tool in the proof of our Liouville theorems is given by the following proposition.

Proposition 4.3 *Let $v \in C^2(\mathbb{R}^n, \mathbb{R})$. If*

$$v \in L^1(\mathbb{R}^n) \quad \text{and} \quad \mathcal{L}v \geq 0 \text{ in } \mathbb{R}^n$$

then

$$\mathcal{L}v = 0 \text{ in } \mathbb{R}^n.$$

Proof. The representation formula of Theorem 2.3 gives

$$v = M(v) - N(\mathcal{L}v).$$

Now, being $v \in L^1(\mathbb{R}^n)$, Lemma 3.1 implies $M(v) \in L^1(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} v(x) dx = \int_{\mathbb{R}^n} M(v)(x) dx.$$

Then, $N(\mathcal{L}v) \in L^1(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} N(\mathcal{L}v)(x) dx = 0.$$

Since $\mathcal{L}v \geq 0$, by Lemma 3.2-(i), $N(\mathcal{L}v) \geq 0$ and the last integral identity implies $N(\mathcal{L}v) = 0$ a.e. in \mathbb{R}^n . On the other hand, $\mathcal{L}v$ is continuous and, by Lemma 3.2-(iii), $N(\mathcal{L}v)$ is continuous. Therefore

$$N(\mathcal{L}v) = 0 \text{ in } \mathbb{R}^n,$$

so that, by Lemma 3.2-(ii),

$$\mathcal{L}v = 0 \quad \text{in } \mathbb{R}^n.$$

□

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $u \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, be a solution to $\mathcal{L}u = 0$ in \mathbb{R}^n and assume, by contradiction, $u \not\equiv 0$. Define

$$v := F(u)$$

with

$$F : \mathbb{R} \longrightarrow \mathbb{R}, \quad F(t) = (\sqrt{1+t^2} - 1)^p.$$

Elementary computations show that $F \in C^2(\mathbb{R}, \mathbb{R})$,

$$0 \leq F(t) = \left(\frac{t^2}{\sqrt{1+t^2} + 1} \right)^p \leq |t|^p \quad (4.3)$$

and

$$F''(t) > 0 \quad \forall t \neq 0. \quad (4.4)$$

Since \mathcal{L} is hypoelliptic the function u is smooth and, by Lemma 4.1, we have

$$\mathcal{L}(F(u)) = F'(u)\mathcal{L}u + F''(u)|\nabla_A u|^2 = F''(u)|\nabla_A u|^2. \quad (4.5)$$

On the other hand $F(u) \in L^1(\mathbb{R}^n)$, since, by (4.3),

$$0 \leq F(u) \leq |u|^p \quad \text{and} \quad u \in L^p(\mathbb{R}^n).$$

From Proposition 4.3 it follows

$$\mathcal{L}(F(u)) = 0 \quad \text{in } \mathbb{R}^n,$$

so that, keeping in mind (4.5),

$$F''(u)|\nabla_A u|^2 = 0.$$

Then, by (4.4),

$$|\nabla_A u|^2 = 0 \quad \text{in } \Omega_0 := \{x \in \mathbb{R}^n \mid u(x) \neq 0\}. \quad (4.6)$$

Ω_0 is an open subset of \mathbb{R}^n which is nonempty because we are assuming $u \not\equiv 0$. Since $A \geq 0$ from (4.6) we obtain

$$A\nabla u = 0 \quad \text{in } \Omega_0,$$

so that

$$\operatorname{div}(A\nabla u) = 0 \quad \text{in } \Omega_0.$$

As a consequence, keeping in mind that $\mathcal{L}u = 0$,

$$\langle b, \nabla u \rangle = 0 \quad \text{in } \Omega_0. \quad (4.7)$$

Identities (4.6) and (4.7) and Lemma 4.2 imply

$$u = \text{const. on every connected component of } \Omega_0.$$

Let O be one of the connected component of Ω_0 . If $O = \mathbb{R}^n$ we have $u = \text{const.}$ in \mathbb{R}^n , so that, since $u \in L^p(\mathbb{R}^n)$, $u = 0$ in O . If $O \neq \mathbb{R}^n$, then $\partial O \neq \emptyset$ and $u = 0$ on ∂O . Being $u = \text{const.}$ in O , this implies $u = 0$ in O .

Thus, we have proved that $u = 0$ on every connected component of Ω_0 , that is

$$u = 0 \text{ in } \Omega_0,$$

in contradiction with the definition of Ω_0 . \square

The previous argument can be easily adapted to prove Theorem 1.2.

Proof of Theorem 1.2. Let $u \geq 0$ be a solution to $\mathcal{L}u = 0$ such that $u^p \in L^1(\mathbb{R}^n)$ for a suitable $p \in]0, 1[$. Define

$$v := F(u)$$

with

$$F : [0, \infty[\longrightarrow \mathbb{R}, \quad F(t) = (1 + t)^p - 1.$$

The function F is smooth,

$$0 \leq F(t) \leq t^p \quad \text{for every } t \geq 0$$

and

$$F''(t) < 0 \quad \text{for every } t \geq 0.$$

Since \mathcal{L} is hypoelliptic the function u is smooth. By Lemma 4.1, we have

$$\mathcal{L}(F(u)) = F'(u)\mathcal{L}u + F''(u)|\nabla_A u|^2 = F''(u)|\nabla_A u|^2.$$

Hence $\mathcal{L}(F(u)) \leq 0$. On the other hand $F(u) \in L^1(\mathbb{R}^n)$, since

$$0 \leq F(u) \leq u^p \quad \text{and} \quad u^p \in L^1(\mathbb{R}^n).$$

Then by Proposition 4.3,

$$\mathcal{L}(F(u)) = 0 \text{ in } \mathbb{R}^n,$$

so that,

$$F''(u)|\nabla_A u|^2 = 0 \text{ in } \mathbb{R}^n.$$

Being $F''(u(x)) < 0$ at any point, from this last identity we obtain

$$|\nabla_A u|^2 = 0 \text{ in } \mathbb{R}^n.$$

Now, arguing as in the proof of Theorem 1.1, we obtain

$$u = \text{const. in } \mathbb{R}^n,$$

so that, since $u^p \in L^1(\mathbb{R}^n)$,

$$u = 0 \text{ in } \mathbb{R}^n.$$

5 Proof of Theorem 1.3

We start by proving Theorem 1.3 in the case of u smooth. Thus, let $u \in C^\infty(\mathbb{R}^n, \mathbb{R})$ be such that

$$\mathcal{L}u \geq 0 \text{ in } \mathbb{R}^n \quad \text{and} \quad u \in L^p(\mathbb{R}^n), \quad 1 \leq p < \infty.$$

We want to prove that $u \leq 0$ in \mathbb{R}^n . Arguing by contradiction, we assume

$$\Omega_0 := \{x \in \mathbb{R}^n \mid u(x) > 0\} \neq \emptyset.$$

Let us consider the function

$$F : \mathbb{R} \longrightarrow \mathbb{R}, \quad F(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ ((1 + t^4)^{\frac{1}{4}} - 1)^p & \text{if } t > 0. \end{cases}$$

Then:

- (i) $F \in C^2(\mathbb{R}, \mathbb{R})$;
- (ii) F is increasing and convex;
- (iii) $F''(t) > 0$ if $t > 0$;
- (iv) $0 \leq F(t) \leq t^p$ for every $t > 0$.

We let

$$v := F(u).$$

From the properties of F we get that $v \in C^2(\mathbb{R}, \mathbb{R})$ and $0 \leq v \leq C|u|^p$, so that $v \in L^1(\mathbb{R}^n)$. Moreover

$$\mathcal{L}v = F'(u)\mathcal{L}u + F''(u)|\nabla_A u|^2 \geq 0.$$

Then, by Proposition 4.3, $\mathcal{L}v = 0$ hence, in particular

$$F''(u)|\nabla_A u|^2 = 0 \text{ in } \mathbb{R}^n. \tag{5.1}$$

Since $F''(u(x)) > 0$ for every $x \in \Omega_0$, (5.1) implies

$$|\nabla_A u|^2 = 0 \text{ in } \Omega_0.$$

Starting from this identity and arguing as in the proof of Theorem 1.1 (see Section 4), we obtain

$$u = 0 \text{ in } \Omega_0,$$

in contradiction with the definition of Ω_0 . This proves Theorem 1.3 in the case u smooth. We will remove this restriction by using the following lemma.

Lemma 5.1 *Let $u \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, be such that $\mathcal{L}u \geq 0$ in \mathbb{R}^n in the weak sense of distributions. Then there exists a sequence of functions (u_k) such that*

- (i) $u_k \in C^\infty(\mathbb{R}^n, \mathbb{R})$ for every $k \in \mathbb{N}$;
- (ii) $\mathcal{L}u_k \geq 0$ in \mathbb{R}^n for every $k \in \mathbb{N}$;
- (iii) $\mathcal{L}u_k \in L^p(\mathbb{R}^n)$ in \mathbb{R}^n for every $k \in \mathbb{N}$;
- (iv) $u_k \longrightarrow u$ in $L^1_{\text{loc}}(\mathbb{R}^n)$.

Proof. The proof is quite standard. We give it in the details for reader convenience. Let $\varepsilon > 0$ be fixed and choose a function $\mathcal{J}_\varepsilon \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$ such that $\text{supp } \mathcal{J}_\varepsilon \subseteq D(0, \varepsilon)$, $\int_{\mathbb{R}^n} \mathcal{J}_\varepsilon(y) dy = 1$, $\mathcal{J}_\varepsilon \geq 0$. Define

$$\hat{u}_\varepsilon : \mathbb{R}^n \longrightarrow \mathbb{R}, \hat{u}_\varepsilon(x) = \int_{\mathbb{R}^n} u(y \circ x) \mathcal{J}_\varepsilon(y) dy.$$

A change of variable in the integral gives

$$\hat{u}_\varepsilon(x) = \int_{\mathbb{R}^n} u(z) \mathcal{J}_\varepsilon(z \circ x^{-1}) dz,$$

showing that $u_\varepsilon \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$. Moreover, for every $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\varphi \geq 0$, we have (\mathcal{L}^* = formal adjoint of \mathcal{L})

$$\begin{aligned} \int_{\mathbb{R}^n} \hat{u}_\varepsilon(x) \mathcal{L}^* \varphi(x) dx &= \int_{\mathbb{R}^n} \left(\mathcal{J}_\varepsilon(y) \int_{\mathbb{R}^n} u(y \circ x) \mathcal{L}^* \varphi(x) dx \right) dy \\ &= \int_{\mathbb{R}^n} \mathcal{J}_\varepsilon(y) \left(\int_{\mathbb{R}^n} u(z) (\mathcal{L}^* \varphi)(y^{-1} \circ z) dz \right) dy \\ &= (\text{since } \mathcal{L}^* \text{ is left translation invariant}) \\ &\quad \int_{\mathbb{R}^n} \mathcal{J}_\varepsilon(y) \left(\int_{\mathbb{R}^n} u(z) (\mathcal{L}^* (\varphi(y^{-1} \circ z))) dz \right) dy. \end{aligned}$$

Since $\mathcal{L}u \geq 0$ in the weak sense of distributions the inner integral at the last right hand side is ≥ 0 . Therefore

$$\int_{\mathbb{R}^n} \hat{u}_\varepsilon(x) \mathcal{L}^* \varphi(x) dx \geq 0 \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n), \varphi \geq 0.$$

Since \hat{u}_ε is smooth we can integrate by parts at the left hand side, getting

$$\int_{\mathbb{R}^n} \mathcal{L} \hat{u}_\varepsilon(x) \varphi(x) dx \geq 0 \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n), \varphi \geq 0.$$

Thus

$$\mathcal{L} \hat{u}_\varepsilon \geq 0 \text{ in } \mathbb{R}^n.$$

Moreover

$$\begin{aligned} \int_{\mathbb{R}^n} |\hat{u}_\varepsilon(x)|^p dx &\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\hat{u}(y \circ x)|^p \mathcal{J}_\varepsilon(y) dy \right) dx \\ &= \int_{\mathbb{R}^n} \mathcal{J}_\varepsilon(y) \left(\int_{\mathbb{R}^n} |\hat{u}(y \circ x)|^p dx \right) dy \\ &= \int_{\mathbb{R}^n} |\hat{u}(x)|^p dx. \end{aligned}$$

Hence

$$\hat{u}_\varepsilon \in L^p(\mathbb{R}^n).$$

Finally, for every fixed compact set $K \subseteq \mathbb{R}^n$,

$$\begin{aligned} \int_K |\hat{u}_\varepsilon(x) - u(x)| \, dx &\leq \int_{\mathbb{R}^n} \mathcal{J}_\varepsilon(y) \left(\int_K |u(y \circ x) - u(x)| \, dx \right) \, dy \\ &\leq \sup_{y \in D(0, \varepsilon)} \int_K |u(y \circ x) - u(x)| \, dx \\ &:= \omega_K(u, \varepsilon). \end{aligned}$$

On the other hand, being $u \in L^1_{\text{loc}}(\mathbb{R}^n)$,

$$\omega_K(u, \varepsilon) \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0.$$

Therefore, a sequence $(u_k)_{k \in \mathbb{N}}$ satisfying (i)–(iv) can be constructed by choosing $u_k = \hat{u}_{\frac{1}{k}}$. \square

We are ready to complete the proof of Theorem 1.3.

Let $u \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, be such that $\mathcal{L}u \geq 0$ in the weak sense of distributions. By the previous Lemma there exists a sequence $(u_k)_{k \in \mathbb{N}}$ of smooth functions such that $\mathcal{L}u_k \geq 0$, $u_k \in L^p(\mathbb{R}^n)$ and $u_k \longrightarrow u$ as $k \longrightarrow \infty$ in $L^1_{\text{loc}}(\mathbb{R}^n)$. For what proved in the first part of this section, $u_k \leq 0$ in \mathbb{R}^n for every $k \in \mathbb{N}$. This implies $u \leq 0$ a.e. in \mathbb{R}^n , and completes the proof.

6 Proof of the Theorem 1.4

We need several prerequisites. First of all, the assumptions on \mathcal{L} and \mathbb{G} imply the existence of a fundamental solution

$$\Gamma : \mathbb{R}^n \longrightarrow [0, \infty]$$

such that

$$(i) \quad \Gamma \in L^1_{\text{loc}}(\mathbb{R}^n), \Gamma \in C^\infty(\mathbb{R}^n \setminus \{0\}) \text{ and}$$

$$\Gamma(x) \longrightarrow 0 \quad \text{as } x \longrightarrow \infty;$$

$$(ii) \quad \int_{\mathbb{R}^n} \Gamma(x) \mathcal{L}^* \varphi(x) \, dx = -\varphi(0) \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n);$$

$$(iii) \quad \Gamma(\delta_\lambda(x)) = \lambda^{2-Q} \Gamma(x) \quad \forall x \in \mathbb{R}^n \setminus \{0\}, \forall \lambda > 0$$

(see e.g. [Fol75]).

Given $f \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$ we let

$$\Gamma * f(x) := \int_{\mathbb{R}^n} \Gamma(y^{-1} \circ x) f(y) \, dy = \int_{\mathbb{R}^n} \Gamma(z) f(x \circ z^{-1}) \, dz.$$

From (i) it follows:

$$\Gamma * f \in C^\infty(\mathbb{R}^n, \mathbb{R}) \quad \text{and} \quad \Gamma * f(x) \longrightarrow 0 \quad \text{as} \quad x \longrightarrow \infty.$$

Moreover, as an elementary computation shows,

$$\int_{\mathbb{R}^n} (\Gamma * f)(x) \mathcal{L}^* \varphi(x) \, dx = - \int_{\mathbb{R}^n} f(y) \varphi(y) \, dy \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n, \mathbb{R}).$$

Hence,

$$\mathcal{L}(\Gamma * f) = -f.$$

The operator \mathcal{L} satisfies the following Maximum Principle on \mathbb{R}^n .

Proposition 6.1 *Let $u \in C^2(\mathbb{R}^n, \mathbb{R})$ be such that*

$$\mathcal{L}u \geq 0 \quad \text{in } \mathbb{R}^n \quad \text{and} \quad \limsup_{x \longrightarrow \infty} u(x) \leq 0. \quad (6.1)$$

Then $u \leq 0$ in \mathbb{R}^n .

Proof. We have already remarked the existence of a bounded neighborhood V of the origin on which \mathcal{L} satisfies the Picone Maximum Principle:

if $v \in C^2(\mathbb{R}^n, \mathbb{R}) \cap C(\overline{V}, \mathbb{R})$, $\mathcal{L}v \geq 0$ in V and $v|_{\partial V} \leq 0$, then $v \leq 0$ in V

(see Section 2). Let $\varepsilon > 0$ be arbitrarily fixed and define

$$v_\lambda(x) := u(\delta_\lambda(x)) - \varepsilon, \quad x \in \mathbb{R}^n, \quad \lambda > 0.$$

The second assumption in (6.1) implies the existence of $\lambda_\varepsilon > 0$ such that

$$v_\lambda(x) \leq 0 \quad \forall x \in \partial V, \quad \forall \lambda > \lambda_\varepsilon.$$

Moreover, since \mathcal{L} is δ_λ -homogeneous of degree two:

$$\mathcal{L}v_\lambda(x) = \lambda^2(\mathcal{L}u)(\delta_\lambda(x)) \geq 0 \quad \forall x \in \mathbb{R}^n, \quad \forall \lambda > 0.$$

As a consequence, by Picone Maximum Principle on V ,

$$v_\lambda \leq 0 \quad \text{in } V \quad \forall \lambda > \lambda_\varepsilon,$$

which means

$$u(\delta_\lambda(x)) \leq \varepsilon \quad \forall x \in V, \quad \forall \lambda > \lambda_\varepsilon. \quad (6.2)$$

On the other hand, since V is a neighborhood of the origin

$$\bigcup_{\lambda > \lambda_\varepsilon} \delta_\lambda(V) = \mathbb{R}^n.$$

Together with (6.2) this implies

$$u \leq \varepsilon \quad \text{in } \mathbb{R}^n, \quad \forall \varepsilon > 0.$$

Hence $u \leq 0$ in \mathbb{R}^n . □

As an application of the previous proposition, we prove the positivity of Γ .

Corollary 6.2 *It is*

$$\Gamma(x) \geq 0 \quad \forall x \in \mathbb{R}^n \setminus \{0\}. \quad (6.3)$$

Proof. For every $f \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$, $f \leq 0$, we have

$$\mathcal{L}(\Gamma * f) = -f \geq 0 \quad \text{and} \quad \Gamma * f|_\infty = 0.$$

Then, by the previous theorem, $\Gamma * f \leq 0$ in \mathbb{R}^n . In particular

$$\Gamma * f(0) = \int_{\mathbb{R}^n} \Gamma(y^{-1})f(y) dy \leq 0 \quad \forall f \in C_0^\infty(\mathbb{R}^n, \mathbb{R}), f \leq 0,$$

from which (6.3) follows, since Γ is smooth out of the origin. \square

Note. If we agree to let

$$\Gamma(0) := \liminf_{x \rightarrow 0} \Gamma(x),$$

then $\Gamma : \mathbb{R}^n \rightarrow [0, \infty]$ is *lower semicontinuous*.

Given a function $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$, $f \geq 0$, we agree to let

$$\Gamma * f = \lim_{m \rightarrow \infty} \Gamma * f_m \quad (6.4)$$

where $f_m = f\varphi_m$ and $\varphi_m \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$ satisfies

$$\varphi_m = 1 \text{ in } D(0, m), \varphi_m = 0 \text{ in } \mathbb{R}^n \setminus D(0, m+1) \quad \text{and} \quad 0 \leq \varphi_m \leq 1.$$

Since $0 \leq \varphi_m \leq \varphi_{m+1}$, the sequence $(f_m)_{m \in \mathbb{N}}$ is nonnegative and increasing, so that (6.4) is meaningful by Beppo Levi Theorem. It is also easy to recognize that the left hand side of (6.4) is independent of the choice of the sequence $(\varphi_m)_{m \in \mathbb{N}}$.

The proof of Theorem 1.4 relies on the following Lemma 6.3 and Lemma 6.4.

Lemma 6.3 *Let $u \in C^\infty(\mathbb{R}^n, \mathbb{R})$ be such that*

$$u \leq 0 \quad \text{and} \quad \mathcal{L}u \geq 0 \text{ in } \mathbb{R}^n.$$

Then

$$u = -\Gamma * \mathcal{L}u + \hat{w} \quad \text{a.e. in } \mathbb{R}^n,$$

where $\hat{w} \in C^\infty(\mathbb{R}^n, \mathbb{R})$, $\mathcal{L}\hat{w} = 0$ and $\hat{w} \leq 0$ in \mathbb{R}^n .

Proof. Let $(\varphi_m)_{m \in \mathbb{N}}$ be a sequence as above and let

$$f_m = (\mathcal{L}u)\varphi_m.$$

Then $(f_m)_{m \in \mathbb{N}}$ is an increasing sequence of nonnegative $C_0^\infty(\mathbb{R}^n, \mathbb{R})$ -functions such that

$$f_m \nearrow f := \mathcal{L}u.$$

Define

$$w_m := u + \Gamma * f_m. \quad (6.5)$$

Then $w_m \in C^\infty(\mathbb{R}^n, \mathbb{R})$ and

$$\mathcal{L}(w_m) = \mathcal{L}u - f_m = f(1 - \varphi_m).$$

Hence,

$$\mathcal{L}(w_m) \geq 0 \text{ in } \mathbb{R}^n \quad \text{and} \quad \mathcal{L}(w_m) = 0 \text{ in } D(0, m). \quad (6.6)$$

Moreover, since $u \leq 0$,

$$\limsup_{x \rightarrow \infty} w_m(x) \leq \limsup_{x \rightarrow \infty} \Gamma * f_m(x) = 0.$$

The Maximum Principle of Proposition 6.1 gives

$$w_m \leq 0 \text{ in } \mathbb{R}^n.$$

On the other hand, (w_m) is increasing so that

$$w := \lim_{m \rightarrow \infty} w_m$$

is well defined and satisfies

$$w_1 \leq w \leq 0. \quad (6.7)$$

This implies $w \in L^1_{\text{loc}}(\mathbb{R}^n)$ and, keeping in mind the second statement in (6.6),

$$\int_{\mathbb{R}^n} w \mathcal{L}^* \varphi \, dx = \lim_{m \rightarrow \infty} \int_{\mathbb{R}^n} w_m \mathcal{L}^* \varphi \, dx = 0$$

for every $\varphi \in C_0^\infty(\mathbb{R}^n)$. As a consequence, since \mathcal{L} is hypoelliptic, there exists $\hat{w} \in C^\infty(\mathbb{R}^n, \mathbb{R})$ such that

$$\hat{w} = w \text{ a.e.} \quad \text{and} \quad \mathcal{L}\hat{w} = 0 \text{ in } \mathbb{R}^n.$$

Obviously, \hat{w} also satisfies

$$w_1 \leq \hat{w} \leq 0 \text{ in } \mathbb{R}^n.$$

Letting m go to infinity in (6.5) we obtain

$$w = u + \Gamma * f, \quad (6.8)$$

so that

$$u = -\Gamma * \mathcal{L}u + \hat{w} \text{ a.e..}$$

□

Note. (6.8) and (6.7) imply $\Gamma * f(x) < \infty$ for every $x \in \mathbb{R}^n$. We complete our prerequisites by proving next lemma.

Lemma 6.4 *Let $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$, $f \geq 0$ and such that*

$$\Gamma * f \in L^p(\mathbb{R}^n) \quad \text{for a suitable } p \in \left[1, 1 + \frac{2}{Q-2}\right]. \quad (6.9)$$

Then $f \equiv 0$.

Moreover, for every $f \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$,

$$\Gamma * f \in L^p(\mathbb{R}^n) \quad \text{for every } p \in \left]1 + \frac{2}{Q-2}, \infty\right[.$$

Proof. For every $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ define

$$\|x\| = \sum_{j=1}^n |x_j|^{\frac{1}{\sigma_j}}$$

where the σ_j 's are the exponents related to the dilation δ_λ in (1.2). Then $x \rightarrow \|x\|$ is δ_λ -homogeneous of degree one:

$$\delta_\lambda(x) = \lambda \|x\| \quad \forall x \in \mathbb{R}^n, \forall \lambda > 0.$$

Let $\Sigma := \{x \in \mathbb{R}^n \mid \|x\| = 1\}$. Since $\Gamma \geq 0$ and, obviously, $\Gamma \not\equiv 0$, there exists a (relatively) open subset Σ_0 of Σ such that

$$\Gamma(x) \geq 2\sigma > 0 \quad \forall x \in \Sigma_0,$$

for a suitable $\sigma > 0$. Then, there exists $\rho > 0$ such that

$$\Gamma(y^{-1} \circ x) \geq \sigma \quad \forall x \in \Sigma_0, \forall y \in \mathbb{R}^n : \|y\| < \rho. \quad (6.10)$$

Consider the *open δ_λ -cone*

$$K := \{\delta_\lambda(x) \mid x \in \Sigma_0, \lambda > 0\}.$$

Now, assume by contradiction $f \not\equiv 0$. Then there exist a bounded open set $B \neq \emptyset$ such that $f(x) \geq \varepsilon$ for every $x \in B$ and a suitable $\varepsilon > 0$. As a consequence, for every $x \in \mathbb{R}^n$,

$$\begin{aligned} \Gamma * f(x) &\geq \int_B \Gamma(y^{-1} \circ x) f(y) dy \geq \varepsilon \int_B \Gamma(y^{-1} \circ x) dy \\ &= \varepsilon \|x\|^{2-Q} \int_B \Gamma \left(\left(\delta_{\frac{1}{\|x\|}}(y) \right)^{-1} \circ \delta_{\frac{1}{\|x\|}}(x) \right) dy. \end{aligned} \quad (6.11)$$

On the other hand, for a suitable $M > 1$, $\|\delta_{\frac{1}{\|x\|}}(y)^{-1}\| = \frac{1}{\|x\|} \|y^{-1}\| < \rho$ for every $y \in B$ and $\|x\| \geq M$. Moreover $\delta_{\frac{1}{\|x\|}}(x) \in \Sigma_0$ if $x \in K$. Then, by (6.10),

$$\Gamma \left(\left(\delta_{\frac{1}{\|x\|}}(y) \right)^{-1} \circ \delta_{\frac{1}{\|x\|}}(x) \right) \geq \sigma \quad \forall x \in K, \|x\| \geq M \quad \text{and} \quad \forall y \in B.$$

Using this estimate in (6.11) we get

$$\Gamma * f(x) \geq \varepsilon \sigma \|x\|^{2-Q} \quad \forall x \in K, \|x\| \geq M.$$

Therefore:

$$\begin{aligned} \int_{\mathbb{R}^n} (\Gamma * f(x))^p &\geq (\varepsilon \sigma)^p \int_{K \cap \{\|x\| \geq M\}} \|x\|^{p(2-Q)} dx \\ &= (\varepsilon \sigma)^p \sum_{k=1}^{\infty} \int_{K \cap \{M^k \leq \|x\| < M^{k+1}\}} \|x\|^{p(2-Q)} dx \\ &= (\text{using the change of variable } x = \delta_{M^k}(y)) \\ &\quad (\varepsilon \sigma)^p \left(\int_{K \cap \{1 \leq \|y\| \leq M\}} \|y\|^{p(2-Q)} dy \right) \sum_{k=1}^{\infty} M^{k(p(2-Q)+Q)} \\ &= \infty \quad \text{if } p \leq \frac{Q}{Q-2} = 1 + \frac{2}{Q-2}. \end{aligned}$$

This contradicts the assumption (6.9) and proves the first part of the lemma.

To prove the second part we argue as follows. If $f \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$ then $f \in L^q(\mathbb{R}^n)$ for every $q \in]1, \frac{Q}{2}[$. As a consequence, since Γ is δ_λ -homogeneous of degree $2 - Q$, hence $\Gamma \in L_{\text{deb}}^r$ with $r = \frac{Q-2}{Q}$, one has

$$\Gamma * f \in L^p(\mathbb{R}^n) \quad \text{with} \quad \frac{1}{p} = \frac{1}{r} + \frac{1}{q} - 1 = \frac{1}{q} - \frac{2}{Q}.$$

Since we can choose any $q \in]1, \frac{Q}{2}[$, this gives

$$\Gamma * f \in L^p(\mathbb{R}^n) \quad \forall p \in \left] 1 + \frac{2}{Q-2}, \infty \right[.$$

□

We are ready to prove Theorem 1.4. Let $u \in L^p(\mathbb{R}^n)$, with $p \in [1, 1 + \frac{2}{Q-2}]$, be such that $\mathcal{L}u \geq 0$ in \mathbb{R}^n in the weak sense of distributions. We have to prove that $u = 0$ a.e. in \mathbb{R}^n .

We will prove the theorem on the extra assumption $u \in C^\infty(\mathbb{R}^n, \mathbb{R})$. This restriction can be removed with an approximation argument like the one used in the proof on Theorem 1.3. By Theorem 1.3 we already know that $u \leq 0$ so that, from Lemma 6.3, we get

$$u = -\Gamma * \mathcal{L}u + \hat{w} \quad \text{a.e. in } \mathbb{R}^n,$$

where $\hat{w} \in C^\infty(\mathbb{R}^n, \mathbb{R})$, $\mathcal{L}\hat{w} = 0$ and $\hat{w} \leq 0$. Then, since $\mathcal{L}u \geq 0$, $u \leq \hat{w} \leq 0$. Hence, being $u \in L^p(\mathbb{R}^n)$ with $1 \leq p \leq 1 + \frac{2}{Q-2}$, we also have

$$\hat{w} \in L^p(\mathbb{R}^n) \quad \text{for a suitable } p \in \left[1, 1 + \frac{2}{Q-2} \right].$$

Theorem 1.1 implies $\hat{w} \equiv 0$, so that

$$u = -\Gamma * \mathcal{L}u, \quad \text{in } \mathbb{R}^n.$$

From Lemma 6.4 it follows $\mathcal{L}u = 0$, hence $u = 0$ in \mathbb{R}^n . This completes the proof of the first part of Theorem 1.4. The second part directly follows from the second part of Lemma 6.4.

7 Proof of Theorem 1.5

Let $u \in C^2(\mathbb{R}^n, \mathbb{R})$ be a classical solution to the equation (1.5) and define

$$v = F(u)$$

with F given by (1.4). Then $v \in C^2(\mathbb{R}^n, \mathbb{R})$ and, by Lemma 4.1,

$$\begin{aligned} \mathcal{L}v &= F'(u)\mathcal{L}u + F''(u)|\nabla_A u|^2 \\ &= (f(u))^2 + F''(u)|\nabla_A u|^2 \geq 0. \end{aligned}$$

Since $v \in L^1(\mathbb{R}^n)$, by Proposition 4.3, it follows $\mathcal{L}v = 0$, i.e.,

$$(f(u))^2 + F''(u)|\nabla_A u|^2 = 0 \quad \text{in } \mathbb{R}^n.$$

Being $F'' = f' \geq 0$, from this identity we obtain $f(u) = 0$, hence $u = 0$ in \mathbb{R}^n .

8 Some examples

8.1

Let $\mathbb{G} = (\mathbb{R}^n, \circ, \delta_\lambda)$ be a stratified Lie group¹ and let X_1, \dots, X_p be a basis of the first layer of its Lie algebra. The *sub-Laplacian*

$$\mathcal{L} = \sum_{j=1}^p X_j^2 \tag{8.1}$$

is left translation invariant on \mathbb{G} and δ_λ -homogeneous of degree two. Then Theorems 1.1–1.5 and Corollaries 1.6 and 1.7 apply to \mathcal{L} in (8.1).

8.2

Let \mathbb{G} as above and consider in $\mathbb{R}^{n+1} := \mathbb{R}_x^n \times \mathbb{R}_t$ the *heat-type operator*

$$\mathcal{H} := \sum_{j=1}^p X_j^2 - \partial_t. \tag{8.2}$$

¹We refer to the monograph [BLU07] for notions and results recalled in this section.

This operator is left translation invariant and homogeneous of degree two with respect to the stratified Lie group

$$\hat{\mathbb{G}} = \mathbb{G} \oplus \mathbb{R} = (\mathbb{R}^{n+1}, \hat{\circ}, \hat{\delta}_\lambda),$$

where $\hat{\circ}$ and $\hat{\delta}_\lambda$ are defined as follow

$$\begin{aligned} (x, t) \hat{\circ} (x', t') &= (x \circ x', t + t') \\ \hat{\delta}_\lambda(x, t) &= (\delta_\lambda(x), \lambda^2 t). \end{aligned}$$

The *homogeneous dimension* of $\hat{\mathbb{G}}$ is

$$\hat{Q} = Q + 2$$

being Q the homogeneous dimension of \mathbb{G} .

To the operator \mathcal{H} in (8.2) Theorems 1.1–1.5 and Corollaries 1.6 and 1.7 apply.

8.3

Let us consider in $\mathbb{R}^{n+1} := \mathbb{R}_x^n \times \mathbb{R}_t$ the *Kolmogorov-type operators*

$$\mathcal{L} = \operatorname{div}(A\nabla) + \langle Bx, \nabla \rangle - \partial_t, \quad (8.3)$$

where A and B are constant $n \times n$ real matrices, A symmetric and ≥ 0 .

Define

$$E(s) := \exp(-sB), \quad s \in \mathbb{R}.$$

Then the operator \mathcal{L} in (8.3) is left translation invariant on the Lie group

$$\mathbb{K} = (\mathbb{R}^{n+1}, \circ)$$

with composition law

$$(x, t) \circ (x', t') = (x' + E(t')x, t + t'). \quad (8.4)$$

The Lebesgue measure is both left and right invariant on \mathbb{K} if and only if

$$\operatorname{trace}(B) = 0. \quad (8.5)$$

Moreover, if we assume

$$C(t) := \int_0^t E(s) A E^T(s) ds > 0 \quad \forall t > 0, \quad (8.6)$$

then \mathcal{L} is hypoelliptic (see e.g. [LP94], see also [BLU07, Sections 4.1.3, 4.3.4]). Then, under the assumptions (8.5) and (8.6), Theorems 1.1, 1.2, 1.3, 1.5 and Corollaries 1.6 and 1.7 apply to \mathcal{L} in (8.3). On the other hand, if the matrix B takes the particular block form fixed in [LP94], then there exists a family of dilations $(\delta_\lambda)_{\lambda>0}$ in \mathbb{R}^{n+1} making

$$\mathbb{K} = (\mathbb{R}^{n+1}, \circ, \delta_\lambda)$$

a homogeneous Lie group and the operator \mathcal{L} in (8.3) is δ_λ -homogeneous of degree two. Therefore under this extra assumption, also Theorem 1.4 apply to \mathcal{L} .

Remark 8.1 Consider the *stationary* counterpart of \mathcal{L} in (8.3), i.e., the degenerate Ornstein-Uhlenbeck operator

$$\mathcal{L} = \operatorname{div}(A\nabla) + \langle Bx, \nabla \rangle. \quad (8.7)$$

Priola and Zabczyk in [PZ04, Theorem 3.1] proved that \mathcal{L}_0 has the L^∞ -Liouville property if and only if

$$\operatorname{Re}(\lambda) \leq 0 \text{ for every } \lambda \text{ eigenvalue of } B.$$

Then, if B has an eigenvalue with real part strictly positive, there exists a bounded solution v to $\mathcal{L}_0 v = 0$ in \mathbb{R}^n which is not constant. Hence

$$u(x, t) = v(x)$$

is a bounded nonconstant solution to

$$\mathcal{L}u = 0 \text{ in } \mathbb{R}^{n+1}.$$

Thus \mathcal{L} does not have the L^∞ -Liouville property. An explicit example is given by the operator \mathcal{L} in Remark 1.8, which can be written as in (8.3) by taking

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & -1 \end{pmatrix}.$$

The eigenvalues of B are $-\frac{\sqrt{3}}{2}$ and $\frac{\sqrt{3}}{2}$. Moreover

$$\operatorname{trace}(B) = 0,$$

so that the Lebesgue measure is both left and right invariant w.r.t. the composition law in (8.4). Finally, (8.6) can be verified by a direct computation or simply recognizing that, letting

$$X = \partial_{x_1} \quad \text{and} \quad Y = \left(x_1 - \frac{1}{2}x_2\right) \partial_{x_1} + \left(\frac{1}{2}x_1 - x_2\right) \partial_{x_2} - \partial_t,$$

the hypoellipticity Hörmander rank condition

$$\operatorname{Lie}\{X, Y\}(x, t) = 3 \quad \forall (x, t) \in \mathbb{R}^3$$

is satisfied. Then: *the operator \mathcal{L} in Remark 1.8 has the L^p -Liouville property for every $p \in [0, \infty[$, but it has not the L^∞ -Liouville property.*

8.4

Let us consider the operator in \mathbb{R}^n , $n \geq 3$,

$$\mathcal{L} = \partial_{x_1}^2 + u_1(x_1)\partial_{x_2} + \cdots + u_{n-1}(x_1)\partial_{x_n} \quad (8.8)$$

where $\{u_1, \dots, u_{n-1}\}$ is a real fundamental system of solutions of the ODE $P(u) = 0$, being

$$P(u) := u^{(n-1)} + a_{n-2}u^{(n-2)} + \dots + a_1u^{(1)} + a_0u,$$

with $a_0, \dots, a_{n-2} \in \mathbb{R}$ and $a_{n-2} = 1$. In [BL12] it is proved that \mathcal{L} is hypoelliptic and left translation invariant on a Lie group $\mathbb{G}(P) = (\mathbb{R}^n, \circ)$ which, in [BL12], is called *P-group*. Due to the condition $a_{n-2} = 1$, the Lebesgue measure is both left and right invariant on $\mathbb{G}(P)$. Then Theorems 1.1, 1.2, 1.3, 1.5 and Corollaries 1.6 and 1.7 apply to the operator \mathcal{L} in (8.8).

Remark 8.2 An explicit example of an operator \mathcal{L} as in (8.8) is given by the *Mumford operator*

$$\mathcal{M} := \partial_{x_1}^2 + \cos x_1 \partial_{x_2} + \sin x_1 \partial_{x_3} \quad \text{in } \mathbb{R}^3$$

which is left invariant on $\mathbb{G}(P)$ with P given by

$$P(u) = u'' + u.$$

Remark 8.3 Theorems 1.1, 1.2, 1.3, 1.5 and Corollaries 1.6 and 1.7 also apply to the *evolution counterpart* of the operator \mathcal{L} in (8.8), i.e., to

$$\mathcal{L} - \partial_t = \partial_{x_1}^2 + u_1(x_1)\partial_{x_2} + \dots + u_{n-1}(x_1)\partial_{x_n} - \partial_t.$$

This operator is hypoelliptic and left translation invariant w.r.t. the group composition law

$$(x, t) \hat{\circ} (x', t') = (x \circ x', t + t'),$$

where \circ is the composition law of $\mathbb{G}(P)$. More formally, $\mathcal{L} - \partial_t$ is left invariant on

$$\mathbb{G}(P) \oplus \mathbb{R}.$$

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