Infinitely many solutions for the stationary Kirchhoff problems involving the fractional p–Laplacian

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Abstract

The aim of this paper is to establish the multiplicity of weak solutions for a Kirchhoff type problem driven by a fractional $p-$Laplacian operator with homogeneous Dirichlet boundary conditions:

$$\begin{cases} M\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy\right)(-\Delta)_p^s u(x) = f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ with Lipshcitz boundary $\partial\Omega$, $(-\Delta)_p^s$ is the fractional $p-$Laplacian operator with $0 < s < 1 < p < N$ such that $sp < N$, M is a continuous function and f is a Carathéodory function satisfying Ambrosetti–Rabinowitz type condition. When f satisfies suplinear growth condition, we obtain the existence of a sequence of nontrivial solutions by using the symmetric mountain pass theorem, while f satisfies sublinear growth condition, we obtain infinitely many pairs of nontrivial solutions by applying the Krasnoselskii genus theory. Our results cover the degenerate case in the fractional setting: the Kirchhoff function M can be zero at zero.

Keywords: Kirchhoff type problem; Fractional p–Laplacian; Infinitely many solutions; Symmetric mountain pass theorem; Genus theory.

2010 MSC: 35R11, 35A15, 47G20.

1 Introduction

In recent years, a great attention has been focused on the study of problems involving fractional and non-local operators. This type of operators arises in a quite natural way in many different applications, such as, continuum mechanics, phase transition phenomena, population dynamics, minimal surfaces and game theory, as they are the typical outcome of stochastically stabilization of Lévy processes, see [1, 7, 21] and the references therein. The literature on non–local operators and their applications is

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interesting, for example, we refer the interested reader to [3, 23, 24, 25] and the references therein. For the basic properties of fractional Sobolev spaces, we refer the reader to [14].

In this paper we investigate the existence of infinitely many solutions for elliptic problems of Kirchhoff type involving fractional $p-$Laplacian operator. More precisely, we consider

$$\begin{cases} M\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy\right)(-\Delta)_p^s u(x) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where $0 < s < 1 < p < N$ such that $sp < N$, $\Omega \subset \mathbb{R}^N$ is an open bounded set with Lipschitz boundary $\partial\Omega$, $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a continuous function, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $(-\Delta)_p^s$ is the fractional $p-$Laplace operator which (up to normalization factors) may be defined as

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}} dy$$

for $x \in \mathbb{R}^N$, see for example [17, 44] and the references therein for further details on the fractional $p-$Laplacian operator. Here $B_\varepsilon(x)$ denotes the ball in $\mathbb{R}^N$ of radius $\varepsilon > 0$ at the center $x \in \mathbb{R}^N$.

Obviously, when $p = 2$ and $M \equiv 1$, problem (1.1) becomes the fractional Laplacian problem

$$\begin{cases} (-\Delta)^s u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.2}$$

One typical feature of problem (1.2) is the nonlocality, in the sense that the value of $(-\Delta)^s u(x)$ at any point $x \in \Omega$ depends not only on the values of u on Ω, but actually on the entire space $\mathbb{R}^N$. The functional framework that takes into account problem (1.2) with Dirichlet boundary condition was introduced in [35]. In [36], *Servadei* and *Valdinoci* obtained the existence of nontrivial weak solutions of problem (1.2) by using the Mountain Pass Theorem. See also [8, 35, 37, 38] and the references therein for related discussions.

Very recently, *Fiscella* and *Valdinoci* in [15] proposed firstly a stationary Kirchhoff variational equation (1.1) which models the *nonlocal* aspect of the tension arising from nonlocal measurements of the fractional length of the string. Indeed, problem (1.1) is a fractional version of a model, the so–called Kirchhoff equation, introduced by Kirchhoff [19]. More precisely, Kirchhoff established a model given by the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{p_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0 \tag{1.3}$$

where ρ, p_0, h, E, L are constants, which extends the classical D'Alambert wave equation by considering the effects of the changes in the length of the strings during the vibrations. Especially, the Kirchhoff equation (1.3) contains a nonlocal coefficient $p_0/h + (E/2L)\int_0^L |\frac{\partial u}{\partial x}|^2 dx$, which depends on the average (1/L)$\int_0^L |\frac{\partial u}{\partial x}|^2 dx$ of the kinetic energy $|\frac{\partial u}{\partial x}|^2$ on $[0, L]$ and hence the equation is no longer a pointwise identity. Moreover, nonlocal boundary problems like equation 1.3 can be used to model several physical and biological systems where u describes a process, which depend on the average of itself, such as the population density, see [10]. It is worth pointing out that equation (1.3) received much attention only after Lions [20] proposed an abstract framework to the problem. For some motivation in the physical background for the fractional kirchhoff equation, we refer to [15, Appendix A]. Also, see [3, 27, 30] for some recent results in this direction.

Inspired by the above works, we suppose that $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a continuous function satisfying the following condition:

$$\begin{aligned}&\text{there exists } \theta > 0 \text{ such that } M(t)t \leq \theta \mathscr{M}(t) \ \text{ for all } t \in [0,\infty), \quad \text{where } \mathscr{M}(t) = \int_0^t M(\tau)d\tau,\\ &\text{and for all } \sigma > 0 \text{ there exists } \kappa = \kappa(\sigma) > 0 \text{ such that } M(t) \geq \kappa \ \text{ for all } t \geq \sigma.\end{aligned} \tag{M}$$

A very special Kirchhoff function M is given by

$$M(t) = a + bt^{m-1}, \quad a, \ b \geq 0, \quad a + b > 0, \quad t \geq 0,$$

and

$$m \begin{cases} \in (1,\infty) & \text{if } b > 0,\\ = 1 & \text{if } b = 0.\end{cases}$$

When M is this type, problem (1.1) is said to be non-degenerate if $a > 0$ and $b \geq 0$, while it is called degenerate if $a = 0$ and $b > 0$. On this subject, we would like to quote some recent works, see [6, 13, 27, 28, 29, 30, 31, 40, 43] for non-degenerate Kirchhoff-type problems and [2, 11, 22, 32, 41, 42] for degenerate Kirchhoff-type problems. It is worthy mentioning that, in order to investigate the existence of solutions to fractional problems of Kirchhoff type, the authors in [15, 27, 30] assumed that M is an increasing function on $\mathbb{R}_0^+$. However, it is national to enlarge the effective range of the Kirchhoff function M, for this some researchers are dedicated to dropping monotonicity assumption on M and furthermore also covering the interesting case $M(0) = 0$, for example, we refer to [2, 32, 42] for some updated results in the fractional Laplacian setting. In this paper, the degenerate case of problem (1.1) is covered in the fractional p–Laplacian context. More precisely, here we suppose that M satisfies assumption (M). Under hypothesis (M), we can also deal with cases in which M is not monotone as $M(t) = (1 + t)^k + (1 + t)^{-1}$ for $t \geq 0$, with $0 < k < 1$.

Next we assume that the nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying:

$$f \text{ is odd in } \xi, \text{ i.e. } f(x,-\xi) = -f(x,\xi) \text{ for all } x \in \Omega \text{ and } \xi \in \mathbb{R}; \tag{f_1}$$

there exist $a_1 > 0$ and $1 < q < p_s^*$ such that

$$|f(x,\xi)| \leq a_1(1 + |\xi|^{q-1}) \ \text{ a.e. } x \in \Omega, \ \xi \in \mathbb{R}; \tag{f_2}$$

there exist $\mu > \theta p$, $c_0 > 0$ and $r > 0$ such that for a.e. $x \in \Omega$ and $\xi \in \mathbb{R}$, $|\xi| > r$,

$$\mu F(x,\xi) \leq \xi f(x,\xi) \ \text{ and } \inf_{x\in\Omega} F(x,r) = c_0 > 0 \tag{f_3}$$

where $F(x,\xi) = \int_0^\xi f(x,\tau)d\tau$;

$$\lim_{\xi\to 0} \frac{F(x,\xi)}{|\xi|^{\theta p}} = 0 \ \text{ uniformly for a. e. } x \in \Omega. \tag{f_4}$$

Note that assumption (f_3) is not the usual Ambrosetti–Rabinowitz condition, since here we suppose that $\mu > \theta p$. This difference is caused by the Kirchhoff function M in problem (1.1). As a model for f we can take $f(x,\xi) = a(x)|\xi|^{q-2}\xi$, with $0 \leq a \in L^\infty(\Omega)$ and $1 < q < \infty$.

Now we give the definition of weak solutions for problem (1.1).

Definition 1.1. We say that $u \in W_0$ is a weak solution of problem (1.1), if

$$M\left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy\right) \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}dxdy$$
$$= \int_\Omega f(x,u(x))\varphi(x)dx,$$

for any $\varphi \in W_0$, where space W_0 will be introduced in Section 2.

The first result of our paper is concerned with the case that $\theta p < q < p_s^*$.

Theorem 1.1. *Suppose that $0 < \theta < N/(N-ps)$, M satisfies (M) and f satisfies (f_1)–(f_4). If $\theta p < q < p_s^*$, then problem (1.1) has infinitely many nontrivial solutions $\{u_k\}$ with unbounded energy.*

In the second part of this paper, we are devoted to the existence of infinitely many solutions for problem (1.1) as the nonlinearity satisfies sublinear growth condition, that is $1 < q < \alpha p$. To this aim, we first assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{aligned}&\text{there exist } a_2 > 0,\ \gamma_0 \in (1,p) \text{ and an open set } \Omega_0 \subset \Omega \text{ such that}\\ &\qquad F(x,\xi) \geq a_2|\xi|^{\gamma_0} \ \text{ for all } (x,\xi) \in \Omega_0 \times \mathbb{R},\end{aligned} \tag{f_5}$$

then we obtain the following result.

Theorem 1.2. *Suppose that M satisfies (M), f satisfies (f_1), (f_2) and (f_5). If $1 < q < p$, Then problem (1.1) has infinitely many pairs of nontrivial solutions $\{\pm u_k : k = 1, 2, \cdots\}$ with negative energy.*

Furthermore, if we replace the assumption (M) with the stronger condition, that is, $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a continuous function and satisfies the (polynomial growth) condition:

$$m_1 t^\alpha \leq \mathscr{M}(t) \leq m_2 t^\alpha \ \text{ for any } t \in \mathbb{R}_0^+, \tag{M_0}$$

where $0 < m_1 \leq m_2 < \infty$ are constants and exponent $1 < \alpha < \infty$, then we have the following desired result.

Theorem 1.3. *Suppose that M satisfies (M_0), f satisfies (f_1), (f_2) and (f_5) with $1 < \gamma_0 < \alpha p$. If $1 < q < \alpha p$, then problem (1.1) has infinitely many pairs of nontrivial weak solutions $\{\pm u_k : k = 1, 2, \cdots\}$ with negative energy.*

Remark 1.1. a) Consider the function $M(t) = t^{\theta-1}$ for $t \geq 0$ with $\theta > 1$, if we take $\alpha = \theta$ in (M_0) and apply (M), then the existence result for the case $1 < q < \theta p$ is obtained, see Example 4.1 for further details. From the point of view, similarly to the Laplacian case, it is reasonable to say that the condition $1 < q < \theta p$ characterizes the problem (1.1) as sublinear and the condition $\theta p < q < p_s^*$ characterizes the problem (1.1) as superlinear. For the superlinear case, many important and interesting results have been established , see for example [3, 15, 30] and the references therein. To our best knowledge, there are few papers to deal with the sublinear case for the fractional Kirchhoff problems.

b) The condition (M_0) introduced by the authors in [4] was used to investigate the existence of infinitely many solutions for the $p(x)$–Kirchhoff type equation via genus theory.

c) In [11], by using a variant of symmetric mountain pass theorem of Ambrosetti and Rabinowitz, *Colasuonno* and *Pucci* obtained the existence of infinitely many solutions for a class of *possibly degenerate* $p(x)$–polyharmonic Kirchhoff equations of elliptic type as the nonlinearity satisfies suplinear growth condition. From this point of view, Theorem 1.1 can be viewed as a complete extension of their result to the fractional Laplacian setting. A natural question arises: what about if the nonlinearity satisfies sublinear growth condition? Theorem 1.2 and Theorem 1.3 are in this attempt to makes some contribution.

Last but not least, concerning the existence of infinitely many solutions for elliptic problems related to our study, we would like to give a short description. Using the symmetric mountain pass theorem, Molica Bisci in [26] obtained the existence of infinite many solutions for problem (1.1) with $M = 1$, see also [45] for an application of the (variant) symmetric mountain pass theorem to a class of Kirchhoff-Schrödinger-Poisson system. With the help of the (variant) Fountain Theorem, infinitely many solutions for a Kirchhoff type equation involving a nonlocal operator in bounded domains were studied in [13]. Applying the Fountain Theorem, the authors in [5] established the existence of infinite many solutions for some kinds of superlinear fractional Laplacian problems, see also [40] for recent results obtained by the Fountain Theorem and the dual Fountain Theorem. In [39], Shen and Qian considered the existence of infinite many solutions by combining the Fountain Theorem with the symmetric mountain pass theorem. By using Krasnoselskii's genus theory, Pucci et al. in [32] proved the existence of infinite many solutions for the Kirchhoff type equations involving the fractional $p-$Laplacian in the whole space, see also [33] for a related result exploited by the genus theory.

This paper is organized as follows. In Section 2, we give some necessary definitions and properties of space W_0. In Section 3, by using the symmetric mountain pass theorem, we establish the existence of infinitely many nontrivial solutions for problem (1.1). In Section 4, by applying Krasnoselskii's genus theory, we obtain the existence of infinitely many nontrivial solutions for problem (1.1) .

2 Variational framework

In this section we first recall the variational framework for problem (1.1), in which most of results can be referred to [16, 43]. It is worth mentioning that the functional setting was firstly introduced by Servadei and Valdinoci in [35, 36] as $p = 2$.

Let $0 < s < 1 < p < \infty$ be real numbers with $sp < N$, and let p_s^* be the fractional Sobolev critical exponent defined by $p_s^* = Np/(N - sp)$. In the following, we denote $Q = \mathbb{R}^{2N} \backslash \mathcal{O}$, where

$$\mathcal{O} = \mathcal{C}(\Omega) \times \mathcal{C}(\Omega) \subset \mathbb{R}^{2N},$$

and $\mathcal{C}(\Omega) = \mathbb{R}^N \backslash \Omega$. W is a linear space of Lebesgue measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that the restriction to Ω of any function u in W belongs to $L^p(\Omega)$ and

$$\iint_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dxdy < \infty.$$

The space W is equipped with the norm

$$\|u\|_W = \|u\|_{L^p(\Omega)} + \left(\iint_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dxdy \right)^{\frac{1}{p}}.$$

It is readily seen that $\| \cdot \|_W$ is a norm on W and $C_0^\infty(\Omega) \subset W$ (see [43, Lemma 2.1]). We shall work in the closed linear subspace

$$W_0 = \left\{ u \in W : u(x) = 0 \text{ a.e. in } \mathbb{R}^N \backslash \Omega \right\}.$$

Lemma 2.1. (see [43, Lemma 2.3]) *The following two assertions hold:*
(1) *there exists a positive constant* $C_0 = C_0(N, p, s)$ *such that for any* $v \in W_0$ *and* $1 \le q \le p_s^*$

$$\|v\|_{L^q(\Omega)}^p \le C_0 \iint_Q \frac{|v(x) - v(y)|^p}{|x - y|^{N+ps}} dxdy;$$

(2) *there exists a constant* $C_1 = C_1(N, p, s, \Omega)$ *such that for any* $v \in W_0$

$$\iint_Q \frac{|v(x) - v(y)|^p}{|x - y|^{N+ps}} dxdy \le \|v\|_W^p \le C_1 \iint_Q \frac{|v(x) - v(y)|^p}{|x - y|^{N+ps}} dxdy.$$

Remark 2.1. By (2), we get an equivalent norm on W_0 defined as

$$\|v\|_{W_0} = \left(\iint_Q \frac{|v(x) - v(y)|^p}{|x - y|^{N+ps}} dxdy \right)^{\frac{1}{p}}, \quad \textit{for each } v \in W_0.$$

Note that $(W_0, \| \cdot \|_{W_0})$ is a uniformly convex Banach space, see [43, Lemma 2.4], and hence W_0 is a reflexive Banach space.

Lemma 2.2. (see [43, Lemma 2.5]) *Let* $\{v_j\}$ *be a bounded sequence in* W_0*. Then, there exists* $v \in L^\nu(\mathbb{R}^N)$ *with* $v = 0$ *a.e. in* $\mathbb{R}^N \backslash \Omega$ *such that up to a subsequence,*

$$v_j \to v \ \text{ strongly in } L^\nu(\Omega), \quad \text{as } j \to \infty,$$

for any $\nu \in [1, p_s^*)$*.*

For $u \in W_0$, we define

$$J(u) = \frac{1}{p} \mathscr{M} \left(\iint_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dxdy \right), \quad H(u) = \int_\Omega F(x, u) dx,$$

and

$$I(u) = J(u) - H(u).$$

Obviously, the energy functional $I : W_0 \to \mathbb{R}$ associated with problem (1.1) is well defined.

Lemma 2.3. (see [43, Lemma 3.2]) *Let* (M) *hold. Then the functional* $J \in C^1(W_0, \mathbb{R})$ *and*

$$\langle J'(u), v \rangle = M \left(\|u\|_{W_0}^p \right) \iint_Q \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+ps}} dxdy,$$

for all $u, \ v \in W_0$*. Moreover, for each* $u \in W_0$*,* $J'(u) \in W_0^*$*, where* W_0^* *denotes the dual space of* W_0*.*

Lemma 2.4. (see [43, Lemma 3.1]) *If* f *satisfies assumption* (f_1)*, then the functional* $H \in C^1(W_0, \mathbb{R})$ *and*

$$\langle H'(u), v \rangle = \int_\Omega f(x, u) v dx \ \text{ for all } u, \ v \in W_0.$$

Combining Lemma 2.3 and Lemma 2.4, we get that $I \in C^1(W_0, \mathbb{R})$ and

$$\langle I'(u), v \rangle = M \left(\|u\|_{W_0}^p \right) \iint_Q \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+ps}} dxdy - \int_\Omega f(x, u) v dx,$$

for all $u, \ v \in W_0$. Obviously, the critical points of functional I are weak solutions of problem (1.1).

3 Proof of Theorem 1.1

Theorem 3.1. (see [11, Theorem 2.2]) *Let X be a real infinite dimensional Banach space and $K \in C^1(X)$ a functional satisfying the* (PS) *condition as well as the following three properties:*
(1) $K(0) = 0$ *and there exist two constant $\rho, \alpha > 0$ such that $K|_{\partial B_\rho} \geq \alpha$;*
(2) K *is even;*
(3) *for all finite dimensional subspaces $\overline{X} \subset X$ there exists $R = R(\overline{X}) > 0$ such that*

$$K(u) \leq 0 \quad \text{for all } u \in X \setminus B_R(\overline{X}),$$

where $B_R(\overline{X}) = \{u \in \overline{X} : \|u\| \leq R\}$. Then K posses an unbounded sequence of critical values characterized by a minimax argument,

Lemma 3.1. *Let (M) and (f_2) be satisfied. Then the functional $I \in C^1(W_0, \mathbb{R})$ is weakly lower semi–continuous.*

Proof. First we notice that the map $v \mapsto \|v\|^p_{W_0}$ is lower semi–continuous in the weak topology of W_0 and $\mathscr{M}$ is a nondecreasing continuous function, so that $v \mapsto \mathscr{M}(\|v\|^p_{W_0})$ is lower semi–continuous in the weak topology of W_0. Indeed, we define a functional $\psi : W_0 \to \mathbb{R}$ as

$$\psi(v) = \iint_Q \frac{|v(x) - v(y)|^p}{|x - y|^{N+ps}} dxdy.$$

Similar to Lemma 2.3, we obtain $\psi \in C^1(W_0)$ and

$$\langle \psi'(w), v\rangle = p \iint_Q \frac{|w(x) - w(y)|^{p-2}(w(x) - w(y))(v(x) - v(y))}{|x - y|^{N+ps}} dxdy,$$

for all $w,\ v \in W_0$. Notice that

$$\begin{aligned}\psi\left(\frac{w+v}{2}\right) &\leq 2^{-1} \iint_Q \frac{|w(x) - w(y)|^p}{|x - y|^{N+ps}} + \frac{|v(x) - v(y)|^p}{|x - y|^{N+ps}} dxdy \\ &= \frac{1}{2}\psi(w) + \frac{1}{2}\psi(v).\end{aligned}$$

Thus, ψ is a convex functional in W_0. Furthermore, ψ is subdifferentiable and the subdifferential denoted by $\partial\psi$ satisfies $\partial\psi(u) = \{\psi'(u)\}$ for each $u \in W_0$ (see [12]). Now, let $\{v_n\} \subset W_0, v \in W_0$ with $v_n \rightharpoonup v$ weakly in W_0 as $n \to \infty$. Then it follows from the definition of subdifferential that

$$\psi(v_n) - \psi(v) \geq \langle \psi'(v), v_n - v\rangle.$$

Hence, we obtain $\psi(v) \leq \liminf_{n\to\infty} \psi(v_n)$, i.e., the map $v \mapsto \|v\|^p_{W_0}$ is weakly lower semi–continuous.

Let $u_n \rightharpoonup u$ weakly in W_0. By assumption (f_2) and Lemma 2.2, up to a subsequence, $u_n \to u$ strongly in $L^q(\Omega)$. Without loss of generality, we assume that $u_n \to u$ a.e. in Ω. Assumption (f_2) implies

$$F(x,t) \leq a_1 \left(|t| + q^{-1}|t|^q\right) \leq 2a_1(|t|^q + 1).$$

Thus, for any measurable subset $\Omega_1 \subset \Omega$,

$$\int_U |F(x, u_n)|dx \leq 2a_1 \int_{\Omega_1} |u_n|^q dx + 2a_1|\Omega_1|.$$

By $1 < q < p_s^*$, Lemma 2.1 and the Hölder inequality, we have

$$\int_{\Omega_1} |F(x,u_n)|dx \leq 2a_1 \||u_n|^q\|_{L^{\frac{p_s^*}{q}}(\Omega_1)} \|1\|_{L^{\frac{p_s^*}{p_s^*-q}}(\Omega_1)} + 2a_1|\Omega_1|$$
$$\leq 2a_1 C\|u_n\|_{W_0}^q |\Omega_1|^{\frac{p_s^*-q}{p_s^*}} + 2a_1|\Omega_1|.$$

Similar to Lemma 2.4, we obtain

$$\lim_{n\to\infty} \int_\Omega F(x,u_n)dx = \int_\Omega F(x,u)dx.$$

Thus the functional H is weakly continuous. Further, we get that I is weakly lower semi–continuous. □

Lemma 3.2. *Suppose that M satisfies (M) and f satisfies (f_1)–(f_4). Then there exist $\rho > 0$ and $\alpha > 0$ such that*

$$I(u) \geq \alpha > 0,$$

for any $u \in W_0$ with $\|u\|_{W_0} = \rho$.

Proof. By (f_4), for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $|F(x,u)| \leq \varepsilon|u|^{\theta p}$ for all $x \in \Omega$ and $0 \leq u < \delta$. On the other hand, by (f_2) for $u > \delta$ we get

$$|F(x,u)| \leq \int_0^u |f(x,\xi)|d\xi \leq a_1\left(u + \frac{u^q}{q}\right) \leq c_\varepsilon|u|^q,$$

where $c_\varepsilon = 2a_1 \max\{\delta^{1-q}, 1/q\}$. Hence, being $F(x,\cdot)$ even, we have shown that for all $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$|F(x,u)| \leq \varepsilon|u|^{\theta p} + c_\varepsilon|u|^q \quad \text{for all } (x,u) \in \Omega \times \mathbb{R}. \tag{3.1}$$

Furthermore, (M) implies that $M(t) > 0$ for all $t > 0$ and

$$\mathscr{M}(t) \geq \mathscr{M}(1)t^\theta \quad \text{for all } t \in [0,1]. \tag{3.2}$$

By Lemma 2.1–1, (3.1) and (3.2), we obtain for all $u \in W_0$ with $\|u\|_{W_0} \leq 1$

$$I(u) \geq \frac{1}{p}\mathscr{M}\left(\iint_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy\right) - \varepsilon\int_\Omega |u(x)|^{\theta p}dx - c_\varepsilon\int_\Omega |u(x)|^q dx$$
$$\geq \frac{1}{p}\mathscr{M}(1)\|u\|_{W_0}^{\theta p} - \varepsilon C_0^\theta\|u\|_{W_0}^{\theta p} - c_\varepsilon C_0^{\frac{q}{p}}\|u\|_{W_0}^q. \tag{3.3}$$

Choosing $\varepsilon = \mathscr{M}(1)/(2pC_0^\theta)$ in (3.3), we get

$$I(u) \geq \frac{\mathscr{M}(1)}{2p}\|u\|_{W_0}^{\theta p} - c_\varepsilon C_0^{\frac{q}{p}}\|u\|_{W_0}^q$$
$$\geq \|u\|_{W_0}^{\theta p}\left(\frac{\mathscr{M}(1)}{2p} - c_\varepsilon C_0^{\frac{q}{p}}\|u\|_{W_0}^{q-\theta p}\right).$$

Then for all $u \in W_0$ with $\|u\|_{W_0} = \rho$, let $\rho \in (0,1)$ small enough so that $\mathscr{M}(1)/(2p) - c_\varepsilon C_0^{q/p}\rho^{q-\theta p} > 0$. Thus the lemma is proved by taking $\alpha = \rho^{\theta p}\left(\mathscr{M}(1)/(2p) - c_\varepsilon C_0^{q/p}\rho^{q-\theta p}\right)$. □

Lemma 3.3. *Suppose that M satisfies (M) and f satisfies (f_1)–(f_4). Then for any finite dimensional subspace E of W_0 there exists $R_0 = R_0(E) > 0$ such that*

$$I(u) \le 0 \quad \text{for all } u \in W_0 \setminus B_{R_0}(E),$$

where $B_{R_0}(E) = \{u \in E : \|u\|_{W_0} < R_0\}$.

Proof. First, we get by assumption (M) that

$$\mathscr{M}(t) \le \mathscr{M}(1)t^{\theta}, \tag{3.4}$$

for any $t \ge 1$. For all $u > r$, (f_3) yields that

$$\frac{\partial(F(x,\xi)/\xi^{\mu})}{\partial \xi} \ge 0 \quad \text{for a.e } x \in \Omega \text{ and all } \xi \ge r.$$

This means that $F(x,\xi)/\xi^{\mu}$ is nondecreasing with respect to $\xi \in [r,\infty)$ for a.e. $x \in \Omega$. From which we can get that for each $u > r$

$$F(x,u) \ge \frac{F(x,r)}{r^{\mu}}u^{\mu} \ge \frac{c_0}{r^{\mu}}u^{\mu}.$$

By (f_1), the function $f(x,\cdot)$ is odd in $\mathbb{R}$, so that $F(x,\cdot)$ is even. Therefore

$$F(x,u) \ge \frac{c_0}{r^{\mu}}u^{\mu} \quad \text{for a.e. } x \in \Omega \text{ and all } |u| > r.$$

Note that (f_2) gives that for $|u| \le r$

$$\overline{C} = \sup_{x\in\Omega}[F(x,u) - \frac{c_0}{r^{\mu}}u^{\mu}] < \infty.$$

Hence, we have shown that for all $x \in \Omega$ and $u \in \mathbb{R}$

$$F(x,u) \ge \frac{c_0}{r^{\mu}}|u|^{\mu} - \overline{C}.$$

Let E be a fixed finite dimensional subspace of W_0. For any $u \in E$ with $\|u\|_{W_0} = 1$, and for all $t \ge 1$ we get

$$I(tu) \le \frac{1}{p}\mathscr{M}(t^p) - \frac{c_0 C_E^{\mu}}{r^{\mu}}t^{\mu} + \overline{C}|\Omega|,$$

where $C_E > 0$ such that $\|u\|_{L^{\mu}(\Omega)} \ge C_E\|u\|_{W_0}$ for all $u \in E$. Moreover, by (3.4) and $\mu > \theta p$, we have

$$I(tu) \le \frac{1}{p}\mathscr{M}(1)t^{\theta p} - \frac{c_0 C_E^{\mu}}{r^{\mu}}t^{\mu} + \overline{C}|\Omega| \quad \to -\infty, \quad \text{as } t \to \infty.$$

Therefore, as $R \to \infty$,

$$\sup_{\|u\|_{W_0}=R,\ u\in E} I(u) = \sup_{\|u\|_{W_0}=1,\ u\in E} I(Ru) \to -\infty.$$

Hence, there exists $R_0 > 0$ so large such that $I(u) \le 0$ for all $u \in E$, with $\|u\|_{W_0} = R$ and $R \ge R_0$. This ends the proof. $\square$

Definition 3.1. Let X be a real Banach space and $K : X \to \mathbb{R}$ be a functional of class $C^1(X)$. A sequence $\{u_n\}_n \subset X$ is said to be a Palais-Smale sequence of for K, (PS) sequence for shortness, if $\{K(u_n)\}_n$ is bounded and $K'(u_n) \to 0$ as $n \to \infty$. The functional K satisfies (PS) condition, if any (PS) sequence admits a convergent subsequence.

Lemma 3.4. *Suppose that M satisfies (M) and f satisfies (f_2)–(f_3). Then the functional I satisfies* (PS) *condition.*

Proof. For any sequence $\{u_n\} \subset W_0$ such that $I(u_n)$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$, there exits $C > 0$ such that $|\langle I'(u_n), u_n\rangle| \le C\|u_n\|_{W_0}$ and $|I(u_n)| \le C$. Two possible cases arises: either $\inf_n \|u_n\|_{W_0} = d > 0$ or $\inf_n \|u_n\|_{W_0} = 0$, so that we divide the proof in two cases.

Case 1. $\inf\limits_n \|u_n\|_{W_0} = d > 0$. We begin with proving that $\{u_n\}_n$ is bounded. Denote by $\kappa = \kappa(d)$ the number corresponding to $\sigma = d^p$ in (M), so that

$$M(\|u_n\|^p_{W_0}) \ge \kappa \quad \text{for all } n. \tag{3.5}$$

By assumption (f_1), we have

$$\left| \int_{\Omega \bigcap \{|u_n| \le r\}} \left(F(x, u_n) - \mu^{-1} f(x, u_n)u_n\right) dx \right| \le (a_1 + \mu^{-1})(r + r^q)|\Omega| \le C, \tag{3.6}$$

where $\{|u_n| \le r\} = \{x \in \Omega : |u_n(x)| \le r\}$. By (f_3), (3.5) and (3.6), we get

$$\begin{aligned}
C + C\|u_n\|_{W_0} &\ge I(u_n) - \frac{1}{\mu}\langle I'(u_n), u_n\rangle \\
&= \frac{1}{p}\mathscr{M}(\|u_n\|^p_{W_0}) - \frac{1}{\mu}M(\|u_n\|^p_{W_0})\|u_n\|^p_{W_0} - \frac{1}{\mu}\int_\Omega (\mu F(x, u_n) - f(x, u_n)u_n)dx \\
&\ge \left(\frac{1}{\theta p} - \frac{1}{\mu}\right) M(\|u_n\|^p_{W_0})\|u_n\|^p_{W_0} - C \\
&\ge \kappa \left(\frac{1}{\theta p} - \frac{1}{\mu}\right) \|u_n\|^p_{W_0} - C,
\end{aligned}$$

where C denotes various positive constants. Hence, $\{u_n\}$ is bounded in W_0.

To prove that $\{u_n\}_n$ converges strongly to u in W_0, we first introduce a simple notation. Let $\varphi \in W_0$ be fixed and denote by B_φ the linear functional on W_0 defined by

$$B_\varphi(v) = \iint_{\mathbb{R}^{2N}} \frac{|\varphi(x) - \varphi(y)|^{p-2}(\varphi(x) - \varphi(y))}{|x - y|^{N+ps}}(v(x) - v(y))dxdy$$

for all $v \in W_0$. Clearly, by the Hölder inequality, B_φ is also continuous, being

$$|B_\varphi(v)| \le \|\varphi\|^{p-1}_{W_0}\|v\|_{W_0} \quad \text{for all } v \in W_0.$$

Hence, the weak convergence of $\{u_n\}$ in W_0 gives that

$$\lim_{n\to\infty} B_u(u_n - u) = 0. \tag{3.7}$$

Since W_0 is a reflexive Banach space, up to a subsequence, still denoted by $\{u_n\}_n$, such that $u_n \rightharpoonup u$ weakly in W_0. Clearly, $\langle I'(u_n), u_n - u\rangle \to 0$, since $u_n \rightharpoonup u$ in W_0 and $I'(u_n) \to 0$ in W_0^*. Thus, we obtain

$$\langle I'(u_n), u_n - u\rangle = M\left(\|u_n\|_{W_0}^p\right) B_{u_n}(u_n - u) - \int_\Omega f(x, u_n)(u_n - u)dx \to 0 \tag{3.8}$$

as $n \to \infty$.

Moreover, by Lemma 2.2, up to a subsequence,

$$u_n \to u \ \text{ strongly in } L^q(\Omega) \ \text{ and } \ \text{a.e. in } \Omega.$$

Thus, $f(x, u_n)(u_n - u) \to 0$ a.e. in Ω. It is easy to check that sequence $\{f(x, u_n)(u_n - u)\}_n$ is uniformly bounded and equi–integrable in $L^1(\Omega)$. Hence, the Vitali Convergence Theorem implies

$$\lim_{n\to\infty} \int_\Omega f(x, u_n)(u_n - u)dx = 0.$$

Therefore, by (3.8), we have

$$M\left(\|u_n\|_{W_0}^p\right) B_{u_n}(u_n - u) \to 0, \quad \text{as } n \to \infty.$$

It follows from (3.5) and the boundedness of $M([u_n]_{W_0}^p)$ that

$$B_{u_n}(u_n - u) \to 0, \quad \text{as } n \to \infty. \tag{3.9}$$

Combining (3.9) with (3.7), we have

$$[B_{u_n}(u_n - u) - B_u(u_n - u)] \to 0, \quad \text{as } n \to \infty. \tag{3.10}$$

Let us now recall the well–known Simon inequalities:

$$|\xi - \eta|^p \le \begin{cases} C_p\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right)\cdot(\xi - \eta) & \text{for} \quad p \ge 2 \\ \widetilde{C}_p\left[\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right)\cdot(\xi - \eta)\right]^{p/2}\left(|\xi|^p + |\eta|^p\right)^{(2-p)/2} & \text{for } 1 < p < 2, \end{cases}$$

for all $\xi, \eta \in \mathbb{R}^N$, where C_p and $\widetilde{C}_p$ are positive constants depending only on p. Then it follows from (3.10) that Assume first that $p \ge 2$. Then as $n \to \infty$

$$\begin{aligned} \|u_n - u\|_{W_0}^p &\le C_p \iint_{\mathbb{R}^{2N}} \left[|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y)) - |u(x) - u(y)|^{p-2}(u(x) - u(y))\right] \\ &\qquad \times (u_n(x) - u(x) - u_n(y) + u(y))\,|x - y|^{-(N+ps)}dxdy \\ &= C_p\left[B_{u_n}(u_n - u) - B_u(u_n - u)\right] = o(1). \end{aligned} \tag{3.11}$$

In conclusion, $\|u_n - u\|_{W_0} \to 0$ as $n \to \infty$, as desired.

Finally, it remains to consider the case when $1 < p < 2$. Now by the Simon inequality, the Hölder inequality and (3.10) as $n \to \infty$

$$\begin{aligned} \|u_n - u\|_{W_0}^p &\le \widetilde{C}_p\left[B_{u_n}(u_n - u) - B_u(u_n - u)\right]^{p/2}\left(\|u_n\|_{W_0}^p + \|u\|_{W_0}^p\right)^{(2-p)/2} \\ &\le \widetilde{C}_p\left[B_{u_n}(u_n - u) - B_u(u_n - u)\right]^{p/2}\left(\|u_n\|_{W_0}^{p(2-p)/2} + \|u\|_{W_0}^{p(2-p)/2}\right) \\ &\le C[B_{u_n}(u_n - u) - B_u(u_n - u)]^{p/2} = o(1) \end{aligned} \tag{3.12}$$

where C is positive constant. Combing (3.11) with (3.12), we get that $u_n \to u$ strongly in W_0 as $n \to \infty$.
Case 2. $\inf_n \|u_n\|_{W_0} = 0$. Either 0 is an accumulation point of the sequence $\{u_n\}_n$ and so there exists a subsequence of $\{u_n\}_n$ strongly converging to $u = 0$, or 0 is an isolated point of the sequence $\{u_n\}_n$ and so there exists a subsequence, still denoted by $\{u_n\}_n$, such that $\inf_n \|u_n\|_{W_0} > 0$. In the first case we are done, while in the latter case we can proceed as *Case 1.* $\square$

Proof of Theorem 1.1. By (f_1) we know that $F(x, \cdot)$ is even in $\mathbb{R}$, so that I is also even. Since $I(0) = 0$, we conclude from Lemma 3.2–Lemma 3.4 and Theorem 3.1,that there exists an unbounded sequence of weak solutions of problem (1.1). $\square$

4 Proof of Theorem 1.2 and 1.3

In this section we will use Krasnoselskii's genus theory to get the proof of our second result. First we recall some basic notations of Krasnoselskii's genus, which can be found in [12, 18, 34].

Let X be a real Banach space. Set

$$\Re = \{E \subset X \backslash \{0\} : E \text{ is compact and } E = -E.\}$$

Definition 4.1. Let $E \in \Re$ and $X = \mathbb{R}^k$. The genus $\gamma(E)$ of E is defined by

$$\gamma(E) = \min\{k \geq 1 : \text{there exists an odd continuous mapping } \phi : E \to \mathbb{R}^k \backslash \{0\}\}.$$

If such a mapping does not exist for any $k > 0$, we set $\gamma(E) = \infty$. Note that if E is a subset, which consists of finitely many pairs of points, then $\gamma(E) = 1$. Moreover, from definition, $\gamma(\emptyset) = 0$. A typical example of a set of genus k is a set, which is homeomorphic to a $(k-1)$ dimensional sphere via an odd map.

Now, we give some results of Krasnoselskii's genus, which are necessary throughout this section.

Lemma 4.1. *Let $X = \mathbb{R}^N$ and $\partial\Omega$ be the boundary of an open, symmetric, and bounded subset $\Omega \subset \mathbb{R}^N$ with $0 \in \Omega$. Then $\gamma(\partial\Omega) = N$.*

It follows from Lemma 4.1 that $\gamma(S^{N-1}) = N$. Now we present a results due to Clarke [9].

Theorem 4.1. *Let $G \in C^1(X, \mathbb{R})$ be functional satisfying the* (PS) *condition. Furthermore, let us suppose that*
(i) *G is bounded from below and even;*
(ii) *there is a compact set $K \subset \Re$ such that $\gamma(K) = k$ and $\sup_{x \in K} G(x) < J(0)$.*
Then G possesses at least k pairs of distinct critical points, and their corresponding critical values are less that $G(0)$.

Lemma 4.2. *Suppose that (M) and (f_2) are satisfied. Then I is bounded from blow and satisfies the* (PS) *condition.*

Proof. By assumption (f_2), we have

$$I(u) \geq \frac{1}{p}\mathscr{M}(\|u\|_{W_0}^p) - a_1 \int_\Omega |u(x)|^q dx - a_1|\Omega|,$$

for all $u \in W_0$. If $0 \leq \|u\|_{W_0} \leq 1$, then by (3.2) and Lemma 2.1–1 we get

$$\begin{aligned} I(u) &\geq \frac{\mathscr{M}(1)}{p}\|u\|_{W_0}^{\theta p} - a_1 C_0^{\frac{q}{p}}\|u\|_{W_0}^q - a_1|\Omega| \\ &\geq -a_1 C_0^{\frac{q}{p}} - a_1|\Omega|. \end{aligned} \tag{4.1}$$

If $\|u\| > 1$, then by (M) we obtain

$$I(u) \geq \frac{\kappa}{p}\|u\|_{W_0}^p - a_1 C_0^{\frac{q}{p}}\|u\|_{W_0}^q - a_1|\Omega|, \tag{4.2}$$

where $\kappa = \kappa(1)$ is the number corresponding to $\sigma = 1$ in (M). Since $q < p$, it follows from (4.2) and the Young inequality that

$$I(u) \geq \frac{\kappa}{2p}\|u\|_{W_0}^p - C, \tag{4.3}$$

where C is a positive constant. Combining (4.1) with (4.3), we get that the functional I is coercive. Hence the functional I is bounded from below and satisfies the (PS) condition. □

Proof of Theorem 1.2. Set

$$\Re_k = \{E \subset \Re : \Gamma(E) \geq k\}, \quad c_k = \inf_{E\in\Re_k} \sup_{u\in E} I(u), \ k = 1, 2, \cdots,$$

then by Lemma 4.2 we get

$$-\infty < c_1 \leq c_2 \leq \cdots \leq c_k \leq c_{k+1} \leq \cdot.$$

Now we show that $c_k < 0$ for every $k \in \mathbb{N}$. Following the ideas of [28], for each k, we take k disjoint open sets K_i such that $\bigcup_{i=1}^k K_i \subset \Omega_0$. For $i = 1, \cdots, k$, let $u_i \in (W_0 \bigcap C_0^\infty(K_i))\backslash\{0\}$, with $\|u_i\|_{W_0} = 1$, and

$$E_k = \text{span}\{u_1, u_2, \cdots, u_k\}, \quad S_k = \{u \in E_k : \|u\|_{W_0} = 1\}.$$

For each $u \in E_k$, there exist $\mu_i \in \mathbb{R}$, $i = 1, 2, \cdots, k$ such that

$$u(x) = \sum_{i=1}^k \mu_i u_i(x) \ \text{ for } x \in \Omega. \tag{4.4}$$

Then

$$\|u\|_{L^{\gamma_0}(\Omega)} = \left(\sum_{i=1}^k |\mu_i|^{\gamma_0} \int_{K_i} |u_i(x)|^{\gamma_0} dx\right)^{\frac{1}{\gamma_0}} \tag{4.5}$$

and

$$\|u\|_{W_0}^p = \sum_{i=1}^k |\mu_i|^p \iint_Q \frac{|u_i(x) - u_i(y)|^p}{|x-y|^{N+ps}} dxdy \tag{4.6}$$
$$= \sum_{i=1}^k |\mu_i|^p \|u_i\|_{W_0}^p = \sum_{i=1}^k |\mu_i|^p.$$

Since all norms on finite dimensional normed spaces are equivalent, there is a constant $C > 0$ such that

$$\|u\|_{W_0} \leq C\|u\|_{L^{\gamma_0}(\Omega)} \ \text{ for all } u \in E_k. \tag{4.7}$$

By (f_5), (4.4), (4.5) and (4.7), we have

$$\begin{aligned}
I(tu) &= \frac{1}{p}\mathscr{M}\left(\|tu\|_{W_0}^p\right) - \int_\Omega F(x, tu)dx \\
&\leq \frac{1}{p}\max_{0\leq\tau\leq 1} M(\tau)t^p - \sum_{i=1}^k \int_{K_i} F(x, tu_i)dx \\
&\leq \frac{1}{p}\max_{0\leq\tau\leq 1} M(\tau)t^p - a_2 t^{\gamma_0} \sum_{i=1}^k \mu_i^{\gamma_0} \int_{K_i} |u_i(x)|^{\gamma_0} dx \\
&= \frac{1}{p}\max_{0\leq\tau\leq 1} M(\tau)t^p - a_2 t^{\gamma_0} \|u\|_{L^{\gamma_0}(\Omega)}^{\gamma_0} \\
&\leq \frac{1}{p}\max_{0\leq\tau\leq 1} M(\tau)t^p - a_2 t^{\gamma_0} C^{-\gamma_0}.
\end{aligned}$$

for all $u \in S_k$ and $0 \leq t \leq 1$. Since $\gamma_0 < p$, we can find $t_0 = t(k) \in (0, 1)$ and $\varepsilon_0 = \varepsilon(k) > 0$ such that

$$I(t_0 u) \leq -\varepsilon_0 < 0 \ \text{ for all } u \in S_k.$$

that is,

$$I(u) \leq -\varepsilon_0 < 0 \ \text{ for all } u \in S_k^{t_0}.$$

where $S_k^{t_0} = \{t_0 u : u \in S_k\}$. Clearly,

$$S_k^{t_0} = \left\{ \sum_{i=1}^k \mu_i u_i : \sum_{i=1}^k |\mu_i|^p = t_0^p \right\}.$$

So we define a map $\Phi : S_k^{t_0} \to \partial\Sigma$ as follows:

$$\Phi(u) = (\mu_1, \mu_2, \cdots, \mu_k), \quad \forall u \in S_k^{t_0},$$

where $\Sigma = \{(\mu_1, \mu_2, \cdots, \mu_k) \in \mathbb{R}^k : \sum_{i=1}^k |\mu_i|^p < t_0^p\}$. It is easy to verify that $\Phi : S_k^{t_0} \to \partial\Sigma$ is an odd homeomorphic map. By Proposition 7.7 in [34], we get $\gamma(S_k^{t_0}) = k$ and so $c_k \leq -\varepsilon_0 < 0$. Finally, by Lemma 4.2 and the results above, we can apply Theorem 4.1 to obtain that I admits at least k pairs of distinct critical points. Since k is arbitrary, we obtain infinitely many critical points of I. □

Proof of Theorem 1.3. By (M_0) and (f_2), with $1 < q < \theta p$, we can prove that functional I is bounded from below and satisfies (PS) condition. The rest proof of Theorem 1.3 is similar to that of Theorem 1.2, so we omit the details. $\square$

Now, we consider the following example which is a direct application of the main results.

Example 4.1. Let $0 < s < 1 < p < \infty$ with $sp < N$ and Ω be an open bounded set of $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$. We consider problem

$$\begin{cases} \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy\right)^{\theta-1} (-\Delta)_p^s u = |u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{4.8}$$

where $\theta > 1$ is a constant. It is easy to see that for each $\sigma > 0$

$$M(t) = t^{\theta-1} > \sigma^{\theta-1} =: \kappa(\sigma) \text{ for all } t > \sigma,$$

and

$$\theta \mathscr{M}(t) = \theta \int_0^t M(\tau)d\tau = t^\theta = M(t)t \text{ for all } t \geq 0.$$

Let $f(x,\xi) = |\xi|^{q-2}\xi$ and hence $F(x,\xi) = \frac{1}{q}|\xi|^q$. Obviously, f satisfies (f_1) and F satisfies

$$qF(x,\xi) = f(x,\xi)\xi \text{ for all } x \in \Omega \text{ and } |\xi| > 0.$$

When $\theta p < q$, we have

$$\lim_{\xi \to 0} \frac{F(x,\xi)}{|\xi|^{\theta p}} = \lim_{\xi \to 0} \frac{|\xi|^q}{q|\xi|^{\theta p}} = 0 \text{ uniformly in } x \in \Omega.$$

If $\theta p < q < p_s^*$ and $1 < \theta < N/(N-ps)$, then Theorem 1.1 implies that problem (4.8) admits infinitely many weak solutions with unbounded energy;

If $1 < q < \theta p$, then Theorem 1.3 implies that problem (4.8) has infinitely many nontrivial weak solutions with negative energy.

Acknowlegements. M. Xiang was supported by Fundamental Research Funds for the Central Universities (No. 3122015L014) and National Natural Science Foundation of China (No. 11501565). B. Zhang was supported by Natural Science Foundation of Heilongjiang Province of China (No. A201306) and Research Foundation of Heilongjiang Educational Committee (No. 12541667) and Doctoral Research Foundation of Heilongjiang Institute of Technology (No. 2013BJ15).

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