GROUND STATE SOLUTIONS OF SCALAR FIELD FRACTIONAL SCHRÖDINGER EQUATIONS

(Giovanni Molica Bisci & Vicenţiu D. Rădulescu)

To access the original version see: https://doi.org/10.1007/s00526-015-0891-5

(Article begins on next page)

GROUND STATE SOLUTIONS OF SCALAR FIELD FRACTIONAL SCHRÖDINGER EQUATIONS

GIOVANNI MOLICA BISCI AND VICENŢIU D. RĂDULESCU

Abstract. In this paper, we study the existence of multiple ground state solutions for a class of parametric fractional Schrödinger equations whose simplest prototype is

$$(-\Delta)^s u + V(x)u = \lambda f(x,u) \quad \text{in } \mathbb{R}^n,$$

where $n > 2$, $(-\Delta)^s$ stands for the fractional Laplace operator of order $s \in (0,1)$, and λ is a positive real parameter. The nonlinear term f is assumed to have a superlinear behaviour at the origin and a sublinear decay at infinity. By using variational methods, we establish the existence of a suitable range of positive eigenvalues for which the problem admits at least two nontrivial solutions in a related weighted fractional Sobolev space.

Key words: Fractional Schrödinger equation, ground state, variational analysis, fractional Sobolev space, multiple solutions.

2010 AMS Subject Classifications. Primary: 35A15, 35S15, 49J35. Secondary: 45G05, 47G20.

Contents

1. Introduction

In this paper, we study ground state solutions for a nonlinear problem driven by the fractional Laplace operator $(-\Delta)^s$ of order $s \in (0,1)$.

Let $\mathcal{S}$ be the Schwartz space of rapidly decaying $C^\infty(\mathbb{R}^n)$ functions. For every $u \in \mathcal{S}$, we recall that the fractional Laplace operator acting on u is defined by

$$(-\Delta)^s u = \mathfrak{F}^{-1}(|\xi|^{2s}(\mathfrak{F}u)(\xi)), \tag{1.1}$$

where

$$\mathfrak{F}u(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x)\, dx, \quad \xi \in \mathbb{R}^n \tag{1.2}$$

denotes the Fourier transform of u.

From a probabilistic point of view, the fractional Laplace operator is the infinitesimal generator of a Lévy process, cf. Bertoin [8]. This operator arises in the description of various phenomena in the applied sciences, such as plasma physics (Kurihura [31]), flame propagation (Caffarelli, Roquejoffre & Sire [12]), free boundary obstacle problems (Caffarelli, Salsa & Silvestre [13]), finance (Cont & Tankov [21]), Signorini problems (Silvestre [52]), Hamilton-Jacobi equation with critical fractional diffusion (Silvestre [53]), or phase transitions in the Gamma convergence framework (Alberti, Bouchitté & Seppecher [1]).

We start focusing our attention on the fractional Schrödinger equation (briefly fractional NSE) of the form

$$i\frac{\partial \psi}{\partial t} = (-\Delta)^s \psi + V(x)\psi - |\psi|^{p-1}\psi \qquad \text{in } \mathbb{R}^n \times (0, +\infty), \tag{1.3}$$

where $V : \mathbb{R}^n \to \mathbb{R}$ is a given potential. Here $\psi = \psi(x,t)$ represents the quantum mechanical probability amplitude for a given unit-mass particle to have position x at time t (the corresponding probability density is $|\psi|^2$), under a confinement due to the potential V. Equation (1.3) comes from an expansion of the Feynman path integral from Brownian-like to Lévy-like quantum mechanical paths and was considered for the first time in literature by Laskin [35, 36].

When $s = 1$, problem (1.3) gives back the classical Schrödinger equation. In this case, standing wave solutions are of the form

$$\psi(x,t) = e^{-i\omega t} u(x),$$

where ω is a suitable constant and u solves the nonlinear elliptic equation

$$-\Delta u + V(x)u - |u|^{p-1}u = 0. \tag{1.4}$$

We do not intend to review the huge bibliography of equations like (1.4), we just emphasize that the potential $V : \mathbb{R}^n \to \mathbb{R}$ has a crucial role concerning the existence and the asymptotic behaviour of solutions. For instance, when V is a positive constant, or V is radially symmetric, it is natural to look for radially symmetric solutions, see Strauss [54] and Willem [57]. On the other hand, after the seminal paper of Rabinowitz [42] where the potential V is assumed to be coercive, several different assumptions are adopted in order to obtain existence and multiplicity results or qualitative properties of the solutions, see Bartsch *et al.* [5, 6, 7], Furtado, Maya & Silva [28], and Gazzola & Rădulescu [29].

We also point that sublinear problems on the whole space do not have necessarily a solution. In fact, the existence of solutions is in relationship not only with the nonlinearity but also with the behaviour of a certain potential. Brezis and Kamin [10] pointed out a striking phenomenon, which asserts that a *sublinear* problem on the whole space has a solution if and only if a *linear* equation depending only on the potential has a solution. They considered the nonlinear problem

$$-\Delta u = \rho(x)\, u^p, \quad x \in \mathbb{R}^n \ (n \geqslant 3), \tag{1.5}$$

with $0 < p < 1$, $\rho \in L^{\infty}_{\rm loc}(\mathbb{R}^n) \setminus \{0\}$, $\rho \geqslant 0$. Brezis and Kamin [10] proved that the *nonlinear* problem (1.5) has a bounded positive solution if and only if the *linear* equation

$$-\Delta u = \rho(x), \quad x \in \mathbb{R}^n$$

has a bounded solution. Their analysis showed that such a solution exists for potentials like

$$\rho(x) = \frac{1}{1+|x|^\alpha} \quad \text{or} \quad \rho(x) = \frac{1}{(1+|x|^2)\,|\log(2+|x|)|^\alpha} \quad (\alpha > 2),$$

while no solution exists if

$$\rho(x) = \frac{1}{1+|x|^\alpha} \quad \text{with } \alpha \leqslant 2\,.$$

Contrary to the classical Laplacian case that is widely investigated, the situation seems to be in a developing state when $s \in (0,1)$ and of an increasing interest, see Cheng [19], Dipierro, Palatucci & Valdinoci [23], Felmer, Quaas & Tan [26], and Frank & Lenzmann [27]).

In this spirit, Secchi [47] is interested in the existence of standing wave solutions for a more general equation than (1.3). More precisely, in the cited paper it is studied the following nonlocal fractional equation

$$(-\Delta)^s u + V(x)u = f(x,u), \quad x \in \mathbb{R}^n \tag{1.6}$$

under certain hypotheses on the potential V and the nonlinearity f (see also Secchi [48, 49]). Moreover, fractional Schrödinger-type problems have been considered in some interesting papers, see Autuori & Pucci [3], Coti Zelati & Nolasco [22], Chang [18]. In addition, nonlocal fractional equations appear in many fields and a lot of interest has been devoted to this kind of problems and to their concrete applications; see, for instance the seminal papers by Caffarelli & Silvestre [14, 15, 16]. We also refer to Barrios, Colorado, De Pablo & Sanchez [4], Cabré & Tan [11], Capella [17], Dipierro & Pinamonti [24], Molica Bisci & Servadei [40], and Tan [55], as well as the references therein. For completeness we mention the interesting regularity results for fractional problems proved recently by Kuusi, Mingione & Sire [32, 33].

Motivated by this large interest in the current literature, under suitable conditions on the potential V and exploiting variational methods, we are concerned in the present paper with the study of multiple solutions for the following fractional parametric problem

$$(-\Delta)^s u + V(x)u = \lambda(f(x,u) + \mu g(x,u)), \quad x \in \mathbb{R}^n \ (n \geqslant 3). \tag{1.7}$$

We assume that $V : \mathbb{R}^n \to \mathbb{R}$ is a positive potential and f, $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ are given functions, while λ and μ are real parameters. The nonlinearities f and g are assumed to have a superlinear behaviour at the origin and a sublinear decay at infinity.

A key role in our analysis is played by the combined effects of the nonlinear terms f and g and the parameters λ and μ. In the first main result of this paper, we show that for each μ belonging to a certain neighbourhood of the origin, there exists an open interval $\Sigma_\mu \subset (0,\infty)$ such that problem (1.7) admits at least two solutions for all $\lambda \in \Sigma_\mu$. In the second case it is examined the situation when λ lies in a neighbourhood of $+\infty$. In this framework, we prove that there is an open interval Λ_λ such that problem (1.7) has at least two solutions, provided that $\mu \in \Lambda_\lambda$. In both cases, we provide estimates of the energy of solutions.

We refer to Ciarlet [20] as basic reference for the mathematical methods employed in this paper.

1.1. **Technical assumptions on the potential.** We assume that the potential V satisfies the following hypotheses:

(p_1) $V \in C(\mathbb{R}^n)$ *with* $\inf_{x\in\mathbb{R}^n} V(x) > 0$;

(p_2) *for any* $M > 0$ *there exists* $r_0 > 0$ *such that*:
$$\lim_{|y|\to+\infty} |\{x \in B(y, r_0) : V(x) \leqslant M\}| = 0,$$
where $B(y,r)$ *denotes the open ball in* $\mathbb{R}^n$ *with center* y *and radius* $r > 0$, *and* $|\cdot|$ *stands for the Lebesgue measure in* $\mathbb{R}^n$.

Note that conditions (p_1) and (p_2) are not new in the literature, see for instance Chang [18]. Assumption (p_1) is used in the seminal paper by Rabinowitz [42] on nonlinear Schrödinger equations, while hypothesis (p_2) asserts that V can have large values at $+\infty$ only on sets of smaller and smaller measure. In particular, condition (p_2) forces the potential V to not be coercive, like in Rabinowitz [42].

We also require that f, $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying the following hypotheses:

(h_1) *there exist* $W \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \setminus \{0\}$ *and* $q \in (0, 1)$ *such that*
$$\max\{|f(x,t)|, |g(x,t)|\} \leqslant W(x)|t|^q \quad \textit{for all } (x,t) \in \mathbb{R}^n \times \mathbb{R};$$

(h_2) $\lim_{t\to 0} \dfrac{f(x,t)}{t} = \lim_{t\to 0} \dfrac{g(x,t)}{t} = 0$ *uniformly for all* $x \in \mathbb{R}^n$;

(h_3) *there exists* $s_0 \in \mathbb{R}$ *such that*
$$\sup_{\sigma>0} \left(\min_{|x|\leqslant\sigma} \int_0^{s_0} f(x,\tau)\, d\tau \right) > 0.$$

Let $H^s(\mathbb{R}^n)$ denote the fractional Sobolev space of functions $u \in L^2(\mathbb{R}^n)$ such that
$$\text{the map } (x,y) \longmapsto \frac{u(x) - u(y)}{|x-y|^{(n+2s)/2}} \text{ is in } L^2(\mathbb{R}^{2n}, dxdy).$$
On $H^s(\mathbb{R}^n)$ we consider the norm
$$\|u\|_{H^s(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |u(x)|^2\, dx + \int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dxdy \right)^{1/2}. \tag{1.8}$$

Consider the weighted Sobolev fractional space
$$E_s^n(V) := \left\{ u \in H^s(\mathbb{R}^n);\ \int_{\mathbb{R}^n} |\xi|^{2s} |\mathfrak{F}u(\xi)|^2\, d\xi + \int_{\mathbb{R}^n} V(x)|u(x)|^2 dx < \infty \right\}. \tag{1.9}$$
We endow this function space with the norm
$$\|u\|_{E_s^n(V)} := \left(\int_{\mathbb{R}^n} |\xi|^{2s} |\mathfrak{F}u(\xi)|^2\, d\xi + \int_{\mathbb{R}^n} V(x)|u(x)|^2 dx \right)^{1/2}.$$

The nonlocal analysis that we perform in order to use our abstract approach is quite general and it is successfully exploited for other goals in recent works. We refer to Chang

[19], Cheng [18], Di Nezza, Palatucci & Valdinoci [25], and Teng [56] for an introduction to this topic and for a list of related references.

By a *solution* of problem (1.7) we mean a function $u \in E^n_s(V)$ such that

$$\left\{ \begin{aligned} & \int_{\mathbb{R}^n} |\xi|^{2s} \mathfrak{F}u(\xi) \mathfrak{F}v(\xi)\, d\xi + \int_{\mathbb{R}^n} V(x) u(x) v(x)\, dx \\ & \qquad\qquad = \lambda \int_{\mathbb{R}^n} f(x, u(x)) v(x)\, dx + \lambda\mu \int_{\mathbb{R}^n} g(x, u(x)) v(x)\, dx\,, \\ & \forall\, v \in E^n_s(V)\,. \end{aligned} \right. \tag{1.10}$$

Let

$$F(x,t) := \int_0^t f(x,\tau) d\tau \quad \text{and} \quad G(x,t) := \int_0^t g(x,\tau)\, d\tau,$$

for every $(x,t) \in \mathbb{R}^n \times \mathbb{R}$.

By direct computation we observe that Problem (1.10) represents the Euler–Lagrange equation of the C^1-functional $\mathcal{J}_{\lambda,\mu} : E^n_s(V) \to \mathbb{R}$ defined as

$$\begin{aligned} \mathcal{J}_{\lambda,\mu}(u) := \frac{1}{2} \left(\int_{\mathbb{R}^n} |\xi|^{2s} |\mathfrak{F}u(\xi)|^2\, d\xi + \int_{\mathbb{R}^n} V(x) |u(x)|^2 dx \right) & \\ - \lambda \int_{\mathbb{R}^n} F(x, u(x))\, dx - \lambda\mu \int_{\mathbb{R}^n} G(x, u(x))\, dx, & \end{aligned} \tag{1.11}$$

for every $u \in E^n_s(V)$.

1.2. **The main results.** The first result establishes that if the parameters lie in a certain range, then the problem has at least two solutions.

Theorem 1. *Assume that f, g, and V satisfy hypotheses* (h_1)–(h_3) *and* (p_1)–(p_2).

Then there exists $\mu_0 > 0$ such that for all $\mu \in [-\mu_0, \mu_0]$ there are an open interval $\emptyset \neq \Sigma_\mu \subset (0, \infty)$ and $\kappa_\mu > 0$ with the property that problem (1.7) has at least two nontrivial solutions $v_{\lambda\mu}$ and $w_{\lambda\mu}$ satisfying

$$\max\{\|v_{\lambda\mu}\|_{E^n_s(V)}, \|w_{\lambda\mu}\|_{E^n_s(V)}\} \leqslant \kappa_\mu\,.$$

provided that $\lambda \in \Sigma_\mu$.

This theorem will be proved by using variational techniques, in particular by using some results due to Ricceri [44, 45], which assures the existence of multiple critical points for a functional under suitable regularity assumptions (see Theorem 9 below).

Furthermore, by using the notations adopted along the paper, we give additional information for the values of

$$\mu_0 := \frac{\displaystyle\int_{\mathbb{R}^n} F(x, u_0(x))\, dx}{\displaystyle\left| \int_{\mathbb{R}^n} G(x, u_0(x))\, dx \right| + 1}\,,$$

where $u_0 \in E^n_s(V)$ is an element given in Lemma 10. We also provide the localization of the interval Σ_μ. More precisely, we show that

$$\Sigma_\mu \subset \left[0, \frac{2\|u_0\|^2_{E^n_s(V)}}{\int_{\mathbb{R}^n} F(x,u_0(x))\,dx}\left(1+\left|\int_{\mathbb{R}^n} G(x,u_0(x))\,dx\right|\right)\right],$$

for suitable $\mu \in \mathbb{R}$; see Remark 17 in Section 3 for a detailed proof.

From the point of view of the eigenvalues, the counterpart of Theorem 1 is the following property, which establishes a "concentration at infinity" property for the eigenvalues associated to problem (1.7).

Theorem 2. *Assume that f, g, and V satisfy hypotheses* (h_1)–(h_3) *and* (p_1)–(p_2).

Then there exists $\lambda_0 > 0$ such that for all $\lambda \in (\lambda_0, \infty)$ there are an open interval $\emptyset \neq \Lambda_\lambda \subset \mathbb{R}$ and $\kappa_\lambda > 0$ with the property that problem (1.7) has at least two distinct nontrivial solutions $v_{\lambda\mu}$ and $w_{\lambda\mu}$ satisfying $\mathcal{J}_{\lambda,\mu}(v_{\lambda\mu}) < 0 < \mathcal{J}_{\lambda,\mu}(w_{\lambda\mu})$ and $\max\{\|v_{\lambda\mu}\|_{E^n_s(V)}, \|w_{\lambda\mu}\|_{E^n_s(V)}\} \leqslant \kappa_\lambda$*, provided that $\mu \in \Lambda_\lambda$.*

The proof of Theorem 2 is based on the mountain pass theorem of Ambrosetti & Rabinowitz [2]. In the framework of Theorem 2 we have

$$\lambda_0 := \frac{\|u_0\|^2_{E^n_s(V)}}{2\displaystyle\int_{\mathbb{R}^n} F(x,u_0(x))\,dx}.$$

Moreover if we set for all $\lambda > \lambda_0$

$$\mu^*_\lambda := \frac{1}{1+\int_{\mathbb{R}^n}|G(x,u_0(x))|\,dx}\left(1-\frac{\lambda_0}{\lambda}\right)\int_{\mathbb{R}^n} F(x,u_0(x))\,dx\,, \tag{1.12}$$

we have

$$\Lambda_\lambda \equiv (-\mu^*_\lambda, \mu^*_\lambda).$$

Although the above two theorems are completely independent, as a simple byproduct of Theorem 2 we obtain the following result whose conclusion partially goes back to Theorem 1.

Theorem 3. *Assume that f, g, and V satisfy hypotheses* (h_1)–(h_3) *and* (p_1)–(p_2).

Then there exists $\overline{\mu} > 0$ such that for every $\mu \in [-\overline{\mu}, \overline{\mu}]$ the set

$$\Sigma := \{\lambda > 0;\ \textit{problem (1.7) has at least two nontrivial solutions}\}$$

contains an interval.

We note that the above theorems do not work in general for every $\lambda \in \mathbb{R}$. For instance, consider

$$f(x,t) := \frac{\sin^2 t}{(1+|x|^\alpha)^\beta}, \quad \forall\,(x,t) \in \mathbb{R}^n \times \mathbb{R}$$

and

$$g(x,t) := \frac{\arctan^2 t}{(1+|x|^\alpha)^\beta}, \quad \forall\,(x,t) \in \mathbb{R}^n \times \mathbb{R}$$

with $\alpha, \beta > 0$ such that $\alpha\beta > n \geqslant 3$. In such a case, an easy computation shows that the following problem

$$(-\Delta)^s u + V(x)u = \frac{\lambda}{(1+|x|^\alpha)^\beta}(\sin^2 u + \mu \arctan^2 u), \quad x \in \mathbb{R}^n \tag{1.13}$$

has only the trivial solution, whenever

$$0 < \lambda < \frac{1}{(1+|\mu|\pi)S_2^2}$$

and μ is arbitrary. Here S_2 denotes the best Sobolev embedding constant of the injection $E_s^n(V) \hookrightarrow L^2(\mathbb{R}^n)$.

More generally, the following non-existence result holds.

Proposition 4. *Assume that f, g, and V satisfy hypotheses* (h_1)–(h_3) *and* (p_1)–(p_2). *Assume that*

$$f(x,t) := W(x)h(t),$$

and

$$g(x,s) := W(x)k(t)$$

for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, where h and k are Lipschitz continuous functions with Lipschitz constants L_h and L_k, and $W \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ satisfying (h_1).

Then for every $\mu \in \mathbb{R}$ and for every λ with

$$0 < \lambda < \frac{1}{\|W\|_{L^\infty(\mathbb{R}^n)}(L_h + |\mu| L_k)S_2^2},$$

the following nonlinear problem

$$(-\Delta)^s u + V(x)u = \lambda W(x)(h(u) + \mu k(u)), \quad x \in \mathbb{R}^n \tag{1.14}$$

admits only the trivial solution.

The final part of the paper is dedicated to the existence of multiple solutions to the following nonlocal Schrödinger equation

$$(-\Delta)^s u + V(x)u = \lambda W(x)f(u) + \mu g(x,u), \quad x \in \mathbb{R}^n \tag{1.15}$$

where

(H) $V : \mathbb{R}^n \to \mathbb{R}$ *is a function satisfying* (p_1) *and* (p_2);
$W \in L^1(\mathbb{R}^n) \cap L^\infty_+(\mathbb{R}^n)$ *and*

$$\sup_{\sigma>0}\left(\inf_{|x|\leqslant\sigma} W(x)\right) > 0\,;$$

λ and μ are real parameters.

Assuming that the nonlinear term f is superlinear at zero and sublinear at infinity, the main result ensures that for λ large enough, problem (1.15) admits at least two nontrivial solutions, as well as the stability of this problem with respect to an arbitrary subcritical perturbation g of the Schrödinger equation (see Section 4).

The paper is organized as follows. In Section 2 we give some notations and we recall some qualitative properties of the associated functional spaces. In Section 3 we study problem (1.7) and we prove our existence and non-existence results. Finally, in the last section we study the existence of multiple solutions to the problem (1.15).

2. Some preliminaries

The Hilbert space $H^s(\mathbb{R}^n)$ defined in Introduction can be described by means of the Fourier transform as follows:

$$H^s(\mathbb{R}^n) := \left\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\mathfrak{F}u(\xi)|^2 \, d\xi < +\infty \right\}.$$

In this case, the inner product and the norm are given by

$$(u,v)_{H^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} (1+|\xi|^2)^s \mathfrak{F}u(\xi)\mathfrak{F}v(\xi)\, d\xi$$

for every $u, v \in H^s(\mathbb{R}^n)$, respectively

$$\|u\|_{H^s} := \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\mathfrak{F}u(\xi)|^2 \, d\xi \right)^{1/2}. \tag{2.1}$$

In order to give the relationship between the norms defined in relations (1.8) and (2.1), we recall the definition of the fractional Laplace operator $(-\Delta)^s$ acting on the rapidly decreasing $C^\infty(\mathbb{R}^n)$ functions (that is, the space of Schwartz functions $\mathcal{S}$).

For fixed $s \in (0,1)$ we define the operator $(-\Delta)^s : \mathcal{S} \to L^2(\mathbb{R}^n)$ by

$$(-\Delta)^s u(x) := C(n,s) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy,$$

where $B(x,\varepsilon)$ is the open ball centered at $x \in \mathbb{R}^n$ and radius ε and $C(n,s)$ is the following normalization constant

$$C(n,s) := \left(\int_{\mathbb{R}^n} \frac{1-\cos\zeta_1}{|\zeta|^{n+2s}}\, d\zeta \right)^{-1}.$$

The operator $(-\Delta)^s$ is a pseudo-differential operator with symbol $|\eta|^{2s}$, where η denotes the variable in the frequency space. This nonlocal operator can also be defined by the formula

$$(-\Delta)^s u(x) = C(n,s)\, \mathrm{P.V.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy\, .$$

The proof of this formula, as well as the computation of the constant $C(n,s)$, can be found in the book of Landkof [34].

Di Nezza, Palatucci & Valdinoci [25] proved that for all $u \in H^s(\mathbb{R}^n)$

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s}(\mathcal{F}u)(\xi)), \tag{2.2}$$

$$[u]^2_{H^s(\mathbb{R}^n)} := \int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dxdy = 2C(n,s)^{-1} \int_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}u(\xi)|^2\, d\xi\, , \tag{2.3}$$

and

$$[u]^2_{H^s(\mathbb{R}^n)} = 2C(n,s)^{-1} \|(-\Delta)^{s/2}u\|^2_{L^2(\mathbb{R}^n)}. \tag{2.4}$$

By (2.2)–(2.4) and using the Plancherel formula, the following norms

$$u \mapsto \left(\int_{\mathbb{R}^n} |u(x)|^2 \, dx + \int_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dxdy \right)^{1/2},$$

$$u \mapsto \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\mathfrak{F}u(\xi)|^2 \, d\xi \right)^{1/2},$$

$$u \mapsto \left(\int_{\mathbb{R}^n} |u(x)|^2 \, dx + \int_{\mathbb{R}^n} |\xi|^{2s} |\mathfrak{F}u(\xi)|^2 \, d\xi \right)^{1/2},$$

$$u \mapsto \left(\int_{\mathbb{R}^n} |u(x)|^2 \, dx + \|(-\Delta)^{s/2} u\|^2_{L^2(\mathbb{R}^n)} \right)^{1/2},$$

are equivalent. As a direct consequence of the above remarks, we obtain that the space $E^n_s(V)$ can be defined in other (equivalent) ways.

In order to prove the compactness embedding property given in Proposition 7, the following auxiliary results are necessary.

Theorem 5. *Let $s \in (0,1)$ be such that $n > 2s$. Then there exists a positive constant $K = K(n,s)$ such that for all $u \in H^s(\mathbb{R}^n)$*

$$\|u\|^2_{L^{2^*}(\mathbb{R}^n)} \leqslant K [u]^2_{H^s(\mathbb{R}^n)},$$

where

$$2^*_s := \frac{2n}{n - 2s}$$

denotes the fractional critical Sobolev exponent. Consequently, the space $H^s(\mathbb{R}^n)$ is continuously embedded in $L^p(\mathbb{R}^n)$ for all $p \in [2, 2^*_s]$.

Moreover, the embedding $H^s(\mathbb{R}^n) \hookrightarrow L^p_{\text{loc}}(\mathbb{R}^n)$ is compact for every $p \in [2, 2^*_s)$.

We refer to Di Nezza, Palatucci & Valdinoci [25] for a detailed proof.

By using Theorem 5, the following fractional Gagliardo-Nirenberg inequality should be proved.

Lemma 6. *Let $s \in (0,1)$, $p \in [1, \infty)$ and $n > 2s$. Then for every $u \in H^s(\mathbb{R}^n)$ we have*

$$\|u\|_{L^p(\mathbb{R}^n)} \leqslant C(n,s,\alpha) [u]^{\alpha}_{H^s(\mathbb{R}^n)} \|u\|^{1-\alpha}_{L^r(\mathbb{R}^n)}, \tag{2.5}$$

with

$$\frac{n}{p} = \alpha \, \frac{n - 2s}{2} + (1 - \alpha) \, \frac{n}{r},$$

where $r \geqslant 1$, $\alpha \in [0,1]$ and $C(n,s,\alpha)$ is a positive constant.

Proof. The conclusion is trivial if $p = 1$. Indeed, in such a case, it suffices to take $r = 1$ and $\alpha = 0$. Hence, let us suppose that $p > 1$. Since

$$\frac{1}{p} = \frac{\alpha}{2^*_s} + \frac{1 - \alpha}{r},$$

by the Hölder inequality and Theorem 5 it follows that

$$\|u\|_{L^p(\mathbb{R}^n)} \leqslant \|u\|^{\alpha}_{L^{2^*_s}(\mathbb{R}^n)} \|u\|^{1-\alpha}_{L^r(\mathbb{R}^n)} \leqslant C(n,s,\alpha) [u]^{\alpha}_{H^s(\mathbb{R}^n)} \|u\|^{1-\alpha}_{L^r(\mathbb{R}^n)}, \tag{2.6}$$

where we set

$$C(n,s,\alpha) := K(n,s)^{\alpha/2}.$$

This concludes the proof. □

As a byproduct of Theorem 5 and Proposition 6 we obtain the following result that is crucial in the sequel.

Proposition 7. *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a potential that satisfies hypotheses* (p_1) *and* (p_2)*. Then the embedding*

$$E^n_s(V) \hookrightarrow L^p(\mathbb{R}^n)$$

is compact for every $p \in [2, 2^*_s)$.

Proof. By Chang [18], we know that the Hilbert space $E^n_s(V)$ is compactly embedded into $L^2(\mathbb{R}^n)$. Therefore, we only consider the case $p \in (2, 2^*_s)$. In order to do this, we use the fractional Gagliardo-Nirenberg inequality proved in Lemma 6.

Hence, let $\{u_j\}_{j\in\mathbb{N}} \subset E^n_s(V)$ be a sequence such that $u_j \rightharpoonup u_0$ in $E^n_s(V)$, i.e. $\{u_j\}_{j\in\mathbb{N}}$ weakly converges to u_0 in $E^n_s(V)$. Then $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $E^n_s(V)$ and, by Lemma 6, for $r = 2$ and

$$\alpha := \frac{(p-2)n}{2sp} \in (0,1),$$

there exists a constant $C_1(n,s,\alpha) > 0$ such that

$$\|u_j - u_0\|_{L^p(\mathbb{R}^n)} \leqslant C_1(n,s,\alpha)\|u_j - u_0\|^{\alpha}_{E^n_s(V)}\|u_j - u_0\|^{1-\alpha}_{L^2(\mathbb{R}^n)}, \tag{2.7}$$

for every $p \in (2, 2^*_s)$.

Thus since $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $E^n_s(V)$, relation (2.7) yields

$$\|u_j - u_0\|_{L^p(\mathbb{R}^n)} \leqslant C_1(n,s,\alpha)(M + \|u_0\|^{\alpha}_{E^n_s(V)})\|u_j - u_0\|^{1-\alpha}_{L^2(\mathbb{R}^n)} \to 0,$$

hence $u_j \to u_0$ in $L^p(\mathbb{R}^n)$. This completes the proof. □

Remark 8. By Proposition 7 we deduce that for all $p \in [2, 2^*_s]$, there exists a positive constant S_p such that

$$\|u\|_{L^p(\mathbb{R}^n)} \leqslant S_p\|u\|_{E^n_s(V)}, \tag{2.8}$$

for any $u \in E^n_s(V)$.

2.1. **Some useful tools.** A basic tool used along this paper in order to prove our multiplicity result stated in Theorem 1 is given by a direct consequence of some general theorems due to Ricceri [44, 45], which we recall in what follows.

Theorem 9. *Let $(E, \|\cdot\|)$ be a separable and reflexive real Banach space and let $\Phi, \Psi : E \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals. Assume that there exists $z_0 \in E$ such that $\Phi(z_0) = \Psi(z_0) = 0$ and $\inf_{z\in E} \Phi(z) \geqslant 0$ and that there exist $z_1 \in E$, $\varrho > 0$ such that*

(i) $\varrho < \Phi(z_1)$;

(ii) $\sup_{\Phi(z)<\varrho} \Psi(z) < \varrho\dfrac{\Psi(z_1)}{\Phi(z_1)}$.

Set

$$\bar{a} := \frac{\zeta\varrho}{\varrho\dfrac{\Psi(z_1)}{\Phi(z_1)} - \sup\limits_{\Phi(z)<\varrho}\Psi(z)},$$

with $\zeta > 1$ *and assume that the functional*

$$J_\lambda(z) := \Phi(z) - \lambda\Psi(z), \quad \forall\, z \in E$$

is sequentially weakly lower semicontinuous, satisfies the (PS) *condition, and*

(iii) $\lim\limits_{\|z\|\to+\infty} J_\lambda(z) = +\infty,$

for every $\lambda \in [0, \bar{a}]$.

Then there is an open interval $\Lambda \subset [0, \bar{a}]$ *and a number* $\kappa > 0$ *such that for each* $\lambda \in \Lambda$, *the equation* $J'_\lambda(z) = 0$ *admits at least three solutions in* E *having norm less than* κ.

Some details and related topics on the above result can be found in the monograph of Kristály, Rădulescu & Varga [30].

For the sake of completeness, we also recall that the C^1-functional $J_\lambda : E \to \mathbb{R}$ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$ when

$(\mathrm{PS})_c$ *Every sequence* $\{z_j\}_{j\in\mathbb{N}} \subset E$ *such that*

$$J_\lambda(z_j) \to c, \qquad \text{and} \qquad \|J'_\lambda(z_j)\|_{E'} \to 0,$$

as $j \to \infty$, *possesses a convergent subsequence in* E.

Here E' denotes the topological dual of E. Finally, we say that J_λ satisfies the Palais-Smale condition (in short (PS)) if $(\mathrm{PS})_c$ holds for every $c \in \mathbb{R}$.

Fix $\sigma > 0$ and $\eta \in \mathbb{R}$. For every $\varepsilon > 0$, define $u^\eta_\varepsilon \in E^n_s(V)$ as follows

$$u^\eta_\varepsilon(x) := \begin{cases} 0 & \text{if } x \in \mathbb{R}^n \setminus B(0,\sigma) \\ \dfrac{\eta}{\varepsilon}(\sigma - |x|) & \text{if } x \in B(0,\sigma) \setminus B(0,\sigma-\varepsilon) \\ \eta & \text{if } x \in B(0,\sigma-\varepsilon). \end{cases} \tag{2.9}$$

Note that $u^\eta_\varepsilon \in E^n_s(V)$, cf. Di Nezza, Palatucci & Valdinoci [25, Proposition 2.2 and Lemma 5.1]. We also observe that

$$\|u^\eta_\varepsilon\|_\infty := \max_{x\in\mathbb{R}^n} |u^\eta_\varepsilon(x)| \leqslant |\eta|.$$

This function will be useful in the sequel in the proof of our theorem as well as the next auxiliary results.

Lemma 10. *There exists* $u_0 \in E^n_s(V)$ *such that*

$$\int_{\mathbb{R}^n} F(x, u_0(x))\, dx > 0.$$

Proof. By (h_3) there are $\sigma_0 > 0$ and $s_0 \in \mathbb{R}$ such that

$$\min_{|x|\leqslant\sigma_0} \int_0^{s_0} f(x,\tau)\, d\tau > 0\,.$$

Fix $\varepsilon \in (0, \sigma_0/2)$ and denote $\omega_0 := \min_{|x|\leqslant\sigma_0} F(x, s_0) > 0$. Further, let $u_\varepsilon^{s_0} \in E_s^n(V)$ be the function obtained by (2.9) replacing σ with σ_0, as well as η with s_0. We have

$$\begin{aligned}\int_{\mathbb{R}^n} F(x, u_\varepsilon^{s_0}(x))\,dx &= \int_{|x|\leqslant\sigma_0-\varepsilon} F(x, u_\varepsilon^{s_0}(x))\,dx + \int_{|x|\geqslant\sigma_0} F(x, u_\varepsilon^{s_0}(x))\,dx \\ &\quad + \int_{\sigma_0-\varepsilon<|x|<\sigma_0} F(x, u_\varepsilon^{s_0}(x))\,dx \\ &\geqslant \omega_0|B_{\sigma_0/2}| - \int_{\sigma_0-\varepsilon<|x|<\sigma_0} |F(x, u_\varepsilon^{s_0}(x))|\,dx \\ &\geqslant \omega_0|B_{\sigma_0/2}| - \max_{|x|\in[\sigma_0/2,\sigma_0],\,|t|\leqslant|s_0|} |F(x,t)|(|B_{\sigma_0}| - |B_{\sigma_0-\varepsilon}|),\end{aligned}$$

where $|B_r|$ denotes the Lebesgue measure of the ball $B(0, r)$.

Since

$$\max_{|x|\in[\sigma_0/2,\sigma_0],\,|t|\leqslant|s_0|} |F(x,t)|(|B_{\sigma_0}| - |B_{\sigma_0-\varepsilon}|) \to 0 \quad \text{as } \varepsilon \to 0^+,$$

there exists $\varepsilon_0 > 0$ such that

$$\omega_0|B_{\sigma_0/2}| > \max_{|x|\in[\sigma_0/2,\sigma_0],\,|t|\leqslant|s_0|} |F(x,t)|(|B_{\sigma_0}| - |B_{\sigma_0-\varepsilon_0}|).$$

Thus the function $u_0 := u_{\varepsilon_0}^{s_0} \in E_s^n(V)$ verifies the requirement. □

Setting

$$\lambda_1^B := \inf_{u\in H_0^1(B(0,\sigma_0))\setminus\{0\}} \frac{\|\nabla u\|^2_{L^2(B(0,\sigma_0))}}{\|u\|^2_{L^2(B(0,\sigma_0))}}, \tag{2.10}$$

the following result holds.

Lemma 11. *Let $u_0 \in E_s^n(V)$ be the function defined in Lemma 10. Then*

$$\|u_0\|^2_{E_s^n(V)} < \pi^{\frac{n}{2}} \frac{\sigma_0^2}{\varepsilon_0^2} \frac{(\sigma_0^n - (\sigma_0 - \varepsilon_0)^n)}{\Gamma\left(1 + \frac{n}{2}\right)} S_0, \tag{2.11}$$

where

$$S_0 := \left(1 + \frac{1}{\lambda_1^B}\right) \max\left\{\left(1 + \frac{1}{\lambda_1^B}\right), \|V\|_\infty\right\}.$$

Proof. Computing the standard seminorm of the function u_0 in $H^1(\mathbb{R}^n)$, we easily have

$$\begin{aligned}[u_0]^2_{H^1(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |\nabla u_0(x)|^2\,dx = \frac{\sigma_0^2}{\varepsilon_0^2} \int_{B(0,\sigma_0)\setminus B(0,\sigma_0-\varepsilon_0)} dx \\ &= \frac{\sigma_0^2}{\varepsilon_0^2} (|B_{\sigma_0}| - |B_{\sigma_0-\varepsilon_0}|) \\ &= \pi^{\frac{n}{2}} \frac{\sigma_0^2}{\varepsilon_0^2} \frac{(\sigma_0^n - (\sigma_0 - \varepsilon_0)^n)}{\Gamma\left(1 + \frac{n}{2}\right)}.\end{aligned} \tag{2.12}$$

Hence, since $s \in (0,1)$ and bearing in mind that

$$\|u_0\|^2_{L^2(\mathbb{R}^n)} = \|\mathfrak{F}u_0\|^2_{L^2(\mathbb{R}^n)},$$

as well as

$$\|\nabla u_0\|^2_{L^2(\mathbb{R}^n)} = \||\xi|\mathfrak{F}u_0\|^2_{L^2(\mathbb{R}^n)},$$

we deduce that

$$\begin{aligned} \|u_0\|^2_{E^n_s(V)} &= \int_{\mathbb{R}^n} |\xi|^{2s}|\mathfrak{F}u_0(\xi)|^2\,d\xi + \int_{\mathbb{R}^n} V(x)|u_0(x)|^2 dx \\ &< \int_{\mathbb{R}^n} (1+|\xi|^2)|\mathfrak{F}u_0(\xi)|^2\,d\xi + \int_{\mathbb{R}^n} V(x)|u_0(x)|^2 dx \\ &= \|\nabla u_0\|^2_{L^2(\mathbb{R}^n)} + \|u_0\|^2_{L^2(\mathbb{R}^n)} + \int_{\mathbb{R}^n} V(x)|u_0(x)|^2 dx. \end{aligned} \tag{2.13}$$

Thus since $u_0 \in H^1_0(B(0,\sigma_0))$, inequality (2.13) yields

$$\|u_0\|^2_{E^n_s(V)} < \max\left\{\left(1+\frac{1}{\lambda_1^B}\right), \|V\|_\infty\right\}\left(1+\frac{1}{\lambda_1^B}\right)[u_0]^2_{H^1(\mathbb{R}^n)}. \tag{2.14}$$

Substituting (2.12) in (2.14), the conclusion is achieved. □

Finally, a standard computation ensures that our assumptions give a natural control on the growth of the nonlinearities f and g, as well as of their potentials F and G.

Lemma 12. *Let $p \in (2, 2^*_s)$. Then for each $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that*

(i) $\max\{|f(x,t)|, |g(x,t)|\} \leqslant \varepsilon|t| + c(\varepsilon)|t|^{p-1}$ *for every* $(x,t) \in \mathbb{R}^n \times \mathbb{R}$;
(ii) $\max\{|F(x,t)|, |G(x,t)|\} \leqslant \varepsilon t^2 + c(\varepsilon)|t|^p$ *for every* $(x,t) \in \mathbb{R}^n \times \mathbb{R}$.

2.2. **The Palais-Smale condition.** Our purpose in what follows is to show that the energy functional $J_{\lambda,\mu}$ satisfies the Palais-Smale condition.

Lemma 13. *Let $\lambda, \mu \in \mathbb{R}$ be arbitrary fixed. Then every bounded sequence $\{u_j\}_{j\in\mathbb{N}} \subset E^n_s(V)$ such that*

$$\|\mathcal{J}'_{\lambda,\mu}(u_j)\|_{(E^n_s(V))'} := \sup\left\{|\langle \mathcal{J}'_{\lambda,\mu}(u_j), \varphi\rangle| : \varphi \in E^n_s(V),\ \|\varphi\|_{E^n_s(V)} = 1\right\} \to 0, \tag{2.15}$$

as $j \to +\infty$, contains a strongly convergent subsequence.

Proof. Let $\{u_j\}_{j\in\mathbb{N}} \subset E^n_s(V)$ be a bounded sequence such that condition (2.15) holds.

By Proposition 7, due to the compact embedding $E^n_s(V) \hookrightarrow L^p(\mathbb{R}^n)$ for every $p \in (2, 2^*_s)$, we may assume, taking a subsequence if necessary, that $\{u_j\}_{j\in\mathbb{N}}$ converges to u weakly in $E^n_s(V)$ and strongly in $L^p(\mathbb{R}^n)$, for every $p \in (2, 2^*_s)$.

Therefore, fixing $p \in (2, 2^*_s)$ and bearing in mind the regularity assumptions on the function W, we have

$$\begin{aligned} \|u_j - u\|^2_{E^n_s(V)} &= (u, u-u_j)_{E^n_s(V)} + \mathcal{J}'_{\lambda,\mu}(u_j)(u_j - u) \\ &\quad - \lambda \int_{\mathbb{R}^n} f(x, u_j(x))(u-u_j)(x)\,dx - \lambda\mu \int_{\mathbb{R}^n} g(x, u_j(x))(u-u_j)(x)\,dx \\ &\leqslant (u, u-u_j)_{E^n_s(V)} + \|\mathcal{J}'_{\lambda,\mu}(u_j)\|_{(E^n_s(V))'}\|u_j - u\|_{E^n_s(V)} \\ &\quad + |\lambda|(1+|\mu|)\|W\|_{L^{\frac{p}{p-q-1}}(\mathbb{R}^n)}\|u_j\|^q_{L^p(\mathbb{R}^n)}\|u-u_j\|_{L^p(\mathbb{R}^n)}, \end{aligned}$$

where q is given in our hypothesis (h_1).

We deduce that $\|u_j - u\|_{E^n_s(V)} \to 0$ as $j \to \infty$. The proof is complete. □

As a direct consequence of the above result, the following compactness property is valid.

Lemma 14. *For every $\lambda, \mu \in \mathbb{R}$, the functional $\mathcal{J}_{\lambda,\mu}$ is coercive and bounded from below on $E^n_s(V)$. Moreover, $\mathcal{J}_{\lambda,\mu}$ satisfies the* (PS) *condition.*

Proof. Exploiting condition (h_1) we obtain

$$\mathcal{J}_{\lambda,\mu}(u) \geqslant \frac{1}{2}\|u\|^2_{E^n_s(V)} - |\lambda|(1+|\mu|)\|W\|_{L^{\frac{2}{1-q}}(\mathbb{R}^n)} S_2^{q+1}\|u\|^{q+1}_{E^n_s(V)}, \tag{2.16}$$

for every $u \in E^n_s(V)$. Since $q \in (0,1)$, the functional $\mathcal{J}_{\lambda,\mu}$ is bounded from below in $E^n_s(V)$.

Now, let us prove that the (PS) condition is verified. For this purpose, let $\{u_j\}_{j\in\mathbb{N}} \subset E^n_s(V)$ be a sequence such that

$$\{\mathcal{J}_{\lambda,\mu}(u_j)\}_{j\in\mathbb{N}} \ \text{ is bounded in } E^n_s(V) \tag{2.17}$$

and for which condition (2.15) holds. Since the functional $\mathcal{J}_{\lambda,\mu}$ is coercive, we deduce by (2.17) that the sequence $\{u_j\}_{j\in\mathbb{N}} \subset E^n_s(V)$ is bounded.

In conclusion, on account of Lemma 13, the functional $\mathcal{J}_{\lambda,\mu}$ satisfies the compactness (PS) condition. □

3. Proof of the main results

In order to prove Theorem 1 we start with some auxiliary results. For every $\mu \in \mathbb{R}$, let $L_\mu : E^n_s(V) \to \mathbb{R}$ be the functional defined as follows

$$u \longmapsto \int_{\mathbb{R}^n} F(x,u(x))\,dx + \mu\int_{\mathbb{R}^n} G(x,u(x))\,dx.$$

With the above notation we exhibit the following asymptotic property.

Lemma 15. *For every $\mu \in \mathbb{R}$, setting*

$$\chi(\varrho) := \frac{\sup\left\{L_\mu(u) : \|u\|_{E^n_s(V)} < \sqrt{2\varrho}\right\}}{\varrho},$$

we have

$$\lim_{\varrho\to 0^+} \chi(\varrho) = 0.$$

Proof. Fix arbitrarily $\varepsilon > 0$ and $p \in (2, 2^*_s)$. Due to Lemma 12, we obtain

$$L_\mu(u) \leqslant (1+|\mu|)(\varepsilon S_2^2\|u\|^2_{E^n_s(V)} + c(\varepsilon)S_p^p\|u\|^p_{E^n_s(V)}),$$

for every $u \in E^n_s(V)$.

Therefore, for every $\varrho > 0$,

$$0 \leqslant \chi(\varrho) \leqslant (1+|\mu|)(2\varepsilon S_2^2 + c(\varepsilon)2^{\frac{p}{2}}S_p^p\varrho^{\frac{p}{2}-1}).$$

When $\varrho \to 0^+$, the right-hand side of the above inequality tends to zero, due to the arbitrariness of $\varepsilon > 0$, which concludes the proof. □

Lemma 16. *The functional $\mathcal{J}_{\lambda,\mu}$ is sequentially weakly lower semicontinuous in $E^n_s(V)$, for every $\lambda, \mu \in \mathbb{R}$.*

Proof. The function

$$u \longmapsto \int_{\mathbb{R}^n} |\xi|^{2s} |\mathfrak{F}u(\xi)|^2 \, d\xi + \int_{\mathbb{R}^n} V(x) u(x)^2 dx$$

is sequentially weakly lower semicontinuous on $E^n_s(V)$. Thus it is enough to prove that the map L_μ is sequentially weakly continuous on $E^n_s(V)$, for every $\mu \in \mathbb{R}$.

On the contrary, let us suppose that $\{u_j\}_{j\in\mathbb{N}}$ is a sequence in $E^n_s(V)$ which converges weakly to $u_\infty \in E^n_s(V)$ and such that the sequence $\{L(u_j)\}_{j\in\mathbb{N}}$ does not converge to $L(u_\infty)$ as $j \to \infty$. Therefore there exist $\varepsilon_0 > 0$ and a subsequence of $\{u_j\}_{j\in\mathbb{N}}$, denoted again by $\{u_j\}_{j\in\mathbb{N}}$, such that

$$0 < \varepsilon_0 \leqslant |L_\mu(u_j) - L_\mu(u_\infty)| \tag{3.1}$$

for every $j \in \mathbb{N}$, and $u_j \to u$ strongly in $L^p(\mathbb{R}^n)$ for every $p \in (2, 2^*_s)$.

Since the functionals

$$\Psi_1(u) := \int_{\mathbb{R}^n} F(x, u(x)) dx, \quad \text{and} \quad \Psi_2(u) := \int_{\mathbb{R}^n} G(x, u(x)) dx,$$

are smooth with derivatives given by

$$\Psi_1'(u)(v) = \int_{\mathbb{R}^n} f(x, u(x)) v(x) dx, \quad \text{and} \quad \Psi_2'(u)(v) = \int_{\mathbb{R}^n} g(x, u(x)) v(x) dx, \quad (v \in E^n_s(V))$$

the mean value theorem, hypothesis (h_1) and inequality (3.1) yield together

$$\begin{aligned} 0 < \varepsilon_0 &\leqslant |L'_\mu(u + \theta_j(u_j - u))(u_j - u)| \\ &\leqslant (1 + |\mu|) \int_{\mathbb{R}^n} W(x) |u(x) + \theta_j(u_j - u)(x)|^q |(u_j - u)(x)| \, dx \\ &\leqslant (1 + |\mu|) \|W\|_{L^{\frac{p}{p-q-1}}(\mathbb{R}^n)} (\|u\|_{L^p(\mathbb{R}^n)} + \|u_j - u\|_{L^p(\mathbb{R}^n)})^q \|u_j - u\|_{L^p(\mathbb{R}^n)}, \end{aligned}$$

for some $\theta_j \in (0, 1)$.

As $j \to \infty$ the last term of the above inequality tends to zero, which is a contradiction. □

Proof of Theorem 1. Let $u_0 \in E^n_s(V)$ be the element from Lemma 10 and fix

$$\mu_0 := \frac{\displaystyle\int_{\mathbb{R}^n} F(x, u_0(x)) \, dx}{\left| \displaystyle\int_{\mathbb{R}^n} G(x, u_0(x)) \, dx \right| + 1}.$$

Now we apply Theorem 9 by choosing $E := E^n_s(V)$, $\Phi(u) := \frac{1}{2} \| \cdot \|^2_{E^s_n(V)}$, and $\Psi := L_\mu$ for every fixed $\mu \in [-\mu_0, \mu_0]$. It is clear that

$$\mathcal{J}_{\lambda,\mu} = \Phi - \lambda J_\mu.$$

A simple calculation shows that for every $\mu \in [-\mu_0, \mu_0]$ we have

$$L_\mu(u_0) = \int_{\mathbb{R}^n} F(x, u_0(x)) \, dx + \mu \int_{\mathbb{R}^n} G(x, u_0(x)) \, dx \geqslant \mu_0 > 0. \tag{3.2}$$

Using (3.2) and Lemma 15, there exists $\varrho_\mu > 0$ such that

$$\varrho_\mu < \min\left\{1, \frac{\|u_0\|^2_{E^n_s(V)}}{2}\right\}; \tag{3.3}$$

$$\chi(\varrho_\mu) := \frac{\sup\{L_\mu(u) : \ \|u\|_{E^n_s(V)} < \sqrt{2\varrho_\mu}\}}{\varrho_\mu} < \frac{L_\mu(u_0)}{\|u_0\|^2_{E^n_s(V)}}. \tag{3.4}$$

Now, choosing $z_1 := u_0$, $z_0 := 0$, $\zeta := 1 + \varrho_\mu$ and

$$\overline{a} = \overline{a}_\mu = \frac{1 + \varrho_\mu}{2L_\mu(u_0)\|u_0\|^{-2}_{E^n_s(V)} - \chi(\varrho_\mu)},$$

all the assumptions of Theorem 9 are verified, cf. Lemmas 16 and 14, respectively.

Then there is an open interval $\Sigma_\mu \subset [0, \overline{a}_\mu]$ and a number $\kappa_\mu > 0$ such that for any $\lambda \in \Sigma_\mu$, the functional $\mathcal{J}_{\lambda,\mu} = \Phi - \lambda L_\mu$ admits at least three distinct critical points $u^i_{\lambda,\mu} \in E^n_s(V)$ ($i \in \{1, 2, 3\}$), having norm less than κ_μ. This concludes the proof of Theorem 1. □

Remark 17. Thanks to (3.3), (3.4) and (3.2), it follows that

$$\overline{a}_\mu < \frac{2\|u_0\|^2_{E^n_s(V)}}{L_\mu(u_0)} \leqslant \frac{2\|u_0\|^2_{E^n_s(V)}}{\mu_0} = \frac{2\|u_0\|^2_{E^n_s(V)}}{\displaystyle\int_{\mathbb{R}^n} F(x, u_0(x))\, dx}\left(1 + \left|\int_{\mathbb{R}^n} G(x, u_0(x))\, dx\right|\right),$$

for every parameter $\mu \in [-\mu_0, \mu_0]$. Since the right-hand side does not depend on $\mu \in \mathbb{R}$, we have a uniform estimation of Λ_μ, that is,

$$\Sigma_\mu \subset [0, \overline{a}_\mu] \subset \left[0, \frac{2\|u_0\|^2_{E^n_s(V)}}{\displaystyle\int_{\mathbb{R}^n} F(x, u_0(x))\, dx}\left(1 + \left|\int_{\mathbb{R}^n} G(x, u_0(x))\, dx\right|\right)\right],$$

for every $\mu \in [-\mu_0, \mu_0]$.

Proof of Theorem 2. Define

$$c_0 := \int_{\mathbb{R}^n} |G(x, u_0(x))| dx \quad \text{and} \quad \lambda_0 := \frac{\|u_0\|^2_{E^n_s(V)}}{2\displaystyle\int_{\mathbb{R}^n} F(x, u_0(x))\, dx},$$

where $u_0 \in E^n_s(V)$ comes from Lemma 10. Further, for every $\lambda > \lambda_0$, we set

$$\mu^*_\lambda := \frac{1}{1 + c_0}\left(1 - \frac{\lambda_0}{\lambda}\right)\int_{\mathbb{R}^n} F(x, u_0(x))\, dx. \tag{3.5}$$

Remark 18. Note that an explicit estimate of the number λ_0 appearing in the main results can be obtained by using relation (2.11). More precisely, a direct computation gives

$$\lambda_0 < \frac{\sigma_0^2}{2\varepsilon_0^2}\frac{(\sigma_0^n - (\sigma_0 - \varepsilon_0)^n)}{\Gamma\left(1 + \dfrac{n}{2}\right)}\frac{S_0 \pi^{\frac{n}{2}}}{\omega_0 |B_{\sigma_0/2}| - \displaystyle\max_{|x| \in [\sigma_0/2, \sigma_0], |t| \leqslant |s_0|} |F(x, t)|(|B_{\sigma_0}| - |B_{\sigma_0 - \varepsilon}|)}.$$

Lemma 19. *Assume that* $\lambda > \lambda_0$ *and* $\mu \in (-\mu_\lambda^*, \mu_\lambda^*)$. *Then*

$$\inf_{u \in E_s^n(V)} \mathcal{J}_{\lambda,\mu}(u) < 0.$$

Proof. It is sufficient to show that $\mathcal{J}_{\lambda,\mu}(u_0) < 0$ whenever $\lambda > \lambda_0$ and $|\mu| < \mu_\lambda^*$. Due to the choice of λ_0 and μ_λ^*, we obtain

$$\begin{aligned}\mathcal{J}_{\lambda,\mu}(u_0) &= \frac{1}{2}\|u_0\|^2_{E_s^n(V)} - \lambda \int_{\mathbb{R}^n} F(x, u_0(x))\, dx - \lambda\mu \int_{\mathbb{R}^n} G(x, u_0(x))\, dx \\ &\leqslant (\lambda_0 - \lambda) \int_{\mathbb{R}^n} F(x, u_0(x))\, dx + \lambda|\mu| c_0 \\ &= -\lambda(1 + c_0)\mu_\lambda^* + \lambda|\mu| c_0 < 0.\end{aligned}$$

The proof is complete. □

Lemma 20. *For every* $\lambda > \lambda_0$ *and* $\mu \in (-\mu_\lambda^*, \mu_\lambda^*)$, *the functional* $\mathcal{J}_{\lambda,\mu}$ *has the mountain pass geometry.*

Proof. Fix $p \in (2, 2_s^*)$ arbitrary. By Lemma 12 we deduce that for every $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that

$$\max\left\{\left|\int_{\mathbb{R}^n} F(x, u(x))\, dx\right|, \left|\int_{\mathbb{R}^n} G(x, u(x))\, dx\right|\right\} \leqslant \varepsilon S_2^2 \|u\|^2_{E_s^n(V)} + c(\varepsilon) S_p^p \|u\|^p_{E_s^n(V)},$$

for every $u \in E_s^n(V)$.

Thus for every $u \in E_s^n(V)$ we have

$$\begin{aligned}\mathcal{J}_{\lambda,\mu}(u) &\geqslant \frac{1}{2}\|u\|^2_{E_s^n(V)} - \lambda \left|\int_{\mathbb{R}^n} F(x, u(x))\, dx\right| - \lambda|\mu| \left|\int_{\mathbb{R}^n} G(x, u(x))\, dx\right| \\ &\geqslant \left(\frac{1}{2} - \lambda(1 + |\mu|)\varepsilon S_2^2\right)\|u\|^2_{E_s^n(V)} - \lambda(1 + |\mu|)c(\varepsilon) S_p^p \|u\|^p_{E_s^n(V)}.\end{aligned}$$

Set

$$\varepsilon := \frac{1}{4\lambda(1 + |\mu|)S_2^2}.$$

Then

$$\mathcal{J}_{\lambda,\mu}(u) \geqslant \left(\frac{1}{4} - \lambda(1 + |\mu|)c(\lambda, \mu) S_p^p \varrho^{p-2}\right)\varrho^2 > 0$$

provided that

$$\|u\|_{E_s^n(V)} = \varrho < \min\left\{(4\lambda(1 + |\mu|)c(\lambda, \mu) S_p^p)^{\frac{1}{2-p}}, \|u_0\|_{E_s^n(V)}\right\},$$

where, for simplicity, we set

$$c(\lambda, \mu) := c\left(\frac{1}{4\lambda(1 + |\mu|)S_2^2}\right).$$

Moreover, by construction, $\varrho < \|u_0\|_{E_s^n(V)}$ and by the proof of Lemma 19 we have $\mathcal{J}_{\lambda,\mu}(u_0) < 0$. Thus, $\mathcal{J}_{\lambda,\mu}(u)$ verifies the mountain pass geometry. □

Proof of Theorem 2. Fix $\lambda > \lambda_0$ and $\mu \in (-\mu_\lambda^*, \mu_\lambda^*) \equiv \Lambda_\lambda$. Lemma 14 ensures in particular that there exists an element $v_{\lambda\mu} \in E_s^n(V)$ such that

$$\mathcal{J}_{\lambda,\mu}(v_{\lambda\mu}) = \inf_{u \in E_s^n(V)} \mathcal{J}_{\lambda,\mu}(u).$$

By using Lemma 19, we obtain $\mathcal{J}_{\lambda,\mu}(v_{\lambda\mu}) < 0$.

On the other hand, by Lemma 20 we find $w_{\lambda\mu} \in E_s^n(V)$ such that $\mathcal{J}'_{\lambda,\mu}(w_{\lambda\mu}) = 0$ and

$$\mathcal{J}_{\lambda,\mu}(w_{\lambda\mu}) \geqslant \Big(\frac{1}{4} - \lambda(1+|\mu|)c(\lambda,\mu)S_p^p \varrho^{p-2}\Big)\varrho^2 > 0,$$

see for instance [43, Theorem 2.2]). The mountain pass level $\mathcal{J}_{\lambda,\mu}(w_{\lambda\mu})$ is characterized as

$$\mathcal{J}_{\lambda,\mu}(w_{\lambda\mu}) = \inf_{g\in\Gamma} \max_{t\in[0,1]} \mathcal{J}_{\lambda,\mu}(g(t)), \tag{3.6}$$

where

$$\Gamma := \{g \in C([0,1]; E_s^n(V)) : g(0) = 0,\ g(1) = u_0\}.$$

Let $g_0 : [0,1] \to E_s^n(V)$, defined by $g_0(t) := tu_0$. Since $g_0 \in \Gamma$, by using (3.6), we obtain

$$\begin{aligned}\mathcal{J}_{\lambda,\mu}(w_{\lambda\mu}) &\leqslant \max_{t\in[0,1]} \mathcal{J}_{\lambda,\mu}(tu_0)\\ &\leqslant \frac{1}{2}\|u_0\|^2_{E_s^n(V)} + \lambda \max_{t\in[0,1]} \left\{ \left| \int_{\mathbb{R}^n} F(x, tu_0(x))\, dx \right| + \mu_\lambda^* \left| \int_{\mathbb{R}^n} G(x, tu_0(x))\, dx \right| \right\},\end{aligned}$$

for every $\mu \in \Lambda_\lambda$.

By using (2.16), for every $\mu \in \Lambda_\lambda$ we have

$$\begin{aligned}\|w_{\lambda\mu}\|^2_{E_s^n(V)} &\leqslant 2\lambda(1+\mu_\lambda^*)\|W\|_{L^{\frac{2}{1-q}}(\mathbb{R}^n)} S_2^{q+1} \|w_{\lambda\mu}\|^{q+1}_{E_s^n(V)}\\ &\quad + \|u_0\|^2_{E_s^n(V)} + 2\lambda \max_{t\in[0,1]} \left\{ \left| \int_{\mathbb{R}^n} F(x, tu_0(x))\, dx \right| + \mu_\lambda^* \left| \int_{\mathbb{R}^n} G(x, tu_0(x))\, dx \right| \right\}.\end{aligned}$$

Since $q + 1 < 2$ we have at once that there exists $\kappa_\lambda^1 > 0$ such that

$$\|w_{\lambda\mu}\|_{E_s^n(V)} \leqslant \kappa_\lambda^1$$

for every $\mu \in \Lambda_\lambda$.

Moreover, since $\mathcal{J}_{\lambda,\mu}(v_{\lambda\mu}) < 0$ for every $\mu \in \Lambda_\lambda$, a similar argument as above (put formally $C_\lambda = 0$) shows the existence of $\kappa_\lambda^2 > 0$ such that

$$\|v_{\lambda\mu}\|_{E_s^n(V)} \leqslant \kappa_\lambda^2,$$

for every $\mu \in \Lambda_\lambda$. Thus letting $\kappa_\lambda := \max\{\kappa_\lambda^1, \kappa_\lambda^2\}$, the proof is completed. □

Remark 21. On account of (3.5), for every $\lambda > \lambda_0$ we have

$$\mu_\lambda^* < \frac{\displaystyle\int_{\mathbb{R}^n} F(x, u_0(x))\, dx}{1 + c_0}.$$

Since the right-hand side does not depend on $\lambda \in \mathbb{R}$, we have a uniform estimation of Λ_λ, that is, for every $\lambda > \lambda_0$,

$$\Lambda_\lambda \subset \left[-\frac{\displaystyle\int_{\mathbb{R}^n} F(x, u_0(x))\, dx}{1 + c_0}, \frac{\displaystyle\int_{\mathbb{R}^n} F(x, u_0(x))\, dx}{1 + c_0} \right].$$

Proof of Theorem 3 and Proposition 4.

Proof of Theorem 3. Fix $\overline{\lambda} > \lambda_0$, $\gamma \in (0, \overline{\lambda} - \lambda_0)$ and define

$$\overline{\mu} := \frac{\displaystyle\int_{\mathbb{R}^n} F(x, u_0(x))\, dx}{1 + c_0} \left(1 - \frac{\lambda_0}{\overline{\lambda} + \gamma}\right) \frac{\overline{\lambda} - \lambda_0 - \gamma}{\overline{\lambda} - \lambda_0 + \gamma}.$$

Let us consider $\mu \in \mathbb{R}$ with $\mu \in (-\overline{\mu}, \overline{\mu})$. Then the inequality $\overline{\mu} < \mu^*_\lambda$ holds for every $\lambda \in (\overline{\lambda} - \gamma, \overline{\lambda} + \gamma)$. Thus for every

$$\lambda \in (\overline{\lambda} - \gamma, \overline{\lambda} + \gamma),$$

we have $\mu \in (-\mu^*_\lambda, \mu^*_\lambda) = \Lambda_\lambda$.

By Theorem 2, for each $\lambda \in (\overline{\lambda} - \gamma, \overline{\lambda} + \gamma)$, problem (1.7) has at least two nontrivial solutions. Consequently

$$(\overline{\lambda} - \gamma, \overline{\lambda} + \gamma) \subset \Sigma,$$

which completes the proof. □

Proof of Proposition 4. Arguing by contradiction, let us assume that there is a solution $\bar{u} \in E^n_s(V) \setminus \{0\}$ of the problem (1.7), that is,

$$\left\{ \begin{aligned} &\int_{\mathbb{R}^n} |\xi|^{2s} \mathfrak{F}\bar{u}(\xi) \mathfrak{F}v(\xi)\, d\xi + \int_{\mathbb{R}^n} V(x) \bar{u}(x) v(x)\, dx \\ &\qquad = \lambda \int_{\mathbb{R}^n} W(x) h(\bar{u}(x)) v(x)\, dx + \lambda\mu \int_{\mathbb{R}^n} W(x) k(\bar{u}(x)) v(x)\, dx, \\ &\forall\, v \in E^n_s(V). \end{aligned} \right. \tag{3.7}$$

In particular, testing (3.7) with $v := \bar{u}$, we have

$$\|\bar{u}\|^2_{E^n_s(V)} = \lambda \int_{\mathbb{R}^n} W(x) h(\bar{u}(x)) \bar{u}(x)\, dx + \lambda\mu \int_\Omega W(x) k(\bar{u}(x)) \bar{u}(x)\, dx. \tag{3.8}$$

Hence, by (3.8), since h and k are Lipschitz continuous functions with $h(0) = k(0) = 0$, it follows that

$$\begin{aligned} \|\bar{u}\|^2_{E^n_s(V)} &\leqslant \lambda \|W\|_{L^\infty(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} h(\bar{u}(x)) \bar{u}(x)\, dx + \mu \int_{\mathbb{R}^n} k(\bar{u}(x)) \bar{u}(x)\, dx \right) \\ &\leqslant \lambda \|W\|_{L^\infty(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} |h(\bar{u}(x))| |\bar{u}(x)|\, dx + |\mu| \int_{\mathbb{R}^n} |k(\bar{u}(x))| |\bar{u}(x)|\, dx \right) \\ &\leqslant \lambda \|W\|_{L^\infty(\mathbb{R}^n)} (L_h + |\mu| L_k) \|\bar{u}\|^2_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{3.9}$$

By (3.9), since $E^n_s(V) \hookrightarrow L^2(\mathbb{R}^n)$, bearing in mind that

$$0 < \lambda < \frac{1}{\|W\|_{L^\infty(\mathbb{R}^n)}(L_h + |\mu| L_k) S_2^2},$$

we obtain

$$\|\bar{u}\|^2_{E^n_s(V)} \leqslant \lambda \|W\|_{L^\infty(\mathbb{R}^n)}(L_h + |\mu| L_k) S_2^2 \|\bar{u}\|^2_{E^n_s(V)} < \|\bar{u}\|^2_{E^n_s(V)},$$

which is a contradiction.

In conclusion problem (1.7) admits only the trivial solution. □

4. A stability property for fractional NSE

In this section we prove a multiplicity result to the problem (1.15). We assume that the nonlinear term $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$|f(t)| \leqslant |t|^q \quad \textit{for some } q \in (0,1) \textit{ and every } t \in \mathbb{R}, \tag{4.1}$$

and

$$\lim_{t \to 0} \frac{f(t)}{t} = 0. \tag{4.2}$$

Further, we require that the perturbation term g belongs to the class $\mathcal{C}$ of the continuous functions $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that

there exist $Z \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \setminus \{0\}$ *and* $r \in (0,1)$ *such that*

$$|g(x,t)| \leqslant Z(x)|t|^r \quad \textit{for each } (x,t) \in \mathbb{R}^n \times \mathbb{R}\,.$$

As pointed out in Introduction, the main result in this section (Theorem 23) ensures that for $\lambda > 0$ large enough, problem (1.15) admits at least two nontrivial solutions, as well as the stability of this problem with respect to an arbitrary subcritical perturbation of the Schrödinger equation.

The key tool will be the following abstract critical point theorem for differentiable functionals (cf. Ricceri [46, Theorem 2] for a detailed proof).

Theorem 22. *Let E be a separable and reflexive real Banach space, $\Phi : E \to \mathbb{R}$ be a coercive, sequentially weakly lower semicontinuous C^1 functional, belonging to $\mathcal{W}_E$, bounded on each bounded subset of E and whose derivative admits a continuous inverse on E'. Let $J : E \to \mathbb{R}$ be a C^1 functional with compact derivative and assume that Φ has a strict local minimum z_0 with $\Phi(z_0) = J(z_0) = 0$. Finally, assume that*

$$\max\left\{\limsup_{\|z\| \to +\infty} \frac{J(z)}{\Phi(z)}, \limsup_{z \to z_0} \frac{J(z)}{\Phi(z)}\right\} \leqslant 0 \tag{4.3}$$

and

$$\sup_{z \in E} \min\{\Phi(z), J(z)\} > 0.$$

Set

$$\theta := \inf\left\{\frac{\Phi(z)}{J(z)} : z \in E,\ \min\{\Phi(z), J(z)\} > 0\right\}. \tag{4.4}$$

Then for each compact interval $[a, b] \subset (\theta, +\infty)$ *there exists a number* $\varrho > 0$ *with the following property:*
for every $\lambda \in [a, b]$ *and every* C^1 *functional* $\Psi : E \to \mathbb{R}$ *with compact derivative, there exists* $\tilde{\mu} > 0$ *such that for all* $\mu \in [0, \tilde{\mu}]$*, the equation*

$$\Phi'(z) - \lambda J'(z) - \mu \Psi'(z) = 0 \tag{4.5}$$

has at least three solutions whose norms are less than ϱ*.*

Here $\mathcal{W}_E$ denotes the class of all functionals $I : E \to \mathbb{R}$ with the following property: *if* $u_j \rightharpoonup u$ *in* E *and*

$$\liminf_{j\to\infty} I(u_j) \leqslant I(u),$$

then $u_j \to u$ *up to a subsequence.*

The main result of this section reads as follows.

Theorem 23. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function satisfying* (4.1) *and* (4.2)*. Assume that condition* (H) *hold.*
Set

$$\theta := \frac{1}{2} \inf \left\{ \frac{\|u\|^2_{E^n_s(V)}}{\displaystyle\int_{\mathbb{R}^n} W(x)F(u(x))dx} : \ u \in E^n_s(V), \int_{\mathbb{R}^n} W(x)F(u(x))dx > 0 \right\}, \tag{4.6}$$

Then for each compact interval $[a, b] \subset (\theta, +\infty)$*, there exists a number* $\varrho > 0$ *with the following property*:
for every $\lambda \in [a, b]$ *and every* $g \in \mathcal{C}$ *there exists* $\tilde{\mu} > 0$ *such that, for each* $\mu \in [0, \tilde{\mu}]$*, problem* (1.15) *has at least three weak solutions whose norms in* $E^n_s(V)$ *are less than* ϱ*.*

Sketch of the Proof of Theorem 23. We apply Theorem 9 by choosing $E := E^n_s(V)$ and the functionals Φ and J defined respectively by

$$\Phi(u) := \frac{1}{2}\|u\|^2_{E^s_n(V)},$$

and

$$J(u) := \int_{\mathbb{R}^n} W(x)F(u(x))dx,$$

for every $u \in E^n_s(V)$.
Under our hypotheses direct computations ensure that

$$\lim_{\|u\|_{E^n_s(V)}\to 0} \frac{J(u)}{\|u\|^2_{E^n_s(V)}} = \lim_{\|u\|_{E^n_s(V)}\to+\infty} \frac{J(u)}{\|u\|^2_{E^n_s(V)}} = 0. \tag{4.7}$$

By (4.7) it follows that (4.3) holds true.
Hence, since also all the regularity assumptions on Φ and J are verified, our conclusions are easily achieved. □

Remark 24. Collecting the estimates of Lemmas 10 and 11 we obtain a concrete upper bound for the parameter θ in (4.6). More precisely we have $\theta \in (0, \theta^\star)$, where

$$\theta^\star := \frac{\sigma_0^2}{2\varepsilon_0^2} \frac{(\sigma_0^n - (\sigma_0 - \varepsilon_0)^n)}{\Gamma\left(1 + \dfrac{n}{2}\right)} \frac{S_0 \pi^{\frac{n}{2}}}{\omega_0 |B_{\sigma_0/2}| - \displaystyle\max_{|x| \in [\sigma_0/2, \sigma_0]} W(x) \max_{|t| \leqslant |s_0|} |F(t)|(|B_{\sigma_0}| - |B_{\sigma_0 - \varepsilon}|)}.$$

Indeed, recalling the definition of θ, in view of inequality (2.11) and Lemma 10 we obtain

$$\begin{aligned}\theta &\leqslant \frac{\|u_0\|^2_{E^n_s(V)}}{2\displaystyle\int_{\mathbb{R}^n} W(x) F(u_0(x)) dx} \\ &< \frac{\sigma_0^2}{2\varepsilon_0^2} \frac{(\sigma_0^n - (\sigma_0 - \varepsilon_0)^n)}{\Gamma\left(1 + \dfrac{n}{2}\right)} \frac{S_0 \pi^{\frac{n}{2}}}{\displaystyle\int_{\mathbb{R}^n} W(x) F(u_0(x)) dx} \leqslant \theta^\star.\end{aligned}$$

Example 25. Fix $r, q \in (0,1)$ and consider the following problem

$$(-\Delta)^s u + V(x) u = \frac{\lambda |u|^q \sin u + \mu |\sin u|^r}{(1 + |x|^n)^2}, \quad x \in \mathbb{R}^n. \tag{4.8}$$

Owing to Theorem 23, there exists $\theta > 0$ such that for each compact interval $[a, b] \subset (\theta, +\infty)$, there is some $\varrho > 0$ with the following property: for every $\lambda \in [a, b]$ there exists $\tilde{\mu} > 0$ such that for all $\mu \in [0, \tilde{\mu}]$, problem (4.8) admits at least three weak solutions whose norms in $E^n_s(V)$ are less than ϱ.

Remark 26. For completeness we just mention here that by using a similar variational approach to the one adopted here, Molica Bisci & Pansera [37, 38, 39] proved the existence and the multiplicity of weak solutions for nonlocal problems involving regional fractional Laplacian operators in a suitable abstract setting previously introduced by Servadei & Valdinoci [50, 51].

Acknowledgements. The paper is realized with the auspices of the INdAM–GNAMPA Project 2014 titled: *Proprietà geometriche ed analitiche per problemi non–locali*. This work has been completed while V. Rădulescu was visiting the City University of Hong Kong in February 2015. He would like to express his gratitude to Professor Philippe G. Ciarlet for this invitation and for enlightened discussions.

References

[1] G. Alberti, G. Bouchitté, and P. Seppecher, *Phase transition with the line-tension effect*, Arch. Ration. Mech. Anal. **144** (1998), 1-46.

[2] A. Ambrosetti and P. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Functional Analysis **14** (1973), 349-381.

[3] G. Autuori and P. Pucci, *Elliptic problems involving the fractional Laplacian in $\mathbb{R}^N$*, J. Differential Equations **255** (2013), 2340-2362.

[4] B. Barrios, E. Colorado, A. De Pablo, and U. Sanchez, *On some critical problems for the fractional Laplacian operator*, J. Differential Equations **252** (2012), 6133-6162.

[5] T. Bartsch and Z.-Q. Wang, *Existence and multiplicity results for some superlinear elliptic problems in $\mathbb{R}^N$*, Comm. Partial Differential Equations **20** (1995), 1725-1741.

[6] T. Bartsch, A. Pankov, and Z.-Q. Wang, *Nonlinear Schrödinger equations with steep potential well*, Comm. Contemp. Math. **4** (2001), 549-569.

[7] T. Bartsch, Z. Liu, and T. Weth, *Sign-changing solutions of superlinear Schrödinger equations*, Comm. Partial Differential Equations **29** (2004), 25-42.

[8] J. Bertoin, Lévy Processes, *Cambridge Tracts in Math.*, vol. 121, Cambridge Univ. Press, Cambridge, 1996.

[9] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, *Universitext, Springer*, New York, 2011.

[10] H. Brezis and S. Kamin, *Sublinear elliptic equations in* $\mathbb{R}^n$, Manuscripta Math. **74** (1992), 87-106.

[11] X. Cabré and J. Tan, *Positive solutions of nonlinear problems involving the square root of the Laplacian*, Adv. Math. **224** (2010), 2052-2093.

[12] L. Caffarelli, J.-M. Roquejoffre, and Y. Sire, *Variational problems for free boundaries for the fractional Laplacian*, J. Eur. Math. Soc. (JEMS) **12** (2010), 1151-1179.

[13] L. Caffarelli, S. Salsa, and L. Silvestre, Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian, Invent. Math. **171** (2008), 425-461.

[14] L. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations **32** (2007), 1245-1260.

[15] L. Caffarelli and L. Silvestre, *Regularity theory for fully nonlinear integro-differential equations*, Comm. Pure Appl. Math. **62** (2009), 597-638.

[16] L. Caffarelli and L. Silvestre, *Regularity results for nonlocal equations by approximation*, Arch. Ration. Mech. Anal. **200** (2011), 59-88.

[17] A. Capella, *Solutions of a pure critical exponent problem involving the half-Laplacian in annular-shaped domains*, Comm. Pure Appl. Anal. **10** (2011), 1645-1662.

[18] X. Chang, *Ground state solutions of asymptotically linear fractional Schrödinger equations*, J. Math. Phys. **54**, 061504 (2013).

[19] X. Cheng, *Bound state for the fractional Schrödinger equation with unbounded potential*, J. Math. Phys. **53**, 043507 (2012).

[20] P.G. Ciarlet, Linear and Nonlinear Functional Analysis with Applications, *Society for Industrial and Applied Mathematics (SIAM)*, Philadelphia, 2013.

[21] R. Cont and P. Tankov, Financial Modelling with Jump Processes, *Chapman & Hall/CRC Financ. Math. Ser., Chapman & Hall/CRC*, Boca Raton, FL, 2004.

[22] V. Coti Zelati and M. Nolasco, *Existence of ground states for nonlinear, pseudo-relativistic Schrödinger equations*, Atti Accad. Naz. Lincei, Cl. Sci. Fis., Mat. Nat., Rend. Lincei, Mat. Appl. **22** (2011), 51-72.

[23] S. Dipierro, G. Palatucci, and E. Valdinoci, *Existence and symmetry results for a Schrödinger-type problem involving the fractional Laplacian*, Le Matematiche **68** (2013), 201-216.

[24] S. Dipierro and A. Pinamonti, *A geometric inequality and a symmetry result for elliptic systems involving the fractional Laplacian*, J. Differential Equations **255** (2013), 85-119.

[25] E. Di Nezza, G. Palatucci, and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), 521-573.

[26] P. Felmer, A. Quaas, and J. Tan, *Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian*, Proc. Roy. Soc. Edinburgh Sect. A **142** (2012), 1237-1262.

[27] R.L. Frank and E. Lenzmann, *Uniqueness and nondegeneracy of ground states for* $(-\Delta)^s Q + Q - Q^{\alpha+1} = 0$ *in* $\mathbb{R}$, Acta Math. **210** (2013), 261-318.

[28] M.F. Furtado, L.A. Maia, and E.A.B. Silva, *On a double resonant problem in* $\mathbb{R}^N$, Differential Integral Equations **15** (2002), 1335-1344.

[29] F. Gazzola and V. Rădulescu, *A nonsmooth critical point theory approach to some nonlinear elliptic equations in* $\mathbb{R}^N$, Differential Integral Equations **13** (2000), 47-60.

[30] A. Kristály, V. Radulescu, and Cs. Varga, Variational Principles in Mathematical Physics, Geometry, and Economics. Qualitative Analysis of Nonlinear Equations and Unilateral Problems. With a foreword by Jean Mawhin, *Encyclopedia of Mathematics and its Applications*, **136**, *Cambridge University Press*, Cambridge (2010).

[31] S. Kurihura, *Large-amplitude quasi-solitons in superfluid films*, J. Phys. Soc. Japan **50** (1981), 3262-3267.

[32] T. Kuusi, G. Mingione, and Y. Sire, *Nonlocal equations with measure data*, Comm. Math. Phys., to appear.

[33] T. Kuusi, G. Mingione, and Y. Sire, *Nonlocal self-improving properties*, Analysis & PDE, to appear.

[34] N.S. Landkof, Osnovy Sovremennoi Teorii Potentsiala, *Nauka*, Moscow, 1966.

[35] N. Laskin, *Fractional Schrödinger equation*, Phys. Rev. E **66** (2002), 056108.

[36] N. Laskin, *Fractional quantum mechanics and Lévy path integrals*, Phys. Lett. A **268** (2000), 298-305.

[37] G. Molica Bisci, *Fractional equations with bounded primitive*, Appl. Math. Lett. **27** (2014), 53-58.

[38] G. Molica Bisci, *Sequences of weak solutions for fractional equations*, Math. Res. Lett. **21** (2014), 1-13.

[39] G. Molica Bisci and B.A. Pansera, *Three weak solutions for nonlocal fractional equations*, Adv. Nonlinear Stud. **14** (2014), 591-601.

[40] G. Molica Bisci and R. Servadei, *A bifurcation result for nonlocal fractional equations*, Anal. Appl., to appear (DOI:10.1142/S0219530514500067).

[41] P. Pucci and S. Saldi, *Critical stationary Kirchhoff equations in* $\mathbb{R}^n$ *involving nonlocal operators*, Rev. Mat. Iberoam., to appear.

[42] P.H. Rabinowitz, *On a class of nonlinear Schrödinger equations*, Z. Angew. Math. Phys. **43** (1992), 270-291.

[43] P.H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Reg. Conf. Ser. in Math., Amer. Math. Soc., Providence, 1986.

[44] B. Ricceri, *On a three critical points theorem*, Archiv der Mathematik (Basel) **75** (2000), 220-226.

[45] B. Ricceri, *Existence of three solutions for a class of elliptic eigenvalue problems*, Mathematical and Computer Modelling **32** (2000), 1485-1494.

[46] B. Ricceri, *A further three critical points theorem*, Nonlinear Anal. **71** (2009), 4151-4157.

[47] S. Secchi, *Ground state solutions for nonlinear fractional Schröinger equations in* $\mathbb{R}^N$, J. Math. Phys. **54** (2013), 031501.

[48] S. Secchi, *Perturbation results for some nonlinear equations involving fractional operators*, Differ. Equ. Appl. **5** (2013), 221-236.

[49] S. Secchi, *On fractional Schrödinger equations in* $\mathbb{R}^N$ *without the Ambrosetti-Rabinowitz condition*, Topol. Methods Nonlinear Anal., in press.

[50] R. Servadei and E. Valdinoci, *Mountain pass solutions for non-local elliptic operators*, J. Math. Anal. Appl. **389** (2012), 887-898.

[51] R. Servadei and E. Valdinoci, *Variational methods for non-local operators of elliptic type*, Discrete Contin. Dyn. Syst. **33** (2013), 2105-2137.

[52] L. Silvestre, *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Thesis (Ph.D.)-The University of Texas at Austin, 2005, 95 pp. ISBN: 978-0542-25310-2.

[53] L. Silvestre, *On the differentiability of the solution to the Hamilton-Jacobi equation with critical fractional diffusion*, Adv. Math. **226** (2011), 2020-2039.

[54] W.A. Strauss, *Existence of solitary waves in higher dimensions,* Comm. Math. Phys. **55** (1977), 149-162.

[55] J. Tan, *The Brezis-Nirenberg type problem involving the square root of the Laplacian*, Calc. Var. Partial Differential Equations, **36** (2011), 21-41.

[56] K. Teng, *Multiple solutions for a class of fractional Schrödinger equations in* $\mathbb{R}^N$, Nonlinear Analysis: Real World Applications **21** (2015), 76-86.

[57] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1995.

Dipartimento PAU, Università 'Mediterranea' di Reggio Calabria, Via Graziella, Feo di Vito, 89124 Reggio Calabria, Italy
Email address: gmolica@unirc.it

Department of Mathematics, Faculty of Sciences, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia & Institute of Mathematics "Simion Stoilow" of the Romanian Academy, 014700 Bucharest, Romania
Email address: vicentiu.radulescu@imar.ro