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On a fractional degenerate Kirchhoff-type problem

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Abstract

In this paper we study a highly nonlocal parametric problem involving a fractional-type operator combined with a Kirchhoff-type coefficient. The latter is allowed to vanish at the origin (degenerate case). Our approach is of variational nature; by working in suitable fractional Sobolev spaces which encode Dirichlet homogeneous boundary conditions, and exploiting an abstract critical point theorem for smooth functionals, we derive the existence of at least three weak solutions to our problem for suitable values of the parameters. Finally, we provide a concrete estimate of the range of these parameters in the autonomous case, by using some properties of the fractional calculus on a specific family of test functions. This estimate turns out to be deeply related to the geometry of the domain. The methods adopted here can be exploited to study different classes of elliptic problems in presence of a degenerate nonlocal term.

Keywords: Kirchhoff equation, vibrating string, fractional Laplacian, variational methods, multiple weak solutions.
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1 Introduction

The recent years have witnessed an ever more increasing attention to problems driven by nonlocal operators, notably of fractional type. This interest

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is motivated by both the theoretical research and the large number of applications (see, for instance, [19] and the references therein for a list of these applications). Moving along this direction, we are interested here in the existence of weak solutions to the following fractional Kirchhoff-type problem:

$$\begin{cases} -M(\|u\|_{X_0}^2)\mathcal{L}_K u = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{$P_{\lambda,\mu}$}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $2s < n < 4s$ with $s \in]0,1[$, $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are two suitable Carathéodory functions with subcritical growth, λ, μ are two positive real parameters and

$$\|u\|_{X_0}^2 := \int_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 K(x-y)\,dxdy,$$

where, from now on, we set $\mathbb{R}^{2n} := \mathbb{R}^n \times \mathbb{R}^n$.

Further, $\mathcal{L}_K$ is a nonlocal operator of fractional type defined as follows:

$$\mathcal{L}_K u(x) := \int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x))K(y)\,dy, \quad x \in \mathbb{R}^n,$$

where the function $K : \mathbb{R}^n \setminus \{0\} \to]0, +\infty[$ satisfies:

(k_1) $\gamma K \in L^1(\mathbb{R}^n)$, *where* $\gamma(x) := \min\{|x|^2, 1\}$;

(k_2) *there exists* $\alpha > 0$ *such that*

$$K(x) \geq \alpha |x|^{-(n+2s)},$$

for any $x \in \mathbb{R}^n \setminus \{0\}$.

The paradigm of the above K is given by the singular kernel $K(x) := |x|^{-(n+2s)}$; in this case $\mathcal{L}_K$ reduces to the fractional Laplace operator defined, up to normalization factor, by

$$-(-\Delta)^s u(x) := \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}}\,dy, \quad x \in \mathbb{R}^n.$$

In our setting the Kirchhoff function $M : [0, +\infty[\to [0, +\infty[$ is assumed to be continuous and to verify the structural assumptions:

(m_1) M *is non-decreasing*;

(m_2) *there exists* $m_1 > 0$ *such that*

$$M(t) \geq m_1 t,$$

for every $t \in [0, +\infty[$;

(m_3) *there exists* $\sigma > 1$ *such that*

$$m_\sigma := \liminf_{t \to +\infty} \frac{\widehat{M}(t)}{t^\sigma} > 0,$$

where

$$\widehat{M}(t) := \int_0^t M(\xi) d\xi,$$

for every $t \in [0, +\infty[$.

A typical Kirchhoff function verifying the above hypotheses is given by

$$M(t) := m_0 + m_1 t, \text{ where } m_0 \geq 0 \text{ and } m_1 > 0; \tag{1}$$

when M is of the type (1), problem $(P_{\lambda,\mu})$ is said to be non-degenerate when $m_0 > 0$, while it is called degenerate if $m_0 = 0$. The novelty of this paper is that we manage to treat the degenerate case, namely we allow the Kirchhoff function to take the zero value. This case is quite delicate and not very covered in literature, even more in the fractional setting, as explicitly pointed out in [21].

We also mention, for the sake of completeness, that degenerate Kirchhoff-type problems driven by non-homogeneous elliptic operators have been recently taken into account, for instance, in [13, 14, 15, 17, 34, 35].

The analogous and classical counterpart of our problem models several interesting phenomena studied in mathematical physics, even in the one-dimensional case. Its origins, as well known, date back to 1883 when G. Kirchhoff proposed his celebrated equation

$$\rho \partial_{tt}^2 u - \Big(\frac{P_0}{h} + \frac{E}{2L} \int_0^L |\partial_x u(x)|^2 \, dx \Big) \partial_{xx}^2 u = 0 \tag{2}$$

as a nonlinear extension of D'Alambert's wave equation for free vibrations of elastic strings, where the constants have the following meaning: $u = u(x,t)$ is the transverse string displacement at the space coordinate x and time t, L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension (see the classical manuscript [22]). In the very recent paper [21], the authors have proposed an interesting and fascinating physical interpretation of Kirchhoff's equation in the fractional scenario. In their correction of the early (one-dimensional) model, the tension of the string, which has classically a "nonlocal" nature arising from the average of the kinetic energy $|\partial_x u|^2/2$ on $[0, L]$, possesses a further nonlocal behavior provided by the H^s-norm (or other more general fractional norms) of the function u.

From a purely mathematical point of view, it is worth mentioning that the early classical studies dedicated to Equation (2) were given by Bernstein [12] and Pohozaev [33]. An important incentive to its study was provided by Lions's work [25], where a functional analysis approach was proposed to attack it. We cite, amid the wide literature on the subject, the works [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 16, 18, 20] where Kirchhoff-type problems (also in their stationary versions) were studied by exploiting different methods and new technical approaches.

The aim of the present paper is to study $(P_{\lambda,\mu})$ under the variational viewpoint. Our approach employs a functional framework in appropriate fractional Sobolev spaces and an abstract multiplicity result of the critical point theory (see Section 4 for details). Thanks to these ingredients we manage to derive the existence of three weak solutions to problem $(P_{\lambda,\mu})$. Our analytical context is inspired by (but not equivalent to) the fractional Sobolev spaces, in order to correctly encode the Dirichlet boundary datum in the variational formulation. The nonlocal analysis that we perform here is quite general and exploited for other goals in several recent contributions; see [26, 27, 28, 31, 32]. The papers [23, 24] contain some recent nice results on nonlocal fractional problems as well.

Now, let us denote

$$X_0^{n,s} := \big\{ u \in H^s(\mathbb{R}^n) : u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \big\}, \tag{3}$$

where the functional space $H^s(\mathbb{R}^n)$ is the fractional Sobolev space of the functions $u \in L^2(\mathbb{R}^n)$ such that the map

$$(x, y) \mapsto \frac{u(x) - u(y)}{|x - y|^{\frac{n+2s}{2}}} \in L^2\big(\mathbb{R}^{2n}, dxdy\big).$$

With this notation, a particular case of our main result assumes the following form.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and fix $s \in]3/4, 1[$. Moreover, let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\sup_{t\in\mathbb{R}} F(t) > 0$ in addition to*

(c_2') $\displaystyle\lim_{t\to 0} \frac{F(t)}{t^4} \leq 0,$

where $F(t) := \int_0^t f(\xi)d\xi$, for every $t \in \mathbb{R}$. Further, assume that

(c_3') $\displaystyle\lim_{|t|\to+\infty}\frac{|f(t)|}{|t|^q}<+\infty$, *for some* $q\in[0,3[$.

Then, for each compact interval $[a,b]\subset\,]\theta,+\infty[$, *with*

$$\theta:=\frac{1}{4}\inf\left\{\frac{\|u\|^4}{\displaystyle\int_\Omega F(u(x))dx}\;:\;u\in X_0^{3,s},\int_\Omega F(u(x))dx>0\right\},$$

there exists $\varrho>0$ *such that, for every* $\lambda\in[a,b]$, *the problem*

$$\left\{\begin{array}{ll}\displaystyle\left(\int_{\mathbb{R}^6}\frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}dxdy\right)(-\Delta)^s u=\lambda f(u) & \text{in } \Omega\\ u=0 & \text{in } \mathbb{R}^3\setminus\Omega,\end{array}\right. \tag{$P_{s,\lambda}$}$$

admits at least three weak solutions $\{u_j\}_{j=1}^3\subset X_0^{3,s}$ *such that*

$$\left(\int_{\mathbb{R}^6}\frac{|u_j(x)-u_j(y)|^2}{|x-y|^{3+2s}}dxdy\right)^{1/2}<\varrho,$$

for every $j\in\{1,2,3\}$.

The structure of the parameter θ displayed in Theorem 1.1 is not simple and its value depends on the geometry of the domain Ω and on the datum f. As proved in Section 5, an explicit estimate of θ (an upper bound) can be determined by using suitable test functions belonging to the fractional Sobolev space $H^s(\mathbb{R}^n)$ and some technical lemmas proved in [19].

For instance, if Ω is an open ball of radius $r>1$ and $F(t_0)>0$ for some $t_0\in\mathbb{R}$, then $\theta<\theta^\star$, where

$$\theta^\star:=\min_{\eta\in\Sigma_3}\left\{\frac{3\pi^2t_0^4}{4\left(F(t_0)\eta^3-(1-\eta^3)\displaystyle\max_{|t|\le|t_0|}|F(t)|\right)}\left(\frac{(1-\eta^3)\kappa_1\kappa_2}{(1-\eta)^2}\right)^2\right\}, \tag{4}$$

with

$$\Sigma_3:=\left]\left(\frac{\displaystyle\max_{|t|\le|t_0|}|F(t)|}{F(t_0)+\displaystyle\max_{|t|\le|t_0|}|F(t)|}\right)^{1/3},1\right[,$$

$$\kappa_1:=\frac{8}{3}\sqrt{\pi}\left(\frac{\pi}{4}+\frac{1}{1+2s}\right)\left(1+\frac{1}{\lambda_1}\right),\quad \kappa_2:=\frac{1}{2(1-s)}+\frac{2}{s}, \tag{5}$$

and

$$\lambda_1 := \inf_{u\in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}}.$$

In such a case, Theorem 1.1 ensures that for each compact interval $[a, b] \subset]\theta^\star, +\infty[$, there exists $\varrho > 0$ such that, for every $\lambda \in [a, b]$, our fractional problem on the ball admits at least three weak solutions in $X_0^{3,s}$, uniformly bounded in norm by ϱ.

We point out that the assumption $2s < n < 4s$ is essential in our technical approach in order to guarantee the embedding of the working space $X_0^{n,s}$ (or, more generally, the space X_0 defined in Section 2) in the Lebesgue space $L^r(\mathbb{R}^n)$, where

$$\max\{4, q+1\} < r < \frac{2n}{n-2s},$$

with $q \in \left[0, \dfrac{n+2s}{n-2s}\right[$. This embedding seems to be crucial in the proof of the main result (see Theorem 4.2).

The paper is organized as follows. In Section 2 we recall some basic tools and illustrate the main features of our nonlocal setting. Subsequently, some peculiar properties of the functionals involved in the weak formulation of problem $(P_{\lambda,\mu})$ are proved in Section 3 (see Proposition 3.2 and Remark 3.5). In Section 4 we prove the main Theorem 4.2 (and some of its consequences) by making use of the previous preparatory results and the already mentioned abstract critical point theorem (Theorem 4.1). In the last section a concrete upper bound for the parameter θ of Theorem 4.2 is presented. In such a case a restriction on the structure of the kernel function K is required.

We cite the recent monograph [30] as a general reference on nonlocal fractional problems and the variational methods used in this paper.

2 Variational setting

In this section we describe the variational background of problem $(P_{\lambda,\mu})$, starting with the space X_0 where the weak solutions are going to be sought. We recall that this space was introduced in [40].

To begin with, we define X to be the linear space of all Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that the restriction to Ω of any function u in X belongs to $L^2(\Omega)$ and the map

$$(x, y) \mapsto (u(x) - u(y))\sqrt{K(x-y)} \in L^2(Q, dxdy),$$

where $Q := \mathbb{R}^{2n} \setminus (\Omega^c \times \Omega^c)$ and $\Omega^c := \mathbb{R}^n \setminus \Omega$.

Denote by X_0 the following linear subspace of X,

$$X_0 := \{u \in X : u = 0 \text{ a.e. in } \Omega^c\}.$$

We point out that X and X_0 are non-empty since, for instance, $C_0^2(\Omega) \subseteq X_0$ by [40, Lemma 11]. It is easily seen that

$$\|u\|_X := \|u\|_{L^2(\Omega)} + \left(\int_Q |u(x)-u(y)|^2 K(x-y)dxdy\right)^{1/2} \tag{6}$$

defines a norm on X (see [39]); in addition, the function

$$X_0 \ni u \mapsto \|u\|_{X_0} := \left(\int_Q |u(x)-u(y)|^2 K(x-y)dxdy\right)^{1/2} \tag{7}$$

represents a norm on X_0 equivalent to (6) (see [39, Lemma 6]).

The space $(X_0, \|\cdot\|_{X_0})$ turns out to be a Hilbert space with scalar product

$$\langle u, v\rangle_{X_0} := \int_Q (u(x)-u(y))\,(v(x)-v(y))\,K(x-y)dxdy, \tag{8}$$

see [39, Lemma 7].

Notice that in (7) and (8) the integral can be extended to the whole of $\mathbb{R}^{2n}$, since $u \in X_0$ (and so $u = 0$ a.e. in Ω^c). In the sequel, in order to simplify the notation, we will denote $\|\cdot\|_{X_0}$ and $\langle\cdot,\cdot\rangle_{X_0}$ simply by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively.

In the model case $K(x) := |x|^{-(n+2s)}$, the space X_0 can be characterized as follows

$$X_0 = \big\{u \in H^s(\mathbb{R}^n) : u = 0 \text{ a.e. in } \Omega^c\big\},$$

(see [42, Lemma 7(b)]), i.e. $X_0 = X_0^{n,s}$. Here $H^s(\mathbb{R}^n)$ denotes the usual fractional Sobolev space endowed with the so-called Gagliardo norm (not equivalent to (6)):

$$\|u\|_{H^s(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)} + \left(\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy\right)^{1/2}.$$

Under assumptions (k_1) and (k_2) the space X_0 embeds in the canonical Lebesgue spaces; more specifically, the embedding $X_0 \hookrightarrow L^p(\mathbb{R}^n)$ is continuous for any $p \in [1, 2^*]$, while it is compact whenever $p \in [1, 2^*[$, where

$$2^* := \frac{2n}{n-2s}$$

denotes the fractional critical Sobolev exponent (see [39, 41] for details).

The reader interested in knowing more about fractional Sobolev spaces is invited to consult the very nice survey paper [19] and the references therein, while for more details on X and X_0 we refer to the papers [38, 39, 40, 41, 42], where these functional spaces were introduced and analysed in their basic properties.

Now, let us see how $(P_{\lambda,\mu})$ is endowed with a variational structure. Denote by $\mathcal{A}$ the class of all admissible nonlinearities for our problem, i.e. the class of all Carathéodory functions $h : \Omega \times \mathbb{R} \to \mathbb{R}$ such that

$$\sup_{(x,t)\in\Omega\times\mathbb{R}} \frac{|h(x,t)|}{1+|t|^q} < +\infty \tag{9}$$

for some $q \in [0, 2^* - 1[$. Further, let $f, g \in \mathcal{A}$ and put

$$F(x,t) := \int_0^t f(x,\xi)d\xi \quad \text{and} \quad G(x,t) := \int_0^t g(x,\xi)d\xi, \tag{10}$$

for every $(x,t) \in \Omega \times \mathbb{R}$.

Fixed the real parameters λ and μ, it is a standard matter to prove that problem $(P_{\lambda,\mu})$ is the Euler-Lagrange equation of the C^1-functional $\mathcal{E}_{\lambda,\mu} : X_0 \to \mathbb{R}$ defined by

$$\mathcal{E}_{\lambda,\mu}(u) := \Phi(u) - \lambda J_f(u) - \mu J_g(u) \tag{11}$$

where

$$\Phi(u) := \frac{1}{2}\widehat{M}(\|u\|^2), \tag{12}$$

$$J_f(u) := \int_\Omega F(x,u(x))dx, \quad J_g(u) := \int_\Omega G(x,u(x))dx, \tag{13}$$

for any $u \in X_0$.

As a result, the search for weak solutions to problem $(P_{\lambda,\mu})$, namely functions $u \in X_0$ such that

$$M(\|u\|^2)\,\langle u, v\rangle = \lambda \int_\Omega f(x,u(x))v(x)dx + \mu \int_\Omega g(x,u(x))v(x)dx$$

for every $v \in X_0$, reduces to the search for critical points of the energy functional $\mathcal{E}_{\lambda,\mu}$.

The variational tool used to derive the existence of such critical points is a multiplicity result established in [36] that we recall for ease of reference in Section 4.

3 Some properties of the functionals Φ and J

For a generic Banach space E, denote by "$\to$" and "$\rightharpoonup$" the strong and the weak convergence in E, respectively. Further, let E^* be the topological dual of E. Denote by $\mathcal{W}_E$ the class of all functionals $I : E \to \mathbb{R}$ with the following property: *if $u_j \rightharpoonup u$ in E and*

$$\liminf_{j\to\infty} I(u_j) \leq I(u),$$

then $u_j \to u$ up to a subsequence.

In this section, by using the above notations, we focus our attention on some properties of the functionals involved in the weak formulation of $(P_{\lambda,\mu})$, starting with Φ.

First, let us recall the definition of a mapping of type (S_+) which, as known, generalizes the notion of uniform monotonicity.

Definition 3.1. *Given a real Banach space E, a mapping $I : E \to E^*$ is said to be of type (S_+) if for any sequence $\{u_j\}_{j\in\mathbb{N}}$ in E such that $u_j \rightharpoonup u \in E$ and*

$$\limsup_{j\to\infty} I(u_j)(u_j - u) \leq 0,$$

then $u_j \to u$.

Proposition 3.2. *Assume that the kernel K satisfies conditions (k_1) and (k_2). Further, suppose that the Kirchhoff term M verifies (m_1) and (m_2). Then, the following facts hold:*

i) Φ *is a C^1 sequentially weakly lower semicontinuous functional*;

ii) $\Phi \in \mathcal{W}_{X_0}$;

iii) $\Phi' : X_0 \to X_0^*$ *is strictly monotone*;

iv) Φ' *is of type (S_+)*;

v) Φ' *is invertible on X_0 with continuous inverse.*

Proof. i) It is easy to see that $\Phi \in C^1(X_0, \mathbb{R})$. Moreover, the non-negativity of M forces $\widehat{M}$ to be increasing. The conclusion then follows by the sequential weak lower semicontinuity of $u \mapsto \|u\|$.
ii) Let $u_j \rightharpoonup u$ in X_0 and

$$\liminf_{j\to\infty} \Phi(u_j) \leq \Phi(u).$$

The sequential weak lower semicontinuity of Φ yields $\Phi(u_j) \to \Phi(u)$ as $j \to \infty$ up to a subsequence. Since $\widehat{M}$ and $t \mapsto t^2$ are continuous and strictly increasing in $[0, +\infty[$, then $\|u_j\| \to \|u\|$. By a classical result holding in any uniformly convex space, we obtain that $u_j \to u$, as desired.
iii) Let $u, v \in X_0$ with $u \neq v$ and $\alpha_1, \alpha_2 \in]0, 1[$ with $\alpha_1 + \alpha_2 = 1$. Of course $u \mapsto \|u\|^2$ is strictly convex in X_0 and by (m_1), $\widehat{M}$ is convex in $[0, +\infty[$. Thus we have

$$\begin{aligned}\Phi(\alpha_1 u + \alpha_2 v) &< \frac{1}{2}\widehat{M}\left(\alpha_1 \|u\|^2 + \alpha_2 \|v\|^2\right) \\ &\leq \frac{1}{2}\alpha_1 \widehat{M}(\|u\|^2) + \frac{1}{2}\alpha_2 \widehat{M}(\|v\|^2) \\ &= \alpha_1 \Phi(u) + \alpha_2 \Phi(v);\end{aligned}$$

so Φ is strictly convex and, in view of Proposition 25.10 of [43], Φ' is strictly monotone.
iv) Let us identify X_0 with X_0^* via the canonical isomorphism. So, let $u_j \rightharpoonup u$ in X_0 and assume that

$$\limsup_{j\to\infty} \left\langle \Phi'(u_j), u_j - u \right\rangle \leq 0.$$

We will prove that any subsequence of $\{u_j\}_{j\in\mathbb{N}}$ has in turn a strongly convergent subsequence that converges to u. Pick such a subsequence and denote it again by $\{u_j\}_{j\in\mathbb{N}}$; since it is bounded there will exist a further subsequence, still relabeled as $\{u_j\}_{j\in\mathbb{N}}$, and a non-negative number c so that

$$\lim_{j\to\infty} M(\|u_j\|^2) = c.$$

If $c = 0$, in view of assumption (m_2) we get

$$0 \leq m_1 \|u_j\|^2 \leq M\left(\|u_j\|^2\right)$$

for any $j \in \mathbb{N}$ and hence $u_j \to 0$ as $j \to \infty$.
Assume now $c > 0$. Since Φ' is a monotone operator then

$$\begin{aligned}\lim_{j\to\infty} \left\langle \Phi'(u_j), u_j - u \right\rangle &= \lim_{j\to\infty} M\left(\|u_j\|^2\right) \langle u_j, u_j - u\rangle \\ &= c \lim_{j\to\infty} \langle u_j, u_j - u\rangle = 0.\end{aligned}$$

Therefore, by exploiting also the fact that $u_j \rightharpoonup u$, we obtain

$$\lim_{j\to\infty} \|u_j - u\|^2 = \lim_{j\to\infty} \left(\langle u_j, u_j - u\rangle - \langle u, u_j - u\rangle\right) = 0.$$

Hence $u_j \to u$, as claimed.

v) We prove the assertion by using monotone operator methods. For any $u \in X_0 \setminus \{0\}$, by (m_2) we have

$$\frac{\langle \Phi'(u), u\rangle}{\|u\|} = \frac{M\left(\|u\|^2\right)\|u\|^2}{\|u\|} \geq m_1 \|u\|^3$$

and thus Φ' is coercive (in the sense of monotone operators), while the function

$$t \mapsto M\left(\|u+tv\|^2\right)\langle u+tv, w\rangle$$

is continuous in $[0,1]$ for any $u, v, w \in X_0$, meaning that Φ' is hemicontinuous.

Taking also iii) into account, we obtain via Minty-Browder's theorem (Theorem 26.A(d) of [43]) that Φ' is invertible in X_0 and Φ'^{-1} is, in particular, bounded. Let us prove the continuity of Φ'^{-1}.

Let $\{v_j\}_{j\in\mathbb{N}} \subset X_0$ be a sequence such that $v_j \to v \in X_0$ and let $u_j := \Phi'^{-1}(v_j)$, $u := \Phi'^{-1}(v)$. The boundedness of $\{u_j\}_{j\in\mathbb{N}}$ yields that $u_j \rightharpoonup u_0 \in X_0$. So, thanks also to Cauchy-Schwartz's inequality, we obtain

$$\begin{aligned}
|\langle v_j, u_j - u_0\rangle| &\leq |\langle v_j - v, u_j - u_0\rangle| + |\langle v, u_j - u_0\rangle| \\
&\leq \|v_j - v\|\,\|u_j - u_0\| + |\langle v, u_j\rangle - \langle v, u_0\rangle|
\end{aligned}$$

and therefore

$$\lim_{j\to\infty} \left\langle \Phi'(u_j), u_j - u_0 \right\rangle = \lim_{j\to\infty} \langle v_j, v_j - v_0\rangle = 0.$$

Being Φ' of type (S_+), we get $u_j \to u_0$ and, being Φ' continuous and one-to-one, we finally get $u_j \to u$. $\square$

Remark 3.3. The delicate point in the previous proposition is the proof of the property (S_+) (carried out by passing to subsequences) which permits to guarantee the continuity of the inverse of Φ'. The crucial fact is that, due to (m_2), M has a superlinear (or, at most, linear) behavior along all the positive reals.

Remark 3.4. In [37] Ricceri dealt with a Kirchhoff-type problem driven by the classical Laplace operator (which $(P_{\lambda,\mu})$ reduces to when $K(x) = |x|^{-(n+2s)}$ and $s \to 1^-$). Instead of standard monotonicity assumptions on the Kirchhoff term M, taking advantage of the Hilbert space setting, he used the following assumption:

(h_1) *there exists a continuous function* $h : [0, +\infty[\to \mathbb{R}$ *such that*

$$h(tM(t^2)) = t,$$

for every $t \in [0, +\infty[$.

This hypothesis is crucial along the paper [37] to deduce regularity properties on the functional Φ analogous to the ones of Proposition 3.2. This approach is not feasible in our case as it requires that

$$\inf_{t\in[0,+\infty[} M(t) > 0,$$

and hence rules the degenerate case out.

Remark 3.5. As concerns the functional J, the fact that $f \in \mathcal{A}$ and the embeddings in the canonical L^p spaces lead to the conclusion, through very standard arguments, that $J_f \in C^1(X_0, \mathbb{R})$ and that J' is compact.

4 Abstract approach and main results

In this section we prove our main multiplicity results. The key tool will be the following abstract critical point theorem for differentiable functionals (cf. [36, Theorem 2]).

Theorem 4.1. *Let* E *be a separable and reflexive real Banach space;* $\Phi : E \to \mathbb{R}$ *a coercive, sequentially weakly lower semicontinuous* C^1 *functional, belonging to* $\mathcal{W}_E$*, bounded on each bounded subset of* E *and whose derivative admits a continuous inverse on* E^**;* $J : E \to \mathbb{R}$ *a* C^1 *functional with compact derivative. Assume that* Φ *has a strict local minimum* z_0 *with* $\Phi(z_0) = J(z_0) = 0$*. Finally, assume that*

$$\max\left\{\limsup_{\|z\|\to+\infty} \frac{J(z)}{\Phi(z)}, \limsup_{z\to z_0} \frac{J(z)}{\Phi(z)}\right\} \leq 0 \tag{14}$$

and

$$\sup_{z\in E} \min\{\Phi(z), J(z)\} > 0. \tag{15}$$

Set

$$\theta := \inf\left\{\frac{\Phi(z)}{J(z)} : z \in E,\ \min\{\Phi(z), J(z)\} > 0\right\}. \tag{16}$$

Then, for each compact interval $[a, b] \subset]\theta, +\infty[$ *there exists a number* $\varrho > 0$ *with the following property: for every* $\lambda \in [a, b]$ *and every* C^1 *functional*

$\Psi : E \to \mathbb{R}$ *with compact derivative, there exists* $\tilde{\mu} > 0$ *such that, for each* $\mu \in [0, \tilde{\mu}]$, *the equation*

$$\Phi'(z) - \lambda J'(z) - \mu \Psi'(z) = 0 \tag{17}$$

has at least three solutions whose norms are less than ϱ.

The main result of this manuscript reads as follows.

Theorem 4.2. *Let* $s \in]0,1[$, $2s < n < 4s$ *and let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set with Lipschitz boundary* $\partial\Omega$. *Assume that* $K : \mathbb{R}^n \setminus \{0\} \to]0, +\infty[$ *and* $M : [0, +\infty[\to [0, +\infty[$ *are two functions satisfying* $(k_1) - (k_2)$ *and* $(m_1) - (m_3)$, *respectively. Further, let* $f \in \mathcal{A}$ *such that:*

(a_1) $\displaystyle\sup_{u \in X_0} \int_\Omega F(x, u(x))dx > 0,$

(a_2) $\displaystyle\limsup_{t \to 0} \frac{\sup_{x\in\Omega} F(x,t)}{t^4} \leq 0,$

(a_3) $\displaystyle\limsup_{|t| \to +\infty} \frac{\sup_{x\in\Omega} F(x,t)}{|t|^{2\sigma}} \leq 0,$

with F *defined by* (10). *Under such hypotheses, if we set*

$$\theta := \frac{1}{2} \inf \left\{ \frac{\widehat{M}\left(\|u\|^2\right)}{\displaystyle\int_\Omega F(x, u(x))dx} : \ u \in X_0, \int_\Omega F(x, u(x))dx > 0 \right\},$$

then for each compact interval $[a, b] \subset]\theta, +\infty[$, *there exists a number* $\varrho > 0$ *with the following property: for every* $\lambda \in [a, b]$ *and every* $g \in \mathcal{A}$, *there exists* $\tilde{\mu} > 0$ *such that, for each* $\mu \in [0, \tilde{\mu}]$, *problem* $(P_{\lambda,\mu})$ *has at least three weak solutions whose norms in* X_0 *are less than* ϱ.

Proof. Let us invoke the abstract Theorem 4.1 with the choices $E := X_0$, Φ as in (12) and $J := J_f$. In view of assumption (m_2) Φ is coercive, as

$$\Phi(u) \geq \frac{m_1}{4} \|u\|^4$$

for every $u \in X_0$.

Moreover, the increasingness of $\widehat{M}$ implies that Φ is bounded on any bounded subset of X_0. The rest of the properties Φ is required to fulfill follows from Proposition 3.2. It is straightforward to realize that $u_0 = 0$ is the only global minimum of Φ and that $\Phi(u_0) = J(u_0) = 0$.

Now, the membership of f in $\mathcal{A}$ implies that

$$|f(x,t)| \leq c\,(1 + |t|^q) \tag{18}$$

for all $(x,t) \in \Omega \times \mathbb{R}$, for some $c > 0$ and for some $q \in [0, 2^* - 1[$.

Since $2s < n < 4s$, then $4 < 2^*$ and it is possible to pick $r \in \mathbb{R}$ such that

$$\max\{4, q+1\} < r < 2^*.$$

Now, fix $\varepsilon > 0$. Due to (a_2) there exists $T > 0$ so that

$$F(x,t) \leq \varepsilon t^4$$

for each $(x,t) \in \Omega \times [-T, T]$; on the other hand,

$$|F(x,t)| \leq c(1 + |t|^r)$$

for each $(x,t) \in \Omega \times \mathbb{R}$ and the function $\mathbb{R} \setminus [-T,T] \ni t \mapsto |t|^r$ attains its minimum at $t = T$, so

$$|F(x,t)| \leq c|t|^r$$

for each $(x,t) \in \Omega \times (\mathbb{R} \setminus [-T,T])$.

As a result we get the estimate

$$|F(x,t)| \leq \varepsilon t^4 + c|t|^r,$$

holding for each $(x,t) \in \Omega \times \mathbb{R}$. So, by the embeddings $X_0 \hookrightarrow L^4(\Omega)$, $X_0 \hookrightarrow L^r(\Omega)$ and by (m_2) we obtain

$$\begin{aligned}
J(u) &\leq \varepsilon \|u\|^4_{L^4(\Omega)} + c\,\|u\|^r_{L^r(\Omega)} \\
&\leq \varepsilon c_1 \|u\|^4 + c_2 \|u\|^r \\
&\leq \frac{2\varepsilon c_1}{m_1} \widehat{M}\left(\|u\|^2\right) + c_2 \left(\frac{2}{m_1} \widehat{M}\left(\|u\|^2\right)\right)^{\frac{r}{4}} \\
&\leq \frac{4\varepsilon c_1}{m_1} \Phi(u) + c_2 \left(\frac{4}{m_1} \Phi(u)\right)^{\frac{r}{4}}
\end{aligned}$$

for any $u \in X_0$ and for suitable positive constants c_1 and c_2.

Due to the choice of r one has

$$\limsup_{u \to 0} \frac{J(u)}{\Phi(u)} \leq \frac{4\varepsilon c_1}{m_1}. \tag{19}$$

Next, define β by

$$\beta := 2\min\left\{\sigma, \frac{n}{n-2s}\right\}.$$

When $\sigma < n/(n-2s)$, thanks to (a_3) it is easy to obtain the estimate

$$|F(x,t)| \leq \varepsilon|t|^{2\sigma} + c_3$$

for all $(x,t) \in \Omega \times \mathbb{R}$, for some $c_3 > 0$; on the other hand, when $\sigma > n/(n-2s)$, owing to the fact that $|t|^{r-2^*} \to 0$ as $|t| \to +\infty$, one gets

$$|F(x,t)| \leq \varepsilon|t|^{2^*} + c_4$$

for all $(x,t) \in \Omega \times \mathbb{R}$ and some $c_4 > 0$.

As a byproduct we obtain

$$|F(x,t)| \leq \varepsilon|t|^{\beta} + c_5$$

for all $(x,t) \in \Omega \times \mathbb{R}$ and some $c_5 > 0$, and

$$J(u) \leq c_6(1 + \varepsilon\|u\|^{\beta}) \tag{20}$$

for every $u \in X_0$ and some $c_6 > 0$.

Moreover, by (m_3) one has

$$\widehat{M}(t) \geq m_\sigma|t|^\sigma - c_7$$

for every $t \in [0, +\infty[$ and for some $c_7 > 0$.

So, taking account of (20), we obtain

$$\frac{J(u)}{\Phi(u)} \leq \frac{c_6(\varepsilon\|u\|^{\beta} + 1)}{m_\sigma\|u\|^{2\sigma} - c_8}$$

for every $u \neq 0$ and some $c_8 > 0$ and, due to the definition of β,

$$\limsup_{\|u\| \to +\infty} \frac{J(u)}{\Phi(u)} \leq \frac{c_6\varepsilon}{m_\sigma}. \tag{21}$$

Conditions (19), (21), together with the arbitrariness of ε, yield

$$\max\left\{\limsup_{u \to 0} \frac{J(u)}{\Phi(u)}, \limsup_{\|u\| \to +\infty} \frac{J(u)}{\Phi(u)}, \right\} \leq 0,$$

and all the assumptions of Theorem 4.1 are satisfied. So, for each compact interval $[a, b] \subset]\theta, +\infty[$ there exists a number $\varrho > 0$ with the property described in the conclusion of Theorem 4.1.

Fix $\lambda \in [a, b]$, $g \in \mathcal{A}$ and set

$$\Psi(u) := \int_\Omega G(x, u(x))dx$$

for all $u \in X_0$; clearly $\Psi \in C^1(X_0, \mathbb{R})$ and Ψ' is compact, so there exists $\tilde{\mu} > 0$ such that, for each $\mu \in [0, \tilde{\mu}]$, the equation

$$\mathcal{E}'_{\lambda,\mu}(u) = \Phi'(u) - \lambda J'(u) - \mu\Psi'(u) = 0$$

has at least three solutions whose X_0-norms are less than ϱ.

This ends the proof. $\square$

The simplest example of a function M satisfying the set of assumptions $(m_1) - (m_3)$ is clearly a line passing through the origin, to wit $M(t) := m_1 t$, $m_1 > 0$. In this regard, it is immediate to obtain the coming theorem from the previous one.

Corollary 4.3. *Let s, n, Ω, K be as in Theorem 4.2 and let $f \in \mathcal{A}$ be such that:*

(b_1) $\displaystyle\sup_{u \in X_0} \int_\Omega F(x, u(x))dx > 0,$

(b_2) $\displaystyle\max\left\{\limsup_{t \to 0} \frac{\sup_{x\in\Omega} F(x,t)}{t^4}, \limsup_{|t| \to +\infty} \frac{\sup_{x\in\Omega} F(x,t)}{t^4}\right\} \leq 0.$

Then, if we pick $m_1 > 0$ and set

$$\theta := \frac{m_1}{4} \inf\left\{\frac{\|u\|^4}{\displaystyle\int_\Omega F(x, u(x))dx} \,:\, u \in X_0, \int_\Omega F(x, u(x))dx > 0\right\},$$

for each compact interval $[a, b] \subset]\theta, +\infty[$, there exists a number $\varrho > 0$ with the following property: for every $\lambda \in [a, b]$ and every $g \in \mathcal{A}$ there exists $\tilde{\mu} > 0$ such that, for each $\mu \in [0, \tilde{\mu}]$, the problem

$$\begin{cases} -m_1 \|u\|^2 \mathcal{L}_K u = \lambda f(x, u) + \mu g(x, u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \qquad (\widetilde{P}_{\lambda,\mu})$$

admits at least three weak solutions whose norms in X_0 are less than ϱ.

Proof. Fix $m_1 > 0$ and apply Theorem 4.2 with $M(t) := m_1 t$ for all $t \geq 0$. Condition (m_3) follows at once with the choice $\sigma = 2$; this choice itself implies (a_3) via assumption (b_2). $\square$

Finally, if we drop the dependence of f from x, we obtain the following meaningful result.

Corollary 4.4. *Let s, n, Ω, K be as in Theorem 4.2, and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

(c_1) $\sup\limits_{t \in \mathbb{R}} F(t) > 0,$

(c_2) $\limsup\limits_{t \to 0} \dfrac{F(t)}{t^4} \leq 0.$

Further, assume that

(c_3) $\limsup\limits_{|t| \to +\infty} \dfrac{|f(t)|}{|t|^q} < +\infty$, *for some $q \in [0, 3[$.*

Then, if we pick $m_1 > 0$ and set

$$\theta := \frac{m_1}{4} \inf \left\{ \frac{\|u\|^4}{\displaystyle\int_\Omega F(u(x))dx} \; : \; u \in X_0, \int_\Omega F(u(x))dx > 0 \right\}, \tag{22}$$

for each compact interval $[a, b] \subset]\theta, +\infty[$, there exists a number $\varrho > 0$ with the following property: for every $\lambda \in [a, b]$ and every $g \in \mathcal{A}$, there exists $\tilde{\mu} > 0$ such that, for each $\mu \in [0, \tilde{\mu}]$, the problem

$$\begin{cases} -m_1 \|u\|^2 \mathcal{L}_K u = \lambda f(u) + \mu g(x, u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{$\widehat{P}_{\lambda,\mu}$}$$

admits at least three weak solutions whose norms in X_0 are less than ϱ.

Proof. The continuity of f and assumption (c_3) imply that $f \in \mathcal{A}$ and, in turn, that

$$\frac{F(t)}{t^4} \leq c_9 |t|^{-3} + c_{10} |t|^{q-3}$$

for all $t \in \mathbb{R} \setminus \{0\}$ and for some $c_9, c_{10} > 0$. Passing to the $\limsup$ for $|t| \to +\infty$ in the last inequality we obtain that (b_2) is completely satisfied and hence, we get the thesis from Corollary 4.3. $\square$

Remark 4.5. Theorem 1.1 stated in Introduction immediately follows by Corollary 4.4 bearing in mind that, as pointed out in Section 2, in the classical fractional Laplacian setting one has $X_0 = X_0^{n,s}$.

5 An upper bound for θ and some consequences

In this final section, by following the approach adopted in [29], we provide an explicit estimate of the number θ appearing in Corollary 4.4.

To reach our goal we need to require that the kernel, instead of (k_1) and (k_2), satisfies the following more restrictive hypothesis:

(k_3) *there exists* $\alpha \in]0,1]$ *such that*

$$\alpha|x|^{-(n+2s)} \leq K(x) \leq \alpha^{-1}|x|^{-(n+2s)},$$

for any $x \in \mathbb{R}^n \setminus \{0\}$.

It is clear that (k_3) implies (k_1) and (k_2). Moreover, for $\alpha = 1$ we recover exactly the singular Riesz kernel and hence the operator $(-\Delta)^s$. We also point out that assumption (k_3) is not new in the literature; see, for instance, the already cited paper [21] by Fiscella and Valdinoci.

In the one-dimensional setting, a function K satisfying the above assumption is given by the coming

Example 5.1. Choose $s = 3/8$, $n = 1$, $\alpha = 1/2$ and $K : \mathbb{R} \setminus \{0\} \to]0, +\infty[$ defined by

$$K(x) := \begin{cases} \dfrac{1}{2|x|^{\frac{7}{4}}} & \text{for } x \in [-x_0, x_0] \setminus \{0\} \\ -x^2 + \dfrac{3}{2} & \text{for } x \in [-1, -x_0[\cup [x_0, 1] \\ \dfrac{2}{|x|^{\frac{7}{4}}} & \text{for } x \in]-\infty, -1[\cup]1, +\infty[, \end{cases}$$

where $x_0 \in]0,1[$ is the root of the irrational equation

$$2x^{\frac{15}{4}} = 3x^{\frac{7}{4}} - 1.$$

Simple calculations show that K fulfills (k_3).

The key point to estimate θ is the construction of a suitable function in X_0 at which evaluating the norm and the functional J_f. To this end, denote by $B(x_0, r)$ (respectively $\overline{B}(x_0, r)$) the n-dimensional open (respectively closed) ball centered at $x_0 \in \mathbb{R}^n$ and of radius $r > 0$. As Ω is open, we can certainly choose a point $x_0 \in \Omega$ and a number $\tau > 0$ so that $\overline{B}(x_0, \tau) \subset \Omega$.

Fix such x_0 and τ and, for any $\eta \in]0,1[, t \in \mathbb{R}$, define u^t_η to be

$$u^t_\eta(x) := \begin{cases} 0 & \text{if } x \in \mathbb{R}^n \setminus B(x_0,\tau) \\ \dfrac{t}{(1-\eta)\tau}\,(\tau - |x-x_0|) & \text{if } x \in B(x_0,\tau) \setminus B(x_0,\eta\tau) \\ t & \text{if } x \in B(x_0,\eta\tau), \end{cases} \tag{23}$$

for every $x \in \mathbb{R}^n$, where $|\cdot|$ denotes the usual Euclidean norm in $\mathbb{R}^n$.

Moreover, for $n \in \{1,2,3\}$, set

$$\nu := \left(1 + \frac{1}{\lambda_1}\right), \tag{24}$$

where

$$\lambda_1 := \inf_{u \in W^{1,2}_0(\Omega)\setminus\{0\}} \frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}}. \tag{25}$$

The next result gives a localization of u^t_η in X_0.

Lemma 5.2. *Let $\tau > 0$ be defined as before. For any $\eta \in]0,1[$ and $t \in \mathbb{R}$ one has that $u^t_\eta \in X_0$ and*

$$\|u^t_\eta\| < \frac{|t|}{(1-\eta)}\sqrt{\frac{2\pi^{\frac{n}{2}}\tau^{n-2}(1-\eta^n)}{\alpha}\kappa_1\kappa_2}, \tag{26}$$

where

$$\kappa_1 := \begin{cases} \dfrac{2}{\sqrt{\pi}}\nu & \text{if } n=1 \\ \left(\dfrac{\pi}{2} + \dfrac{2}{1+2s}\right)\nu & \text{if } n=2 \\ \dfrac{8}{3}\sqrt{\pi}\left(\dfrac{\pi}{4} + \dfrac{1}{1+2s}\right)\nu & \text{if } n=3 \end{cases} \quad \text{and} \quad \kappa_2 := \frac{1}{2(1-s)} + \frac{2}{s}.$$

Proof. We argue by following the procedure developed in [29, Lemma 3]. More precisely, the direct computation of the (standard) $H^1(\mathbb{R}^n)$-seminorm of u^t_η leads to

$$\begin{aligned} [u^t_\eta]^2_{H^1(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |\nabla u^t_\eta(x)|^2\,dx = \int_{B(x_0,\tau)\setminus B(x_0,\eta\tau)} \frac{t^2}{(1-\eta)^2\tau^2}dx \\ &= \frac{t^2}{(1-\eta)^2\tau^2}(|B(x_0,\tau)| - |B(x_0,\eta\tau)|) \\ &= \frac{t^2\pi^{\frac{n}{2}}\tau^{n-2}(1-\eta^n)}{(1-\eta)^2\Gamma\left(1+\frac{n}{2}\right)}, \end{aligned} \tag{27}$$

where $|B(x_0,\tau)|$ and $|B(x_0,\eta\tau)|$ denote respectively the Lebesgue measure of $B(x_0,\tau)$ and $B(x_0,\eta\tau)$ and

$$\Gamma(t) := \int_0^{+\infty} z^{t-1}e^{-z}dz, \quad \text{for all } t>0,$$

is the usual Gamma function.

Now, since $u_\eta^t \in W_0^{1,2}(\Omega) \subset W^{1,2}(\Omega)$, by [19, Proposition 2.2] it follows that $u_\eta^t \in W^{s,2}(\Omega)$. Moreover, the boundary $\partial\Omega$ is Lipschitz, $\overline{B}(x_0,\tau) \subset \Omega$, and $u_\eta^t = 0$ in $\Omega \setminus \overline{B}(x_0,\tau)$: by [19, Lemma 5.1] one has that $u_\eta^t \in H^s(\mathbb{R}^n)$.

Hence, thanks also to Proposition 3.4 of [19], we have

$$\begin{aligned}\|u_\eta^t\|^2 &\le \frac{1}{\alpha}\int_{\mathbb{R}^{2n}} \frac{|u_\eta^t(x)-u_\eta^t(y)|^2}{|x-y|^{n+2s}}dxdy \\ &\le \frac{2}{\alpha}\left(\int_{\mathbb{R}^n}\frac{1-\cos x_1}{|x|^{n+2s}}dx\right)\int_{\mathbb{R}^n}|\xi|^{2s}|\mathcal{F}u_\eta^t(\xi)|^2d\xi,\end{aligned} \tag{28}$$

where

$$\mathcal{F}u_\eta^t(\xi) := \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-i\xi\cdot x}u_\eta^t(x)dx$$

stands for the classical Fourier transform of u_η^t. Now, since $s \in \,]0,1[$, one has

$$\int_{\mathbb{R}^n}|\xi|^{2s}|\mathcal{F}u_\eta^t(\xi)|^2d\xi < \int_{\mathbb{R}^n}\left(1+|\xi|^2\right)|\mathcal{F}u_\eta^t(\xi)|^2d\xi$$

and furthermore

$$\begin{aligned}\int_{\mathbb{R}^n}\left(1+|\xi|^2\right)|\mathcal{F}u_\eta^t(\xi)|^2d\xi &= \int_{\mathbb{R}^n}\left(|u_\eta^t(x)|^2+|\nabla u_\eta^t(x)|^2\right)dx \\ &\le \nu\int_\Omega |\nabla u_\eta^t(x)|^2dx = \nu\left[u_\eta^t\right]^2_{H^1(\mathbb{R}^n)}.\end{aligned} \tag{29}$$

Indeed

$$u_\eta^t \in L^2(\mathbb{R}^n) \quad \text{if and only if} \quad \mathcal{F}u_\eta^t \in L^2(\mathbb{R}^n)$$

and

$$\|u_\eta^t\|^2_{L^2(\mathbb{R}^n)} = \|\mathcal{F}u_\eta^t\|^2_{L^2(\mathbb{R}^n)}; \tag{30}$$

further, for every $j \in \{1,...,n\}$, one has

$$\partial_j u_\eta^t \in L^2(\mathbb{R}^n) \quad \text{if and only if} \quad \xi_j\mathcal{F}u_\eta^t \in L^2(\mathbb{R}^n)$$

and

$$\|\nabla u_\eta^t\|_{L^2(\mathbb{R}^n)}^2 = \||\xi|\mathcal{F}u_\eta^t\|_{L^2(\mathbb{R}^n)}^2. \tag{31}$$

Relations (30) and (31) give

$$\int_{\mathbb{R}^n} (1+|\xi|^2)|\mathcal{F}u_\eta^t(\xi)|^2\, d\xi = \|u_\eta^t\|_{L^2(\mathbb{R}^n)}^2 + \|\nabla u_\eta^t\|_{L^2(\mathbb{R}^n)}^2. \tag{32}$$

Hence, by (24) and (25) one has that inequality (29) is a direct consequence of (32). Then by virtue of (29) relation (28) becomes

$$\|u_\eta^t\|^2 \le \frac{2\nu}{\alpha}\left(\int_{\mathbb{R}^n} \frac{1-\cos x_1}{|x|^{n+2s}}dx\right)\left[u_\eta^t\right]_{H^1(\mathbb{R}^n)}^2.$$

If $n=3$ then

$$\int_{\mathbb{R}^3} \frac{1-\cos x_1}{|x|^{3+2s}}dx = \left(\int_{\mathbb{R}^2} \frac{1}{(1+|x|^2)^{\frac{3+2s}{2}}}dx\right)\left(\int_{\mathbb{R}} \frac{1-\cos t}{|t|^{1+2s}}dt\right).$$

The conclusion follows by (27) and the estimates

$$\int_{\mathbb{R}^2} \frac{1}{(1+|x|^2)^{\frac{3+2s}{2}}}dx < 2\pi\left(\frac{\pi}{4}+\frac{1}{1+2s}\right),$$

$$\int_{\mathbb{R}} \frac{1-\cos t}{|t|^{1+2s}}dt < \frac{1}{2(1-s)}+\frac{2}{s}.$$

On the other hand, we also have

$$\int_{\mathbb{R}^n} \frac{1-\cos x_1}{|x|^{n+2s}}dx < \begin{cases} \dfrac{1}{2(1-s)}+\dfrac{2}{s} & \text{if } n=1 \\ \left(\dfrac{\pi}{2}+\dfrac{2}{1+2s}\right)\left(\dfrac{1}{2(1-s)}+\dfrac{2}{s}\right) & \text{if } n=2 \end{cases}$$

and hence inequality (26) is completely proved.

Clearly $u_\eta^t : \mathbb{R}^n \to \mathbb{R}$ is a continuous function, $u_\eta^t \in L^2(\Omega)$ and by the above computations it follows that $u_\eta^t \in X$. Finally $u_\eta^t = 0$ in $\mathbb{R}^n \setminus \Omega$ and thus $u_\eta^t \in X_0$. □

The following preparatory result will be crucial in the proof of Proposition 5.4.

Lemma 5.3. *Let τ be as before and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (c_1). Then there exist $t_0 \in \mathbb{R}$ and $\eta_0 \in]0,1[$ such that*

$$\int_\Omega F(u_{\eta_0}^{t_0}(x))dx \geq \left(F(t_0)\eta_0^n - (1-\eta_0^n) \max_{|t|\leq|t_0|} |F(t)| \right) \omega_n \tau^n > 0, \tag{33}$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$.

Proof. On account of (c_1), there exists $t_0 \in \mathbb{R}$ such that $F(t_0) > 0$. Now, let $\eta_0 \in]0,1[$ be such that

$$F(t_0)\eta_0^n - (1-\eta_0^n) \max_{|t|\leq|t_0|} |F(t)| > 0$$

and, with the same notation as (23), consider the function $u_{\eta_0}^{t_0}$, namely

$$u_{\eta_0}^{t_0}(x) := \begin{cases} 0 & \text{if } x \in \mathbb{R}^n \setminus B(x_0,\tau) \\ \dfrac{t_0}{(1-\eta_0)\tau}(\tau - |x-x_0|) & \text{if } x \in B(x_0,\tau) \setminus B(x_0,\eta_0\tau) \\ t_0 & \text{if } x \in B(x_0,\eta_0\tau), \end{cases} \tag{34}$$

for every $x \in \mathbb{R}^n$.

Since

$$\max_{x\in\overline{\Omega}} |u_{\eta_0}^{t_0}(x)| \leq |t_0|,$$

it follows that

$$\int_{B(x_0,\tau)\setminus B(x_0,\eta_0\tau)} F(u_{\eta_0}^{t_0}(x))dx \geq -(1-\eta_0^n) \max_{|t|\leq|t_0|} |F(t)|\omega_n\tau^n$$

and therefore

$$\begin{aligned} \int_\Omega F(u_{\eta_0}^{t_0}(x))dx &= \int_{B(x_0,\eta_0\tau)} F(u_{\eta_0}^{t_0}(x))dx + \int_{B(x_0,\tau)\setminus B(x_0,\eta_0\tau)} F(u_{\eta_0}^{t_0}(x))dx \\ &\geq F(t_0)\eta_0^n\tau^n\omega_n + \int_{B(x_0,\tau)\setminus B(x_0,\eta_0\tau)} F(u_{\eta_0}^{t_0}(x))dx \\ &\geq \left(F(t_0)\eta_0^n - (1-\eta_0^n) \max_{|t|\leq|t_0|} |F(t)| \right) \omega_n\tau^n, \end{aligned}$$

as desired. □

Collecting the estimates of Lemmas 5.2 and 5.3 we obtain a concrete upper bound for the parameter θ, as previously claimed.

Proposition 5.4. *Let θ be defined by (22), t_0, η_0 as in Lemma 5.3 and κ_1, κ_2 as in Lemma 5.2. Then, one has*

$$\theta < \theta_K(f, \Omega), \tag{35}$$

with

$$\theta_K(f,\Omega) := \frac{m_1 \pi^n t_0^4 \tau^{n-4}}{\left(F(t_0)\eta_0^n - (1-\eta_0^n) \max\limits_{|t|\le|t_0|} |F(t)|\right)\omega_n} \left(\frac{(1-\eta_0^n)\kappa_1\kappa_2}{\alpha(1-\eta_0)^2}\right)^2.$$

Proof. Let $u_{\eta_0}^{t_0}$ be the function defined in (34). By Lemma 5.2 $u_{\eta_0}^{t_0} \in X_0$ and by Lemma 5.3, due to the choice of t_0 and η_0, it follows that

$$\int_\Omega F(u_{\eta_0}^{t_0}(x))dx > 0.$$

So, recalling the definition of θ, in view of inequalities (26) and (33) one has

$$\begin{aligned}\theta &\le \frac{m_1 \left\|u_{\eta_0}^{t_0}\right\|^4}{4\displaystyle\int_\Omega F(u_{\eta_0}^{t_0}(x))dx} \\ &< \frac{m_1 \pi^n t_0^4 \tau^{n-4}}{\left(F(t_0)\eta_0^n - (1-\eta_0^n) \max\limits_{|t|\le|t_0|} |F(t)|\right)\omega_n} \left(\frac{(1-\eta_0^n)\kappa_1\kappa_2}{\alpha(1-\eta_0)^2}\right)^2\end{aligned}$$

and the proof is complete. □

Remark 5.5. The upper bound $\theta_K(f,\Omega)$ in Proposition 5.4 is a function of the nonlinearity f through t_0 and η_0; of the geometry of the domain Ω through τ and of the kernel K through α and s.

Remark 5.6. We emphasize that, for a fixed $\bar\tau > 0$ and $t_0 \in \mathbb{R}$ such that $F(t_0) > 0$, the sharp value of the constant $\theta_K(f,\Omega)$ is given by

$$\theta_K^\star(f,\Omega) := \bar\tau^{n-4} \min_{\eta\in\Sigma_n} \lambda^\star(\eta),$$

where

$$\Sigma_n := \left] \left(\frac{\max\limits_{|t|\le|t_0|} |F(t)|}{F(t_0) + \max\limits_{|t|\le|t_0|} |F(t)|} \right)^{1/n}, 1 \right[,$$

and the real function $\lambda^\star : \Sigma_n \to]0, +\infty[$ is defined by

$$\lambda^\star(\eta) := \frac{m_1 \pi^n t_0^4}{\left(F(t_0)\eta^n - (1-\eta^n) \max\limits_{|t|\leq|t_0|} |F(t)|\right)\omega_n} \left(\frac{(1-\eta^n)\kappa_1\kappa_2}{\alpha(1-\eta)^2}\right)^2.$$

Notice that $\min\limits_{\eta\in\Sigma_n} \lambda^\star(\eta)$ exists since $\lambda^\star$ is continuous on Σ_n and, as it is easy to see, one has

$$\lim_{\eta\to\bar{\eta}_0^+} \lambda^\star(\eta) = \lim_{\eta\to 1^-} \lambda^\star(\eta) = +\infty,$$

where

$$\bar{\eta}_0 := \left(\frac{\max\limits_{|t|\leq|t_0|} |F(t)|}{F(t_0) + \max\limits_{|t|\leq|t_0|} |F(t)|}\right)^{1/n}.$$

Remark 5.7. Assume that $\Omega = B(x_0, r) \subset \mathbb{R}^3$ is an open ball centered at $x_0 \in \mathbb{R}^3$ and of radius $r > 1$. Then, for $m_1 = 1$ and $-\mathcal{L}_K = (-\Delta)^s$, clearly $\theta_K(f, \Omega)$ in Theorem 5.4 coincides with $\theta^\star$ defined in (4) by using the test function

$$u_{\eta_0}^{t_0}(x) := \begin{cases} 0 & \text{if } x \in \mathbb{R}^3 \setminus B(x_0, 1) \\ \dfrac{t_0}{(1-\eta_0)}(1 - |x - x_0|) & \text{if } x \in B(x_0, 1) \setminus B(x_0, \eta_0) \\ t_0 & \text{if } x \in B(x_0, \eta_0), \end{cases}$$

for every $x \in \mathbb{R}^3$, where $\eta_0 \in \Sigma_3$ is such that $\lambda^\star(\eta_0) = \min\limits_{\eta\in\Sigma_3} \lambda^\star(\eta)$.

We end this paper by exhibiting the example of a nonlinearity satisfying Theorem 1.1 together with the related estimate of the parameter λ.

Example 5.8. Let $\Omega := B(0, \sqrt{2}) \subset \mathbb{R}^3$, $s = 7/8$, $K(x) = |x|^{-19/4}$ for any $x \in \mathbb{R}^3 \setminus \{0\}$ and take $f(t) := t^2 - t$ for any $t \in \mathbb{R}$. It is clear that

$$\sup_{t\in\mathbb{R}} F(t) = +\infty, \quad \lim_{t\to 0} \frac{F(t)}{t^4} = -\infty,$$

and

$$\lim_{|t|\to+\infty} \frac{|f(t)|}{|t|^{5/2}} = 0.$$

Hence, all the assumptions of Theorem 1.1 are fulfilled. Setting $t_0 := 2$, $\eta_0 := (\sqrt[3]{7} + 2)/4$, and considering the fractional problem

$$\begin{cases} \left(\displaystyle\int_{\mathbb{R}^6} \frac{|u(x) - u(y)|^2}{|x-y|^{19/4}} dxdy\right)(-\Delta)^{7/8} u = \lambda f(u) & \text{in } B(0, \sqrt{2}) \\ u = 0 & \text{in } \mathbb{R}^3 \setminus B(0, \sqrt{2}), \end{cases}$$

we obtain that for each compact interval $[a,b]$ localized in the interval

$$\left]\frac{\pi^3}{8{\eta_0}^3-7}\left(\frac{\nu(1-\eta_0^3)}{(1-\eta_0)^2}\left(\frac{\pi}{4}+\frac{1}{1+2s}\right)\left(\frac{1}{2(1-s)}+\frac{2}{s}\right)\right)^2,+\infty\right[,$$

there exists a number $\varrho>0$ for which there are at least three weak solutions (in $X_0^{3,7/8}$) whose norm is less than ϱ.

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