Università degli Studi di Urbino Carlo Bo

Department of Economics, Society, Politics (DESP)

PhD Program in: Global Studies. Economy, Society and Law

Thematic Area: International Economic Policy, Business and Governance

Cycle XXXV

Title

MONETARY POLICY AND BANKING: NON-LINEAR DYNAMIC MODELS EVOLVING AS ADAPTIVE SYSTEMS

Scientific Disciplinary Sector: SECS-P/01 - Economics

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Academic Year 2021/2022

A DISSERTATION

SUBMITTED TO THE DEPARTMENT OF ECONOMICS, SOCIETY AND POLITICS (DESP)

AND THE COMMITTEE ON GRADUATE STUDIES

OF THE PHD IN GLOBAL STUDIES

IN FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

DECLARATION

I certify that the thesis I have presented for examination for the PhD degree of the University of Urbino Carlo Bo is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).

STATEMENT OF CONJOINT WORK

I confirm that Chapter 1 – "Introduction" – was solely written by me. It is an original work in its entirety and has not been published in any Journal.

I confirm that Chapter 2 - "An optimal control problem of monetary policy" - was jointly co-authored with Germana Giombini, and I contributed to 70% of this work. Chapter 2 was published as an article: Bacchiocchi, A., Giombini, G. (2021). "An Optimal Control Problem of Monetary Policy". In: Discrete and Continuous Dynamical Systems Series-B, 26(11), pp. 5769–5786. DOI: 10.3934/dcdsb.2021224.

I confirm that Chapter 3 - "A non-linear model of public debt with bonds and money finance" - was jointly co-authored with Alessandro Bellocchi, Gian Italo Bischi and Giuseppe Travaglini. I contributed to 60% of this work. I confirm that Chapter 3 is an original work in its entirety and that has not been published in any Journal.

I confirm that Chapter 4 – "The effects of a green monetary policy on firms financing cost" - was jointly co-authored with Germana Giombini and Sebastian Ille. I contributed to 70% of this work. I confirm that Chapter 4 is an original work in its entirety and that has not been published in any Journal.

I confirm that Chapter 5 - "Non-performing loans, expectations and banking stability: a dynamic model" - was jointly co-authored with Germana Giombini and Gian Italo Bischi. I contributed to 70% of this work. Chapter 5 was published as an article: Bacchiocchi, A., Bischi, G. I., Giombini, G. (2022). "Non-performing loans, expectations and banking stability: A dynamic model". In: Chaos, Solitons & Fractals, 157, 111906. DOI: 10.1016/j.chaos.2022.111906.

ACKNOWLEDGEMENTS

This project has greatly benefited from the help and support of several people.
First and foremost, I am deeply indebted to my supervisor, Prof. Germana Giombini. Much of the intellectual work of this thesis took place in conversations with her. She has been my mentor, she followed me since my bachelor thesis, she first introduced me to the main themes addressed in this thesis and collaborated with me in most of the works contained in this project. But even more, I must thank her for her patience and valuable guidance, the encouragement and advice she provided at each stage of the process, and her continuous support throughout the PhD.
I am also very grateful to my co-supervisor (and coauthor) Prof. Gian Italo Bischi for his generosity with his time and thought, for all the fruitful and insightful discussions we have had over these years, for his precious teachings, and for the countless suggestions about the design and the analytical part of my research. I can safely say that this work in its current form would not exist without his support. Therefore, I wish to express both my deepest gratitude.

Many others have helped me during the PhD.
I am very grateful to Prof. Sebastian Ille for making it possible for me to visit the Department of Economics of the New College of the Humanities in London (Northeastern University), for the fruitful discussions we had on several aspects of this project, for the time he dedicated to me also after the visiting period, and for accompanying me in the design and drafting of Chapter 4.
Special thanks also to Prof. Elmer Sterken for making it possible for me to visit the Department of Economics and Business of the University of Groningen, but, most of all, for his precious teachings, kindness, and time he dedicated to me.

During these years, I have had the possibility to discuss my ideas with many colleagues and faculty members in Urbino. I owe them many thanks. I have benefitted from fruitful interactions either in terms of productive conversations, deep insights, and generous advice with all the faculty of the Department of Economics, Society and Politics (DESP): Alessandro Bellocchi, Roberta Bocconcelli, Giorgio Calcagnini, Andrea Coveri, Catherine L. Farwell, Federico Favaretto, Ilario Favaretto, Laura Gardini, Chiara Lodi, Giovanni Marin, Alessandro Pagano, Elena Paglialunga, Rosalba Rombaldoni, Agnese Sacchi, Edgar J. Sánchez Carrera, Laerte Sorini, Luciano Stefanini, Fabio Tramontana, Giuseppe Travaglini, Francesco Vidoli, Antonello Zanfei and many more. The several discussions we have had constituted an essential ingredient of this thesis and have contributed to the improvement of its quality.

I wish also to express my gratitude to all the SIE, MDEF, and NED groups with a special mention to Anastasiia Panchuk and Iryna Sushko, as well as to many other participants at seminars and conferences which I have attended for the useful questions, feedback, and comments provided.

In this regard, a special mention goes also to the director of the PhD in Global Studies Prof. Antonello Zanfei for all the stimulating and useful seminars, and conferences organized within the PhD program, and for all his constant efforts to improve this already great international doctorate. Furthermore, to all the office personnel who have followed and helped us, PhD students, with technical issues and administrative formalities.

I am indebted to many people in Urbino that have accompanied me during my doctoral years. I mention the amazing group composed of PhD students of the XXXIII, XXXIV and XXXV cycles, which have been a source of friendships, good advice, and collaboration. Valentina Acquafredda, Alessandro Bellocchi, Mariagrazia D’Angeli, Selene Righi, Itibar Aydemir-Uslu, have been good friends and a huge support, the PhD would definitively not have been the same without them.

My deepest gratitude goes to my parents, my family, and my girlfriend. Thank you for supporting and helping me in every possible way, for always believing in me, in my work and education.

Abstract

This thesis presents five chapters on different aspects of monetary policy theory with a thorough analysis of the instruments at disposal of central banks (CBs) to stabilize and correct imbalances in globalized economic and financial markets. The recent shocks posed by the Covid-19 recession and energy crisis have changed the interdependencies between key economic actors, heavily affecting the mechanism of transmission of monetary policy.

The aim of the thesis is to investigate the impact of such monetary instruments in a theoretical construct that includes non-linear relationships among variables, agents' heterogeneity and limited rationality, market frictions, and asymmetric information. In particular, the agents' expectations play a crucial role in monetary policy decisions and, in this work, are well represented by adaptive schemes that allow for learning, social interaction, imitation, and changing beliefs. Adaptive schemes are modeled in the form of discrete or continuous dynamical systems and their analysis provides new economic insights into the evolution process that leads to equilibrium or disequilibrium situations. This turns out to be precious from a policy-maker perspective because it helps to understand the intrinsic fragilities of the economic/financial systems, providing appropriate policy measures to mitigate them.

After a brief literature review, chapter two focuses on the identification of an endogenous and dynamic Taylor rule for the short-term interest rate to target inflation and output gaps. The aim is to mitigate temporary economic unbalances and shocks. The results highlight the dilemma faced by the CBs in trade-off scenarios where it is not possible to fully achieve both goals with a unique instrument at their disposal.

The third chapter provides an in-depth analysis of the dynamic relationship between the public debt ratio and the inflation rate. It is explored how different monetary policies (interest rate, quantitative easing, monetization) and active fiscal rules can avoid unsustainable government debt paths and excessive inflation fluctuations. In low inflation scenarios, quantitative easing and moderate money finance can be helpful in stabilizing debt evolution thanks to their role in containing spreads and stimulating growth, while the effect on inflation rise is generally limited. Furthermore, interest-rate-based policy alone is not sufficient to control inflation: the CB's credibility in driving inflation expectations results to be crucial to control price developments and achieving macroeconomic stability. One of the novelties of this analysis is the presence of a threshold level for both debt ratio and inflation, beyond which the debt ratio becomes unsustainable following an explosive path.

Chapter four sheds light on the mechanisms through which a CB can implement the risks related to climate change in its unconventional monetary operations (e.g. a corporate bonds purchase program). The so-called green monetary policy aims to steer or tilt the allocation of assets and collateral toward low-carbon industries. In the model developed, this CB strategy effectively reduces the cost of capital for green bonds as opposed to conventional bonds, and thus favors sustainable investment/technology in the market. However, there still could be technology trap equilibria in which no investment in green technology occurs in the long-run, even if the non-green investment equilibrium is inefficient. The green monetary policy can help firms to leave these technology traps and the degree of market competition and of market imperfections can contribute to amplifying the effects of this instrument by the transmission channel.

The fifth chapter deals with market regulations and imperfections of the banking system. The aim is to increase banks' resilience to adverse shocks and to explore the impact of monetary policy on banks with different degrees of rationality, available information, and characteristics. Banks are modeled, in an oligopolistic market, as boundedly rational agents that can adopt two different adaptive schemes in the lending activity. The model suggests that in the presence of a larger degree of bounded rationality of banks (i.e.: gradient dynamics), the monetary policy set by the CB performs worse than in the presence of more rational agents (i.e.: adaptive best reply). In addition, bank heterogeneity in terms of cost structure and share of non-performing loans can compromise the stability of the market. The financial stress of a credit institution could translate into suffering situations for all the other banks in the market, leading to a possible credit crunch.

Keywords: Monetary policy; Central Banks and Their Policies; Banking; Dynamical systems; Adaptive learning; Non-linearity.

JEL code: E52, E56, E70, C61, C73.

Abstract in italiano

Questa tesi presenta cinque capitoli su diversi aspetti della teoria della politica monetaria con un'analisi approfondita degli strumenti a disposizione delle banche centrali (BC) per stabilizzare e correggere gli squilibri nei mercati economici e finanziari globalizzati. I recenti shock provocati dalla recessione del Covid-19 e dalla crisi energetica hanno modificato le interdipendenze tra i principali attori economici, influenzando pesantemente il meccanismo di trasmissione della politica monetaria.

L'obiettivo della tesi è indagare l'impatto di tali strumenti monetari in un costrutto teorico che include relazioni non-lineari tra variabili, eterogeneità degli agenti e razionalità limitata, frizioni di mercato e asimmetrie informative. In particolare, le aspettative degli agenti giocano un ruolo cruciale nelle decisioni di politica monetaria e, in questo lavoro, sono ben rappresentate da schemi adattivi che consentono l'apprendimento, l'interazione sociale, l'imitazione e il cambiamento di opinioni. Gli schemi adattivi sono modellati nella forma di sistemi dinamici a tempo discreto e continuo e la loro analisi fornisce nuove intuizioni economiche sul processo evolutivo che porta a situazioni di equilibrio o disequilibrio. Ciò si rivela prezioso in una prospettiva di policy-maker poiché aiuta a comprendere le fragilità intrinseche dei sistemi economici/finanziari, fornendo appropriate misure di policy per mitigarle.

Dopo una breve rassegna della letteratura, il capitolo due si concentra sull'identificazione di una regola di Taylor endogena e dinamica per il tasso di interesse a breve termine al fine di ridurre l'inflazione e l'output gap. Lo scopo è quello di mitigare squilibri e shock economici temporanei. I risultati evidenziano il dilemma che le BC si trovano ad affrontare in scenari di trade-off in cui non è possibile raggiungere pienamente entrambi gli obiettivi con un unico strumento a disposizione.

Il terzo capitolo fornisce un'analisi approfondita sulla relazione dinamica tra rapporto debito pubblico PIL e tasso di inflazione. Si esamina come diverse politiche monetarie (tasso di interesse, quantitative easing, monetizzazione) e regole fiscali attive possano evitare percorsi insostenibili del debito pubblico e fluttuazioni eccessive dell'inflazione. In scenari di bassa inflazione, il quantitative easing e una modesta monetizzazione finanziaria possono essere utili a stabilizzare l'evoluzione del debito grazie al loro ruolo di contenimento degli spread e di stimolo alla crescita, mentre l'effetto di incremento dell'inflazione è generalmente limitato. Inoltre, la politica basata sui tassi d'interesse da sola non è sufficiente a controllare l'inflazione: la credibilità della BC nel guidare le aspettative di inflazione risulta essere cruciale per controllare l'andamento dei prezzi e raggiungere la stabilità macroeconomica. Una delle novità di questa analisi è la presenza di un livello soglia sia per il rapporto

debito/PIL che per l'inflazione, oltre il quale il rapporto debito/PIL diventa insostenibile seguendo un percorso esplosivo.

Il quarto capitolo fa luce sui meccanismi attraverso i quali una BC può implementare i rischi legati al cambiamento climatico nelle sue operazioni monetarie non convenzionali (ad esempio, un programma di acquisto di obbligazioni societarie). La cosiddetta politica monetaria verde mira a orientare o a far convergere l'allocazione di attività e garanzie verso i settori industriali a basse emissioni di carbonio. Nel modello sviluppato, questa strategia della BC riduce effettivamente il costo del capitale per le obbligazioni verdi rispetto a quelle convenzionali, favorendo così gli investimenti/tecnologie sostenibili sul mercato. Tuttavia, potrebbero ancora esistere equilibri definiti come trappole tecnologiche in cui non si verificano investimenti in tecnologie verdi nel lungo periodo, anche se l'equilibrio di investimento non-verde risulta essere inefficiente. La politica monetaria verde può aiutare le imprese a uscire da queste trappole tecnologiche e il grado di concorrenza e di imperfezione del mercato può contribuire ad amplificare gli effetti di questo strumento attraverso il canale di trasmissione.

Il quinto capitolo si occupa della regolamentazione del mercato e delle imperfezioni del sistema bancario. L'obiettivo è quello di aumentare la resilienza delle banche agli shock avversi e di esplorare l'impatto della politica monetaria su banche con diversi gradi di razionalità, informazioni disponibili e caratteristiche. Le banche sono modellate, in un mercato oligopolistico, come agenti razionalmente limitati che possono adottare due diversi schemi di adattamento nell'attività di prestito. Il modello suggerisce che, in presenza di un maggior grado di razionalità limitata delle banche (dinamica del gradiente), la politica monetaria stabilita dalla BC ha risultati peggiori che in presenza di agenti più razionali (best reply adattiva). Inoltre, l'eterogeneità delle banche in termini di struttura dei costi e di quota di non-performing loans può compromettere la stabilità del mercato. Lo stress finanziario di un istituto di credito potrebbe tradursi in situazioni di sofferenza per tutte le altre banche del mercato, portando a una possibile stretta creditizia.

Keywords: Politica monetaria; Banche centrali e loro policy; Banking; Sistemi dinamici; Apprendimento adattivo; Non-linearità.

JEL code: E52, E56, E70, C61, C73.

SUMMARY

1 Chapter 1: Introduction

In November 2010, European Central Bank (ECB) Governor J.C. Trichet opened the Central Banking Conference with a plea to the scientific community to develop radically new approaches for the understanding of the economy: "*When the crisis came, the serious limitations of existing economic and financial models immediately became apparent. Macro models failed to predict the crisis and seemed incapable of explaining what was happening to the economy in a convincing manner. As a policy-maker, I found the available models of limited help. We need to develop complementary tools to improve the robustness of our overall framework*".

I would like, insofar as I can in my small way, to take up this invitation with this thesis. These hoped-for models start from incremental modifications of the existing ones.

The aim of the thesis is to present some theoretical advancements in monetary policy theory with a particular focus on the recent challenges posed by the Covid-19 recession and energy crisis. All these challenges are global and affect with different extent many countries in the world. In this respect, trade and financial globalization have changed the dynamic interdependencies of economies: on the negative side by increasing their exposure to foreign shocks, on the positive one facilitating international risk sharing. Because these changes also affect the transmission of monetary policy, understanding the implications of globalization is crucial for central banks.

The work presented in this thesis, based on monetary policy and the role of central banks in stabilizing the economy, is thus strictly related to International Economic Policy and, more in general, to Global Studies.

The global financial crisis and recourse to unconventional monetary policy measures have been creating renewed interest in the international dimension of national monetary policy. It is a commonly held view that globalization has amplified the international impact of the monetary policy of major advanced economies. This view is motivated by some stylized facts such as the increase in global trade and financial integration and the strong comovement of key macroeconomic indicators across countries.

The emergence of countries with large economies, but relatively underdeveloped financial systems, not only changes the impact of capital flows, but also affects the dynamics of exchange rates and more generally of international adjustments, including the response to global imbalances and financial crises.

Central banks need to adapt the conduct of their monetary policy to preserve their ability to anticipate and react to international crises.

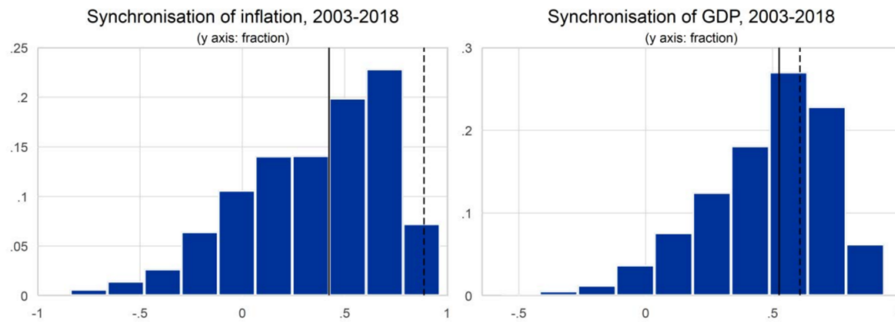
One of the most important implications of globalization for monetary policy is the number of developments in financial asset markets. Financial market innovation, the emergence of new financial institutions, and the increasing globalization of financial markets pose challenges to monetary policy in two spheres. First, asset price dynamics and their possible departure from fundamentals-based valuation must be taken into account when the risks to domestic price stability are assessed. Second, given the global dimension of financial intermediation, multilateral surveillance procedures must be enhanced with a view to incorporating analysis of the international transmission of shocks (including economic policy) that originate in one economy and are propagated to others through financial market linkages (Moutot and Vitale 2009).

This leads to a further motivation to consider global the dimension of national monetary policy: key macroeconomic variables have been co-moving strongly and shocks tend to be more and more transversal, especially across advanced economies (one need only think at the Great Financial Crisis, Covid-19 pandemic, Ukraina-Russia conflict, etc.).

Figure 1 clearly shows that during the period 2003-2018, inflation was strongly positively correlated across country pairs: the median bilateral correlation for all country pairs was around 0.4 across the 53 advanced and emerging countries considered, while the bilateral correlation between inflation in the euro area and the United States was incredibly high at 0.9. Remarkable is also the bilateral real GDP growth correlations in the same period: 0.52 across all countries in the sample, and almost 0.6 for the USA and the euro area (Ca' Zorzi et al. 2020).

Thus, economies are interrelated and national monetary policy responses to the same shocks must certainly be similar. Indeed, data shows they are and, in many cases, are also coordinated between advanced economies (e.g. most of the time the ECB and the Bank of England follow FED monetary policy decisions).

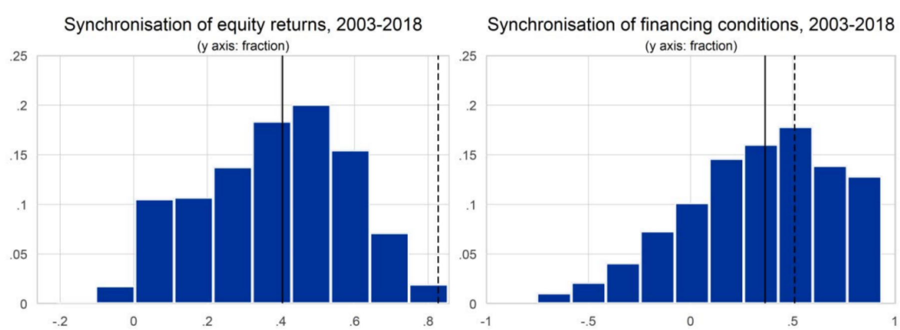
Figure 1: Co-movements of inflation and GDP across world countries, 2003-2018



Source: World Bank (World Development Indicators).
Note: The solid line indicates the median correlation and the dashed line the correlation between the United States and the euro area. The data covers 53 advanced and emerging economies at annual frequency.

Financial variables have also become more synchronized across countries, especially between the euro area and the United States (Figure 2). Over the period 2003-2018, almost all national stock markets were positively correlated. The correlation was again particularly strong between the stock markets of the United States and the euro area, standing at above 0.8. The same holds for the financing conditions indicator, a set of nine variables calculated by ECB, which stood at around 0.5 for both correlations considered.

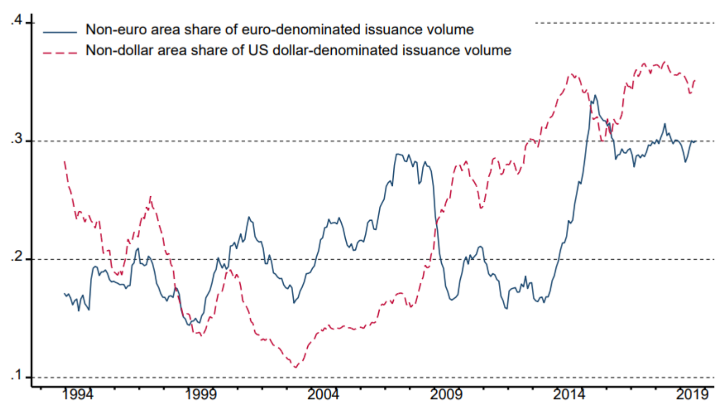
Figure 2: Correlations of equity returns and financing conditions across world countries, 2003-2018



Sources: Bloomberg and IMF Global Financial Stability Report.
Note: The solid line indicates the median correlation and the dashed line the correlation between the United States and the euro area. Equity prices are represented by the S&P 500 index for the United States and the Euro Stoxx index for the euro area. The sample covers 43 countries for financing conditions and 49 countries for equity returns at monthly frequency. Financing conditions indices are calculated by ECB staff extending the IMF Global Financial Stability Report (April 2017) methodology using a set of nine financial variables.

Another example of the increasingly international dimension of financial conditions is the amount of bond issuance in foreign denominations. Globalization has increased the foreign component in both euro and US dollar bond markets, which exceeds 30% of total bonds for both economies in 2019, see Figure 3.

Figure 3: Share of new bonds issued outside the area of the bond's denomination currency *(percentages of total issuance of bonds denominated in that currency, 12-month moving averages)*



Sources: Dealogic and ECB calculations.

It follows that the monetary policies of the ECB and the Federal Reserve matter not only for financial conditions at home, but also for financial conditions in other countries.

The approach used in this thesis includes relatively new features to get as close as possible to stylized and empirical facts and to overcome some of the less realistic tenets of mainstream economics.

Indeed, one of the novelties of the models presented is that they are permeated by non-linear (instead of linear) relationships among the different variables of the system. The idea is to capture the asymmetric responses and threshold levels typical of economic and financial systems. Thinking non-linearly is crucial because not all economic relationships are lines. Numerous empirical works have proven that textbook correlations between unemployment (or growth) and inflation (or wages), budget deficits, and interest rates exhibit several non-linearities and regime shifts. Relevantly for monetary and stabilization policy, in a low inflation context the empirical correlation with the unemployment gap is very mild, while for higher inflation rates, the trade-off comes back to life. Phillips 1958 himself pointed out that the

latter relationship might be highly non-linear and asymmetric as it will be specified more in detail in subsequent section 2.2.
Similarly, interest rate policies have been very little responsive to growth, unemployment, and budget deficits for several years (e.g. 2009-2018 period). That could be changing, and we are already witnessing it, with the recent monetary tightening imposed by the major central banks. Chapter 2 will account for different responses of inflation and growth for changes in the monetary policy stance, with a particular reference to the zero lower bound environments. I claim that for the CB is relatively more difficult/costly to stimulate a price increase with an expansionary measure when interest rates are already low than the opposite (i.e. reduce prices with a restrictive policy). The asymmetric effect is in line with what many advanced economies (especially Europe) have experienced in the aftermath of the financial crisis (i.e. liquidity trap situation), where albeit very low interest rates set by the monetary institutions (almost near to minimum bound of zero), inflation remains at very low values during the 2014-2017 period.
In addition, public debt ratio and real interest rate often exhibit non-linear relationships and thresholds, as higher financial spreads (or risk premium) could impair business cycle stabilization, the debt path of sustainability, and ultimately the monetary policy transmission mechanisms, as demonstrated in Chapter 3. For this reason, unconventional monetary policies have become so widely implemented by central banks during last years thanks to the active role they can have in containing the spread/risk-premium. Other than additional instruments helpful to target inflation and stimulate growth, they have been effective in reducing the interest rates on public bonds, especially for highly indebted countries, making debt ratios more sustainable and more resilient to shocks. In short, in order to better understand the functioning of real and financial systems and the effects of monetary policy on economic agents, it is crucial to consider the potential non-linearity among variables. Other relevant features of the theoretical constructs developed in the following are agents' limited rationality, market frictions, and asymmetric information.
Economic agents' choices are not always completely rational (i.e. they are often bounded). For this reason, all the models presented in this thesis are characterized by some form of bounded rationality in agents' decisions, particularly in the form of adaptive schemes that allow for social interaction, changing expectations, imitation, networks, and learning.
If we just think at inflation expectations, they have been a central factor in models of inflationary dynamics since the 1960s and 1970s, with the seminal work of Phelps 1968, Friedman 1968, and Lucas 1972, and they play a key

role in New Keynesian dynamic stochastic general equilibrium (DSGE) models used to inform and evaluate monetary policy. In many inflation models used by central banks, inflation is driven by three key factors: some measure of a resource utilization gap (for example, the output gap or unemployment rate gap), lagged inflation which captures the inertia in the inflation process, and expectations of inflation. Different models put different weights on these fundamental factors, but household and business expectations matter, since they affect wage demands and offers, and therefore firms' price-setting behavior. For example, in Chapter 3, forward-looking measures of inflation expectations play a larger role in explaining inflation dynamics: generally higher expectations cause firms to raise their prices. This is consistent with the literature on the topic, as shown in section 2.4. Furthermore, inflation expectations also provide an indication of how credible the public finds the central bank's commitment to achieving its policy goals. Indeed, they are determined not only by movements in inflation, but, also, by policymakers' actions to follow through on their stated commitment to return inflation to its longer-run goal, thereby justifying the public's belief in the central bank's commitment.

The market frictions and the heterogeneity of the economic agents in terms of intrinsic characteristics and behaviors affect the conducting and the result of a certain monetary policy and sometimes can pose risks for the well-functioning of the designed instrument. In Chapter 3, different groups of economic agents are modelized with opposite inflation expectations beliefs. A change in the numerosity of each group can have a dramatic impact on the inflation rate in equilibrium. It is demonstrated that a greater share of agents that have expectations disanchored from the CB target makes shocks on inflation more persistent in time with a potentially destabilizing effect on the economic system. The credibility of the central bank in guiding inflation expectations toward the announced objective emerges again as a crucial factor to guarantee the stability of the economy and a more effective interest-rate-based policy.

The heterogeneity in the behavior and expectations of the agents is also relevant to understand the emergence of new aggregate proprieties of the economic system as a whole. Economic agents, as they interact and learn, are affected by the behavior of the system, but, at the same time, agents' interactions can lead to emergent and not predicted behaviors at the aggregate level of the system. In this regard, economics is seen as a complex system of heterogeneous actors characterized by critical points and regime shifts (often in disequilibrium rather than in equilibrium).

For example, in Chapter 5 two boundedly rational banks adopt an adaptive behavior on their loan supply to increase profits under different assumptions of limited information (bounded rationality) and computational ability. Notably, the two alternative dynamic adjustments lead to the same Nash equilibrium of the oligopolistic model under complete information and rational (profit-maximizing) banks. Thus, an incremental (or adaptive) decision based on the repeated strategic interactions between banks (through imitation or learning-by-doing process) could converge to the rational equilibrium, which was not expected from the initial bounded rationality assumptions. This may be seen as an evolutionary interpretation of a rational equilibrium, and some authors say that, in this case, the boundedly rational agents are able to learn, in the long run, what rational agents already know under very pretentious rationality assumptions (see e.g. Fudenberg and Levine 1998).

Another illustration of heterogeneity in agents' characteristics is contained in Chapter 5, where the two banks can have different sizes, diverse cost structures and risk-taking in the lending activity, and different shares of non-performing loans (NPLs). The more heterogeneous they are in terms of these variables, the higher the risk for the stability of the banking sector. Usually, these results cannot be obtained in a model of perfect competition where all agents are considered to be the same (i.e. the representative agent), and that does not include market imperfections or frictions.

The repetition of boundedly rational decisions based on trial and error, imitation of the better, and comparison between expected and realized results, denoted by the general term "adaptive", may be a much more realistic (and even more rational) behavior with respect to a rigid optimizing attitude, based on fixed rules assumed as fundamental axioms of rational behavior.

In Chapter 4, there is a population of firms that makes investment choices under limited information: the firms in the industry do not know exactly what the return on investment of each technology will be and/or are not able to compute the optimal alternative following traditional profit maximization rules. Consequently, the decision is not based on the expected return on investment as in a perfect information setting. Instead, firms imitate the investment behavior of other firms in the sector. More specifically, each company in the industry simply observes a small subset of other firms and replicates the investment strategy of the most successful ones. This generates an evolutionary mechanism in which, eventually, a Nash equilibrium is reached in the long-run.

The initial distribution of green and non-green technology in the industry could also lead to different possible equilibria (e.g. full green investment, full non-green investment, or mixed equilibrium) showing a path-dependency phenomenon. Furthermore, some of the equilibria could be Pareto sub-optimal in terms of industrial profits. In this regard, the analysis of the dynamics provides new economic insights into the evolution process that leads to the selection of a certain equilibrium. This turns out to be precious from a policy-maker perspective.

Adaptive systems can be mathematically modeled in the form of discrete or continuous dynamical systems with the related concepts of stability, bifurcations, attractors, and basins of attraction as the main tool for the analysis of their long-run (or asymptotic) properties.

This new stream of research will be the common denominator of this research project, which will revolve around a dynamic view of monetary economics. This approach could help understand the intrinsic fragilities of the economic/financial systems, providing appropriate policy measures to mitigate them. Acting on models' parameters enables to simulate diverse scenarios: changes in adaptive expectations, policy modifications, agents' evolving behavior, different market, and economy characteristics, etc. The real strength of these dynamic models is that they are versatile and may give rise to a vast range of different situations: convergence towards one or more equilibria, oscillations, disequilibrium dynamics, path dependence phenomenon, irreversibility, hysteresis, periodic cycles, chaotic patterns, and other non-linear and complex phenomena commonly observed in real systems in economics, finance, and social sciences. In addition, for certain parameters value, self-sustained endogenous oscillations may arise that, in some cases (e.g. deterministic chaotic attractor), bring a high level of uncertainty and unpredictability on the future values of the variable due to the associated phenomenon of sensitivity to arbitrarily small perturbations (a quite common situation in economics and social sciences).

The results will be obtained mainly through rigorous analysis to provide equilibrium values, conditions of stability, and local/global proprieties of the models. Furthermore, qualitative study and numerical simulations are offered in the few cases where the analysis of the non-linear dynamical systems becomes too complicated to handle in terms of mathematical tractability. The outcomes often lead to novel and interesting insights in terms of policy implications.

To leave for further research, it will also be interesting to empirically test the value associated with the models' parameters, perform an econometric analysis to improve robustness and provide additional feedback on the models' setup.

The thesis is organized as follows.

The first chapter introduces the literature on the topic. Starting from the stabilization role of the monetary policy advocated by Keynes 1936, I move to the empirical Phillips 1958 trade-off between wage changes and unemployment, to the subsequent Phillips curve critics. Contributions from Phelps 1968, Friedman 1968, Samuelson and Solow 1960, Sargent and Wallace 1973, Okun 1975, Taylor 1993, among the others, are exposed, with a particular focus on the consequences for monetary policy operations. From Friedman's monetarist theory to Phelps' expectation evolution, passing from Sargent's rational expectations and Taylor's rule, these works have shaped the way in which modern monetary policy is conceived and conducted. Recent developments on inflation dynamics, NAIRU, and modern monetary policy responses are the focus of the last section of the chapter with several references to the challenges posed by the recent macroeconomic turmoils: supply-side shocks, built-in inflation, NAIRU hysteresis, market bottlenecks, and imperfections.

The second chapter addresses the issue of the monetary policy effects on macroeconomic variables: inflation and GDP, employing a dynamic model based on optimal control theory.

The research focuses on the role of monetary policy in adjusting and correcting temporary economic unbalances and shocks with a particular focus on the so-called Phillips curve and its recent evolution in the related literature. The aim of the modelized Central Bank (CB) is to identify an appropriate interest rate path over time that minimizes the two main objectives: inflation and output gaps.

In the literature, monetary policy macroeconomic models are often presented in a discontinuous time fashion and they are, in most cases, generalizations of the Taylor rule. I would like to improve in this direction considering an endogenous CB target (or, in this case, loss) function in continuous time.

Recent studies have argued that the dynamics of inflation have changed substantially in many, if not all, advanced economies over the past four decades, showing that various types of structural changes have affected the statistical properties of inflation. This makes increasingly complex the modeling of inflation dynamics. In this respect, a key novelty of the model is the assumption of an augmented non-linear Phillips curve that takes into account the asymmetric responses of inflation to changes in economic activity and in the monetary policy course.

The model can be used to simulate, by changing the initial conditions and the parameters value, a wide range of real economic situations: from inflation demand-pull shocks to supply-side shocks, from recession to sustained economic growth and to ensure an appropriate monetary policy path in time for each of these scenarios. The results highlight the dilemma faced by the CB in trade-off scenarios where it is not possible to fully achieve both goals (i.e. minimize output deviation from a defined growth target and control inflation) with a unique instrument at its disposal (i.e. interest rate). Furthermore, it is emphasized how a different priority in the monetary agenda can change the outcomes towards one objective or the other, reflecting the potentially diverse preference of each policymaker.

The third chapter provides an in-depth analysis of the dynamic relationship between the public debt ratio and the inflation rate. Using a non-linear macroeconomic model of difference equations, I place emphasis on the role of monetary and fiscal policy in influencing the stability of these macroeconomic variables. Specifically, I study the impact of money finance (or debt monetization) on the debt ratio and on the inflation rate, assuming endogenous and non-linear relationships between the variables.

In this model, the government can generate public deficits financed by issuing new debt. The CB sets a target for the interest rate to achieve desired inflation and it can also use a (moderate) monetization to finance public debt if the relative magnitude of it undermines the financial stability of the economy. Consequently, in this 'augmented' version of the inter-temporal government budget constraint, the final effects on changes in government debt stock depend also on CB's monetization actions. The nominal interest rate is determined by a standard Taylor rule defined by the CB plus a financial market component, the risk premium (or spread) on government bonds.

The dynamics of inflation affect the debt ratio through the real interest rate, that is the cost in real terms of government debt. The evolution of inflation is captured referring to a variant of the classical Phillips curve, in which agents' inflation expectations are implemented by considering the presence of both '*fundamentalists*' and '*trend-follower*' economic agents in the markets. This assumption makes it possible to describe complex inflation dynamics (e.g. self-sustained oscillations around the equilibrium value without reaching a steady state) and to include other determinant factors as the credibility of the CB commitment in targeting inflation.

Three main results are obtained. First, in a low inflation scenario, money finance can be helpful in stabilizing debt evolution and the resulting effect on inflation rise is generally limited. Second, in a dynamic setting, standard Taylor rules may not be sufficient to control inflation: the central bank's credibility in driving inflation expectation is, in fact, crucial to control price developments and achieve macroeconomic stability. Finally, an active budget adjustment rule, that aims to target the primary deficit to deviations of the debt ratio from the value perceived as sustainable, has a stabilizing effect on public debt, even if it may not always be enough to avoid explosive patterns. The stability of the steady state, and thus of the economy, crucially depends on the fine-tuning of the policy mix in time.

One of the novelties of this analysis, compared to the benchmark linear model of the debt ratio, is the presence of a threshold level for both debt ratio and inflation, beyond which the debt ratio becomes unsustainable following an explosive path. The distance between this threshold value and the steady state can be considered as a proxy of the robustness of the economy to exogenous shocks.

In addition, another key finding concerns unconventional monetary policy, which may prove to be fundamental in stabilizing the economy thanks to the active role it can have in containing the spread/risk-premium. If the CB succeeds, through a program of government bond purchases (e.g. quantitative easing), to reduce financial spreads, the system becomes much more stable and shocks on both the debt ratio and inflation rate do not alter the convergence toward the equilibrium.

Unconventional monetary policy is also the protagonist of the fourth chapter of this work. In this chapter, I explore the recent developments in the Asset Purchase Programs (APPs) and Corporate Sector Purchase Programme (CSPP) introduced by the CBs lately (for the European Central Bank in 2014) as part of monetary policy operations.

CB monetary policies have been starting to consider risks related to climate change (i.e. physical risk and transaction risk) with the aim to strengthen the role of the financial system to manage these risks and mobilize capital for green and low-carbon investments in the broader context of environmentally sustainable development.

The model built in this chapter sheds light on the mechanisms through which a CB can implement in its CSPP a green monetary policy to steer or tilt the allocation of assets and collateral towards low-carbon industries. The aim is to verify if this strategy can effectively reduce the cost of capital for these sectors in comparison to high-carbon ones and, ultimately, to incentive green investment by firms.

Starting from a CSPP that follows a carbon-neutral monetary policy, I investigate how a shift in the CB portfolio allocation towards bonds issued by low-carbon companies can favor greener firms present in the market. Relying on optimal portfolio theory, I study the way in which the CB might include (or internalize) the risk related to the environmental sustainability of the companies in its balance sheet decisions.

I find that the monetary authority can indeed reduce the financing costs for environmentally sustainable firms and tightens the financing conditions of non-green companies, i.e. increasing the so-called *green premium* or *gree-*

nium, by altering the composition of its balance sheet without modify the latter's total dimension.
In addition, I analyze the interactions between the neutral or green CB rebalancing policy and the evolutionary investment choice (i.e. by means of an exponential replicator dynamics) of a population of firms that can decide to be green or not based on bonds borrowing cost. The investment choice is financed through bonds issue and it is made under limited information: firms in the sector can only observe some competitors and replicate the investment strategy of the most successful ones.
Relevant policy insights are obtained. First, some scenarios are characterized by a strong path dependency in which if a large share of firms employed non-green technology, no investment in green technology occurs in the long-run, even if the non-green investment equilibrium is inefficient. I define this equilibrium *technology trap* and show that CSPP monetary policy helps the industry leave the technology trap. Second, green and non-green bond riskiness is a key factor that impacts borrowing costs. The larger the average financial risk of bonds, the lower the share of bonds in the CB portfolio, and the larger the borrowing cost for firms. Third, the degree of market competition and of market (im)perfections contribute to amplifying the effects of the green monetary policy by affecting the transmission channel. In the presence of imperfect competition and (or) a high degree of market imperfections the *technology trap* is more likely to happen, and the green monetary policy seems to foster the adoption of green technology and stabilize investment decisions.
The last and fifth chapter of this thesis deals with the main market regulations and imperfections of the banking system. The focus is on the banking lending activity with a particular reference to the main risk factors and activities bearers of financial stress. The aim is to increase banks' resilience to adverse shocks and to explore the impact of monetary policy on banks with different degrees of rationality, available information, and intrinsic characteristics.
In this regard, I postulate a dynamic oligopolistic banking market where the presence of a large share of Non-Performing Loans (NPLs) in banks' portfolios could change, or even destabilize, the equilibrium of the entire industry. The model provides policy insights in evolutionary contexts characterized by repeated strategic interactions between agents, information asymmetries, and bounded rationality.
A dynamic model represents a novelty in the investigation of the role of NPLs on market stability and it could provide a better setup to study the complex structures of relationships and equilibria that characterize the banking sys-

tem over time. It allows us to study the evolution of NPLs over a long time horizon, analyzing how the quality and riskiness of assets in banks' portfolios may endanger the capacity of each credit institution to generate profits, and thus be competitive in the banking sector (i.e. financial sustainability). The peculiarities of the banking sector are well represented by an oligopoly system, which allows capturing both the cooperation (i.e. the interbank market) and competition (in the loan market) relationships between credit institutions.

Banks are modeled as boundedly rational agents that can adopt two different adaptive behavior (or dynamic adjustments) to increase their profits.

The first dynamic adjustment proposed, in discrete time, is the *adaptive best reply* approach with naive expectations, i.e. the two banks are assumed to know the demand loan function and solve the profit maximization problem, thus computing the correct reaction functions, but they are not informed about competitor's choices. Consequently, to compute the best reply they assume the currently observed competitor's loan decision as the expected next choice.

The second dynamic adjustment mechanism involves a lower degree of rationality and is obtained by considering the so-called *gradient dynamics*. It is based on the assumption that the banks adjust their loan supply over time proportionally to their marginal profits. In this case, each bank does not have complete knowledge of the demand function or is not able to solve the optimization problem, hence it tries to infer how the market will respond to small changes in loan supply by an empirical estimate of the marginal profit, that may be obtained by market research or by brief experiments.

The strategic interactions between the adaptive banks result in a variety of behavior in the banking services offered.

The initial economic implication is that bank heterogeneity, which in the model derives from either different cost structures, different shares of NPLs, or both, can compromise the stability of the equilibrium.

The models also suggest that in the presence of a larger degree of bounded rationality of banks (i.e.: *gradient dynamics*), the monetary policy set by the Central Bank performs worsen than in the presence of more rational agents (i.e.: *adaptive best reply*). Likely, in the former case, the transmission mechanisms (that work through the price or quantity channels) encounter obstacles related to too limited bank rationality. These obstacles refer to the capacity of banks to modify their loan supply, potentially affecting the potency of forward guidance and leading to powerful mitigation of the effects of monetary policy. This leads the system to diverge or to be unstable for certain levels of interest rates and share of expected NPLs.

In addition, the analyses show that bank interdependence affects the market dynamics so that the financial stress of a credit institution could translate into suffering situations for all the other banks in the market, leading to a possible credit crunch.
Along the thesis, several methodologies have been used: from optimal control theory in continuous time, to discrete dynamical systems, to modern portfolio optimization and evolutionary game theory.
As highlighted in this chapter description, the entire thesis revolves around monetary policy and its implications on both macro and micro variables. For this reason, I will present, in the following, the related literature, foundation of the theoretical works developed.

2 Literature review

2.1 The Keynesian countercyclical monetary policy

The interest in adjusting and correcting temporary economic unbalances generated by different shocks is well-rooted in the policy literature and practice. It dates back to Keynes's *General Theory* (Keynes 1936) where, in the aftermath of the Great Depression, advocated an active intervention of public governments in the economy.
The *General Theory* shows the persistence of equilibria characterized by unemployment and sub-optimal utilization of available resources. As market economies do not have reliable automatic mechanisms to achieve equilibria of high income and employment, fiscal and monetary policies are needed. The monetary-financial lever and the fiscal lever (e.g. public expenditure) become the main instruments to control inflation and stimulate growth.
One of the several contributions of Keynesianism was the definition of a 'point of effective demand', interpreted as the intersection of two curves: an aggregate supply function and an aggregate demand function. These functions relate the number of employed workers to the entrepreneurs' evaluations regarding costs, on the one hand, and revenues on the other, thus being conceptually different from the neoclassical ones.
The entrepreneur's viewpoint is fundamental because she/he decides whether to invest, attempting to evaluate expected returns on investment and comparing them with the monetary rate of interest indicating the return on financial investments, which constitute an alternative usage of available funds.

Keynes indeed assumed that: *"with a given organization, equipment and technique, real wages and the volume of output (and hence of employment) are uniquely co-related, so that, in general, an increase in employment can only occur to the accompaniment of a decline in the rate of wages"* (Keynes 1936, p. 17).

Full employment (i.e. the absence of involuntary unemployment) is obtained by maintaining an adequate effective demand to match the current production capacity of the economy. *"When effective demand is deficient", writes Keynes, "there is underemployment of labour in the sense that there are men unemployed who would be willing to work at less than existing real wage."*

Keynes gave an alternative definition of full employment at another place in his General Theory: *"It is a situation in which aggregate employment is inelastic in response to an increase in the effective demand for its output"* (Keynes 1936, p.26).

In an economy of full employment, any further increase in effective demand is not accompanied by any increase in output.

Therefore, if consumers and entrepreneurs formulate budgets for the period ahead based on incorrect aggregate expectations and remain at least partly stuck to them, a sufficient (unexpected) jump in the price level would curb the real volume of spending to the level of full-employment output. In fact, in the inflationary zone above full employment, incremental aggregate demand merely bid up the price level, leaving production unchanged.

The rate of inflation, then, will depend on the size of the inflationary gap and the length of the budgeting lag (Okun 1975).

For the broad spectrum of output values below the one related to full employment, variations in aggregate demand change the aggregate production maintaining essentially constant prices and wages. Hence, in the Keynesian construct real wages are considered to be stable over time. Exceptions arise when monetary wages and product prices do not move at the same pace, or they do it but with delayed evolution in time. This could potentially generate temporary upward or downward price pressures.

The major role of monetary policy should be to mitigate these occurrences, acting in a counter-cyclical way to offset fluctuations of aggregate demand and potential shocks.

2.2 The Phillips contribution

Several years later, in 1958, the economist A.W. Phillips published an empirical article (Phillips 1958) describing an inverse relationship between rates of unemployment and corresponding rates of rises in monetary wages within an economy. The analysis was based on United Kingdom aggregate data from 1861 to 1957. The work lacked an underlying theory, but it was extremely relevant to develop, in subsequent years, a flourishing debate about this alleged relationship.
The intuition behind the relationship is the following. Focusing on the labor market, the firms' demand for workers is related to the actual economic activity level and this, in turn, influences monetary wages. When labor demand is high and the current unemployment rate is low, firms and industries have to compete in bidding salaries up to attract the best-suited workers.
On the contrary, in periods characterized by low labor demand and high unemployment levels, monetary wages will go down even if at a slower pace than in the opposite situation.
This is because, generally, firms are more reluctant to lower monetary wages in fear that quit rates will go up and career workers will leave. For this reason, the author himself pointed out that this alleged relationship might be highly asymmetric and non-linear.
The Phillips approach departed fundamentally from Keynesianism because instead of relating a given utilization rate to an equilibrium level of prices, it posed a continuous trade-off between monetary wages and unemployment. The actual stage of the business cycle has an impact both on the labor demand and on the rate of unemployment reinforcing the potential relationship with the money wage rate of change (demand-pull inflation).
Following (Phillips 1958, p. 283):
"in a year of rising business activity, with the demand for labour increasing and the percentage unemployment decreasing, employers will be bidding more vigorously for the services of labour than they would be in a year during which the average percentage unemployment was the same but the demand for labour was not increasing. Conversely in a year of falling business activity, with the demand for labour decreasing and the percentage of unemployment increasing, employers will be less inclined to grant wage increases, and workers will be in a weaker position to press for them than they would be in a year during which the average percentage unemployment was the same but the demand for labour was not decreasing."
In addition, the change in prices may influence inflation through the channel

of production cost. Here, a distinction must be made between domestic and import prices. The increase of the former prices can be partially offset by domestic rising productivity, while a variation of import prices, depending on the share of imports in the economy, could be more directly related to the final price of the products. Additional production factors cost may be translated into rising prices for products and services, which in turn might boost wages through the cost of living adjustments, triggering a wage-price spiral (the so-called cost-push inflation).

As intended by Phelps later on (Phelps 1968), cost-push inflation can be interpreted as a kind of inflation that can be offset only by a reduction of the employment rate through lower aggregate demand, raising a cruel dilemma for fiscal and monetary policy. Nonetheless of its simplicity, the original Phillips Curve was a milestone of the economics discipline because it offered an original point of view of the possible interconnections among fundamental macroeconomic variables.

The proposed curve fitted from UK data for the three periods considered separately (1861-1913, 1913-1948, and 1948-1957) has the following functional form:

$$y + a = b\,x^{\,c}$$

which linearized gives:

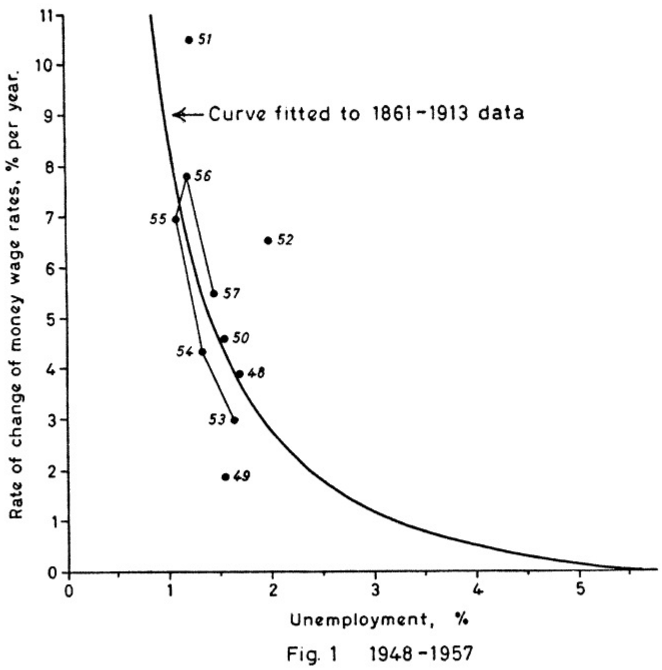
$$log\,(y + a) = log\,b + c\,log\,x$$

Where y is the rate of change of wage rates (i.e. inflation rate) and x is the percentage of unemployment. The estimated coefficients a, b, c are respectively equal to 0.900, 9.638, and -1.394.

Annual increases in prices (and wages) relative to unemployment rates in Great Britain from 1954 to 1957 fit the estimated hyperbola like a glove (see Figure 4) and this occurred until 1968.

The persistence of the 1956-57 inflation into 1958 could be explained by some short lags of a vintage Keynesian type, while the better trade-off performance of the early sixties relative to the fifties could be accounted for by the price-wage guideposts or by "*hidden unemployment*" as shown in (Perry 1967). Empirically, the Phillips curve worked well for the United States too. However, the stubbornness of wage and price inflation in 1970-71 ended the heyday of the Phillips approach.

Figure 4: Phillips' curve, 1948-1957



Perry 1970 sought to save the Phillips curve by reinterpreting the measure of labor-market tightness to reflect the shifting demographic composition of unemployment. The Perry shift is now generally accepted as a constructive refinement, but it explains only a small part of the “new” inflation of the 1970s (Okun 1975).

2.3 The Phillips curve critics

In many neoclassical models (including the Phillips proposal), the real outputs, inputs, and relative prices of goods and factors can be thought of as determined by a set of competitive equations which are independent of the absolute level of prices. There is the implicit assumption that money is valued only for what it will buy and not for its intrinsic utility (i.e. neutral money). This "relative homogeneity" property means that if we exactly double aggregate demand there will be an exact doubling of all prices.
As in a barter system, the absolute level of all prices is inessential because

variations in the total amount of money must necessarily correspond to new equilibria of absolute prices that have moved in exact proportion so that relative prices and all real variables are unaffected. This doctrine is oversimplified and does not take into account the numerous interconnections between the monetary and the economic markets.

To illustrate the danger of such hypothesis that a balanced change in all prices might, in the long run, be consistent with no substantive changes in real relations, an enlightening example is taken from Samuelson and Solow 1960, p. 179:

"A rise in defense expenditure matched by, say, excise taxes cannot raise the price level if the supply of money is held constant; instead it must result in enough decrease in real wage and other factor costs to offset exactly the rise in tax costs. Actually, however, such a fiscal policy change could lead to a strong reduction in the combined public and private savings; with the quantity of money M constant, it would tend to swell the volume of total spending, putting upward pressure on interest rates and inducing a rise in money velocity, and presumably resulting in a higher equilibrium level of prices. To roll back prices to their previous level would take, even within the framework of a strictly competitive neoclassical model, a determined reduction in the previous money supply."

From here, it is clear that variations in money demand and/or supply are not neutral at all and they can permanently change relative prices in an economy (real wages, real interest rates, relative prices between goods and production factors, real exchange rates, etc...).

This assumption has an impact on several business decisions on investment, technology adopted, quantity and quality of services/products offered, as well as on consumption and saving choices from private and public actors.

On the other hand, Keynes departed from the perfect competition models stressing the downward inflexibility of wages and prices to convert any reduction in money spending into a real reduction in output and employment rather than a balanced reduction in all prices and factor costs (Samuelson and Solow 1960). In Keynes's words (Keynes 1936): *"Every trade union will put up some resistance to a cut in money-wages [since such reductions 'are seldom or never of an all-round character']. But ... no trade union would dream of striking on every occasion of a rise in the cost of living"* (pp. 14-15).

The unresponsiveness of unemployment to demand in the full-employment Keynes' theory depended on the notions of money-wage behavior. At more than minimum unemployment, a rise (fall) of demand and employment would produce a once-for-all rise (fall) of the money wage, prices constant.

A rise (fall) in the price level would cause a rise (fall) in the money wage in a smaller proportion. Hence, in a stationary economy at least, his theory did not predict the possibility of a secular rise (fall) of money wage rates at normal unemployment rates except in the phase of transition towards the minimum unemployment, where wages variations can exceed or fall behind productivity gains (Phelps 1967).

A more eclectic model of imperfect competition in the factor and commodity markets is needed to explain the fact of price and wage rises before full employment and full capacity have been reached.

The comparison between the rate of growth of nominal wages and productivity could also be misleading since there exist industries where productivity gains are very dissimilar. If these sectors, due to collective bargaining, receive similar wage updates in time, the assumption behind the neoclassical models of wages equal to labor productivity will surely be violated. Such a persistent and growing differential is likely eventually to alter the skill or quality-mix of the labor force in the different industries, which ultimately casts doubt on the original productivity comparison.

Furthermore, we have witnessed a widespread productivity increase in the last decades, thanks also to numerous technological advancements, which, however, in many cases has not been outweighed by a proportional increase in salary. Thus, relying just upon these differentials to account for the cost-push inflation could result in persistent bias or erroneous cause-effect relationships.

A related problem is that in a closely interdependent economy, effects can precede causes and it could be very difficult to distinguish one from the other. Prices may begin to ease up because wage rates are expected to. And, more importantly, as wage and price increases ripple through the economy, aggregation may easily distort the apparent timing relations.

We should recognize that the same general price increase could easily be the consequence of different causes in different sectors. A monolithic theory may have its attractiveness and simplicity (i.e. Phillips Curve), but it cannot describe and especially forecast accurately the evolution of such variables.

Prices in imperfect commodity markets can respond to changes in costs (which comprehend several factors, not only wages), to changes in competitors' prices, variations of demand for substitutes and complementary products, changes in overall demand or other behavioral parameters, and so on. To make an example, in a single market, prices may rise either because the demand curve shifts to the right or because the supply curve shifts to the left as a consequence of cost increases. But in the first case, the output should increase; in the second case, decline.

As Schultze 1959 has argued in his "demand-shift" theory of inflation, certain sectors may face excess demand, without there being aggregate pressure. Those sectors will show strong price increases and rises in output (or pressure on capacity), but in a real sense, the source of inflation is the failure of other sectors, in which excess capacity develops, and it does not translate into a sufficient decrease of prices due to constraints or rigidities. Thus, when demand shifts between different economic sectors, inflation might result even though aggregate demand remains unchanged.

The presence of rigidities in the product and labor markets is another reason put forward by several economists to explain the possible breakdown of the Phillips relationship.

Dunlop already in 1944 (Dunlop 1944) set forth the concept that a union, to maximize its utility, seeks to "trade-off" the real wage rate against the unemployment of its members, raising the former (relative to productivity) until the gain from a further real wage increase is offset by the utility loss from the increase in unemployment expected to result from it.

At an unemployment level below the unions' optimum, the unions push up wage rates faster than productivity, but firms pass part of these higher costs onto consumers, so real wage gains are frustrated, and as long as the government maintains the low unemployment level the rounds of inflation will continue (Phelps 1968).

Furthermore, after the war, Singer 1947, Bronfenbrenner 1948, Haberler 1948, Brown 1955, Lerner 1967, and many others wrote that at low albeit above-minimum unemployment levels there occurs a process of "cost inflation," "wage-push inflation," "income inflation", "creeping inflation", "sellers' inflation", "dilemma inflation" or the "new inflation", a phenomenon which was attributed to the discretionary power of unions and oligopolies to raise wages (prices) without excess demand.

However, as Phelps 1968, p. 679 pointed out:

"An increase of monopoly power due, say, to increased concentration, will raise prices relative to wages at any given unemployment rate and productivity level; but once, at the prevailing unemployment rate, the real wage has fallen (relative to productivity) enough to accommodate the higher mark-up, this process will stop and any continuation of inflation will depend on other sources."

Thus, to explain persistent inflation or disinflation, other mechanisms on the costs side should be at work. For example, we know that during the seventies, due to mainly a sharp and sustained increase in the price of oil and its derivative, the average inflation rate rose from about 2.5 percent in the 1960s to about 7 percent in the 1970s. However, contrary to the prescriptions of the Phillips curve, the unemployment rate not only did not fall, but it actually rose from about 4 percent to above 6 percent in the U.S., and even higher rates were registered in Europe. However, these cost-push theories/hypotheses are only a partial response to the congruity of the original Phillips proposal.

Okun 1975 stressed that imperfections in the labor market are not only caused by the market power and the presence of unions, but they can be intrinsic proprieties of it, a sort of distinctive features of this particular market. One must not lose sight of the considerable variety of jobs and workers, each one with her/his wealth of expertise, distinguished skills, competence, etc., and consequently, it should postulate imperfect information about their availability and consider the possible difficulties in matching specific labor supply and demand.

As Phelps 1968, p. 683 highlighted with a brilliant metaphor:

"firms must incur search costs to find round pegs to fill round holes, and unemployed workers must also expend money and energy to find suitable employment. As a consequence, positive unemployment and positive job vacancies tend to persist in a growing labor market and even under stationary labor supply because of the turnover or attrition of firms' employment rolls. Total vacancies can be positive for every kind of job and total unemployment can be positive for every type of worker because of spatial mismatching among jobs and people."

And this is even more true in contemporary times where the flourishing of new highly technical jobs requires specialized competence and skills that can bring persistent mismatches between demand and supply. When this occurs frictional unemployment might remain at consistent levels in the economy that cannot be cleared by the standard market mechanism of prices and wages.

Neo-Keynesian economists, Sargan 1964 and Kuh 1967, linked nominal wage changes with the continuous variation of unemployment (instead of using the unemployment level rate as in Phillips 1958), as well with productivity and price level.

The underlying theory is that a rise in aggregate demand creates "bottlenecks" and, hence, a rise of wage rates in certain areas and skills, coupled together with increases in employment. Once these bottlenecks have melted away and employment has reached its new and higher level, there is no longer upward wage pressure. On this theory, money-wage increases go hand in hand with employment growth and not intrinsically with a high level of employment rate (Phelps 1968).

The "bottleneck" theory also helps to explain why wage increases should be associated with rapidly increasing employment. An economy adjusted to one level of aggregate demand, with its peculiar structure, cannot adapt instantaneously to a higher aggregate demand level with its new structure; certain types of labor will be in excess demand, and this will drive up the general wage index. This is related to the argument previously discussed concerning the peculiarity of each job and industry that requires a certain amount of time for the labor offer to adapt to the newly increased sector-specific labor demand.

2.4 The expectations evolution and the role of monetary policy

Other economists, such as Mises 1953, pp. 418-20, looked at the problem from another perspective emphasizing the role of inflationary expectations. Lerner 1949 modeled the so-called "anticipated" inflation showing how high inflation confers no benefits in the form of higher employment if, or as soon as, the inflation rate is fully anticipated by firms and workers.
R. Ball 1964 suggested that firms and workers extrapolate the unemployment trend and set wages based on the projected unemployment rate.
Samuelson and Solow 1960 were two of the first economists to explicitly recognize, in a theoretical construct, the role played by expectations: *"periods of high demand and rising prices molds attitudes, expectations, even institutions in such a way as to bias the future in favor of further inflation"*.
Friedman 1968 and Phelps 1968 in two concomitant and separate works formally defined a revision of the original Phillips curve to take into account the evolving expectations on money wages.
Phelps 1968 began by stating: *"If the economy were always in macroeconomic equilibrium then perhaps the full employment money-and-growth models of recent vintage would suffice to explain the time paths of the money wage and the price level. But since any actual economy is almost continuously out of equilibrium we need also to study wage and price dynamics under arbitrary conditions. I postulated that the Phillips curve, in terms of percentage price increase (or wage increase), shifts uniformly upward by one point with every one-point increase of the expected percentage price increase (or expected wage increase). Then the equilibrium unemployment rate, the rate at which the actual and expected price increases (or wage increases) are equal, is independent of the rate of inflation."*

Friedman 1968 continuing on the same line of thought: *"Implicitly, Phillips wrote his article for a world in which everyone anticipated that nominal prices would be stable and in which that anticipation remained unshaken and immutable whatever happened to actual prices and wages. Suppose, by contrast, that everyone anticipates that prices will rise at a rate of more than 75 percent a year ... Then wages must rise at that rate simply to keep real wages unchanged. An excess supply of labor will be reflected in a less rapid rise in nominal wages than in anticipated prices, not in an absolute decline in wages. Restate Phillips' analysis in terms of the rate of change of real wages (and even more precisely, anticipated real wages) and it all falls into place."*

Both Friedman and Phelps argued that the government could not permanently trade higher inflation for lower unemployment, as simplistic assumed in the original Phillips curve. Imagine that unemployment is at the natural rate. The objective of workers and their representatives (i.e. trade unions) is to keep real wages constant: workers who expect a given rate of price inflation insist that their wages increase at the same rate, to prevent the erosion of their purchasing power.

Now, imagine that the government uses expansionary monetary or fiscal policy in an attempt to lower unemployment below its natural rate. The resulting increase in demand encourages firms to raise their prices faster than workers had anticipated. With higher revenues, firms probably have to increase also their supply and are willing to employ more workers at the old wage rates and even to raise those rates somewhat. For a short time, workers suffer from what economists call money illusion: they see that their money wages have risen and willingly supply more labor. Thus, the unemployment rate falls. They do not realize right away that their purchasing power has fallen because prices have risen more rapidly than they expected. But, over time, as workers come to anticipate higher rates of price inflation, they supply less labor and insist on increases in wages that keep up with inflation. The real wage is restored to its old level, and the unemployment rate returns to the natural rate, but the price inflation and wage inflation brought on by expansionary policies continue at the new higher rates.

Consequently, Friedman's and Phelps's analyses provide a distinction between the "short-run" and "long-run" Phillips curves. As long as the average rate of inflation remains fairly constant for a quite long period, as it did in the 1960s, inflation and unemployment might exhibit a non-linear inverse relationship, *ceteris paribus*. This is because inflation expectations of workers are constantly borne out. Nevertheless, if the average rate of inflation changes due to shocks or other causes: cost-push inflation, excess aggregate demand, or when policymakers persistently try to push unemployment below (or above) the natural rate to increase (or reduce) inflation, after a period of adjustment, unemployment will return to the natural rate. That is, once workers' expectations of price inflation have had time to adjust, the natural rate of unemployment is potentially compatible with any rate of inflation. The long-run Phillips curve or equilibrium steady-state Phillips curve can be then described as a vertical (i.e. completely inelastic) line above the natural rate.

It follows that monetary policy "*cannot peg interest rates for more than very limited periods*". Indeed, the relevant variable is the real rate. Monetary policy will be efficacious only in the short-run, during the lapse of time when private agents fail to correct their price expectations. As soon as expectations are adjusted, as soon as the bargained real wage is the correct one, the expansionary effect of the policy disappears. In the long run, the sole effect of an expansive monetary policy is inflationary. Consequently, the objective assigned to monetary policy is to "*provide a stable background for the economy – keep the machine well oiled*". Therefore individuals "*can proceed with full confidence that the average level of prices will behave in a known way in the future*" (Friedman 1968). Friedman's argument against discretionary monetary policy relies on the following argument: "*We do not know enough to be able to recognize minor disturbances when they occur or to be able to predict either what their effects will be with any precision or what monetary policy is required to offset their effects*". Hence, the monetary authority should "avoid sharp swings in policy" (Friedman 1968). The monetary rule he then advocates consists of a stable monetary framework to anchor private expectations.

Friedman 1967 Presidential Address contained a section that was particularly relevant for the Phillips curve debate because it pointed out that policymakers could not choose any unemployment rate in the long run other than the natural rate of unemployment, the rate that would be 'ground out' by the microeconomic structure of labor and product markets. A more practical interpretation of the level of "natural rate" was the unemployment rate consistent with accurate inflation expectations, which implied a steady rate of inflation (R. J. Gordon 2011).

Accordingly, this propriety of making the rate of inflation constant over time, namely non-accelerating, let, subsequently, Tobin 1980 to define it as NAIRU (acronym of Non-Accelerating Inflation Rate of Unemployment) and defined as u_n.

In Friedman 1967's words (Presidential Address): "*At any moment of time, there is some level of unemployment which has the property that it is con-*

sistent with equilibrium in the structure of real wage rates. At that level of unemployment, real wage rates are tending on the average to rise at a "normal" secular rate..."
The Expectations-Augmented Phillips Curve which arises from both Friedman and Phelps's models can be synthesized by this equation: [1]

$$\pi_t = a\,\pi_t^e - b\,(u_t - u_n)$$

The basic idea behind if policy-makers tried to exploit an apparent Phillips curve trade-off, then the public would get used to high inflation and come to expect it: π_t^e would drift up and the trade-off between inflation and output would worsen. In the long run, you can't go on fooling the public ($\pi_t^e = \pi_t$), and, as a consequence, you can't keep unemployment away from its natural (NAIRU) rate "$u_t = u_n$".
Both Friedman and Phelps's models rely on the concept of adaptive expectations, first used by Cagan 1956, based on the idea that workers adapt the inflation expectation through a comparison of it with the realized value, this process allows adjusting their guesses in time. Therefore, the more quickly workers' expectations of price inflation adapt to changes in the actual rate of inflation, the more quickly unemployment will return to the NAIRU, and the less successful the government will be in reducing unemployment through monetary and fiscal policies.
Friedman's detractors criticized the proposal calling it the "fooling's model". Friedman postulated that employers are always accurate with expectations of the price level, but workers' expected price level does not forecast promptly inflation variations (as if there was asymmetric information between the two categories). This results in a substantial lag in adjusting money wages to the actual price level.
In a business expansion, firms raise the wage but increase also the price level by more, thus reducing the real wage as needed to provide the incentive to hire additional workers. The latter, see the higher nominal wage and interpret it as a higher actual real wage because they fail to adjust their expectation of the price level.
Friedman's model was attacked as grossly implausible because workers have access to monthly announcements of the Consumer Price Index (CPI) and indeed observe actual prices as they shop almost every day.
R. J. Gordon 2011 claimed that: *"in contrast to Friedman's distinction between smart firms and dumb workers, in Phelps's world everyone is equally*

[1] where a and b are coefficients.

fooled. Both firms and workers see the price rise in their industry and produce more, not realizing that the general price level has risen in the rest of the economy (money illusion)".

Phelps 1968 developed one model in which workers are isolated from information about the rest of the economy offering the parable of an economy in which goods are produced in separate "islands," each with its own labor market. Wages and employment decisions must be made on each island without an opportunity to observe what is being done on other islands. An increase in nominal expenditure across all of the islands due to loose monetary policy need not be immediately recognized as such on individual islands, and, as a result, wages and prices need not immediately adjust to the extent required to neutralize any effect upon the real quantities produced and consumed. Thus, the unemployment rate decreases even though, without workers' knowledge, all other firms in the economy have raised the wage by the same amount at the same time.

The workers are fooled into a reduction in frictional unemployment, and the macroeconomic data register a decline in the unemployment rate. Hence, there is a short-term correlation between the rate of wage change and the unemployment rate, but this lasts only as long as expectations are incorrect (R. J. Gordon 2011).

The expectations augmented theory of the two economists argued that the possibility of getting unemployment below the natural rate depends on a process of fooling people, coaxing out higher employment and higher production with rising prices for the things they sell and then surprising them with higher prices than they expected on the things they buy. Through lags in the perception of inflation, these surprises raise output and employment, but as people learn that they are being fooled, the lags shorten.

An alternative, but similar version, of the Expectations-Augmented Phillips Curve is the Accelerationist Philips Curve set forth by Lucas 1972, which takes into account the persistence of inflation when forming expectations. In its most simple form, inflation expectation is the realized value at the preceding time:

$$\pi_t = \pi_{t-1} - b\,(u_t - u_n) \qquad \text{where } \pi_t^e = \pi_{t-1}\text{ , b is a coefficient}$$

When Milton Friedman and Edmund Phelps independently set forth this theory, the Phillips-curve approach seemed still to be working well.

Some of the macroeconomic empirical facts of the early seventies fit the expectations augmented or accelerationist theory. Even though the unem-

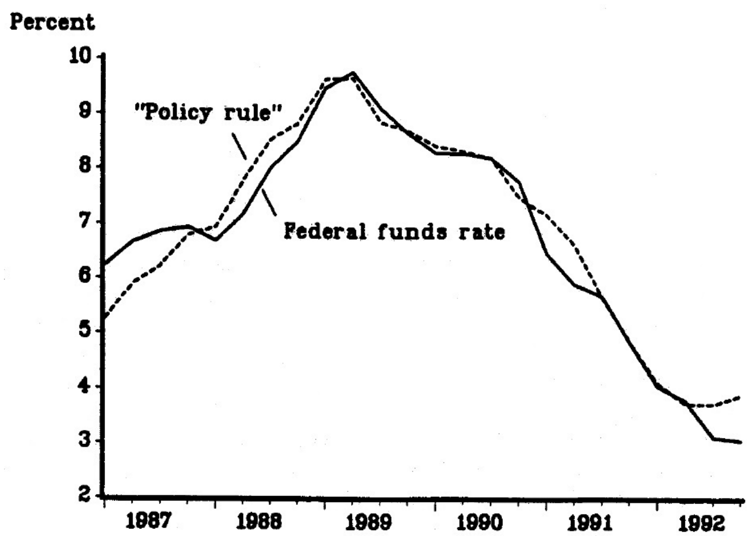
ployment rate exceeded the natural rate in 1970-71, people were, according to the accelerationists, still adapting to the inflationary surprises of 1965-69; hence, inflation decelerated very slowly and only after a lag.
More generally, the unemployment-inflation experience of the first half of the 1970s manifestly reveals a far less favorable trade-off than does that of 1954-68. Clearly, the short-term Phillips curve has shifted upward.
However, Okun 1975 found particularly incredible the clear, though often ignored, implication of the accelerationist view that inflation no longer imposes a cost. He argued that if the American public had fully adapted to some anticipated inflation rate like 6 percent, then not only inflation would have not done any good in expanding output and employment, but it also might have caused harm in distorting distribution or allocation.
He insisted that the microanalytical underpinning of accelerationism is seriously deficient: *"In part, inflation is supposed to distort temporarily the trade-off between work and leisure. According to this story, when people observe a rise in money wages, they believe that real wages are rising too. Consequently, they take jobs and give up leisure, which they now view as more expensive. Ultimately, however, they find that the cost of living has accelerated too, and the labor supply hence gradually shifts back. But why should people take significantly longer to perceive the movement of the cost of living than that of wages? Even more fundamentally, how can the thesis assume a substantial positive elasticity of the supply of labor with respect to the real wage? While that proposition has been widely accepted (by Keynes, among others), the empirical evidence suggests that the elasticity is close to zero and may not even be positive."*
Sargent and Wallace 1973, Sargent 1982 criticized the assumption of adaptive expectations, arguing that expectations should be based on the perceived policy regime and not just on recent history. If the policy regime changes, there is no need for people to use the recent past as their guide (i.e. agents are endowed with rational expectations on inflation).
A core assertion of the rational expectations theory is that actors will seek to "head off" central-bank decisions by acting in ways that fulfill predictions of higher inflation. This means that central banks must establish their credibility in fighting inflation, or economic actors will make bets that the central bank will expand the money supply rapidly enough to prevent a recession, even at the expense of exacerbating inflation.
Thus, if a central bank has a reputation for being "soft" on inflation when it announces a new policy of fighting inflation with restrictive monetary growth, economic agents will not believe that the policy will persist; their inflationary expectations will remain high, and so will inflation.

On the other hand, if the central bank has a reputation for being "tough" on inflation, then such a policy announcement will be believed and inflationary expectations will come down rapidly, thus allowing inflation itself to come down rapidly with minimal economic disruption.
For this reason, several economists advocated credible and publicized policy rules from central banks to target inflation. Among them, probably the most famous one is the Taylor rule. This monetary policy targeting rule was proposed by Taylor 1993 for central banks with the purpose to stabilize inflation and economic activity by appropriately setting short-term interest rates. According to Taylor's original version of the rule, the real policy interest rate $r_t = i_t - \pi_t$ should respond to divergences of actual inflation rates from target inflation rates and to divergences of actual Gross Domestic Product (GDP) from potential GDP:

$$i_t = \pi_t + r_t^* + a_\pi(\pi_t - \pi_t^*) + a_y(y_t - \bar{y}_t)$$

In this equation, i_t, is the target short-term nominal policy interest rate (e.g. the federal funds rate in the US, the Bank of England base rate in the UK), π_t is the rate of inflation as measured by the GDP deflator, π_t^* is the desired rate of inflation, r_t^* is the assumed natural/equilibrium interest rate, y_t is the natural logarithm of actual GDP, and $\bar{y}_t$ is the natural logarithm of potential output, as determined by a linear trend. Thus, $y_t - \bar{y}_t$ represents the output gap. In this equation, both the parameters a_π and a_y should be positive. As a rough rule of thumb, Taylor 1993 paper proposed setting $a_\pi = a_y = 0.5$), see Figure 5 for the empirical estimation found in the original article for the period 1987-1992. That is, the rule produces a relatively high real interest rate (a "tight" monetary policy) when inflation is above its target or when output is above its full-employment level, in order to reduce inflationary pressure. It recommends a relatively low real interest rate ("easy" monetary policy) in the opposite situation, to stimulate output. Sometimes monetary policy goals may conflict, as in the case of stagflation, when inflation is above its target with a substantial output gap. In such a situation, a Taylor rule specifies the relative weights given to reducing inflation versus increasing output.

Figure 5: FED funds rate and Taylor rule, 1987-1992



In the recent period, especially after the Great Financial Crisis of 2007, the Taylor rule has been less adherent to empirical data. Consequently, several other rules or refinements of the Taylor one have been proposed to take into account financial variables such as stock prices, housing prices, interest rate spreads, etc., agent expectations on inflation, GDP, yields on futures markets, etc., other policy instruments such as reserve funds adjustment or balance sheet policies.

The effort to build the analytical base for accelerationism, despite its limitations, has produced constructive research into the "microeconomic foundations" of both employment and inflation theory.

For example, the New Keynesian (NK) models have more explicit microeconomic foundations than its Keynesian ancestor: on the supply side, they consider price and/or wage staggering (Taylor 1979; Calvo 1983). On the demand side, they are composed of an Euler equation and a Taylor rule. Blanchard and Galí 2007 demonstrated that the presence of real imperfections in NK models, such as real wage rigidities, can lead to a trade-off for central banks between stabilizing inflation and stabilizing the welfare-relevant output gap. According to some of these NK macroeconomic models, insofar as the central bank keeps inflation stable, the degree of fluctuation in output will be optimized (Blanchard and Galí 2007 call this property the 'divine coincidence'). In this case, the central bank does not need to take fluctuations in the output gap into account when setting short-term interest rates, e.g. $a_y = 0.5$ in a Taylor rule.

The pervasive issue is the relative role of quantity adjustments and of price (wage) adjustments in different types of markets over different time horizons. The common theme is that, in markets that lack a proper clearing mechanism, quantities vary a lot because prices (wages) vary too little and too late. These non-clearing phenomena are widespread and they might be the heart of the explanation of both inflationary and recessionary processes. Because of them, quantity adjustments may carry the burden for many types of products and factors, e.g. production lags (Muth 1961), leading to the observed sluggishness and persistence of inflation and excessive unemployment.

There are many possible imperfections (i.e. monopoly/oligopoly markets, transactional and friction costs, uncertainties, and various sunk costs) that can explain the narrowness or the sluggishness of market-clearing mechanisms, among them, Okun 1975, decided to focus on the cost of information, interpreting it broadly, to include costs of prediction, of establishing reliability and reputation, etc.

In his vision (pp. 358-359): *"the cost of information leads to implicitly contractual long-term relationships between employees and employers and between customers and suppliers. These relationships create a zone of indeterminacy for wages and prices and a need for 'fair' formulas for the sharing of bilateral monopoly surpluses. By putting price and wage making into a longer-term context, they lengthen the lags and weaken the causal connections between changes in demand and changes in prices or wages. Thus, the welfare costs usually attributed to inflation should be viewed in a broader context as disturbances to a set of institutions that economize on information, prediction, and transaction costs through continuing buyer-seller relationships. Inflation does fool people, as the accelerationists contend. But it does so, not so much by disappointing their point-estimate expectations, as by depriving them of a way of economic life in which they need not depend heavily on the formulation of costly and uncertain point-estimate expectations."*

Moreover, one of the characteristics of a modern industrial economy is that workers do not encounter their employers in an atomized and perfect market. As already remarked, they operate in a complex combination of imperfect markets, monopolies, monopsonies, labor unions, and other institutions. In many cases, employees may lack the bargaining power to act on their expectations, no matter how rational they are, no matter how free of money illusion they are (Okun 1975).

Similarly, built-in inflation is not simply a matter of subjective "*inflationary expectations*", but also reflects the fact that high inflation can gather momentum and continue beyond the time when it was started due to the objective price/wage spiral. And the same holds for low or negative inflation that through pessimistic expectations can generate persistent disinflation or deflation.

Some years later, Akerlof et al. 1996 in their "*The Macroeconomics of Low Inflation*" summarized some of the most relevant determinants of inflation in what was defined as the "*triangle model*":

• Demand-pull inflation is caused by increases in aggregate demand due to increased private and government spending, etc. Demand inflation encourages economic growth since the excess demand and favorable market conditions will stimulate investment and expansion (i.e. traditional Keynesian vision);

• Cost-push inflation, also called "supply shock inflation", is caused by a drop in aggregate supply (potential output). This may be due to natural disasters, or increased prices of inputs. For example, a sudden decrease in the supply of oil, leading to increased oil prices, as we discussed previously, can cause cost-push inflation. Producers for whom oil is a part of their production costs could then pass this onto consumers in the form of increased prices. Another example stems from unexpectedly high insured losses, either legitimate (catastrophes) or fraudulent (which might be particularly prevalent in times of recession). High inflation can prompt employees to demand rapid wage increases to keep up with consumer prices. In the cost-push theory of inflation, rising wages in turn can help fuel inflation;

• Built-in inflation is induced by adaptive expectations, and is often linked to the "price/wage spiral". In the case of collective bargaining, wage growth will be set as a function of inflationary expectations, which will be higher when inflation is high. This can cause a wage spiral. In other words, it involves workers trying to keep their wages up with rising price expectations and firms passing these higher labor costs onto their customers as higher pricing for final products, leading to a feedback loop. In a sense, inflation begets further inflationary expectations, which leads to further price increases. Built-in inflation reflects events in the past, and so might be seen as "hangover inflation".

2.5 The recent theories and the NAIRU debate

Modern macroeconomic models often employ another version of the Phillips curve in which the output gap replaces the unemployment rate as the measure of aggregate demand relative to aggregate supply. The output gap is the difference between the actual level of GDP and the potential (or sustainable) level of aggregate output. The potential output (sometimes called the "natural gross domestic product") and indicated as y^* can be thought of as the level of growth needed to maintain the economy at its optimal level of production given institutional and natural constraints.
This growth level is strictly associated with the NAIRU (Tobin 1980).
If GDP exceeds its potential (and unemployment is below the NAIRU), the theory says that inflation will accelerate as suppliers increase their prices and built-in inflation worsens. If GDP falls below its potential level (and unemployment is above the NAIRU), inflation will decelerate as suppliers attempt to fill excess capacity, cutting prices and undermining built-in inflation.
This formulation may also explain why, at the end of the 1990s boom, when unemployment rates were well below estimates of NAIRU, prices did not accelerate. The reasoning might be the following.
Potential output depends not only on labor inputs, but also on plant and equipment and other capital inputs. At the end of the boom, after nearly a decade of rapid investment, firms found themselves with too much capital. The excess capacity raised potential output, widening a negative output gap and reducing the pressure on prices.
However, one problem with this theory for policy-making purposes is that the exact level of potential output (and of the NAIRU) is generally unknown and tends to change over time (Modigliani and Papademos 1975). In particular, this natural level depends not only on productivity gain, structural parameters, or specific characteristics of the labor market (exogenous to the models) but also on changes in unemployment (endogenous).
Some economists hold that, at best, there is only a weak tendency for an economy to return to NAIRU (Blanchard 2018). The most skeptical ones argue that there is no natural rate of unemployment to which the actual rate tends to return because the economy is in constant disequilibrium (Galbraith 1997). Many others, instead, think that when actual unemployment rises and remains high for some time, NAIRU also increases. This pattern is evident in the numerous recent empirical works (Barro and D. B. Gordon 1983; Jordan 1997; L. Ball and Mazumder 2011) and it has become widely

accepted in the literature.
The dependence of NAIRU on actual unemployment is known as the hysteresis hypothesis. If the unemployment rate exhibits hysteresis, it means it follows a statistically non-stationary process: the expected value of the unemployment rate now and in the future permanently shifts when the rate itself changes (i.e. unit root process).
Moreover, there is also evidence that it could change because of policy. For example, the restrictive monetary and fiscal policy to counteract inflation under British Prime Minister Margaret Thatcher caused persistently high unemployment in the economy that might have led to a rise in the NAIRU (and a fall in potential output). On that occasion, many of the unemployed found themselves as structurally unemployed, unable to find jobs that fit their skills. A rise in structural unemployment implies that a smaller percentage of the labor force can find jobs at the NAIRU, where the economy avoids crossing the threshold into the realm of accelerating inflation.
Another possible explanation for hysteresis is the one attributed to unionization, which could lead to some market rigidities (unemployment insurance and protection, minimum wages, bargaining rules, such as extension agreements, etc.). However, trade unions and labor organizations have lost, in the modern world, most of the power they used to have until the economic liberalization and deregulation of the eighties.
According to this hypothesis, once unemployment becomes high, as it did in Europe in the recession of the 1970s, it becomes relatively impervious to monetary and fiscal stimuli, even in the short run.
The unemployment rate in France in 1968 was 1.8 percent, and in West Germany, 1.5 percent. In contrast, since 1983, both French and West German unemployment rates have fluctuated between 7 and 11 percent. The hypothesis related to the labor market characteristics seemed to be more relevant in Europe, where unionization was relatively higher and where labor laws create some barriers to hiring and firing than it was in the United States with its considerably more flexible labor markets. The unemployment rate in the United States was 3.4 percent in 1968 and peaked just momentarily in the early 1980s at 10.8 percent, falling back around 5 percent in the last eighties. However, unemployment in Europe was not always high. If we refer to recent times, several European countries had low unemployment prior to the start of the financial crisis of 2007, and this was true also for large parts of previous decades. One might argue that high unemployment is more the result of the inefficient application of these policies, rather than the policies per se.

Surely, also the evolution of technology (i.e. breakthrough technological advancements) and the changes in the labor-capital mix have played a role in the permanent change of the NAIRU level, especially in actual times. For instance, in the U.S. boom of the late 90s, unemployment dipped below NAIRU estimates without causing significant increases in inflation. FED Chair Alan Greenspan had judged that the Internet revolution had structurally lowered NAIRU.
A recent paper by Benati 2007 provides a possible explanation for the flattening of the Phillips curve from early 2000, which is rooted in the main thrust of the New Keynesian (NK) framework. In the standard firm pricing rule assumed in such models, higher (lower) trend inflation increases (decreases) the frequency of firms' price adjustments – this being a "deep" parameter in the reduced-form coefficient of the output gap in the Phillips curve – thus increasing (reducing) the sensitivity of domestic inflation to cyclical output fluctuations. Benati 2007 shows that historically, and across a large set of countries including the euro area, the time-varying slope of the Phillips curve is positively correlated to the trend rate of inflation. Hence, according to his interpretation, the decline in the coefficients of the estimated reduced-form Phillips curve for many OECD countries, including the euro area over the last two-three decades, is due to the progressive confirmation of a low-inflation environment.
The standard NK Phillips curve models inflation as a function of past and expected inflation plus some driving variable, which is normally the output gap or firms' marginal costs. Hence, assuming rational expectations and forward-looking behavior, the trend in inflation is affected by the trend in the driving variable. This would imply that globalization may have contributed to flattening the short-term Phillips curve by influencing the trend of the driving variable, for example by affecting the trend of domestic wage dynamics, rather than through integration of the production process and increasing international trade.
Lastly, prolonged supply-side shocks, such as the energy crises that began in 2021 and were further exacerbated by the Russia-Ukraine conflict, can pose risks of a permanent shift in the inflation-unemployment relationship. The shock on natural gas prices, supply bottlenecks, and the subsequent exceptional increases in the prices of commodities that hit the major economies in 2022 led many economists to make analogies with the oil crises of 1973-74. Similarly, it is a supply or price shock, coming from an accumulation of causes, largely external.
This poses an altogether different stabilization problem. In particular, in the case of demand shocks, there exists in principle an adequate monetary policy

that can reduce the social costs imposed by high inflation by dampening effective demand, as advocated by Keynes, and guarding against the risk of a persistent upward shift in inflation expectations. The aim is to offset the shock with a temporary sacrifice in terms of aggregate demand and economic growth that helps to stabilize the price level until unemployment can return to the NAIRU (or natural) level.

As Modigliani 1995 highlighted: "*there may be disagreement as to whether this target can be achieved and how, but not about the target itself. But in the case of supply shocks, there is no miracle cure, there is no macro policy that can both maintain a stable price level and keep employment at its natural rate. To maintain stable prices in the face of the exogenous price shock, say a rise in import prices, would require a fall in all domestic output prices; but we know of no macro policy by which domestic prices can be made to fall except by creating enough slack, thus putting downward pressure on wages. And the amount of slack would have to be substantial in view of the sluggishness of wages in the face of unemployment. If we do not offset the exogenous shock completely, then the initial burst, even if activated by an entirely transient rise in some prices, such as a once and for all deterioration in the terms of trade, will give rise to further increases, as nominal wages rise in a vain attempt at preserving real wages. In short, once a price shock hits, there is no way of returning to the initial equilibrium except after a painful period of both above equilibrium unemployment and inflation*".

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Chapter 2: An optimal control problem of monetary policy

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Abstract

This chapter analyses an optimal monetary policy under a non-linear Phillips curve and linear GDP dynamics. A central bank controls inflation and GDP trends through the adjustment of the interest rate to prevent shocks and deviations from the long-run optimal targets. The optimal control path for the monetary instrument, the interest rate, is the result of a dynamic minimization problem in a continuous-time fashion. The model allows considering various economic dynamics ranging from hyperinflation to disinflation, sustained growth, and recession. The outcomes provide useful monetary policy insights and reveal the dilemma between objectives faced by the monetary authority in trade-off scenarios.

Keywords: Optimal Control, Monetary Policy, Inflation Targeting, Phillips Curve, Non-linearity.
JEL Classification: C61, C63, E52, E58.

Statement of joint authorship and acknowledgments

I confirm that this Chapter - "An optimal control problem of monetary policy" - was jointly co-authored with Germana Giombini, and I contributed to 70% of this work.
This entire chapter was published as an article:
Bacchiocchi, A., Giombini, G. (2021). "An Optimal Control Problem of Monetary Policy". In: *Discrete and Continuous Dynamical Systems Series-B*, 26(11), pp. 5769–5786. DOI: 10.3934/dcdsb.2021224.
This chapter was presented to the 62nd Annual Conference (RSA) of the Italian Economic Association - SIE, online (26-29 Oct. 2021).
I would like to thank the discussants at the 62nd RSA SIE conference, two anonymous referees of "Discrete and Continuous Dynamical Systems Series-B", Gian Italo Bischi, Giovanni Marin (University of Urbino), Fabio Lamantia (University of Calabria), and Davide Radi (University Cattolica Sacro Cuore) for helpful suggestions and comments.

1 Introduction

Most macroeconomists and central bankers agree that the main aim of monetary policy should be to control long-run inflation (e.g. Phillips 1958; Phelps 1967; Friedman 1968). However, many also believe that monetary policy can have a short-run role in helping to stabilize business cycles (e.g. Keynes 1936; Galì 2008). Thus, the ultimate target of a Central Bank (hereinafter CB) is often a mixed signal based both on inflation and the business cycle, generating a potential conflict between short-run and long-run goals.

One of the objectives of the monetary authority is to minimize any deviation, or gap, of the gross domestic product (hereinafter GDP) from its long-term level, i.e. the potential output. As GDP can rise or fall, the output gap can be either positive or negative.

On the one hand, a positive output gap, occurs in periods characterized by particularly high levels of demand so that firms and workers operate close or even above their efficient capacity frontier to meet demand. This situation generates upward pressures on prices leading to a rise in inflation and subsequent negative consequences for the economy. Indeed, high inflation can lead to a spiral of increasing prices, limiting households purchasing power and making firms' investments more complicated.

On the other hand, a negative output gap occurs when the actual output is less than what an economy could potentially produce, given the technology and the production factors, among others. It means that there is a spare capacity or a slack in the economy due to a weak demand compared to the supply, or to frictions and market imperfections. In this case, the economic downturn might drive down inflation, and the latter may create negative feedback loops with the real economy leading to a spiral of falling prices. The burden for debt servicing rises creating additional negative feedback loops between the real economy and the price level as long as firms and households postpone their investment and consumption decisions.

Output fluctuations are intrinsic to any economy and, often, they do not exhibit uniform or predictable dynamics. One of the major tasks of the monetary policy should be to mitigate these temporary downward and upward movements of GDP around its long-term growth trend through expansionary or contractionary monetary measures.

Along with the so-called Keynesian counter-cyclical objective, the core aim of a central bank remains that of stabilizing prices in order to prevent whatsoever relevant deviation from the long-run inflation target.

The existing literature in this area has mainly focused on simple monetary policy rules that are generalizations of the Taylor Rule (Taylor 1993) and has not fully been drawn to the idea of applying optimal control theory to the problem of monetary policy, with few exceptions (Semmler and Zhang 2004; Kodera and Tran 2013).[1]

In this chapter, we improve with respect to the previous literature by addressing the issue of the impact of monetary policy on prices and output by means of a dynamic model based on optimal control theory to identify an appropriate monetary policy path.

We start by defining the aim of the CB as the minimization of a loss function that depends both on output and inflation gaps, the two main objectives. Then, we model GDP and inflation dynamics.

A key novelty of the model is the assumption of an augmented nonlinear Phillips curve, to take into account the possibility that the response of inflation to changes in economic activity may be asymmetric. Indeed, recent studies have argued that the dynamics of inflation have changed substantially in many, if not all, advanced economies over the past four decades, showing that various types of structural changes have affected the statistical properties of inflation, making increasingly complex the modeling of inflation dynamics (Musso et al. 2009).

We obtain a dynamic system that presents several non-linearities, as it is formed by four differential equations, meaning that is a 4-th order or 4-th dimensional system. We explore the model dynamics by means of numerical simulations, and the main findings of the model can be described as follows. Firstly, our model allows considering different phenomena such as inflation and deflation as well as situations of sustained economic growth and periods of recession. Secondly, the model simulations well represent different economic scenarios and trade-off situations. Thirdly, the model reveals the CB's difficulties to reach both the output and the inflation targets with only one instrument, i.e. the interest rate. Finally, the dynamics of the variables heavily depend on the value of the parameters, and the relative priority assigned by the CB to one target relative to the other.

The chapter is organized as follows. In the next Section, we describe the model, while Section 3 solves the optimal control problem. Section 4 analyzes the model dynamics employing numerical simulations, and Section 5 concludes.

[1] Some of the pioneer works in this field are Chow (1976), Tabellini (1986), Svensson (1997) and more recently Evans et al. (2001), Bischi and Marimon (2001), Ferrero (2007), Orphanides and Williams (2008), which addressed the issue even if with different methodologies.

2 The model

The CB minimizes both the output and the inflation gap, over a defined time period $[0, T]$, where 0 can be assumed as the time when the Board of Governors is appointed, and T as the end of the term.[2]

The two objectives can be conflicting not only in attainment but also in time. Variables are expressed as percentage variation from one period of time to the other (i.e. years). Therefore, the current GDP rate of growth and inflation rate are defined, respectively, as follows: $\tilde{y_t} = \frac{GDP_t - GDP_{t-1}}{GDP_{t-1}}$, $\tilde{\pi_t} = \frac{p_t - p_{t-1}}{p_{t-1}}$.

Then, inflation and output gap at time t are defined as: $\pi_t = \tilde{\pi_t} - \bar{\pi}$ and $y_t = \tilde{y_t} - \bar{y}$, where $\bar{\pi}$ and $\bar{y}$ are the inflation target and the potential or natural output growth, which we assume exogenous to the model.

Similarly, the interest rate i is the monetary policy instrument (the control variable of the problem) and might be as well expressed in terms of deviation from the long-run optimal value [3] $\bar{i}$ prevailing in the economy. Thus, $i_t = \tilde{i_t} - \bar{i}$, where the variable with a tilde above indicates the current value of the interest rate at time t.

2.1 The Loss Function

Tinberger (1956) addressed the issue of the controllability of a fixed set of targets by a policymaker endowed with given instruments, stating that when targets exceed instruments, the system is overdetermined and the economic policy model might not allow for solutions. One way to overcome this problem is to reduce the number of targets and make them flexible. Thus, in our model, the authorities can reduce the two targets to a single one by channeling inflation and output into a single function, the 'Loss function' of the policymaker.

[2] The Board of governors of a CB is the executive committee composed of senior members and responsible for the monetary policy stance of the bank. Generally, it remains in charge for a defined period of time that varies from 3 years up to 8 or 10 years, depending on the statute and regulation of each CB.

[3] $\bar{i}$ is the long-run nominal interest rate that allows the economy to grow at the natural output rate $\bar{y}$ and to remain at the inflation target rate $\bar{\pi}$ when no short-term macroeconomic shocks occur.

We assume a quadratic Loss Function specified as follows:[4]

$$L = \frac{1}{2}\,\alpha\,\pi^2 + \frac{1}{2}\,\beta\,y^2 \tag{1}$$

In equation (1) inflation and output gaps are squared, so that positive and negative deviations are counted the same way. This is because, as stated in the introduction, both can be potentially detrimental to the economy.
However, π and y do enter the loss function with different weights, α and β, respectively. The latter parameters reflect the different degrees of preference by CB for the two main goals. Accordingly, $0 \leq \alpha, \beta \leq 1$ always sum to 1, and can be interpreted as behavioral parameters that depend on the policy will of the Board of governors.
In the next Sections, we assume that the political stance on the two final objectives, the control over inflation and the mitigation of output fluctuations, remains unchanged during the period the Board of Governors is in charge.
If the CB ascribes more relevance to the price stabilization objective rather than the mitigation of output fluctuations, α will be greater than 0.5; while $\beta > 0.5$ if the economic stimulus becomes primary in the CB agenda.
In the scenario in which CB gives the same priority, α and β are equal to 0.5, while in the extreme cases where only one objective is pursued at the expense of the other, the weight is set to 1 (and the complementary will be zero, not entering the functional).
Aside from the monetary policy trade-off in terms of preference, the two variables under consideration are also linked to each other, as described in the next sections.

[4]This loss function is known as LQR (linear quadratic regulator) for optimal control systems in engineering theory. It has been widely used also in economic dynamics, particularly in monetary policy problems. Some examples are Svensson (1997), where the same loss function is used in a linear discrete-time optimal control problem, and Tabellini (1986) that models a linear-quadratic game between the fiscal and monetary authorities. In our case, we use this quadratic regulator (i.e. loss function), but with the additional complexity of a non-linear differential equation.

2.2 The inflation dynamics: a non-linear Phillips Curve

The differential equation that describes the evolution of π consists of two parts, as follows:

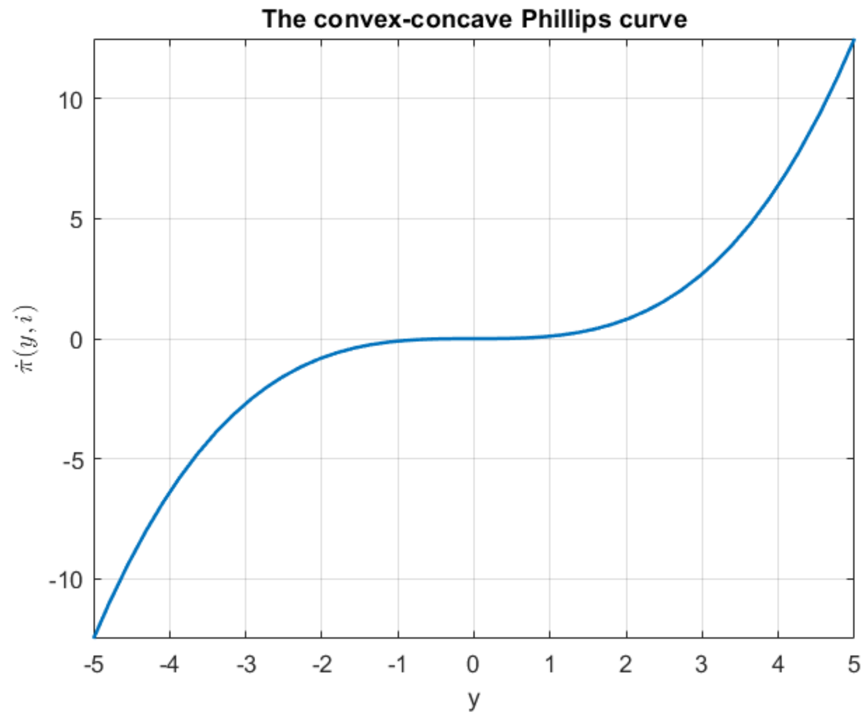
$$\dot{\pi} = \gamma\, y^3 - \frac{1}{2}\,\omega\, i^2 - i \tag{2}$$

with γ, $\omega > 0$.

The first part of equation (2) derives from the non-linear Phillips curve specification proposed by Filardo (1998), which relates the acceleration of the inflation ($\dot{\pi}$) with the deviation of growth from its natural value (y) through a third-degree equation.
When the output gap is $y = 0$, the economy is at its potential output growth $\bar{y}$, that can be interpreted as the level of growth needed to maintain the economy at its optimal level of production given institutional and natural constraints.
This level of growth is associated with the NAIRU, namely the "Non-accelerating inflation rate of unemployment" (Okun, 1975).
As the acronym means, for this particular economic equilibrium value the inflation rate does not accelerate, thus $\dot{\pi} = 0$, given a constant monetary policy.
The shape of this nonlinear Phillips curve implies that when the output gap is positive ($y > 0$), the Phillips curve is convex, and when the output gap is negative ($y < 0$), the Phillips curve is concave, as shown in Figure 1.
The convex-concave Phillips curve implies that the cost of fighting inflation or disinflation rises when the output gap increases, both in positive and negative terms.[5]
The positive parameter γ measures the cost of fighting inflation gaps. It can be also interpreted as the sensitivity of the rate of change of inflation to the output deviations, capturing the impact of the real variable y on prices.

[5] Filardo (1998) explores the problem of the shape of the Phillips curve theoretically and empirically. He proposes that the curve is not purely convex or concave, but a combination of both, namely, the convex-concave curve.

Figure 1: The convex-concave Phillips curve



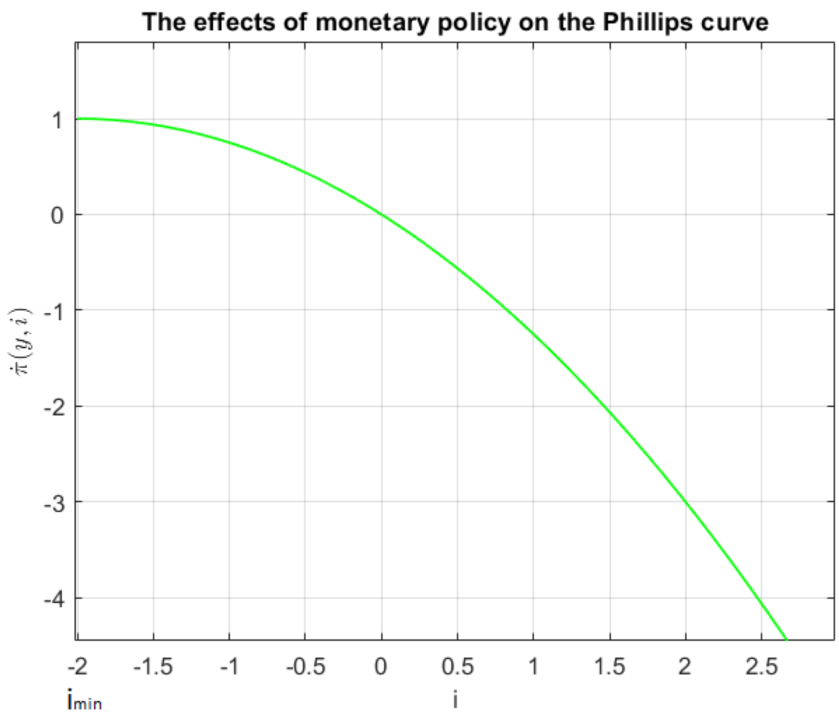
Note: Figure 1 represents the first part of equation (2). Parameters: $\gamma = 0.1$.

In a Keynesian vision, we argue that the higher the output and the associated aggregate demand in the short-run, the greater will be the upward pressure on prices. On the contrary, in a period characterized by a slowdown of growth, the uncertainty and the lack of trust of consumers and economic agents lead to a postponement of the decisions of consumption and investment, which results in downward pressure on prices causing disinflation, or even deflation when the percentage change in prices becomes negative.
For positive output gaps, the convex shape of the Phillips curve is consistent with an economy subject to capacity constraints. Indeed, as the economy becomes stronger and capacity constraints restrict firms' ability to expand output, an increase in demand is more likely to show up as higher inflation than as higher output. Moreover, if firms have some market power, they could benefit from this growing aggregate demand setting increasingly higher prices and mark-ups on their final product, on the margin (Turner, 1995; Clark et al., 1996; Debelle and Laxton, 1997).
For negative output gaps, the concave shape of the Phillips curve describes

an economy where agents are not purely competitive. In this case, price-maker firms will decide to reduce prices in order to stimulate demand for their products during economic downturns (Stiglitz, 1997).

The second part of equation (2) captures the impact of the monetary policy instrument, the interest rate i, on the acceleration of inflation (Figure 2). This introduces a second non-linearity given that the instrument enters in a quadratic form in differential equation (2).

Figure 2: The effects of monetary policy on the Phillips curve



Note: Figure 2 represents the second part of equation (2). Parameters: $\omega = 0.5$, $i_{min} = -2$ and $\bar{i} = 2$.

The positive parameter $\omega > 0$ measures the strength of the monetary policy on inflation variations, modifying the steepness of the parabola in Figure 2. An interest rate gap equal to $i = 0$ means that the CB sets the interest rate at the value $\bar{i}$ coherent to the long-run potential growth of the economy $\bar{y}$ and the steady optimal inflation rate $\bar{\pi}$.

In accordance with the monetarist theory (Friedman 1957), we assume a negative relationship between $\dot{\pi}$ and i. However, we postulate that the impact of a contractionary or expansionary monetary policy on inflation is asymmetric. Indeed, the second part of equation (2) captures the assumption that for the CB is relatively more difficult/costly to stimulate a price increase with an expansionary measure than the opposite (i.e restrictive policy).
A monetary contractionary policy (i.e. an increase of the interest rate over $\bar{i}$) means a positive interest rate gap ($i > 0$), which implies a reduction of the amount of liquidity in circulation in the economy and, as a result, weakens price level over long periods.
A monetary expansionary policy (i.e. a reduction of the interest rate under $\bar{i}$) means a negative interest rate gap ($i < 0$). This implies a boost in the amount of liquidity in circulation in the economy and an increase in the price level over long periods, even though with a lower magnitude than a restrictive policy.
The asymmetric effect is in line with what many advanced economies (especially Europe) have experienced in the aftermath of the financial crisis, where albeit with very low interest rates set by the monetary institutions (almost near to the minimum bound of zero), inflation remains at low values during the 2014-2017 period.
When the interest rate $\tilde{i}$ reaches the minimum level called zero lower bound (ZLB), it cannot be further reduced. In this situation, the negative effect of a lower interest rate on bank profits may lead to a contraction in lending and economic activity (Brunnermeier and Koby, 2016).
In our model, the ZLB is defined by the maximum (or apex) of the parabola depicted in Figure 2 (i_{min}).
The parameter ω, acting on the shape of the parabola, affects both the value of i_{min} and $\bar{i}$.
Coherently with the empirical evidence (Altavilla et al. 2020), an ultra expansive monetary measure, with a reduction of the interest rate near to zero lower bound i_{min}, decreasingly boosts inflation $\frac{\partial^2 \dot{\pi}}{\partial i} < 0$.
In conclusion, equation (2) allows us to embrace the two major macroeconomic contributions and theories that describe the inflation dynamics over time.

2.3 The GDP dynamics

We assume that the GDP dynamics depends on inflation and interest rate, as follows:

$$\dot{y} = \eta\,\pi - \phi\,i \tag{3}$$

The first term in equation (3) assumes that the price dynamics can linearly affect the GDP rate of growth, at least in the short run.
Inflation moderately above the reference ($\pi > 0$), especially if not forecasted by economic agents, may represent a stimulus in the short-run to anticipate some decisions of consumption. In fact, in the cases of positive inflation deviations, it is more convenient for privates and firms to purchase goods, production factors and services now than later at higher prices. This extra consumption and investment lead to an increase in the aggregate demand and ultimately in GDP. [6]
In the cases of negative inflation deviations ($\pi < 0$), the opposite happens: households postpone their consumption decisions waiting for a further decrease in prices, and firms, similarly, delay investment. The parameter $\eta > 0$ measures the magnitude of this impact.
The second term of equation (3) captures the interaction between monetary and real variables. The traditional counter-cyclical role of the monetary policy indicates that a boost in the money supply (i.e. reduction of the interest rate, so $i < 0$) brings a stimulus to the economy.
On the contrary, a monetary tightening (i.e. increase of the interest rate, so $i > 0$) causes a reduction in the aggregate demand and a slowdown of growth in the short-run.
The parameter $\phi > 0$ indicates the effectiveness of the monetary policy on output variations.

[6] However, a clarification must be made. In the long-run, inflation well over the target and persistent in time might be costly in terms of growth because it creates uncertainty over relative prices generating resource misallocations and distortions in economic choices. Some of these costs involve the postponement of relevant investment decisions (i.e. a reduction of investment) that could impair the growth potential of an economy (Briault, 1995), thus highlighting a negative relationship between the variables.
Some empirical works have also tried to estimate this cost. Bruno and Easterly (1996) have demonstrated that growth could relevantly decrease when the inflation gap π is positive and over 20 percent values (hyperinflation), as witnessed by several Latin America developing economies during the early '90s. Barro (1997), similarly, has shown that for very high inflation values ($> +10\%$) the lack of growth could be between 3 and 4 percentage points.

2.4 The minimization problem

The CB faces the following optimal control problem in which it minimizes the Loss Function (1) subject to inflation dynamics (2) and GDP dynamics (3):

$$min_{\,i} \qquad J(t) = \int_0^T \frac{1}{2}\,\alpha\,\pi^2 + \frac{1}{2}\,\beta\,y^2\,dt \tag{4}$$

subject to

$$\dot{\pi} = \gamma\,y^3 - \frac{1}{2}\,\omega\,i^2 - i,$$

$$\dot{y} = \eta\,\pi - \phi\,i$$

where the interest rate i is the control variable that influences the dynamics of both the state variables π and y, and with the following initial and final conditions:

$$\pi(0) = \pi_0 \qquad \pi(T)\,free \qquad \pi_0,\,T\;\,given$$

$$y(0) = y_0 \qquad y(T)\,free \qquad y_0,\,T\;\,given$$

Given that it is a free-terminal-state problem for the two state variables π and y, we must define the transversality conditions needed to solve it. Such conditions only concern what happens at the terminal time T and involve the co-state variables:

$$\lambda_1(T) = 0 \qquad \lambda_2(T) = 0.$$

In this vertical-terminal-line problem, i, namely the control variable, represents the principal instrument at disposal of CB to achieve the two main goals of stabilizing prices and stimulating growth in the short-run, especially in the event of unexpected negative shocks.
The variation of nominal interest rate i is directly controlled by the monetary

authority through open-market operations, interbank rate interventions, and reserve requirements.
For the sake of realism, hereinafter, we assume 5 years as time span T, an average between the duration in charge of the Board of Governors of the main central banks (Federal Reserve 4 years, ECB 8 y., Bank of England 8 y., etc.).

3 The solution

The Hamiltionian of the optimal control problem is defined as:

$$\mathrm{H}\ (t, \pi, y, i, \lambda_1, \lambda_2) = \frac{1}{2}\ \alpha\pi^2 + \frac{1}{2}\ \beta y^2 + \lambda_1\ (\gamma\, y^3 - \frac{1}{2}\,\omega\, i^2 - i) + \lambda_2\ (\eta\,\pi - \phi\, i) \tag{5}$$

We start by deriving (5) with respect to $i(t)$ [7], the control variable:

$$\frac{\partial H}{\partial i} = 0 \qquad -\omega\,\lambda_1\, i - \lambda_1 - \phi\,\lambda_2 = 0$$

$$i^*\ (t) = -\frac{1}{\omega} - \frac{\phi}{\omega}\,\frac{\lambda_2}{\lambda_1} \tag{6}$$

Substituting the optimal control (6) into the equation of motion for π in (4), after some algebraical steps we can obtain the following differential equation for the first state variable:

$$\dot{\pi} = \gamma\, y^3 + \frac{1}{2\,\omega} - \frac{\phi^2}{2\,\omega}\,\frac{{\lambda_2}^2}{{\lambda_1}^2} \tag{7}$$

Considering the law of motion of y in (4) and repeating the substitution of the optimal control (6), we achieve the differential equation for the second state variable:

$$\dot{y} = \ \eta\,\pi + \frac{\phi}{\omega} + \frac{\phi^2}{\omega}\,\frac{\lambda_2}{\lambda_1} \tag{8}$$

[7] We should recall that the variables of the problem $\pi, y, \lambda_1, \lambda_2, i$ are dynamical and hence a function of time, for conciseness we decided to omit throughout the term t.

The optimal co-state paths are derived from the following conditions:

$$\dot{\lambda_1} = -\frac{\partial H}{\partial \pi} = -\alpha\,\pi - \eta\,\lambda_2 \tag{9}$$

$$\dot{\lambda_2} = -\frac{\partial H}{\partial y} = -\beta\,y - 3\,\gamma\,\lambda_1\,y^2 \tag{10}$$

We apply Pontryagin's maximum principle to obtain a system of four differential equations, two for the state variables, (7) and (8), and two for the associated co-state variables, (9) and (10):

$$\begin{cases} \dot{\pi} = \gamma\,y^3 + \frac{1}{2\,\omega} - \frac{\phi^2}{2\,\omega}\,\frac{{\lambda_2}^2}{{\lambda_1}^2} \\ \dot{y} = \ \eta\,\pi + \frac{\phi}{\omega} + \frac{\phi^2}{\omega}\,\frac{\lambda_2}{\lambda_1} \\ \dot{\lambda_1} = -\alpha\,\pi - \eta\,\lambda_2 \\ \dot{\lambda_2} = -\beta\,y - 3\,\gamma\,\lambda_1\,y^2 \end{cases}$$

with again the following initial and final conditions:

$$\pi(0) = \pi_0 \qquad \pi(T)\;free \qquad \pi_0,\,T\;\;given$$

$$y(0) = y_0 \qquad y(T)\;free \qquad y_0,\,T\;\;given$$

and the transversality conditions associated with the two free-terminal state variables:

$$\lambda_1(T) = 0 \qquad \lambda_2(T) = 0.$$

From the solution of this ordinary differential equation (ODE) system, it is possible to trace back the optimal paths for the two state variables π and y, as well as for the two auxiliary variables λ_1 and λ_2.
In addition, it allows us to define from equation (6) the optimal control path for i.
This path represents the optimal dynamic monetary instrument or measure implemented by the CB to drive, in time, the initial deviations/shocks of inflation and growth toward zero.

This system presents several non-linearities and is formed by four differential equations, meaning that is a 4-th order or 4-th dimensional system. Besides, the two transversality conditions on the lambdas make the solutions even more difficult to obtain. In mathematical research, these are known as boundary value problems (BVPs).
The long-run equilibrium of the economy is obtained for $\pi = 0,\ y = 0$ and $i = 0$, when neither the GDP, nor the inflation accelerates (i.e. they remain steady at their pre-fixed target rate of growth), and the CB pursues the optimal long-run monetary policy.
Consequently, we are interested in disequilibrium situations where for some reason (shocks, business cycle fluctuations, etc.) inflation and GDP growth are not in the long-run steady state, calling for an intervention of the CB to restore the long-run trends of the economy.
Therefore, in the next section, we provide numerical simulations to try to understand the behavior of the model and to cover different economic scenarios.[8]

[8]Numerical simulations are performed by means of an algorithm in the software Matlab. In particular, the function 'bvp5c' allows performing numerical computations for solving (non-linear) ODE systems in all those cases in which there are final/transversality conditions on the variables, by using an educated guess of their starting values. These algorithms work for successive approximation through thousands iterations of the derivatives at each point of time (Runge-Kutta methods). As for all the numerical computations the results are not exact, nevertheless, it is possible to achieve on average accuracy of 0.01 percent with an estimated maximum error lower than 0.02 percent.

4 Dynamics and numerical simulations

As previously mentioned, the two main objectives of a monetary policy are to control the evolution of prices, avoid any detrimental effect caused by acceleration or deceleration of inflation in the long-run, and, at the same time, influence the macroeconomic condition in the short-run. This latter function is now widely recognized among economists and policymakers. It has been pursued by the vast majority of central banks in the aftermath of the Great Recession and also recently, during the Coronavirus recession, to alleviate the financial and economic distress through a massive injection of liquidity in the economy and extraordinarily expansive monetary measures. As a consequence, it becomes interesting to analyze situations characterized by an economy located far from its long-run employment and growth equilibrium. This might occur frequently due to the number of shocks and cyclical fluctuations typical of the economic cycle.

Therefore, in this section, we provide some numerical simulations to study the effects of different economic conditions and/or monetary policies on the dynamics of the economic system.

To this purpose, we start by defining inflation and output targets, taking the Eurozone as a reference area. The European Central Bank (ECB) aims at inflation rates below but close to 2% over the medium term, whereas a plausible natural value for annual growth in the Eurozone might be in a range between 1% and 2%. The target nominal interest rate may be associated with the long-run inflation target and the natural rate of growth. These optimal values for inflation, nominal interest rate, and output can be interpreted as the long-run optimum in absence of any shock in the economy. Therefore, in the next figures, all variables are intended as deviations from these target values $(\bar{\pi}, \bar{y}, \bar{i})$, here normalized at zero for convenience.

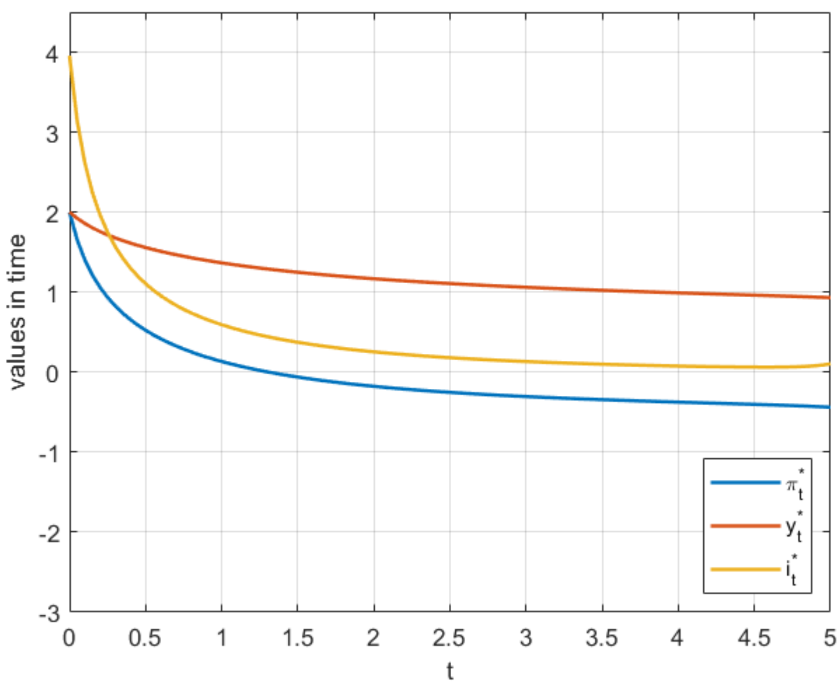
In the following simulations, we assume that at time $t = 0$ a shock affects the economy, so that it moves from its long-run path. Then, we study the proper dynamic monetary response in time $i(t)$ to bring back GDP and inflation to their long-run values.

The blue curve represents the evolution of the inflation gap, while the dynamics of the business cycle and the nominal interest rate are represented by the yellow and orange curves, respectively.

4.1 Sustained economic growth and inflation

Figure 3 represents a situation where both inflation and GDP are 2 percent points over the pre-established target, the CB gives the same priority to the objectives and ϕ is relatively low. As can be expected, as this situation might be related to economic bubbles, the CB reacts by undertaking a restrictive monetary measure, motivated by the need to mitigate the negative effect of an "overheating" situation, maintaining the interest rate above the long-run optimal value for the whole time span considered (from a positive 3.95 % gap at the initial moment $t = 0$ of the shock to a +0.1 % in the last years). The CB effort leads to a strong reduction of aggregate demand that in turn helps to mitigate the initial over-inflation and excessive economic growth. The result is satisfactory as the macroeconomic gaps approach zero, even though GDP remains 0.93 % higher than the target and inflation ends up slightly under the reference (-0.44 %) due to also the output slowdown.

Figure 3: Sustained economic growth and inflation



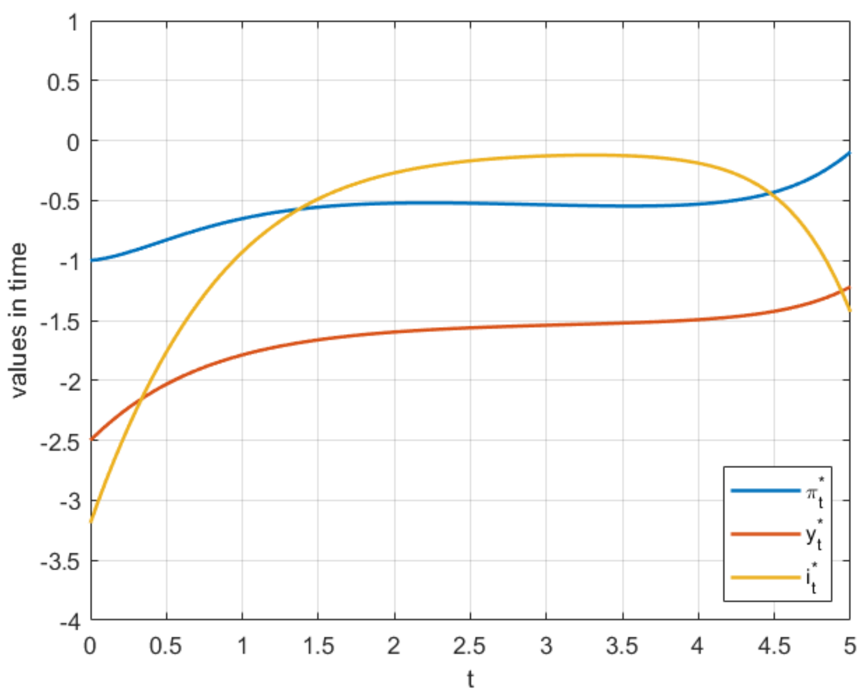
Parameters: $\pi_0 = 2,\ y_0 = 2,\ T = 5,\ \alpha = 0.5,\ \beta = 0.5,\ \gamma = 0.1,\ \eta = 0.05,\ \omega = 0.8,\ \phi = 0.5.$

A different time period from the 5 years used as benchmark, does not qualitatively affect the evolution of the economic variables. A longer time span T simply allows CB to have a more gradual approach in the change of the monetary instrument i to achieve its goals.

4.2 Recession and deflation

Figure 4 simulates an opposite scenario, a negative shock carries the economy into a severe recession (-2.5 %) and to an inflation slowdown (-1 %).
As a consequence, an injection of liquidity through a reduction of the interest rate (at $T = 0,\ i = -3.19$ %) brings back the values closer to their targets. In this case, however, the variables follow a non-linear path.

Figure 4: Recession and deflation



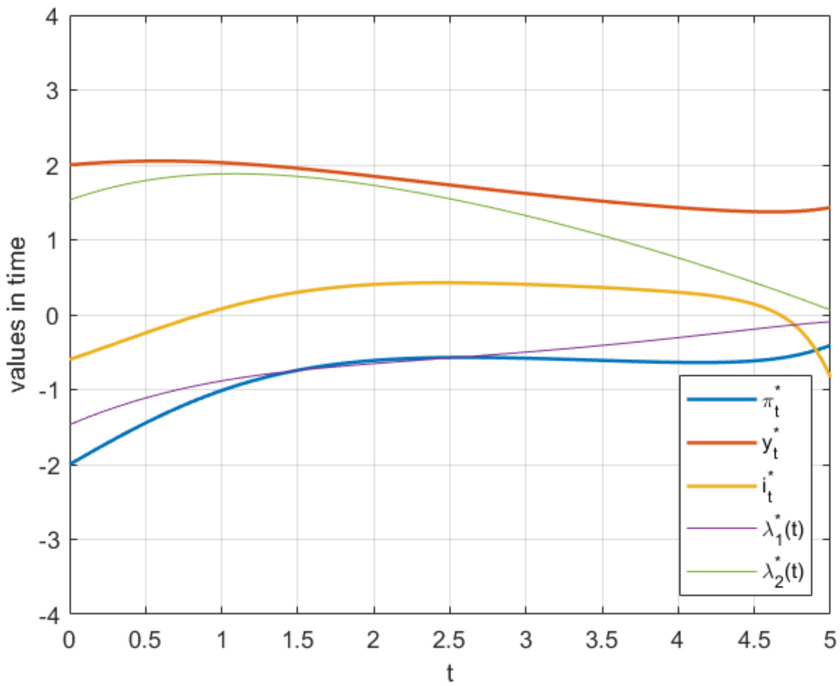
Parameters: $\pi_0 = -1,\ y_0 = -2.5,\ T = 5,\ \alpha = 0.5,\ \beta = 0.5,\ \gamma = 0.1,\ \eta = 0.1,\ \omega = 0.3,\ \phi = 0.5$.

In particular, an ultra-expansive monetary policy, while leading to a relevant increase in output, affects the inflation dynamics both directly and indirectly.
On the one side, we have a direct effect resulting from an expansion of liquidity that boosts prices over time.
On the other, there is an indirect effect arising from the growth in time of GDP and aggregate demand, which, approaching zero value, bears to a rise in money demand for transactional scope and can ultimately strengthen inflation.
As can be noticed, thanks to these two positive effects, the initial disinflation moderately improves and ends up only 0.1 point under the normalized target of zero.
However, as already pointed out the CB effort required to recover from negative inflation deviations is relatively greater than in contexts characterized by positive inflation gaps (such as Figure 3). Coherently, in Figure 4, the improvement is relatively slower and lower in magnitude.
Finally, given the strong original shock on GDP (-2.5%), the economy does not fully recover. Despite the strong upswing brings by the extraordinary expansionary measure, it ends more than a percentage point under the potential growth value (-1.21%).
This confirms and supports the evidence that, in the most severe scenarios, the monetary policy alone is not sufficient to restore the long-run equilibrium trends of the economy and should be accompanied and supported by other types of interventions, such as an adequate counter-cyclical fiscal policy.

4.3 Trade-off scenarios

In Figure 5 we highlight a trade-off scenario characterized by inflation two points under the target and a positive output gap at $t = 0$. [9]

Figure 5: A liquidity trap scenario



Parameters: $\pi_0 = -2,\ y_0 = 2,\ T = 5,\ \alpha = 0.5,\ \beta = 0.5,\ \gamma = 0.1,\ \eta = 0.05,\ \omega = 0.8,\ \phi = 0.5.$

This simulation might be particularly relevant to understand the difficulties faced by the monetary authority to achieve two conflicting objectives with a unique instrument at its disposal.
Furthermore, right after the global financial crisis and the EU debt sovereign crisis, Europe and also many other advanced economies experienced a situation that can be well stylized by Figure 5. In fact, from 2015 onward, inflation remained well under the target for a long period in many OECD

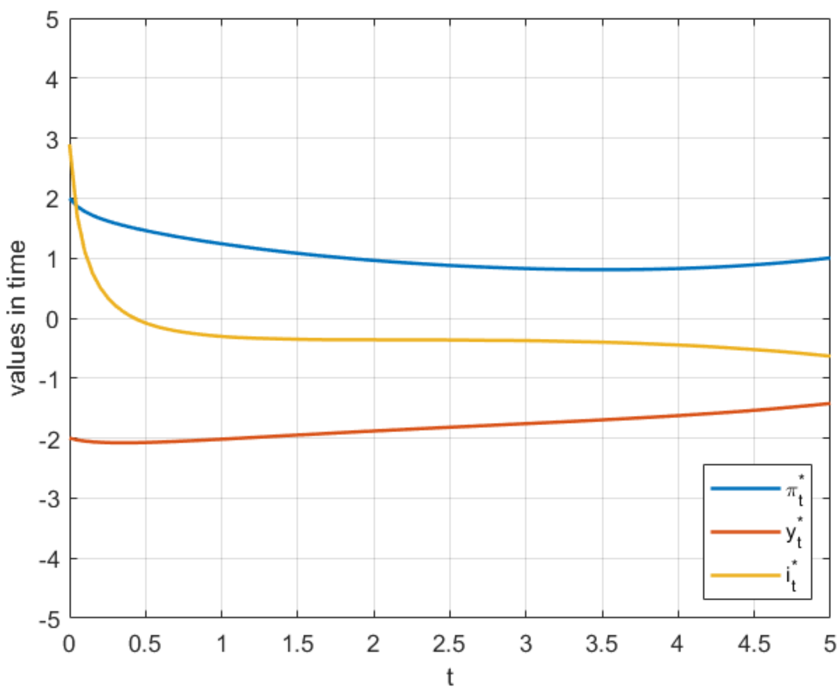
[9] In this figure we also highlight the time path for the auxiliary or co-state variables to show that the transversality conditions of problem (4) are met, and their final value is 0.

countries (including the United States and Japan), while there was an economic upturn after several years of recessions.
Many economists, such as Paul Krugman, started to debate about a possible liquidity trap.[10]
In this context, the exit from a period of deep recession and of prolonged low inflation values (2 percentage points under the target, which means zero inflation, or even deflation) has caused the postponement of several relevant economic decisions that are vital for the well-being of an economy.
Figure 5 is rather explicative on how it can be difficult to find a compromise between the need to slow down the positive output gap and the need of revitalizing inflation.
It shows the powerlessness of the CB in achieving both objectives. Initially, it sets an expansive monetary measure $i < 0$ that improves inflation but slightly worsens the output gap. Subsequently (after the first year to almost the fifth), the CB changes its monetary policy stance through a prolonged tightening measure $i > 0$ that helps to reduce only in part the positive output gap but prevents inflation from further improving. Inflation does not worsen and remains steady at -0.7 % only thanks to the enduring positive output gap (from the non-linear Phillips curve in equation (2)).
In the end, none of the two objectives is entirely met.
The GDP remains at 1.43 % over the long-run optimal, and the prominent disinflation slightly improves, but not sufficiently to reach the target (-0.42%). Consequently, Figure 5 incontrovertibly reveals the dilemma faced by the CB in these scenarios.
Figure 6 shows another relevant trade-off, characterized by an inflation over the target (+2%) and a growth below potential (-2%), that is a situation of stagflation.
We can trace back to a similar situation during the 1970s energy crisis when for the first time it was coined the term stagflation to depict this unusual economic condition. A swift increase in the price of oil and its derivative, in absence of valid energy alternatives, brought a negative spiral of rising inflation and economic stagnation.
Before the supply shocks of the '70s, inflation and recession were usually regarded as mutually exclusive, the relationship between the two being described by the classical Phillips curve (Phillips, 1958).

[10] Theorized by Keynes in his 1936 General Theory, a liquidity trap is a situation where interest rates are close to zero and changes in the money supply fail to translate into movements of the price level. Among the causes, he argued, there is a climate of general uncertainty that pushes people to hoard cash in the fear of an adverse event such as deflation, insufficient aggregate demand, or war.

Nowadays, stagflation or recession-inflation is recognized as a situation in which the inflation rate is high, the economic growth rate slows, and unemployment remains steadily high.
It presents a dilemma for economic policy, since actions intended to lower inflation may exacerbate unemployment.

Figure 6: A stagflation scenario

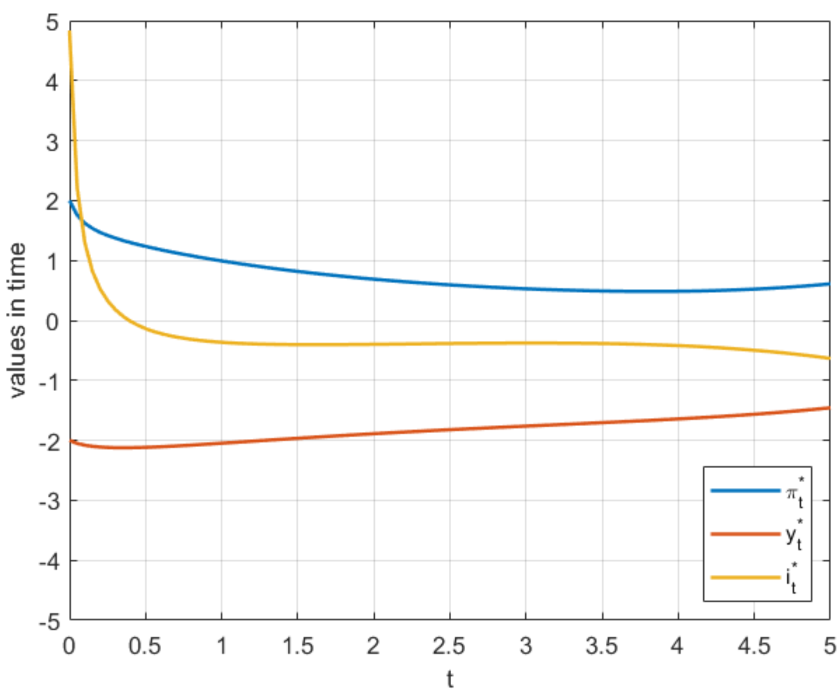


Parameters: $\pi_0 = 2,\ y_0 = -2,\ T = 5,\ \alpha = 0.5,\ \beta = 0.5,\ \gamma = 0.1,\ \eta = 0.1,\ \omega = 1.2,\ \phi = 0.6.$

From Figure 6 we can better appreciate the difficulties and the consequent policy indecision that may arise for the CB.
An initial tightening measure ($i_0 = +2.9$ %) improves inflation partly reducing the upward pressure on prices, but at the cost of worsening the output gap to a value slightly lower than the initial shock. A subsequent mild expansionary policy brings a little stimulus to the output, but prevents inflation to improve further. Also in this scenario, after the 5-year period considered, both the macroeconomic variables end up far from the objective ($\pi_T = +1$ % and $y_T = -1.4$ %).

This again reveals the trade-off pointed out by the political economy theory about policy instruments. When we specifically refer to monetary policy, a situation of dilemma could arise, for the central bank, every time the gap of the two state variables (π, y) is not concordant in sign (i.e. asymmetric shock). In these contexts, a given monetary policy measure (expansive or restrictive) can improve (i.e. minimize) only one of the two initial deviations/shocks on the state variables, and this is unfortunately obtained at the expense of the other variable.
Figures 5 and 6 clearly demonstrate this.
In the next two graphs (7 and 8), starting from the same initial scenario ($\pi_0 = +2, \%$ $y_0 = -2$ %), we try to understand if changing the relative CB priority on goals might help to reach better at least one of them.
In Figure 7, the monetary authority decides to ascribe more relevance to the inflation target ($\alpha = 0.9$, $\beta = 0.1$), *ceteris paribus*.

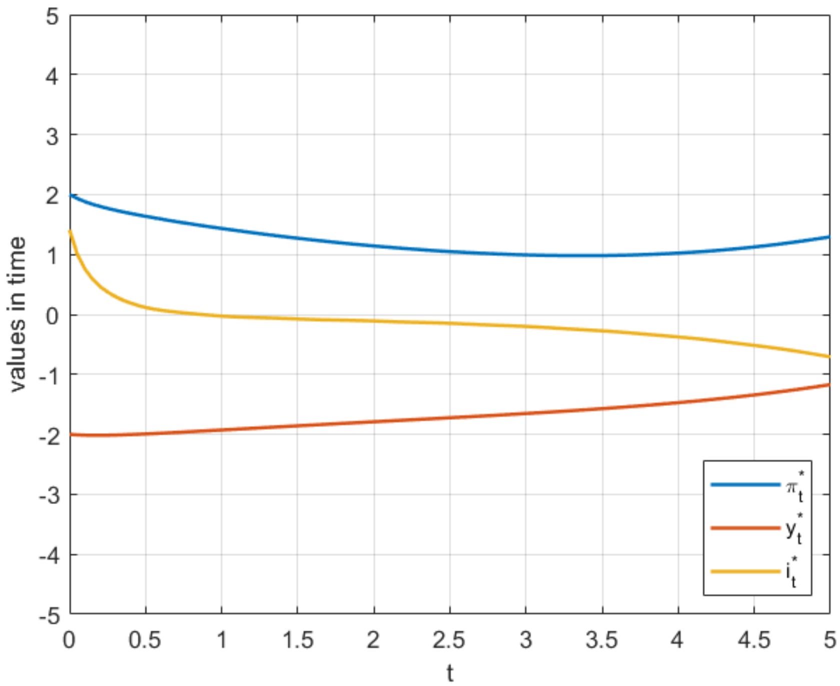
Figure 7: Inflation target priority



Parameters: same parameters of Figure 6, except for $\alpha = 0.9$ and $\beta = 0.1$.

Coherently, the stronger tightening policy ($i = +4.83$ %) leads to an improvement in the price rate of change, which reduces following a non-linear path to end at just $+0.61$ % points above the target compared to $+1$ % of Figure 6.
This positive result is offset by a relatively worst output path with respect to the previous Figure 6 (from -1.40 % to -1.46 % at time T) accordingly to the minor importance ascribed to this objective.
In Figure 8, the CB shifts the priority on the output target ($\alpha = 0.2$, $\beta = 0.8$), *ceteris paribus*.

Figure 8: Output target priority



Parameters: same parameters of Figure 6, except for $\alpha = 0.2$ and $\beta = 0.8$.

As can be expected, the higher priority given to stabilize GDP results in a less restrictive measure ($i = +1.40$ %) that helps y to get closer to the objective after the 5 years considered (-1.16 %), but at the expenses of the other macroeconomic variable π that deteriorates with respect to Figure 6 towards a serious and persistent inflation issue ($+1.3$ %).

5 Conclusion

The present chapter has analyzed the effects of monetary policy, as captured by changes in the nominal interest rate, on the dynamics of GDP and inflation utilizing an optimal control model.

With respect to the previous literature, we improved in at least two directions. First, in the literature, monetary policy macroeconomic models are often presented in a discontinuous time fashion (Galí 2008; Tramontana et al. 2010), while we propose a continuous-time version of a monetary policy model, investigating the impact of the CB instrument (i.e. interest rate) according to a specific loss function.

Secondly, we propose an augmented non-linear Phillips curve to take into account the direct impact of monetary policy on price dynamics.

The results of the analysis are worth stressing.

First of all, the model allows considering in a unified manner different phenomena such as inflation and deflation, as well as situations of sustained economic growth and periods of recession. In the second place, the simulations show that the model might well represent different economic scenarios and trade-off situations, underlining the difficulties of the CB in reaching both the output and inflation targets employing only one instrument. Furthermore, the dynamics of the variables depend on the value of the parameters and the relative relevance assigned by the CB to one target relative to the other. Coherently, a different priority in the monetary agenda can change the outcomes towards one direction or another reflecting the potentially diverse preference of each policymaker.

Future research could focus on at least two directions.

Firstly, the model could specifically consider fiscal policy in order to explicit the interactions between fiscal and monetary policies, and the relationship between interest rates, public deficit, and the two macroeconomic variables considered. Secondly, a line of future research may study the impact of evolving forecasting rules on the model dynamics.

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Chapter 3: A non-linear model of public debt with bonds and money finance

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Abstract

In this chapter, we study the dynamic relationship between the public debt ratio and the inflation rate. Using a non-linear macroeconomic model of difference equations, we analyze the role of monetary and fiscal policy in influencing the stability of the debt ratio and inflation. We get three main results. First, we find that, in a low inflation scenario, money finance can be helpful in stabilizing the debt ratio. Second, we show that, in a dynamic setting, standard Taylor rules may not be sufficient to control inflation. The Central Bank's credibility in driving inflation expectations is in fact crucial to control price developments and achieve macroeconomic stability. Finally, an active budget adjustment rule has a stabilizing effect on the debt ratio, even if it may not be enough to avoid explosive patterns. Notably, the stability of the steady state depends on the fine-tuning of the policy mix. One of the novelties of our analysis is the presence of a threshold level for the debt ratio and inflation, beyond which the debt ratio becomes unsustainable following an explosive path. The distance between this threshold and the steady state can be considered a proxy of the robustness of the economy to exogenous shocks.

Keywords: Public Debt, Monetary Policy, Economic Dynamics.
JEL classification: E52, H62, H63.

Statement of joint authorship and acknowledgments

I confirm that this Chapter - "A non-linear model of public debt with bonds and money finance" - was jointly co-authored with Alessandro Bellocchi, Gian Italo Bischi, and Giuseppe Travaglini. I contributed to 60% of this work.
I confirm that this Chapter is an original work in its entirety and that has not been published in any Journal.
This chapter was presented at the 11th Workshop on Dynamic Models in Economics and Finance (MDEF), Urbino (8-10 Sept. 2022), and at the 63rd Annual Conference (RSA) of the Italian Economic Association - SIE, Turin (20-22 Oct. 2022).
I would like to thank the participants at these workshops for their helpful comments. I also thank Germana Giombini, Laura Gardini, Ilario Favaretto (University of Urbino), Iryna Sushko (National Academy of Sciences of Ukraine) and Anastasiia Panchuk (Institute of Mathematics NAS of Ukraine) for their useful teachings and suggestions.

1 Introduction

The Covid-19 crisis shaped the conduct of monetary and fiscal policies in an unprecedented way. Governments launched massive debt-financed spending programs to counteract the negative consequences of the pandemic shock (Baldwin and Weder Di Mauro 2020). Further, major Central Banks (CBs) implemented a coordinated reduction in policy interest rates, in an attempt to provide a global monetary easing (Unsal and Garbers 2021; Cantú et al. 2021).

As it is well known, in contexts of high public debt burden, the space for further increases in government spending financed by fiscal deficit is however limited (European Commission, 2022). In addition, in the Eurozone, there is widespread fear of a return to stricter fiscal rules in 2024, despite the activation of the General Escape Clause (GEC) of the Stability and Growth Pact (SGP) (Amato and Saraceno 2022).

Besides, the falling of real interest rates towards the zero-lower-bound, in many OECD countries until 2022, made interest-rate-based policies less effective. Therefore, major CBs continued to draw upon alternative policies to spur economic activity during and after the Covid-19 shock, expanding their toolkit of *unconventional* monetary policies, particularly in the form of large-scale asset purchase of government bonds, i.e. quantitative easing (Akovali and Kamil 2021). For instance, while the FED bought an unprecedented amount of public and private debt to flatten the yield curve, the ECB collected the equivalent of about 2.590 billion euros in bonds, mainly as collateral in refinancing facilities (ECB, 2022).[1] Finally, the UK went a step further, with the Treasury and Bank of England jointly announcing the temporary reactivation of a scheme that allows the CB to finance public spending directly (O. Blanchard and Pisani-Ferry 2020). This led some economists to argue that the age of CBs' "independence" from fiscal policy and of monetary policy isolationism was over (see J. Taylor 2013).

These developments have raised concerns about the reappearance of large-scale 'monetization' programs which might result in major inflation episodes and/or threats of *fiscal dominance* of monetary policy (Menuet, Minea, and Villieu 2016). Yet, other commentators would have liked the CBs to do even more and embark on some form of 'helicopter money' (e.g. Galí 2020). Therefore, the issue of monetizing government debts returns to the forefront of both academic and political debate, fuelled by the concrete question of how monetary authorities can, at least indirectly, reduce the burden and cost

[1]Source: https://www.ecb.europa.eu/mopo/implement/app/html/index.en.html.

of government bonds.
Even if debt ratios are expected to stabilize in the future years (IMF 2021) as the real growth rate of economies is currently higher than the real interest rate paid on new debt issues, there are concerns about the ability of governments to continue servicing their debt, particularly in a context of low economic growth and uncertainty following Russia's invasion of Ukraine (O. Blanchard 2022).
As stressed by monetary authorities, Quantitative Easing (QE) programmes do not qualify as debt monetization operations.[2] Indeed, while QE has led to a growth of the monetary base in recent decades, it had limited effect on the money supply (Papadamou, Sidiropoulos, and Vidra 2021). Above all, the nature of QE is only *temporary*, as it is conceived as a non-conventional tool used by the CB to achieve its medium-term inflation target. However, although there appears to be little current interest among CBs for abandoning their policy-making objectives in favor of *debt monetization*, this could change down the road as pressures mount and the appeal of monetization grows (Shahid 2020).
In this perspective, Paris and Wyplosz (2014) argued that the only (politically acceptable) way for fiscally sunk Eurozone countries to escape default might be to sell the monetized debt to the ECB. Similarly, according to De Grauwe (2013) "Ideally, the Eurozone would combine a symmetrical budget policy with debt monetization by the ECB", in such a manner that low-deficit countries like Germany would run a more expansionary fiscal policy and share the burden of the adjustment in the periphery of the Eurozone. Furthermore, the main argument against debt monetization, namely the additional inflation it may create (Sargent and Wallace 1981), does not seem to be very relevant in depressed economies and in the presence of liquidity traps that disconnect inflation from the money stock. As such, a moderate debt monetization could affect inflation only marginally and would allow avoiding deflation and its possible harmful effects on growth (Aron and Muellbauer 2008, O. J. Blanchard 2014).
What relationship does, therefore, exist between debt sustainability and monetization? And how to describe these 'dangerous relations'? In this chapter, we attempt to address these questions by considering the crucial issues that have emerged from this recent debate. Specifically, we study the impact of *money finance* on the debt ratio and the inflation rate, focusing

[2] QE, as practiced by the major CBs, is not a form of monetary finance for two reasons: (1) the fact that these monetary stimulus policies are carried out indirectly (on the secondary market); (2) the fact that these operations are reversible (the CB can always sell the bonds back to the private sector).

on the endogenous relationships between these variables and their potential non-linearities.

To this aim, we develop a macroeconomic model of public debt sustainability formalized by means of a dynamic system of two first-order difference equations, one for the public debt ratio and the other for the inflation rate. Importantly, in order not to stray too far from the standard linear model, we look at the first-order difference equation of the debt ratio as benchmark. In our model, the government can, on the one hand, generate public deficits financed by issuing new debt. However, on the other hand, the CB can set a target for the interest rate and use monetization to finance the public debt, if the (relative) magnitude of the latter undermines the financial stability of the economy. To simplify our analysis, we consider the existence of only one nominal interest rate on government bonds, i.e. the rate on the composite bond. The nominal interest rate is determined by a standard Taylor rule from the CB plus a financial market component, the *risk premium* on government bonds. Lastly, the dynamics of inflation affect the debt ratio through the real interest rate, that is the cost in real terms of government debt. The evolution of inflation is finally captured by reference to a variant of the classical *Phillips curve*, in which agents' inflation expectations are implemented by considering the presence of both 'fundamentalists' and 'trend-follower' economic agents in the markets. This assumption makes it possible to describe complex inflation dynamics that fluctuate around equilibrium values without ever reaching a steady state.

We get three main results. First, in a low inflation scenario, debt monetization can be helpful in stabilizing debt evolution and the resulting effect on inflation rise is generally limited. Then, we show that in a non-linear dynamic setting standard Taylor rules may not be enough to control inflation. In fact, the CB's credibility in affecting inflation expectations is crucial to control price dynamics and achieving macroeconomic stability. Finally, an active budget adjustment rule has a stabilizing effect on the debt ratio, even if, in some critical circumstances, it may not be enough to avoid explosive patterns. Notably, the stability of the steady state(s) depends, to a large extent, on the fine-tuning of the policy mix. Lastly, one of the novelties of our analysis, compared to the benchmark linear model of the debt ratio, is the presence of some 'threshold values' beyond which the debt ratio becomes unsustainable, following an explosive path (default). The distance between the threshold boundary and the steady state can be seen as a proxy of the robustness of the economy to exogenous shocks.

The chapter is organized as follows. Section 2 discusses the relevant literature on the topic. Section 3 describes the dynamics of both the debt ratio and the inflation rate; the main properties of the nonlinear dynamic model are presented in (Section 4). In Section 4.1 we provide the analysis of the system with a description of the fixed points and their stability. Section 4.2 studies a few benchmark cases on the effect of different policy rules. Section 5 uses simulations to provide some relevant policy insights. Section 6 concludes the chapter.

2 Literature

For a long time, the issue of debt sustainability was addressed in terms of the effects of public debt on the economy. According to Hume (1777), public debt could lead to harmful tax increases in the short run and possibly to default in the long run. Adam Smith also considered that debt financing would eventually lead to default. The common view was that debt financing should only be used in exceptional cases. At the beginning of the 1920s, when writing about the public debt problem faced by France, Keynes (1923) mentioned the need for the French government to conduct a sustainable fiscal policy in order to respect its budget constraint. Keynes stated that the absence of sustainability would be evident when "the State's contractual liabilities [...] have reached an *excessive proportion of the national income*". In the literature, there is a large debate and a lack of consensus among economists about the definition of public finance *sustainability*. In fact, many contributions in the field introduce their own - similar but not identical - criteria (Balassone and Franco 2000; Wyplosz 2011; Ghosh et al. 2013). According to O. Blanchard (1990), sustainability is about whether, based on current fiscal policy, a government heads towards excessive debt accumulation. To give effect to this general statement, Blanchard defines a sustainable fiscal policy as a strategy that ensures the convergence of the debt ratio towards its initial level. A similar definition is provided by Buiter (1985), who defines a fiscal policy as sustainable if it maintains the ratio of government net worth to GDP at its current level.
The requirement of convergence of the debt ratio towards its initial level is only the special case of a more general definition, according to which fiscal policy is sustainable if the present value of future primary surpluses is equal to the current level of debt (Chibi, Chekouri, and Benbouziane 2019). These ambiguities led some authors to distinguish between *solvency* and *sustainability* (Artis and Marcellino 2000; IMF 2002). A government is said to be solvent if it is capable, over an infinite time horizon, of paying its debt via future primary surpluses. In other words, the government is solvent if

its inter-temporal budget constraint (IBC) is fulfilled. On the other hand, sustainability is a more imprecise concept that refers to the possibility that the government, with current policies, will reach a pre-determined debt/GDP ratio in a finite time horizon. As it turns out, the latter definition implies the former.

The easiest way to assess a government's fiscal sustainability position is to start with its IBC. The correct implementation of the one-period IBC requires the use of the net market value of government debt. Net debt is defined as gross debt minus financial assets. Dividing each term by nominal GDP the budget constraint can be rewritten as $\Delta b_t = d_t + (r - g)b_{t-1}$. d_t is the government's primary deficit, b_t is the government debt at the end of period t, r is the real interest rate on government debt and g is the growth rate of the economy. This equation is an identity that holds ex-post in time t and says that the interest-inclusive government deficit (right-hand side) is financed by new bond issues (left-hand side). If $(r - g) < 0$ for all t, the result is a stable difference equation that can be solved backward. This implies that the debt–GDP ratio b_t remains finite for any sequence of finite primary deficits. In contrast, if $(r - g) > 0$ for all t, the debt–GDP ratio will eventually explode for $d_t > 0$.

A standard metric for judging debt sustainability has become the gap between the real interest rate on government debt r and the growth rate of real GDP g (Checherita-Westphal and Semeano 2020). For the US and most advanced economies, the cost of servicing public debt $(r-g)$ is currently negative. In this case, the government can run a primary deficit of any size in perpetuity or, equivalently, a government running a primary balance would see its debt-to-GDP ratio shrink to zero (O. Blanchard 2022).

However, interest rates are not constant and, as emphasized in Ball, Elmendorf, and Mankiw (1998), a Ponzi strategy of continuous rolling over the public debt is risky. A sudden rise in interest rates relative to growth with a large stock of debt could quickly result in explosive debt dynamics (Mauro and Zhou 2020; Weicheng, A. Presbitero, and Wiriadinata 2020).

In this regard, more recent works have emphasized the importance of non-linearity in the debt-growth relationship (Eberhardt and A. F. Presbitero 2015). These non-linearities may arise if we expect fiscal authorities to react differently to whether the deficit has reached a certain threshold deemed to be unacceptable or unsustainable. Bertola and Drazen (1993) elaborate a framework that allows for trigger points in the process of fiscal adjustment, such that significant adjustments in budget deficits may take place only when the ratio of deficit output reaches a certain threshold. This may reflect the existence of political constraints that block deficit cuts, which are relaxed only when the budget deficit reaches a sufficiently high level deemed to be unsustainable (Alesina and Drazen 1993). Nevertheless, the presence of a tipping point does not mean that it has to be common across countries. For instance, Ghosh et al. (2013) defines 'debt limit' as the level of debt beyond which fiscal solvency fails and shows that this debt limit is a function of countries' structural characteristics and GDP growth. This argument resembles the idea of country-specific debt 'vulnerability regions', which would be consistent with country-specific non-linearities (Reinhart and Rogoff 2009; Bischi, Giombini, and Travaglini 2022).

Another debate, triggered by the economic challenges posed by the global financial crisis and, more recently, the Covid-19 pandemic, concerns whether CBs should expand their unconventional monetary policy toolbox to include *money finance* (Unsal and Garbers 2021). Money finance is often associated with Milton Friedman's metaphor of a helicopter dropping money from the sky (Friedman 1948). In fact, it is argued that a permanent increase in the monetary base could stimulate aggregate demand even in a severe liquidity trap, that is when interest rates are at zero and prices are stagnant or declining (Galí 2020; De Grauwe 2020; De Grauwe and Diessner 2020; Gürkaynak and Lucas 2020; Kapoor and Buiter 2020; Martin, Monnet, and Ragot 2021). Proponents of money finance argue that it has a stronger effect on aggregate demand than a debt-financed fiscal stimulus (Agur et al. 2022). It could also prevent self-fulfilling runs on government debt should investors suddenly lose confidence in debt sustainability (Corsetti and Dedola 2016; Bacchetta, Perazzi, and Van Wincoop 2018; Camous and Cooper 2019). Yet, calls for CBs to engage in money finance are often seen with skepticism, if not outright rejection. Skeptics argue that money finance involves swapping government debt with CB liabilities and, thus, it does not carry tangible benefits in terms of economic stimulus and debt sustainability (Cecchetti and Schoenholtz 2016; Borio and Zabai 2018; O. Blanchard and Pisani-Ferry 2020). Money finance may also fail to fend off self-fulfilling runs in the sovereign market if it instills concerns about *systematic* actions of debt monetization. Indeed, a permanent debt monetization fuels fears about *fiscal dominance*, loss of CB independence, and run-away inflation (Adrian et al. 2021).

This chapter contributes to the debate on debt sustainability in two ways. First, we introduce *debt monetization* into the standard model of public debt, thus relaxing an important assumption of this literature, namely that public spending is financed by the government through net debt issuance and the CB may under no circumstances intervene permanently in the bond market. Second, we study the effectiveness of fiscal and monetary policy to control the debt ratio and inflation rate in the presence of endogeneity and potential non-linearities. As a first anticipation, in this scenario, the issue of debt sustainability becomes much more slippery and the role of the CB much more complex.

3 Public debt and inflation

3.1 The (augmented) government budget constraint

We start our analysis with the government's intertemporal budget constraint. It states that the total public deficit D_t at any time t (i.e. any year) is equal to:[3]

$$D_t = (G_t - T_t) + rB_{t-1} \tag{1}$$

where G_t denotes government spending on goods and services during the year t, T_t taxes minus transfers in the same year, r is the real interest rate, and B_{t-1} is the amount of government debt at the end of year $t-1$. Thus, rB_{t-1} represents the real interest payments on outstanding government debt. In other words, the total budget deficit in a given year equals spending minus taxes net of transfers (i.e. the primary deficit) plus interest payments on outstanding debt.

When a deficit is budgeted, the government has only one option to finance it, namely issuing new public debt on the bond market $\Delta B_t = B_t - B_{t-1} > 0$. Once the bonds have been issued, the CB may eventually decide to buy them (typically on the secondary market) in exchange for real money $\Delta M_t = M_t - M_{t-1} > 0$, where M_t is the real stock of money at time t. This process is called *money finance* or *debt monetization*. It differs from outright monetization of the deficit since the CB is not mandated by the government to buy or sell these securities and the amount eventually exchanged derives only from monetary policy strategy and not from fiscal policy considerations (Bénassy-Quéré et al. 2010). Putting resources and means of financing together, we rewrite equation (1) as:

$$(G_t - T_t) + rB_{t-1} = \Delta B_t + \Delta M_t \tag{2}$$

The budget constraint in (2) is an extension of those in R. Barro (1990), and Bischi, Giombini, and Travaglini (2022). While R. Barro (1990) considers balanced-budget-rules, Bischi, Giombini, and Travaglini (2022) introduces

[3]In Equation (1) all variables are in *real terms*.

public debt, but without money. By means of simple algebraic operations, equation (2) can be restated as:

$$B_t - B_{t-1} = (G_t - T_t) + rB_{t-1} - \Delta M_t \tag{3}$$

The government budget constraint in (3) links the change in government debt in a given year to the level of debt in the previous year (which affects interest payments), current government spending, and current taxes (i.e. the primary deficit). However, in this 'augmented' version, the final effects on the change in the stock of government debt also depend on CB's monetization actions (ΔM_t). If the government runs a deficit ($G_t + rB_{t-1} > T_t$), which is not covered ex-post by CB's monetization (i.e., $\Delta M_t = 0$), government debt increases as the government borrows on the market to fund the part of spending (including the interest rate on debt) in excess of revenues. If, on the other hand, the government runs a surplus ($G_t + rB_{t-1} < T_t$), government debt decreases as the government uses the budget surplus to repay part of its outstanding debt.

As is well known, in an economy where output grows over time, it makes more sense to focus on the debt-to-GDP ratio. Therefore, we divide both sides of (3) by real output (Y_t) and rewrite $(B_{t-1}/Y_t) = (B_{t-1}/Y_{t-1})(Y_{t-1}/Y_t)$. To simplify the final result, we assume that output growth is constant, denoted by g, so that (Y_{t-1}/Y_t) can be written as $1/(1+g)$. Finally we use the approximation $(1+r)/(1+g) \approx 1+r-g$. This requires some steps, but the final relationship has a simple interpretation.

$$b_t - b_{t-1} = (r - g)\, b_{t-1} + (d_t - \Delta m_t) \tag{4}$$

where $b_t = (B_t/Y_t)$, $b_{t-1} = (B_{t-1}/Y_{t-1})$, $d_t = (G_t - T_t)/Y_t$, $\Delta m_t = \Delta M_t/Y_t$.

The change in the government debt ratio-to-GDP over time (the left side of (4)) is equal to the sum of two terms: (i) the difference between the real interest rate and the growth rate times the initial debt ratio $(r-g)b_{t-1}$, (ii) the ratio of the primary deficit to GDP, minus the ratio of money growth related to CB's monetization operations $(d_t - \Delta m_t)$.

Each period t, the CB independently from government deficit decisions, through its *debt monetization* program, chooses the share of government debt (i.e. government bonds) to be purchased on the secondary market. We call this share η, so that η by definition is strictly lower than one ($0 \leq \eta \leq 1$). The CB expands its portfolio with the acquisition of public debt, injecting real monetary base into the economy, $\Delta m_t > 0$. This means that the net real money created must be sufficient to cover the share of public debt that the CB decides to purchase (i.e. the extent of debt monetization programs): $\eta b_{t-1} = \Delta m_t$. By substituting this condition in (4) we obtain the following 'augmented' budget constraint:

$$b_t = (1 + r - g - \eta)\, b_{t-1} + d_t \tag{5}$$

It is worth noting that equation (5) can be seen as a first-order linear difference equation in b_t if r, g, η and d are considered exogenous parameters (O. Blanchard 2022).

Assuming no monetary financing by the CB (i.e. $\eta = 0$), the standard discussion of public debt dynamics has typically concerned the term $(r - g)$. If the latter is negative $(g > r)$, as is currently the case for many advanced OECD economies, the government can run a primary deficit $(d_t > 0)$ without compromising debt sustainability (IMF 2021). If, however, $g < r$, to stabilize the debt ratio, the government must inevitably run a primary surplus $(d_t < 0)$. Therefore, a sudden surge in $r - g$ is a source of serious concern as it can generate large economic costs (Born et al. 2020) and eventually lead to sovereign debt distress (Mauro and Zhou 2020).

However, when η is greater than zero, i.e. the CB employs debt monetization, the relevant factor for the dynamics of public debt becomes $r - g - \eta$. It is indeed its value that determines the long-run dynamics of the linear model in (5), whose equilibrium point $d/(g - r + \eta)$ is asymptotically stable whenever $r < (g + \eta)$ with either a lender $(d_t < 0)$ or borrower $(d_t > 0)$ government.[4] The solutions of (5) are, instead, unstable whenever $r > (g + \eta)$.

The literature on the sustainability of the debt ratio has provided numerous empirical studies to assess the effectiveness of fiscal policies in controlling the evolution of the debt ratio (Balassone and Franco 2000; Chalk and Hemming 2000; Collignon 2012; Beqiraj, Fedeli, and Forte 2018; Bischi, Giombini, and Travaglini 2022). Still, these quantitative analyses do not consider the role of monetary policy and/or the issue of non-linearities in the relationship between the debt ratio, the real interest rate, and the inflation rate. In fact, as shown by Weicheng, A. Presbitero, and Wiriadinata (2020) high public debts can lead to adverse future $(r - g)$ dynamics. In this scenario, monetary policy can be used to stabilize debt and control inflation expectations.

[4] The other condition for the stability of the equilibrium is $r > g + \eta - 2$, which is always satisfied on economic terms, given that both g and η are much smaller than 1 $(g + \eta \ll 1)$.

3.2 The central bank and inflation dynamics

Traditional prescriptions for monetary policy focus on the money stock (Romer 2012). For instance, Friedman (1960) famously argues that the CB should keep the money stock growing steadily at an annual rate of *k-percent* and renounce stabilizing the economy. However, despite many economists' impassioned advocacy of money-stock rules, CBs have rarely given the behavior of the money stock more than a minor role in policy. In addition, in many countries the relationship between measures of the money stock and aggregate demand has broken down in recent decades, further weakening the case for money-stock rules (Eggertsson 2010). Because of these difficulties, modern CBs almost universally conduct policy by adjusting the short-term interest rate in response to various disturbances (R. J. Barro 1989).

A key fact about conducting policy in terms of interest rates is that interest-rate policies, in contrast to money-supply policies, cannot be passive. C. Taylor (1993) and Bryan, Hooper, and Mann (1993) therefore argue that we should think about the conduct of monetary policy in terms of rules for the short-term nominal interest rate. That is, we should neither think of the CB as choosing a path for the nominal rate that is unresponsive to economic conditions, nor think of it as adjusting the nominal rate on an ad-hoc basis. Instead, we should think of the CB as following a policy of adjusting the nominal rate in a predictable way to economic developments. Therefore interest-rate rules may provide a reasonable approximation to actual CBs behavior and can be analyzed formally (Orphanides 2010).

Following a simple *Taylor rule* we assume that the CB adjusts its interest rate policy instrument in a systematic manner in response to inflation developments (see equation 6). Specifically, the nominal interest rate i_t responds to divergences of the actual inflation rate from a target inflation rate $(\pi_t - \bar{\pi})$. The idea is that when inflationary (or dis-inflationary) pressures develop, a monetary restriction can restore the CB's price stability objective. Therefore, the nominal interest rate must rise when inflation exceeds the CB's current target $(\bar{\pi})$ and reduce when inflation lies below it.[5]

The parameter $\alpha \geq 0$ measures the responsiveness (i.e., the elasticity) of changes in the nominal interest rate to inflation deviations from the target, usually defined at 2% (Krugman 2014). Note that α could be both smaller than 1, less than proportional response, or greater than 1, more than proportional response (Davies 2013). The original Taylor rule assumes that the

[5]Taylor argues that $\alpha = 1.5$ and $\bar{r} = 2\%$ provide a good description of US monetary policy in the period since the FED switched to a clear policy of adjusting interest rates to keep inflation low and the economy stable (J. B. Taylor 1979).

funds rate responds by a half-percentage-point to a one-percentage-point change in either inflation or the output gap (that is, the coefficient $\alpha \approx 1.5$). Likewise, the CB should decrease the real funds rate by the same amount for deviations below either target or potential. Empirical evidence suggests that CBs typically respond to inflation deviations (at least since 1983) a little less than Taylor assumed (Carlstrom and Fuerst 2007).

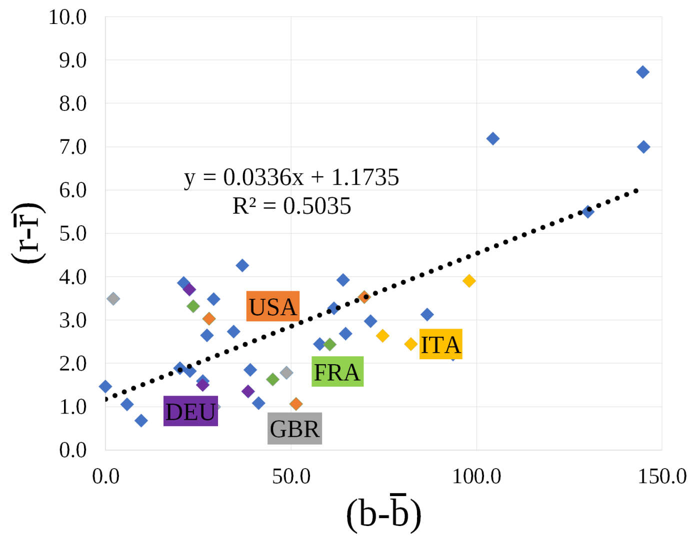
To simplify the analysis we focus exclusively on *pure inflation targeting* without considering the output gap in the objective function of the CB. We do this mainly for two reasons: first, the problems associated with measuring the output gap and hence implementing rules of the 'Taylor' type; second, because the primary objective of any modern CB is to regulate inflation (Bacchiocchi and Giombini 2021). Note, however, that adding the output gap to the CB's objective function does not change the main implications of the analysis.[6]

The nominal interest rate i_t relevant for the calculation of public debt is also determined by the market. Indeed, it depends on both the short-term interest rate set by the CB (from the Taylor rule) and a *risk premium* required by investors to hold public bonds in their portfolio, measured by β (with $\beta \geq 0$). The idea is that the spread between the actual indebtedness of an economy and the level of it considered as 'sustainable' by investors can be seen as a proxy of the risk premium (Von Hagen, Schuknecht, and Wolswijk 2011; Bernoth, Von Hagen, and Schuknecht 2012). This is consistent with the IMF rule of thumb that the interest rate is given by the riskless rate plus a risk premium, which increases by 3.5 basis points for every one percentage point increase in the debt ratio above 90% of GDP (IMF 2017; Alcidi and Gros 2019). This would imply that the interest rate of a country with a debt-to-GDP ratio of 150% would be 2.1 percentage points above the riskless rate (e.g. 10-year German government bonds). If the country had a lower debt ratio (130%), the interest rate would be only 1.4% above the riskless rate. The risk premium may eventually be negative for very virtuous countries, which have lower debt ratios with respect to the benchmark country. Equation 6 represents the market-determined nominal interest rate and Figure 1 the relationship between the real interest rate and the risk premium.

$$i_t = \bar{i} + \alpha\,(\pi_t - \bar{\pi}) + \beta\,(b_t - \bar{b}) \tag{6}$$

[6]When inflation and the output gap send opposite signals, the relevant case being excess inflation and a negative output gap (*stagflation*), the dynamic control of the system requires that the reaction to inflation is greater than the output gap. Therefore, *pure inflation targeting* can be considered as the case where the CB gives predominant weight to inflation in its objective function.

Figure 1: Relationship between real interest rate and risk premium



Note: Real interest rate (deflator GDP) and General government gross debt (% of GDP). Countries included: Austria, Germany, Spain, France, the UK, Greece, Ireland, Italy, Japan, Portugal, Sweden, US. Observations for different countries are identified by *rhombs* with different colors. The interest rate 'spread' is calculated as the excess over the annual average value for the panel of countries considered. The threshold value of public debt is taken at 90% (countries' average). Measures are average growth rates over five-years periods (from 1990 to 2022). Source: Authors' calculations on OECD data.

The dynamics of inflation π_t is defined by an *augmented Phillips Curve* (PC), which relates the next-period inflation to the actual inflation, the output gap, and a "cost-push" effect influenced by expected inflation (Clarida, Gali, and Gertler 2000; Roberts 2006; O. Blanchard and Galí 2007). Practical modern formulations of pricing behavior generally do not assume that price and wage-setters are rational in forming their expectations since this has strong implications that do not appear to be supported by the data. Alternatively, if one assumes that workers and firms do not form their expectations rationally, this would rest on the theory of irrationality (Romer 2012). A natural compromise is to assume that core inflation is a weighted average of current inflation (π_t) and expected inflation ($E_{t+1}[\pi_t]$), as in equation (7) where, for the sake of simplicity, these weights are not specified and thus are equal to 1.

When prices are sticky, there is a positive relationship between the rate of inflation and a proxy of real economic activity. In practice, the output gap, or a measure of real marginal cost is used as a proxy of real economic activity. Here, we rely on Roberts (2006) and use the real interest rate.[7] Therefore, a rise in the real interest rate can increase output temporarily (if $r_t < \bar{r}$), but cannot increase it permanently, since in the long-run $\bar{r}$ would prevail. On the other hand, a reduction of the latter can decrease output temporarily (if $r_t > \bar{r}$), but in the long-run again $\bar{r}$ prevails. The magnitude of the effect of the real interest gap $(r_t - \bar{r})$ on next period inflation π_{t+1} is given by the parameter $\gamma > 0$ in eq. 7. The idea is that inflation is a forward-looking phenomenon caused by staggered nominal price setting as developed by Taylor (1979,1980) and Calvo (1983) or quadratic price adjustment cost (J. J. Rotemberg 1982). With these assumptions, we obtain a hybrid Phillips curve (7):

$$\pi_{t+1} = \pi_t + E_{t+1}[\pi_t] - \gamma(r_t - \bar{r}) \tag{7}$$

More specifically, the expectations of the next period's inflation are formulated on the basis of the actual observable inflation, i.e. $E_{t+1}[\pi_t]$. We postulate that heterogeneous economic agents form their subjective beliefs (i.e., forecasts) by making some corrections to this value (i.e. π_t), taking into account whether inflation over the period is currently above or below the target value $\bar{\pi}$ (Hommes 2011; Hommes and Lustenhouwer 2019). Two types of economic agents are included in our model. The *trend-follower* which have a trend-following expectation strategy: they believe that when inflation is over the target or the reference value of the CB, i.e. $\pi_t > \bar{\pi}$, it will continue to increase in the next period, while if it is currently under the target $\pi_t < \bar{\pi}$, it will also reduce in $t + 1$. For this reason, these agents are defined *trend-follower*, and their share in the economy is equal to the parameter $0 \leq \mu \leq 1$. The *fundamentalists*, on the contrary, base their expectations strategy on the existence of a fundamental value for inflation, consistent with the objective pursued by the CB (i.e. $\bar{\pi}$), thus behaving oppositely to *trend-follower agents*. Indeed, in each period, they bet on inflation returning to its fundamental value. Therefore, when $\pi_t > \bar{\pi}$ fundamentalists think that inflation will drop in the next period, approaching the CB target, whereas if $\pi_t < \bar{\pi}$ they expect inflation to rise in $t + 1$, so as to reach the CB objective. In other words, fundamentalists fully trust the ability of the CB to bring back inflation to its target $\bar{\pi}$ by clearing out any possible shocks or deviations from the fundamental/target. The share of *fundamentalist agents* in the economy is complementary to the share of *trend-follower agents*, and

[7] A higher real interest rate tends to lower the output gap because it encourages households to save and discourages households' consumption and aggregate demand (Occhino 2019).

therefore equal to $1-\mu$. Consequently, the overall effect on inflation expectations in each period t is a weighted average (i.e., a convex combination) of these two effects: $E_{t+1}[\pi_t] = \mu\,(\pi_t - \bar{\pi}) + (1-\mu)\,(\bar{\pi} - \pi_t)$.
If inflation shocks are not persistent (i.e. transitory phenomenon), this year's inflation is not a good predictor of inflation next year. Therefore, under the rational agents' hypothesis, fundamentalists prevail (i.e., μ reduces) by driving inflation, in the next periods, to the reference level $\bar{\pi}$. On the contrary, if inflation shocks become more persistent, agents start to take into account this persistence when forming their expectations, and trend-following behavior would prevail (μ increases). Hence, a high level of inflation in one year becomes likely to be followed by high inflation values also in the next periods. In macroeconomic jargon, expectations that were previously anchored (i.e., roughly constant around the reference or CB target value $\bar{\pi}$) suddenly become de-anchored (Baumann et al. 2021). Different from (Hommes and Lustenhouwer 2019), we do not assume a heuristic switching model which would allow for endogenous credibility of CB targeting. Rather in Section 4.2.4, we exploit this setting to simulate how the equilibrium of the system changes when μ varies.
Finally, to complete our *inflation equation* we also consider a '*monetarist effect*'. As is well known, the Quantity Theory of Money (QTM) predicts a positive relationship between the money supply and the general price level of goods and services. Therefore, monetarists contend that "inflation is always and everywhere a monetary phenomenon" (Friedman 1989). Thus, at any given time, the actual rate of inflation is seen as a function of monetary expansion. In line with the success of Neo-Keynesian models of monetary policy, the importance of monetary aggregates has declined in CB modeling (Julio J. Rotemberg and Woodford 1997; Goodfriend and King 1997; Woodford 2003). However, since its establishment, the ECB has conducted a 'two-pillar' monetary policy with a 'leading role' for the growth rate of monetary aggregates and the output gap (ECB 1999; Assenmacher-Wesche and Gerlach 2008). Recently, several authors have presented empirical models that provide a formal interpretation of the two pillars by incorporating money growth in a reduced-form Phillips-curve model for inflation (Assenmacher-Wesche and Gerlach 2007). The monetary and the economic pillars of the ECB's framework are in these models viewed as reflecting different time perspectives in the determination of inflation. While money growth impacts inflation in the long run, real economic indicators such as the output gap and cost-push factors influence inflation mainly in the short run.[8] Considering

[8]The reason for such an eclectic policy is explained by ECB as: "[In the Euro Area] the

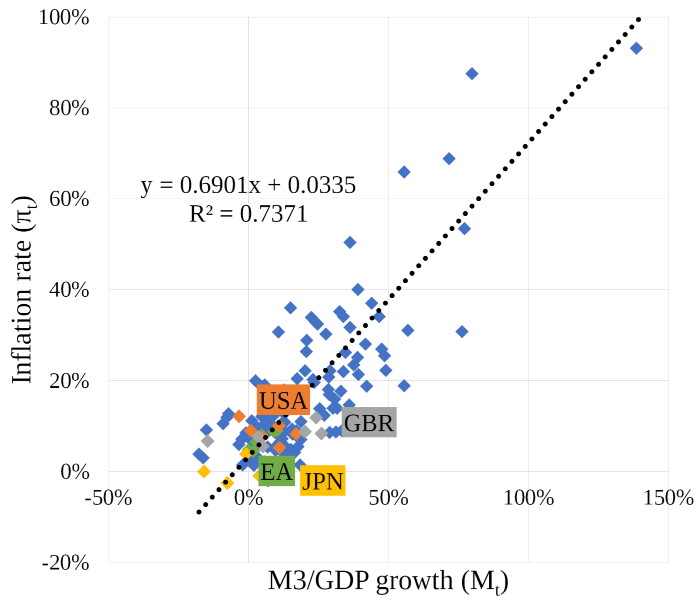
this additional long-term effect, the corresponding *inflation equation*, under the usual assumption that $\eta b_t = \Delta m$ (i.e., monetary financing of public debt), takes the following form (8):

$$\pi_{t+1} = \pi_t - \gamma(r_t - \bar{r}) + \mu\left(\pi_t - \bar{\pi}\right) + (1-\mu)\left(\bar{\pi} - \pi_t\right) + \delta\eta b_t \qquad (8)$$

Where the term ηb_t reflects the extent to which the next period inflation rate is affected by a new issue of real money used to buy government bonds at time t, and the parameter $\delta > 0$ measures the intensity of this relationship (Gerlach 2003, 2004).
As a cross-country long-run regularity, the link between money growth and inflation raises little discussion (McCandless and Weber 1995; Lucas Jr 1996). In a short horizon and low-inflation context, there is little relationship between these two variables (Bénassy-Quéré et al. 2010). However, it is important to add that the strength of the relationship between money growth and inflation mostly comes in the long run. Indeed, as shown in Figure 2, the annual inflation rate (measured by the GDP deflator) (vertical axis) roughly tracked the average excess growth in the broad real money supply (M3/GDP) (horizontal axis) during the period 1990-2022. Although, in the Euro Area, the high growth rate of M3 in the 2000s was accompanied by subdued headline inflation - hardly more than 2% per year (Assenmacher-Wesche and Gerlach 2007).

inflation process can be broadly decomposed into two components, one associated with *output gap* at high frequency, and the other connected to more *persistent trends*, which is closely associated with the medium-term trend growth of money" (ECB 2003).

Figure 2: Inflation rate versus broad money (M3) growth



Note: Price deflator of gross domestic product at market prices and Broad money (M3). Countries included: Austria, Germany, Spain, France, the UK, Greece, Ireland, Italy, Japan, Portugal, Sweden, US. Observations for different countries are identified as *rhombs* with different colors. Total growth rate over five-years periods (from 1990 to 2022). Source: Authors' calculations on OECD data.

3.3 The government budget adjustment rule

In many countries, fiscal policy decisions are increasingly conditioned by rules and institutions that contribute to limiting the scope for discretionary choices (Wyplosz 2012). The design of rules and the choice of a mandate for institutions determine a country's fiscal regime and contribute to the quality of its policy. The Euro Area has been at the forefront of this trend toward rules-based fiscal policy, but it is by no means the only region of the world where such a move was apparent.[9]

Fiscal rules are legal provisions that impose constraints on fiscal policy through numerical limits on budgetary aggregates. They can target the

[9]In the 1980s, few countries were equipped with fiscal rules: Germany, Indonesia, Japan, Malaysia, Singapore, and the United States. By 2015, more than 93 countries had such rules (Lledó et al. 2017).

deficit, the debt, or public expenditures. They can be couched in nominal terms (such as absolute limits for the fiscal deficit or the primary deficit), in real terms (such as benchmarks for the real growth rate of public spending), or in structural terms (such as thresholds for and minimum annual improvements of the cyclically adjusted balance). They can apply ex-ante or ex-post, to the general government as a whole or to sub-entities. Finally, they can, as in the EU, result from an international treaty and secondary supranational legislation, as well as be part of the national constitution, or simply national law (Schuknecht 2004).

Therefore, countries with a high public debt often target the primary deficit, which is more directly under the control of the government as interest payments on public debt depend on market-determined interest rates. The adjustment programs that IMF negotiates with countries in financial stress also include primary balance targets (Caselli and Wingender 2018).

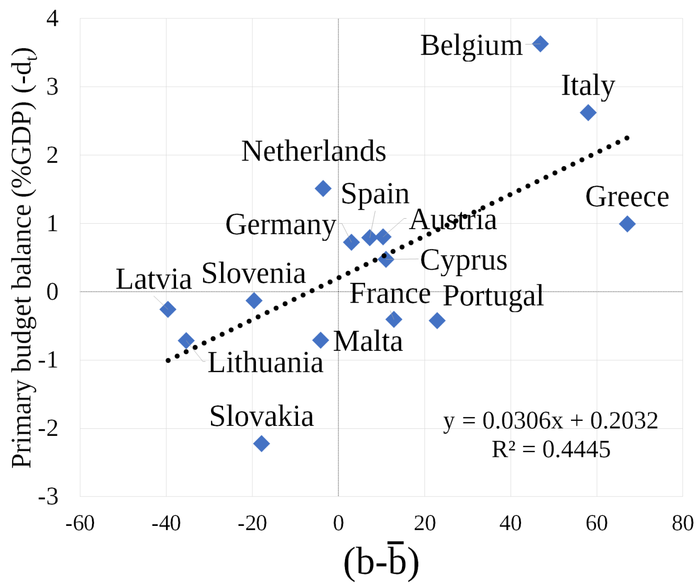
Following these general considerations, to close the model, let us define the dynamics of the government budget balance. To this end, we rely on an active budget adjustment rule in equation (9), that aims to target the *primary deficit* (d_t) to deviations of the public debt ratio from the value perceived as sustainable (i.e. $\bar{b}$):

$$d_t = \lambda - \epsilon \left(b_t - \bar{b}\right) \tag{9}$$

Where λ is the constant value of the government deficit (if $\lambda > 0$) or surplus (if $\lambda < 0$), which is independent of the current debt ratio value, while $\epsilon > 0$ measures the elasticity of adjustment of the primary deficit to debt ratio deviations from the sustainable value. Note how, for virtuous governments ($b_t < \bar{b}$), it is possible to increase the primary deficit d_t, whereas more indebted governments ($b_t > \bar{b}$) are forced to reduce the primary deficit d_t up to potential negative values (primary surplus).

In Figure 3, we show this adjustment process for Euro Area economies from 1990 to 2022: on the y-axis, there is the country's primary budget balance on GDP (expressed in terms of surplus), on the x-axis, there are deviations from the sustainable level of debt ratio (set at the average debt ratio over the period, i.e., 90%). The relationship is positive and the slope measures the intensity of the adjustment (i.e. parameter ϵ).

Figure 3: Primary budget balance and debt ratios in the Euro Area



Note: Net lending/ borrowing excluding interest (% of GDP) and General government gross debt (% of GDP). Total percentage change from 1990 to 2022. We have excluded the economic crisis periods (2009-2013) and (2020-2022) in which the budget balance was determined by the economic downturn rather than by fiscal rules. Source: Authors' calculations on AMECO data.

4 The model

The model is composed of the two dynamic equations (5) and (8) and the auxiliary equations (6) and (9) which express, respectively, the Taylor rule (i.e. the monetary policy rule of the CB) and the budget adjustment rule (i.e. the fiscal policy rule of the government).
The relationship between nominal and real interest rates is eventually given by the Fisher identity ($r_t \approx i_t - \pi_t$) (Fisher and Barber 1907).

$$i_t = \bar{i} + \alpha(\pi_t - \bar{\pi}) + \beta(b_t - \bar{b})$$
$$d_t = \lambda - \epsilon(b_t - \bar{b})$$
$$r_t = i_t - \pi_t \quad \text{and} \quad \bar{r} = \bar{i} - \bar{\pi}$$

$$\begin{cases} b_{t+1} = (1 + r_t - g - \eta)\, b_t + d_t \\ \pi_{t+1} = \pi_t - \gamma(r_t - \bar{r}) + \mu(\pi_t - \bar{\pi}) + (1-\mu)(\bar{\pi} - \pi_t) + \delta\eta b_t \end{cases}$$

After the substitution of the auxiliary equations (6) and (9) into the dynamic equations (5) and (8), and re-arranging the latter for the variables b_t and π_t, we get the complete map T in (10). The time evolution of both the *debt ratio* and the *inflation rate* is expressed by the iteration of the following two-dimensional discrete non-linear map T: $(b_t,\pi_t)\rightarrow(b_{t+1},\pi_{t+1})$.

$$\begin{cases} b_{t+1} = [1 + \bar{i} - g - \eta - \epsilon + \alpha(\pi_t - \bar{\pi}) + \beta(b_t - \bar{b}) - \pi_t]\, b_t + \lambda + \epsilon\bar{b} \\ \pi_{t+1} = (\delta\eta - \beta\gamma)\, b_t + [2\mu + \gamma(1-\alpha)]\,\pi_t + \gamma\,[\beta\bar{b} - \bar{\pi}(1-\alpha)] - \bar{\pi}\,(2\mu - 1) \end{cases} \tag{10}$$

We study the dynamic properties of the map (10) and explore the behavior of the model for economically meaningful values of the parameters. Since we are interested in the sustainability of the debt ratio, we will focus on the case $b_t \geq 0$, even if the dynamic model (10) is feasible for $b_t < 0$ as well. We will highlight the role of some local and global bifurcations that explain the qualitative changes and evolution of the economic system in section 5, including the occurrence of different kinds of instability in the debt ratio and fluctuations in the real interest rate, with worrying default scenarios. Moreover, benchmark cases with $\beta = 0$ (no risk-premium/spread); $\eta = 0$ (no debt monetization by the CB), $\epsilon = 0$ (no government budget adjustment rule), and $\mu = 0$ (no population of trend-followers agents) will be studied in section 4.2. These cases provide some basic mathematical structures of our model and may constitute useful economic scenarios for comparison.

4.1 Fixed points and local stability analysis

Equilibrium (or stationary) situations are obtained by setting $b_{t+1} = b_t = b$ and $\pi_{t+1} = \pi_t = \pi$ in the map in (10). Solving both equations for the variable π, we get $\pi_1(b)$ in (11), and $\pi_2(b)$ in (12) with k, v, z which are aggregations of basic parameters.

$$\pi = \pi_1(b) = \frac{\beta b^2 + \lambda + \epsilon \bar{b}}{(1-\alpha)\, b} + k, \qquad k = \frac{\bar{i} - g - \eta - \epsilon - \alpha\bar{\pi} - \beta\bar{b}}{1-\alpha} \tag{11}$$

$$\pi = \pi_2(b) = z + vb, \qquad v = \frac{\delta\eta - \beta\gamma}{1 - 2\mu - \gamma\,(1-\alpha)}, \quad z = \bar{p} + \frac{\beta\gamma\bar{b}}{1 - 2\mu - \gamma\,(1-\alpha)} \tag{12}$$

Equilibrium points are located at the intersections of the two curves: the hyperbola (11) and the line (12).[10] The graphical representation of these two curves is shown in Figures 4 ($\alpha > 1$) and 5 ($\alpha < 1$). The conditions of existence (C.E.) for the hyperbola are $\alpha \neq 1$ and $b \neq 0$. (11) has a vertical asymptote $b = 0$ and an oblique asymptote $\pi = 1/(1-\alpha)[\beta(b-\bar{b}) + \bar{i} - g - \epsilon - \mu - \alpha\bar{\pi}]$. In addition, for $\alpha > 1$ the hyperbola (11) becomes concave, while for $\alpha < 1$ is convex, as demonstrated in the Mathematical Appendix (A.1). On the other hand, the C.E. for the line (12) is simply $\mu \neq [1 - \gamma(1-\alpha)]/2$. A typical scenario (Figures 4 and 5) is characterized by the presence of two equilibrium points: a lower equilibrium $E_L = (b_L, \pi_L)$ and an upper equilibrium $E_U = (b_U, \pi_U)$ characterized by a low and a high level of public debt respectively, i.e. $b_L < b_U$. Analytically, the equations of the two equilibrium points $E_L(b_L; \pi_L)$ and $E_U(b_U; \pi_U)$ are:

$$E_L : \begin{cases} b_L = \frac{-(1-\alpha)(k-z) - \left\{(1-\alpha)^2(k-z)^2 - 4[\beta - v(1-\alpha)](\lambda + \epsilon\bar{b})\right\}^{1/2}}{2[\beta - v(1-\alpha)]} \\ \pi_L = z + v\frac{(1-\alpha)(k-z) + \left\{(1-\alpha)^2(k-z)^2 - 4[\beta - v(1-\alpha)](\lambda + \epsilon\bar{b})\right\}^{1/2}}{2[\beta - v(1-\alpha)]} \end{cases} \tag{13}$$

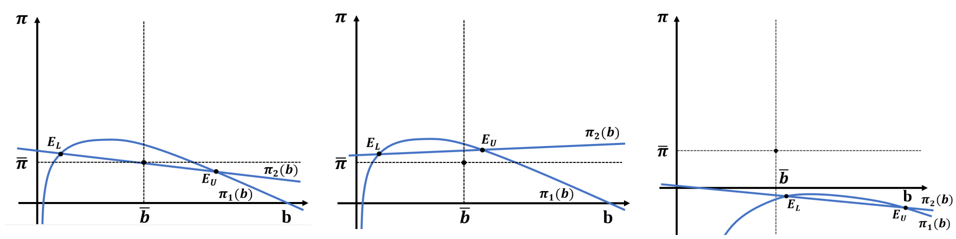
$$E_U : \begin{cases} b_U = \frac{-(1-\alpha)(k-z) + \left\{(1-\alpha)^2(k-z)^2 - 4[\beta - v(1-\alpha)](\lambda + \epsilon\bar{b})\right\}^{1/2}}{2[\beta - v(1-\alpha)]} \\ \pi_U = z - v\frac{(1-\alpha)(k-z) - \left\{(1-\alpha)^2(k-z)^2 - 4[\beta - v(1-\alpha)](\lambda + \epsilon\bar{b})\right\}^{1/2}}{2[\beta - v(1-\alpha)]} \end{cases} \tag{14}$$

[10] Note how the non-linearity in the debt curve (with respect to the standard debt sustainability model) originates from the spread over the interest rate and thus the endogeneity of the latter parameter.

Depending on the C.E. of the curves, cases with one or no equilibria may exist. Specifically, for the condition $W = (1-\alpha)^2(k-z)^2 - 4\left[\beta - v(1-\alpha)\right](\lambda + \epsilon\bar{b}) = 0$ only one (stable/unstable) equilibrium appears, while for $W < 0$, the two curves (11) and (12) do not intersect, hence no equilibrium can be found.

Figure 4: Public debt and the coexistence of two equilibria with $\alpha > 1$

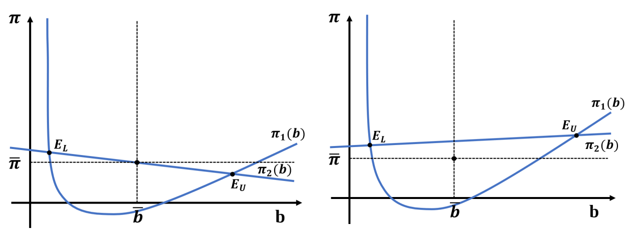
(a) Panel a (b) Panel b (c) Panel c



Note: Two equilibria may exist E_L and E_U, characterized by a low and high level of public debt ratio. Parameters of the model. Panel a: $\alpha = 1.5, \beta = 0.035, \gamma = 0.4, \delta = 0.7, \epsilon = 0.05, \eta = 0.01, \lambda = 0.01, \mu = 0.1, \bar{b} = 0.9, \bar{i} = 0.02, \bar{\pi} = 0.02, g = 0.015$; Panel b: same parameters of Panel a, except for $\eta = 0.03$; Panel c: same parameters of Panel a, except for $\beta = 0.055, \epsilon = 0.09, \lambda = 0.06$ and $\mu = 0.45$.

Figure 5: Public debt and the coexistence of two equilibria with $\alpha < 1$

(a) Panel a (b) Panel b



Note: Two equilibria may exist E_L and E_U, characterized by a low and high level of public debt ratio. Parameters of the model. Panel a: same parameters of Figure 4a, except for $\alpha = 0.9$; Panel b: same parameters of Figure 4b, except for $\alpha = 0.9$.

As already stated, in Figures 4 and 5 we only focused on situations of positive debt ratio for equilibrium points (positive part of the hyperbolic branch). These fixed points can be associated with either positive or negative values for inflation, i.e. $\pi^* > 0$ or $\pi^* < 0$. All the panels in Figure 4 are characterized by $\alpha > 1$, thus representing scenarios in which the CB adjusts the interest rate more than proportionally to deviations from its target. In contrasts the panels in Figure 5 show scenarios in which the CB's interest rate responses to inflation deviations are less than proportional ($\alpha < 1$). The remaining structural parameters were set to the average values identified in the literature.

In particular, Figure 4a depicts a two-equilibrium situation in which public debt ratios are $b_L < \bar{b} < b_U$ and the inflation rates are $\pi_U < \bar{\pi} < \pi_L$. Indeed, the two equilibrium points $E_L(0.71; 0.028)$ and $E_U(2.48; 0.015)$ are characterized, respectively, by low and high debt ratios and, vice versa, high and low inflation rates. In 4b (higher η with respect to 4a), the two fixed points $E_L(0.56; 0.037)$ and $E_U(2.57; 0.051)$ are more distant from each other in terms of public debt ratios with respect to 4a, $b_L < \bar{b} < b_U$, but they show higher rates of inflation: both E_L and E_U have an inflation value greater than $\bar{\pi}$, i.e. $\bar{\pi} < \pi_L < \pi_U$. Lastly, in 4c (higher η, ϵ and μ with respect to 4a), the two equilibrium points $E_L(1.87; -0.008)$ and $E_U(2.51; -0.04)$ are both characterized by debt ratios greater than $\bar{b}$ (i.e. $\bar{b} < b_L < b_U$), but closer to each other, along with negative inflation (a slight deflation), $\pi_U < \pi_L < \bar{\pi}$.

Moving to Figure 5, 5a is obtained with the same set of parameters of 4a with the only exception of $\alpha = 0.9$ (i.e., less aggressive interest rate adjustment by the CB). In comparison to the latter situation, the lower equilibrium point $E_L(0.65; 0.031)$ is characterized by a lower debt ratio and moderately higher inflation rate, while the upper equilibrium $E_U(2.36; 0.015)$ has a slightly lower debt ratio and the same inflation value. To conclude, 5b is obtained with the same set of parameters of 4b and, again, the only exception of $\alpha = 0.9$. Compared to 4b, the lower fixed point $E_L(0.49; 0.041)$ has a smaller debt ratio and moderately higher inflation, while the upper fixed point $E_U(3.27; 0.067)$ has both considerably larger values of debt ratio and inflation.

The local stability of each equilibrium can be determined through the usual linearization procedure based on the Jacobi matrix (J) of the map (10), given by:

$$J(b,\pi) = \begin{bmatrix} 1 + \bar{i} - g - \eta - \epsilon - \pi(1-\alpha) + \beta(2b - \bar{b}) - \alpha\bar{\pi} & -(1-\alpha)b \\ \delta\eta - \beta\gamma & 2\mu + \gamma(1-\alpha) \end{bmatrix} \tag{15}$$

In our model an analytical computation of the conditions of stability of the fixed points is possible. This is done by substituting the equilibrium values

(13) and (14) into the Jacobi matrix in (15). However, due to the complexity in the mathematical tractability of the results, we refer to the numerical values of the equilibria, given the set of parameters identified in Figures 4 and 5, to localize in the complex plane their eigenvalues (Schei 2020). At the same time, in the next Subsection (4.2), we will employ simulations and a few benchmark cases to show the dynamical proprieties of the system when parameters change and to draw some relevant economic implications.

In all the scenarios considered, the lower equilibrium E_L is a *stable node* or a *stable focus* or an *unstable equilibrium* with a bounded attractor around it, whereas the upper equilibrium E_U is a *saddle*. The local stability analysis is included in the Mathematical Appendix at the end of the chapter (A.2).

Figure 6 shows the basins of attraction for the three parameters' set used in panels (a), (b), (c) of Figure 4, respectively, whereas Figure 7 represents the basins for the parameters' set of Figure 5. In these plots the basins of attraction of the stable equilibrium E_L are represented by the red region, whereas the black region shows the basin of divergent trajectories, i.e. the set of initial conditions (b_0, π_0) that generate time evolution leading to public default. The frontier (or watershed) that separates these two basins is formed by the stable set of the saddle point E_U (see e.g. Mira et al. 1996).

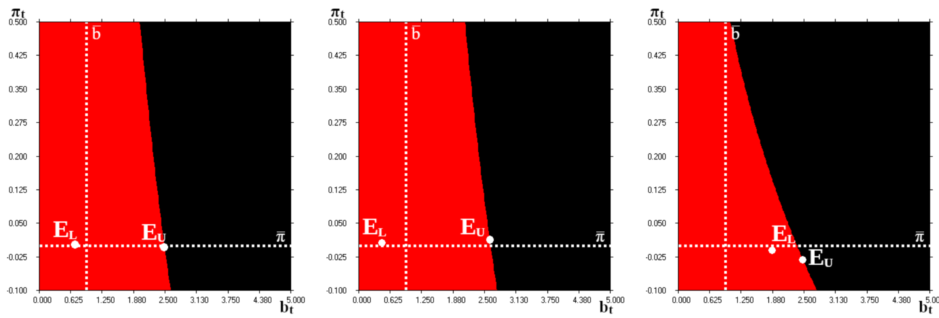
The distance between the two fixed points E_L and E_U constitutes a good proxy of the extension of the basin of the stable equilibrium E_L, thus a measure of the robustness/resilience of the latter to exogenous shocks. Indeed, one of the novelties of this analysis is the presence of a threshold level for debt ratio and inflation, after which the debt ratio becomes unsustainable and takes an explosive path. This is different from the standard model of public debt sustainability represented by the government's intertemporal budget constraint, where debt is always either convergent or divergent (O. Blanchard 2022). On the contrary, it supports the empirical literature that suggests that the identification of a specific debt sustainability threshold should consider a number of country characteristics that might constrain government choices and influence the economy's vulnerability to crises (Eberhardt and A. F. Presbitero 2015). The threshold level here is represented by E_U, while the equilibrium at which the model converges is E_L. Thus, the distance between the two points indicates how much the system is resilient to possible perturbations from the equilibrium (i.e. E_L). In this regard, the basins of attraction of the stable equilibrium in Figures 6a and 6b are quite similar in terms of size and behavior: a higher rate of inflation moderately reduces the stability of the system (i.e. the initial value of debt ratio at which it is possible to converge to E_L). In Figure 6c despite the proximity of the two fixed points E_L and E_U, the width of the basin continues to be almost the same for low inflation values. However, a small perturbation from the equilibrium E_L could lead the debt ratio trajectory towards the black region of default. Moreover, for higher inflation rates the basin tends to shrink very quickly and this makes the debt ratio path unsustainable for even smaller initial values of debt. In this case, the system is clearly less robust to shocks than the two previous panels.

Moving to Figure 7, we recall that these two panels are obtained with the same parameters of Figures 4a and 4b with the exception of $\alpha = 0.9$. For this reason, the equilibrium values are fairly similar, but in terms of stability proprieties, we find an opposite behavior. As long as the inflation rate grows the system becomes more stable, so it is possible to start from higher levels of debt ratio and, nonetheless, to converge towards E_L. In addition, the size of the basin of attraction of Figure 7b is larger than Figure 7a due to the higher level of debt ratio and inflation in E_U. In this latter case, even relevant shocks from the equilibrium E_L can be borne by the system (i.e. more robust).

It is not surprising in economic terms that a higher inflation rate in equilibrium (such as in 5a and 5b) is associated with a lower debt ratio. In fact, inflation erodes the real value of debt, through its effect on the real interest rate. This opens up a trade-off which, depending on the weak (strong) reaction of the CB on interest rates in the face of rising inflation (i.e., α), will lead to the debt ratio stabilizing at a lower (higher) level in equilibrium. The inflation rate, on the other hand, will be higher in the first case and lower in the second. The extent of this trade-off is determined by the structural parameters of the model.

Figure 6: Basins of attractions of equilibrium points in Figure 4.

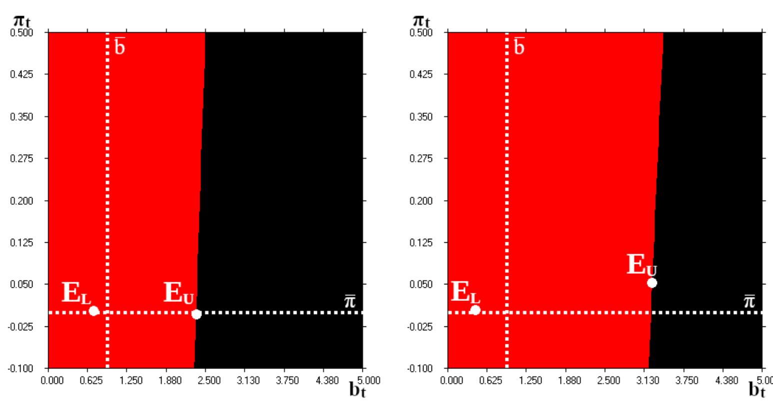
(a) Panel a (b) Panel b (c) Panel c



Note: Figures 6a, 6b and 6c show the basins of attraction for the three parameters' set used in Fig. 4a, 4b and 4c respectively. The red areas represent initial conditions that generate converging trajectories, while the black areas represent initial conditions that generate diverging trajectories. Threshold values for $\bar{b}$ and $\bar{\pi}$ are represented by the white dotted lines.

Figure 7: Basins of attractions of equilibrium points in Figure 5

(a) Panel a (b) Panel b



Note: Figures 7a and 7b show the basins of attraction for the two parameters' set used in Fig. 5a and 5b respectively. The red areas represent initial conditions that generate converging trajectories, while the black areas represent initial conditions that generate diverging trajectories. Threshold values for $\bar{b}$ and $\bar{\pi}$ are represented by the white dotted lines.

By properly tuning the parameters of the model, local bifurcations can occur leading to the creation or disappearance of the equilibrium points, giving rise to several types of self-sustained oscillations and changing the stability properties of the system. These dynamic scenarios will be illustrated in Section 5, by means of numerical simulations, guided by some of the analytically determined conditions on the parameters. In the next section (4.2), we will focus on a few benchmark cases, obtained by *turning off* some crucial parameters to explore the economic implications on the model.

4.2 Some benchmark cases

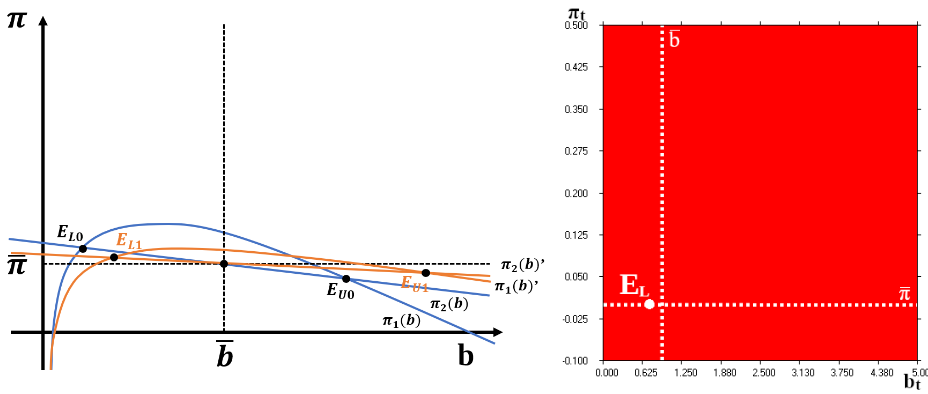
In this section, we study four benchmark cases of the dynamic model in (10). In all the cases considered hereinafter, we take as reference 4a and its parameters' set. In every benchmark case we 'turn off' (i.e. set to zero) one of the parameters in order to compare the difference between the fixed points of 4a, labeled as E_{L0}, E_{U0}, and the new steady states E_{L1}, E_{U1}. From the comparison, it is possible to draw relevant economic insights into the effects of these parameters on the model.

4.2.1 $\beta = 0$: absence of a spread effect (no risk-premium)

First, let us assume $\beta = 0$. In this scenario, there is *no risk-premium* or spread as the nominal interest rate is simply the one set by the CB. We compare the two situations, with the risk-premia (blue curves) and without (orange curves), in Figure 8a. A first notable consequence is that both curves shift: the hyperbolic branch $\pi_1(b)'$ from equation (11) now becomes almost completely flat, while the new line $\pi_2(b)'$, given by equation (12), flattens with a slightly lower intercept.

Figure 8: Benchmark case No. 1, $\beta = 0$: no risk-premium/spread

(a) Panel a (b) Panel b



Note: Parameters are the same as those used in 4a, except for the risk-premium $\beta = 0$. In panel b, the red areas represent initial conditions that generate converging trajectories, while the black areas represent initial conditions that generate diverging trajectories.

For $\beta = 0$, the market does not apply a risk premium on the relevant interest rate for government debt issues, which means that government debt is always considered safe by the market no matter its amount. As it is possible to see from 8a, the lower stable equilibrium does not change much: from $E_{L0} = (0.71; 0.028)$ to $E_{L1} = (0.76; 0.025)$. It shows a small increase in the public debt ratio and a slightly lower inflation rate. However, the basin of attraction of the latter expands enormously, covering a much larger range of initial conditions (up to the saddle E_U), as depicted in 8b, where all the area is covered in red (convergent trajectories). In fact, the saddle point moves from $E_{U0} = (2.48; 0.015)$ to $E_{U1} = (20.67; 0.165)$. It means that from every reasonable economic value of the variables b and π (up to 500% of debt ratio and 50% of the inflation rate - given the structural parameters of the economy in this example), the system converges to the lower equilibrium E_{L1}. This is, of course, an extreme case in which financial operators do not price in the riskiness associated with the sustainability of the debt ratio and thus the probability of a country defaulting. However, even from this extreme scenario, important policy implications can be drawn.

First of all, if investors in government bonds are less risk-averse about the government's ability to repay the debt and/or perceive, on average, a lower probability of default by the country, they will tend to price the risk on government bonds less (i.e. lower β). As a result, the State could borrow relatively more with a lower risk of default (Von Hagen, Schuknecht, and

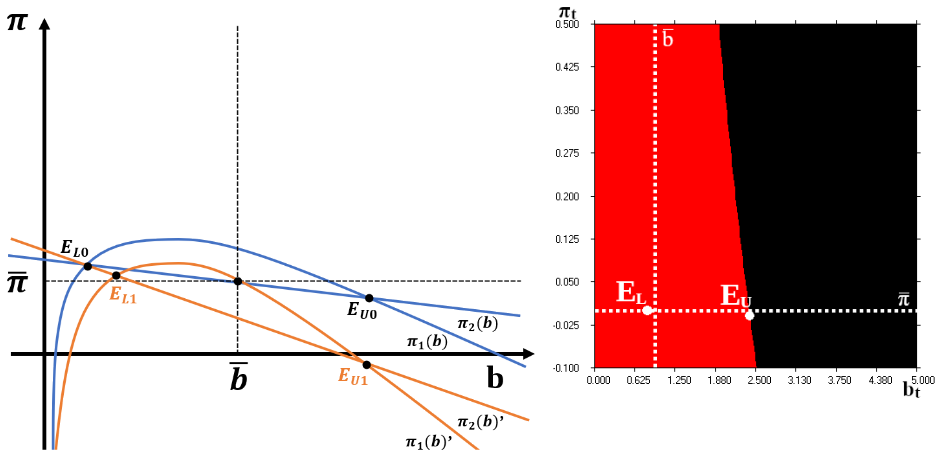
Wolswijk 2011; Du, Pflueger, and Schreger 2020). Secondly, a key finding argues that unconventional monetary policy is fundamental in stabilizing the economy thanks to the active role it can have in containing the spread/risk-premium β. If the CB succeeds, through a program of government bond purchases (e.g. quantitative easing), to reduce financial spreads, the system becomes much more stable and shocks on both the debt ratio and the inflation rate do not alter the convergence toward E_L (Krishnamurthy and Vissing-Jorgensen 2011; Kinateder and Wagner 2017). This is the reason why Q.E. and similar unconventional monetary measures have been so widely used by CBs during the last years. Other than additional instruments helpful to target inflation, they have been effective in reducing spreads and interest rates on public bonds, especially for highly indebted countries, making debt ratios more sustainable as highlighted in these figures.

4.2.2 $\eta = 0$: absence of debt monetization

Now, let us assume $\eta = 0$. In this scenario, there is no debt monetization by the CB. It is possible to compare the two situations, with the debt monetization (curves in blue) and without (curves in orange), in Figure 9a. The hyperbola branch $\pi_1(b)'$ now shifts downwards, whereas the new line $\pi_2(b)'$ is much more downward sloping with an increased intercept, shrinking the distance between the two equilibria.

Figure 9: Benchmark case No. 2, $\eta = 0$: no debt monetization

(a) Panel a (b) Panel b



Note: Parameters are the same as those used in 4a, except for the debt monetization parameter $\eta = 0$. In panel b, the red areas represent initial conditions that generate converging trajectories, while the black areas represent initial conditions that generate diverging trajectories.

For $\eta = 0$, the CB does not implement any debt monetization measures to finance (ex-post) a specific share of the government's outstanding debt ratio. Therefore, the dynamic equation in (5) becomes simply $b_t = (1 + r - g)\, b_{t-1} + d_t$, which is the standard model of public debt sustainability. The evolution of the debt ratio is hence determined only by (exogenous) g and (endogenous) r. In this case, the debt ratio value of the lower stable equilibrium increases from $E_{L0} = (0.71; 0.028)$ to $E_{L1} = (0.82; 0.021)$, as we can expect, since there is no more support of the CB to finance part of it. However, without debt monetization, the inflation rate in equilibrium is quite lower. This is because a debt monetization program generates a persistent shock in terms of inflation due to the increased money base that needs to be injected into the economy to purchase or finance part of the debt. Indeed, as long as debt monetization is in place, the equilibrium value of inflation increases, showing the trade-off of the measure: the CB is able to lower the debt ratio level to preserve the financial stability of the economy, but at the expense of a persistent price increase. However, if the debt monetization measure is *moderate*, as in the example of Figure 4a where $\eta = 0.01$, the adverse effect on the inflation rate is rather limited. We have shown that a permanent government bond purchase program/money financing of 1% of

the public debt ratio leads to a moderate increase of inflation (0.7%) and to a non-negligible reduction in debt ratio (11%). From the point of view of stability of the equilibrium, the distance between the two equilibria moderately decreases for $\eta = 0$, because the saddle point moves to the bottom-left: from $E_{U0} = (2.48; 0.015)$ to $E_{U1} = (2.40; -0.001)$. Consequently, the size of the basin of attraction of the stable equilibrium (in red) becomes a little lower, as depicted in 9b. In contrast, in the absence of debt monetization, the system is slightly less stable.
Thus, in this scenario, a moderate debt monetization might have a positive effect on the stability of the equilibrium, since it increases its basin of attraction making it easier for the government to bear unexpected public debt shocks (for instance when $g \simeq r$) (Agur et al. 2022). In this setting, we conclude that this measure (even if considered controversial) when implemented with caution can effectively reduce the debt burden in the long-run, as well as make the economy more resilient to perturbations with a (relatively) little sacrifice in terms of inflation cost. This makes it more suitable to be carried out in periods of prolonged low inflation rates and liquidity traps such as the situation in the Eurozone in 2013-2018 (Botta, Caverzasi, and Russo 2020). It is not a coincidence that the debate around a possible partial debt monetization of the most indebted member countries came to the fore precisely in those years. Nonetheless, the topic might have a comeback in the near future, as long as the situation could worsen and public debt sustainability becomes again a serious concern for the financial stability of the economy.

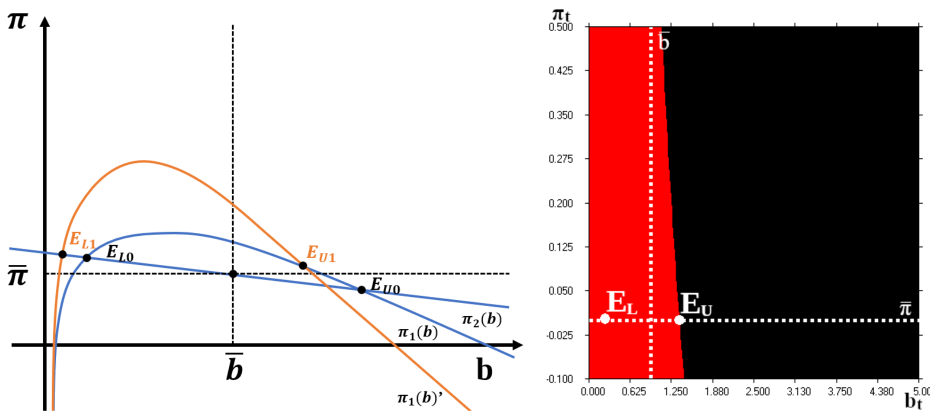
4.2.3 $\epsilon = 0$: no government budget adjustment rule

Here we assume $\epsilon = 0$. As in the standard model, in this scenario, there is no government budget adjustment to the current value of the debt ratio. In other words, it is as if the government in each period of time t (i.e., each year) set the same (constant) amount of public deficit (if $\lambda > 0$) or surplus (if $\lambda < 0$). It is possible to compare the two situations: without the active government budget adjustment rule (orange curves) and with (blue curves) in Figure 10a. In the latter case, from the parameters of Figure 4a, we have that $\lambda = 0.02$, so it means a constant primary deficit of 2%. The hyperbolic branch $\pi_1(b)'$ now warps approaching the origin of the axes, while the line $\pi_2(b)$ does not change as ϵ does not enter in the dynamic equation (8), and thus in (12).

Figure 10: Benchmark case No. 3, $\epsilon = 0$: no government budget adjustment rule.

(a) Panel a

(b) Panel b



Note: Parameters are the same as those used in 4a, except for the government budget elasticity $\epsilon = 0$. In panel b, the red areas represent initial conditions that generate converging trajectories, while the black areas represent initial conditions that generate diverging trajectories.

For $\epsilon = 0$, the government decides to set a constant (i.e. a passive) budget rule without reacting to changes in the public debt ratio. The lower equilibrium is now $E_{L1} = (0.23; 0.031)$. The level of debt ratio (sustainable in equilibrium) decreases considerably (-48%), but the inflation rate spikes at 3.1%. Further, the unstable upper equilibrium, i.e. the saddle point, considerably reduces in terms of debt ratio from $E_{U0} = (2.48; 0.015)$ to $E_{U1} = (1.36; 0.023)$. This translates into a much lower resilience or sustainability of the public debt ratio, which is less robust to perturbations/shocks and it can converge to E_{L1} only from smaller values of initial debt. This can be seen also from 10b where the basin of attraction of the stable equilibrium (in red) is very small. The result is not surprising, since there is no budget adjustment rule to the actual level of the public debt ratio, the system is less sensitive to changes and/or shock of public debt and, as consequence, the stability is compromised.

For this reason, a certain degree of adjustment of the fiscal policy at the current debt ratio may be desirable (Beetsma 2022). Governments have in fact limited control over r and g. The interest rate is under the control of the CB, while potential growth is hard to affect, and structural reforms often have uncertain effects. Thus, the policy focus is on the primary balance

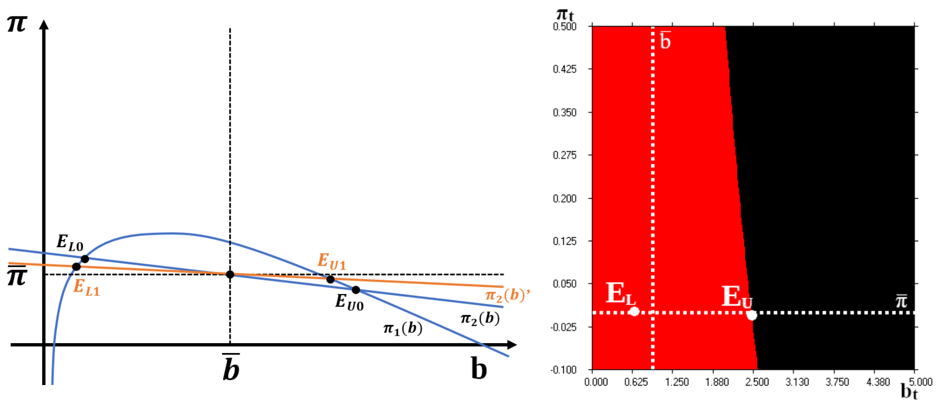
(O. Blanchard, Leandro, and Zettelmeyer 2021). As shown by Bohn (1998), as long as the primary balance reacts sufficiently to debt, any debt ratio is sustainable. However, there are economic and political limits to how large a primary surplus a government can generate (Ghosh et al. 2013). Moreover, if this rule is not imposed on the primary current account balance, fiscal austerity has often led to a decrease in public investment rather than other forms of spending, with the consequence of worsening not only the debt ratio but also long-term economic growth (Cerniglia, Saraceno, and Watt 2020; O. Blanchard 2022).

4.2.4 $\mu = 0$: no population of trend-followers agents in inflation expectations

In the last benchmark case, we assume $\mu = 0$. In this scenario, there is no population of agents that behave as trend-followers in forming expectations on inflation. As in the other benchmarks, we compare the two situations, with the presence of the trend-followers $\mu = 0.1$ (in blue) and without $\mu = 0$ (in orange), in Figure 11a. The curve $\pi_1(b)$ does not change as μ does not enter in the dynamic equation (5), and thus in (11). The line $\pi_2(b)'$ flattens out (i.e. less downward sloping) with a lower intercept.

Figure 11: Benchmark case No. 4, $\mu = 0$: no population of trend-followers agents

(a) Panel a (b) Panel b



Note: Parameters are the same as those used in 4a, except for the share of trend-followers agents $\mu = 0$. In panel b, the red areas represent initial conditions that generate converging trajectories, while the black areas represent initial conditions that generate diverging trajectories.

For $\mu = 0$ there is only the population of fundamentalist agents who believe that inflation will eventually return to its fundamental target (because of the likely intervention of the CB). In other words, there are no trend-followers agents in the economy (who, on the contrary, expect a less aggressive intervention). As it is possible to see from Figure 11a, this has a strong effect on the value of public debt and inflation in equilibrium, which decreases to $E_{L1} = (0.69; 0.026)$, respectively.

A growing number of agents who trust in the CB's ability to influence and bring back inflation to the targeted level (i.e. a small increase of fundamentalist agents: from 90% to the totality of the population) does not alter significantly the stability of the lower equilibrium E_{L1}. Indeed, the saddle point now moves to the left, $E_{U1} = (2.46; 0.016)$, with respect to 4a, but the basin of attraction remains almost the same in Figure 11b.

Inflation expectations of economic agents matters, especially for the equilibrium value of inflation. As one might expect, when the share of fundamentalist agents in the economy increases, and vice versa the share of trend-follower agents decreases (they are complementary), the long-run inflation approaches the 2% value, as more agents behave with expectations anchored to the target. On the contrary, when trend-followers beliefs prevail in the economy, inflation expectations are dis-anchored to the objective (i.e. agents

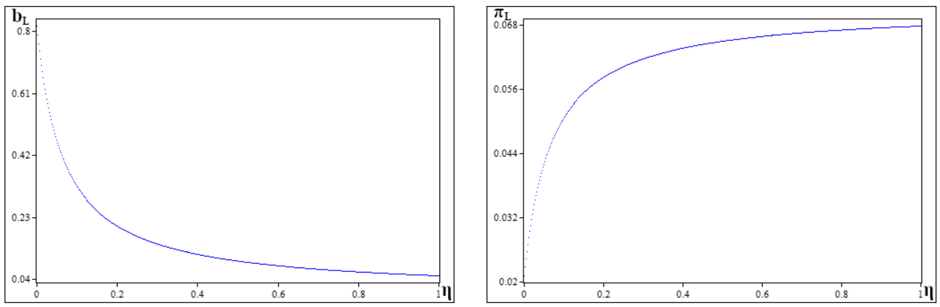
simply follow the previous realized inflation value) and this contributes to moving away the equilibrium value of inflation from the target set by the central bank (i.e. $\bar{\pi} = 2\%$).
Consequently, a relevant policy prescription arises and involves the credibility of the CB in guiding inflation expectations (Woodford 2004). If the CB seems more credible in the eyes of economic agents, it follows that a higher share of economic agents will behave as fundamentalists, anchoring inflation expectations to the value pursued by the CB. If vice versa, the CB starts to lose credibility and efficacy (for whatever reason) in the agents' perception, the share of trend-following agents in the economy will inevitably rise, in the belief that the CB, with its monetary policy, is not able (or not as effective as before) to drive inflation to the target (Hommes and Lustenhouwer 2019). It results that shocks on inflation are more persistent in time as inflation expectations continue to follow past realizations. This leads to a longer time needed for the CB to accommodate the rate of inflation to the objective value and, often, requires more effort in terms of interest-rate-based policy (i.e. changes of i_t) to achieve the same result (i.e. the target $\bar{\pi}$).

5 Numerical simulations

In this final section, we provide some numerical simulations to study the effects of different economic conditions and/or policies on the long-run evolution of the economic system. Let us again take Figure 4a as a reference and let us vary one parameter at a time, *ceteris paribus*. In Figure 12 we show the change of the equilibrium value E_L for both public debt ratio (12a) and inflation rate (12b) when varying the share of outstanding debt monetized η from 0 to 1 (i.e. bifurcation diagram).

Figure 12: Bifurcation diagram for η

(a) Panel a - Debt ratio (b_L) (b) Panel b - Inflation (π_L)



Note: same set of parameters of Figure 4a. Panel a shows the equilibrium value of public debt (b_L) and panel b that of inflation (π_L) as η changes between 0 and 1.

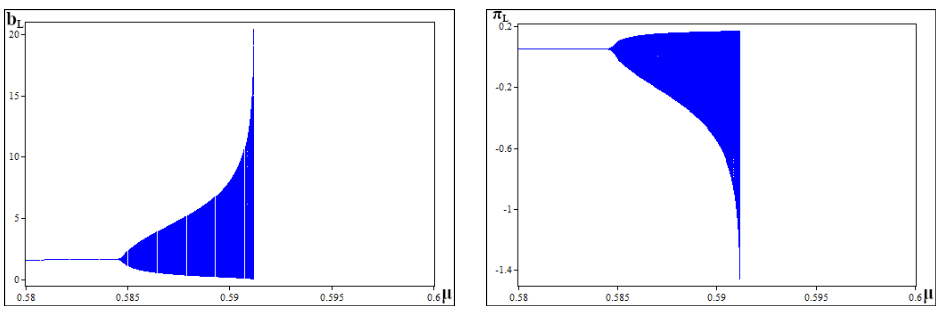
It is evident the trade-off of the debt monetization measure: a higher level of η reduces the equilibrium debt ratio (b_L) in the long run, but has a negative impact on the inflation rate (π_L). In addition, due to the peculiarity of the non-linear map in (10), the effect is highly non-linear: a small variation of η has a large impact on both variables for lower values of debt monetization rather than for larger levels. This suggests that for the policy maker (in this case the CB), it is sufficient to monetize a relatively small amount of debt (e.g. $\eta < 0.03$ per year) to generate a beneficial effect on debt ratio and, at the same time, to avoid strong inflationary pressure.

On the other hand, a stronger measure is not advisable due to the perverse effect it may have on inflation rates and thus on the stability of the economic system. The CBs always face this dilemma when it comes to choosing a debt monetization instrument. For their mandate and role, CBs attribute the maximum priority to inflation targeting. For this reason, these measures have been often considered a 'taboo' in CBs practices. However, we claim that in particular circumstances, such as economic periods characterized by very low rates of inflation (or even negative), a moderate debt monetization measure could be an alternative unconventional instrument (in addition to interest-rate-based policies). This holds especially when the interest rate has already hit the lower bound and conventional monetary levers become no longer effective in stimulating inflation. In fact, this latter measure would increase the inflation rate and simultaneously reduce the debt burden of the economy.

In Figure 13, with the parameters set of 4a, the bifurcation diagram for μ is represented in a relative small range between 0.58 and 0.6, for both debt ratio (13a) and inflation (13b). We focus on this small range of the parameter because for $\mu < 0.5802$ the model always converges to the unique stable equilibrium E_L, while for $\mu = 0.5802$ the system undergoes a Neimark-Sacker (N-S) bifurcation where E_L becomes unstable and an attracting closed orbit around E_L is created along with a quasi-periodic motion occurs, see the phase diagram in Figure 14b. As long as μ increases, the attracting orbit grows in size (i.e. the area in blue in Figures 13a,13b) until, for $\mu \approx 0.5911$, a global (or final) bifurcation occurs. This happens when the closed orbit collides with the boundary of its basin, destroying the latter and driving the system toward public default.

Figure 13: Bifurcation diagram for μ

(a) Panel a - Debt ratio (b_L) (b) Panel b - Inflation (π_L)



Note: same set of parameters of Figure 4a. Panel a represents the equilibrium debt ratio (b_L) for changes of μ between 0.58 and 0.6. Panel b represents the equilibrium inflation rate (π_L) for changes of the same parameter.

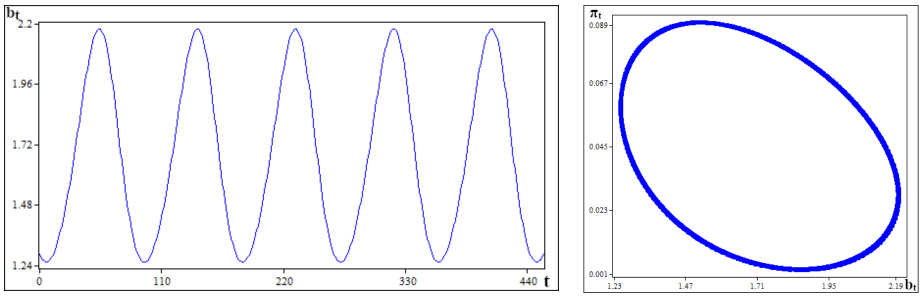
We capture the behavior of the system in this small window of instability (characterized by bounded oscillations that start at $\mu \approx 0.5802$ and ends at $\mu \approx 0.5911$ with the collapse of the orbit) in Figure 14 for the value $\mu = 0.5849$. In 14a, the time series of debt ratio b_L shows a cyclical path of ups and downs between 1.24 and 2.18 over a very long period of time (90 years in this case). The phase diagram, in 14b, confirms that these cycles are generated by an attracting closed orbit located at the debt ratio values highlighted previously and at an inflation range between 0.001 (0.1%) and 0.089 (8.9%). In other words, the parameter μ of the model may generate self-sustained oscillations (Baiardi, Naimzada, and Panchuk 2020).
The dynamics of a cycle can be described as follows. In the first instance, government debt is above the sustainability threshold (125% *against* 90%), while inflation is above the CB's target level (5.79% *against* 2%). As a result,

the government budget balance starts to improve (due to the adjustment rule on the primary deficit), inflation rises (given the push by trend-followers agents) and the nominal interest rate to rise in turn. Notably, the response of the interest rate is stronger than that of inflation, so the real interest rate also rises. These adjustments occur period after period until the inflation rate reaches its maximum. At this point, as a result of the Taylor Rule, the nominal interest rate starts to slow down (with an overshooting period of 4 years), until it reaches its maximum point too. It will finally take 8 more periods before the real interest rate begins its descent (from a max of 6.13%). This moment marks the turning point for the growth of public debt, which starts to slow down its growth rate (following the favorable developments in the interest rate and inflation) but will grow again for the next 15 periods (up to 218% of GDP). When the public debt reaches its maximum, the budget surplus also reaches its maximum (5,04%) and then begins to fall. It will therefore improve from now on, as will public debt until the end of the cycle. However, just 15 periods after the debt peak, inflation will reach a low of 0.29% and start to rise again, driven (now) by the fundamentalist agents, who believe it has fallen too far below its target value, thus expecting a reaction from the CB. The resumption of inflation will end the decline of the nominal interest rate (at 2.55%) after 4 periods. Therefore, the nominal interest rate will start to rise until the end of the cycle, dragging the real interest rate, which will reach a low of 1.82% in just 7 periods. After 90 years (in our example), all variables have returned to their initial value and a new cycle can hence begin. In Appendix (B7b), we report the time series of all variables when $\mu = 0.5849$, and compare them with the previous situation of $\mu = 0.1$ (B7a).

Figure 14: Time series and phase diagram for $\mu = 0.5849$

(a) Panel a - Time series (b_t)

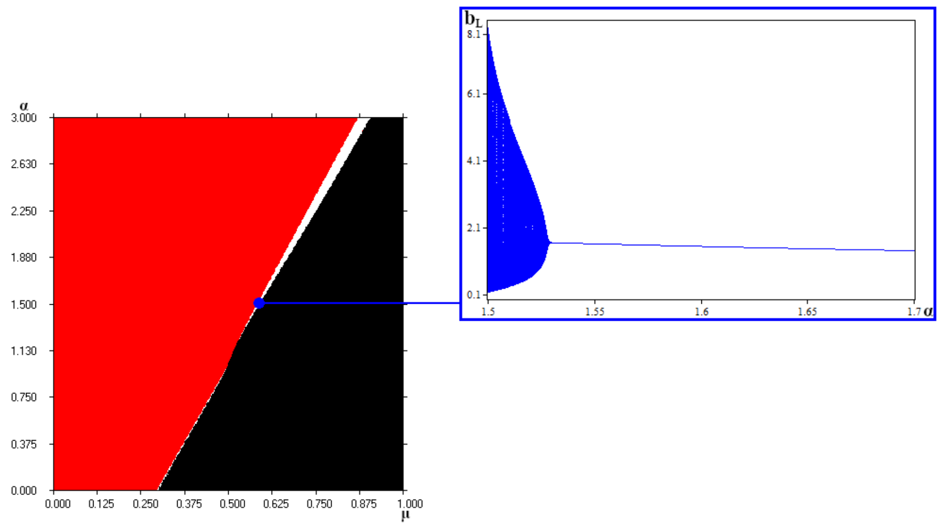
(b) Panel b - Phase plot (b_t, π_t)



Note: same set of parameters of Figure 4a, except for $\mu = 0.5849$. Panel a shows the public debt (b_t) as a time series, while panel b shows the phase diagram with all pairs of public debt ratios and inflation over time (b_t, π_t). In both cases, a transient of 1 million iterations was removed.

The first panel of Figure 15 depicts the two-dimensional bifurcation diagram for μ from 0 (only fundamentalist agents) to 1 (only trend-follower agents), and for α ranging from 0 to 3. In this case, starting from the parameters constellation of Figure 4a, we let vary two parameters to understand their influence on the stability of the system. In red is shown the stability area, in white the area of instability where the N-S bifurcation creates an attracting closed orbit along which a quasi-periodic motion occurs (such as the one in Figure 14b), and in black the area of divergence. The second panel of Figure 15 is an enlargement of a small portion of the parameters basin. This bifurcation diagram is obtained by fixing μ at 0.59 and changing α from 1.5 to 1.7. Thus, we start from a region of instability for $\alpha = 1.5$, and increasing this parameter, *ceteris paribus*, we move to a region of stability where the model converges at E_L.

Figure 15: Two-dimensional bifurcation diagram (α and μ) and one-dimensional bifurcation diagram for only α varying with fixed μ



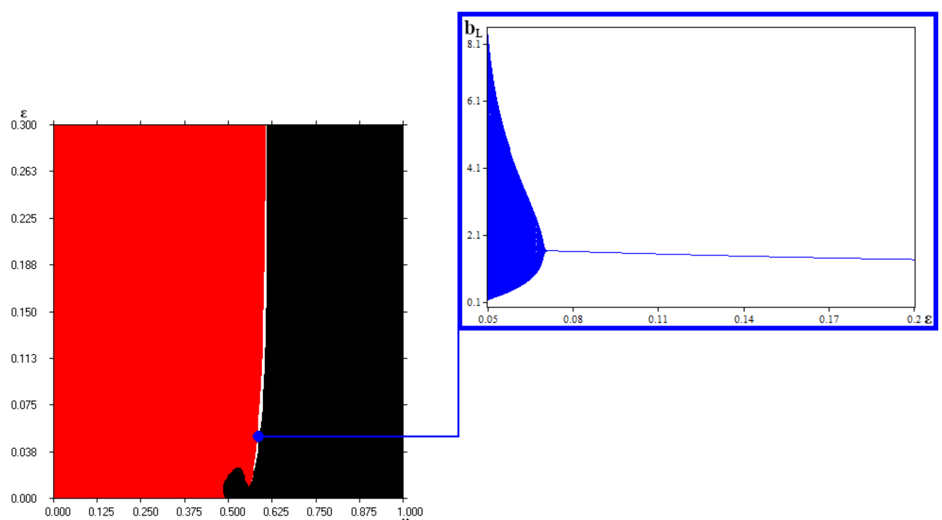
Note: same set of parameters of Figure 4a. Panel a represents the two-dimensional bifurcation diagram with parameters α (vertical axis) and μ (horizontal axis). Panel b represents the transition from instability (Neimark-Sacker area) to stability through changes in monetary policy (α).

It follows that for any level of μ (i.e. share of agents in the economy that follows a given rule in inflation expectation), an increase of α enhances the stability of the model. The parameter α represents the responsiveness of the CB conventional monetary policy (i.e. short-term nominal interest rate) to inflation deviations/shocks from the target. When the CB becomes more aggressive in tackling inflation shocks, a greater share of trend-follower agents is sustainable in the economy.

Thus, the monetary policy plays a relevant role in stabilizing the system: when it is too sluggish in reacting to inflation shocks, there could be the risk that both inflation and debt ratio take an explosive path if agents' beliefs are mainly pessimistic on inflation return to the target. Reversing the perspective, if the CB is more credible in the eyes of economic operators, the share of trend-followers agents in inflation expectation will be relatively less (fundamentalist prevails) and, as a result, the CB can afford to be less reactive in interest rate changes to target inflation. This confirms that the CB credibility in driving agents' inflation expectation is a fundamental variable, as it can reduce the effort required in terms of policy (interest rate change) to achieve the same inflation equilibrium result (Hommes and Lustenhouwer 2019).

In Figure 16, the first panel shows the two-dimensional bifurcation diagram for parameters μ, from 0 to 1, and for ϵ that ranges from 0 to 0.3. Also in this case, starting from the parameters constellation of Figure 4a, we let vary two parameters to understand the influence on the stability of the system. The colors represent the same conditions of Figure 15, and the second panel (i.e. bifurcation diagram) is, as well, an enlargement of a small one-dimensional section for increasing values of ϵ. This bifurcation diagram is obtained by fixing μ at 0.59 and changing ϵ from 0.05 to 0.2. We start from a region of oscillatory instability (quasi-periodic stable oscillations around the unstable equilibrium) for $\epsilon = 0.05$ and increasing this parameter, *ceteris paribus*, we move to a region of stability where the model converges to E_L. It follows that for values of μ in the range between 0.5 and 0.6, an increase of ϵ, i.e. a stronger adjustment of primary deficit to the actual level of country indebtedness, can enhance the stability of the model.

Figure 16: Two-dimensional bifurcation diagram (ϵ and μ) and one-dimensional bifurcation diagram for only ϵ varying with fixed μ



Note: same set of parameters as Figure 4a. Panel a represents the two-dimensional bifurcation diagram with parameters ϵ (vertical axis) and μ (horizontal axis). Panel b represents the transition from instability (Neimark-Sacker area) to stability through changes in fiscal policy (ϵ).

However, the stabilization effect deriving from a more responsive fiscal policy is lower than the one of monetary policy in Figure 15. Nonetheless, fiscal policy decisions on deficit financing are important to avoid embarking on unsustainable debt paths. As already outlined in section 4.2.3, when the debt ratio is already high, particular caution is required in the amount of deficit that can be used.

6 Conclusion

In this chapter, we study the non-linear relationship between public debt and inflation. By means of a macroeconomic model of simultaneous difference equations – one for the debt ratio and the other one for the inflation rate – we focus on the role of monetary and fiscal policy in affecting the stability of the system and the existence of multiple equilibria.
Notably, the non-linearities linking the debt ratio to the real interest rate are important novelties compared to the linear intertemporal budget constraint, in which the interest rate and primary deficit are assumed to be exogenous parameters. Here, we introduce a 'risk premium' that can cause the interest rate on debt issues to deviate from the nominal rate set by the CB and an active adjustment rule for the government's primary balance. Above all, our model provides crucial insights into the relationship between public debt, inflation, and debt monetization, which is not considered by standard models of public debt sustainability.
In this perspective, we show that an indebted economy can easily shift towards divergence regions (default) even for negligible and transitory shocks in some of its policy instruments and behavioral parameters. Accordingly, the creation or disappearance of equilibria or periodic (stable) cycles can generate situations of weak stability of the system. Thus, the distance between these thresholds and the steady state is a good proxy for the economy's robustness to exogenous shocks.
We obtain clear evidence that, in a dynamic setting, an active monetary policy (implemented by means of a standard Taylor Rule) has a stabilizing effect on the inflation rate. On the other hand, we also find that debt monetization can be effective to reduce the spread and stabilize debt dynamics with limited effects on inflation. In this context, we show that - under heterogeneous expectations - the credibility of the CB vis-à-vis economic agents and its ability to guide inflation expectations are key to controlling price

developments and achieving macroeconomic stability. In addition, we also highlight the role of an active primary deficit adjustment rule, which may have a stabilizing effect on the debt ratio, even though it may not be a sufficient instrument to avoid explosive patterns of the latter.

As far as fiscal rules are concerned, our model is also able to nest more general ones. For instance, fiscal policy adjustments can be related to changes in $(r-g)$ which has a more direct impact on public debt sustainability. This analysis can be done by providing additional instruments to affect the dynamics of both the debt ratio and the economy, but the system of equations must satisfy more complicated conditions. Therefore, this chapter is a starting point for further studies into the many possible extensions and applications of the model. These studies are left to our future research.

Our analytical results have some crucial normative implications. Indeed, it emerges that mixed policies would be more effective in stabilizing the debt ratio and inflation at the same time. The topic of coordination between monetary and fiscal policy has become the focus of policy discussion in recent years (Draghi 2014; Lagarde 2020; Schnabel 2021). One reason is that there is limited room for traditional monetary policy based on targeting the short-term interest rate when the latter is at or near the effective lower bound (ELB). Therefore, the way European policymakers will solve the policy mix trilemma of asymmetric fiscal rules, no central fiscal capacity, and constrained monetary policy in the post-pandemic economy will define the resilience of the Euro Area in the face of future shocks.

A Mathematical Appendix

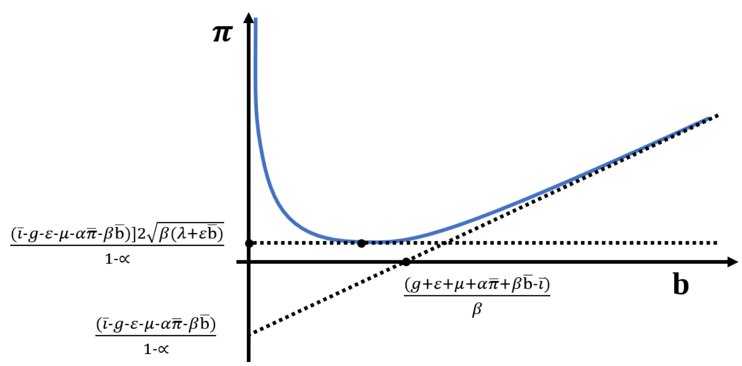
A.1 $\pi_1(b)$ for $\alpha \lesseqgtr 1$

With a simple algebraic reformulation, $\pi_1(b)$ in (11) can be rewritten as:

$$\pi = \pi_1(b) = \frac{1}{1-\alpha}\left[(\bar{i} - g - \epsilon - \mu - \alpha\bar{\pi} - \beta\bar{b}) + \frac{\beta b^2 + \lambda + \epsilon\bar{b}}{b}\right] \tag{A.1}$$

This results in two possible cases. The first occurs when $\alpha < 1$, in which case the hyperbolic branch for $b > 0$ is convex (Figure A1).

Figure A1: $\pi_1(b)$ for $\alpha < 1$.



The following properties apply:

$$\lim_{b\to 0^+} \pi_1(b) = \frac{\lambda + \epsilon\bar{b}}{0^+} = +\infty \tag{A.2}$$

which means a vertical asymptote at $b = 0$. Moreover

$$\lim_{b\to +\infty} \frac{\pi_1(b)}{b} = \frac{\beta}{(1-\alpha)} \tag{A.3}$$

and

$$\lim_{b\to +\infty} \pi_1(b) - \frac{\beta}{(1-\alpha)}b = \frac{(\bar{i} - g - \epsilon - \mu - \alpha\bar{\pi} - \beta\bar{b})}{(1-\alpha)} \tag{A.4}$$

It follows that there is an oblique asymptote represented by the (dotted) line in Figure A1 and by the following equation:

$$y = \frac{[\bar{i} - g - \epsilon - \mu - \alpha\bar{\pi} + \beta(b - \bar{b})]}{(1-\alpha)} \tag{A.5}$$

The first derivative of $\pi_1(b)$ in eq. (11) is equal to:

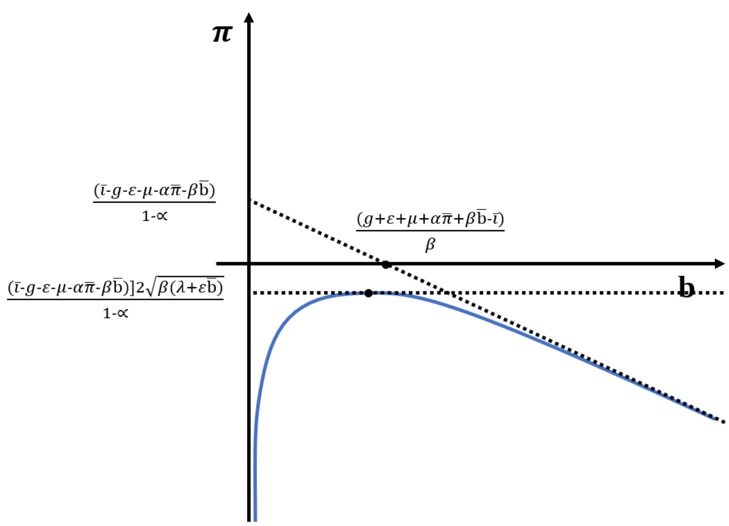
$$\pi_1^{'}(b) = \frac{\beta b^2 - \lambda - \epsilon \bar{b}}{(1-\alpha)b^2} \tag{A.6}$$

Hence, the hyperbola is decreasing before $b_{min} = \sqrt{(\lambda + \epsilon \bar{b})/\beta}$ and increasing after it. The point of minimum for the inflation rate $\pi_1(b_{min})$ (first derivative equal to zero) is equal to:

$$\pi_1(b_{min}) = \frac{1}{(1-\alpha)} \left[(\bar{i} - g - \epsilon - \mu - \alpha\bar{\pi} - \beta\bar{b}) + 2\sqrt{\beta(\lambda + \epsilon\bar{b})} \right] \tag{A.7}$$

The second case occurs when $\alpha > 1$, in which case the hyperbolic branch becomes concave for $b > 0$ (Figure A2).

Figure A2: $\pi_1(b)$ for $\alpha > 1$.



A.2 Jacobian matrix in equilibrium points

Let us first analyze the situation of Figure 4a.

The J matrix of the lower equilibrium $E_L(0.71; 0.028)$ is $\begin{bmatrix} 0.95 & 0.35 \\ -0.007 & 0 \end{bmatrix}$.

It has trace $Tr = 0.95$ and determinant $Det = 0.002$. The discriminant $Tr^2 - 4Det = 0.89$ of the characteristic equation is positive. The eigenvalues are $z_1 = 0.0026$ and $z_2 = 0.9941$ (i.e. they are inside the unit circle), thus E_L is a *stable node*.

The J matrix of the upper equilibrium $E_U(2.48; 0.015)$ is $\begin{bmatrix} 1.06 & 1.24 \\ -0.007 & 0 \end{bmatrix}$. It has trace $Tr = 1.06$ and determinant $Det = 0.009$. The discriminant $Tr^2 - 4Det = 1.1$ of the characteristic equation is positive. The eigenvalues are $z_1 = 0.0082$ and $z_2 = 1.0562$ (i.e. z_1 is inside and z_2 is outside the unit circle), thus E_U is a *saddle*.

In Figure 4b, the J matrix of the lower equilibrium $E_L(0.56; 0.037)$ is $\begin{bmatrix} 0.92 & 0.28 \\ 0.007 & 0 \end{bmatrix}$. It has trace $Tr = 0.92$ and determinant $Det = -0.002$. The discriminant $Tr^2 - 4Det = 0.86$ of the characteristic equation is positive. The eigenvalues are $z_1 = -0.0021$ and $z_2 = 0.9228$, thus E_L is a *stable node*.

The J matrix of the upper equilibrium $E_U(2.57; 0.051)$ is $\begin{bmatrix} 1.07 & 1.28 \\ 0.007 & 0 \end{bmatrix}$. It has trace $Tr = 1.07$ and determinant $Det = -0.009$. The discriminant $Tr^2 - 4Det = 1.18$ of the characteristic equation is positive. The eigenvalues are $z_1 = -0.0083$ and $z_2 = 1.0766$, thus E_U is a *saddle*.

In Figure 4c, the J matrix of the lower equilibrium $E_L(1.87; -0.008)$ is $\begin{bmatrix} 1.03 & 0.94 \\ -0.015 & 0.7 \end{bmatrix}$. It has trace $Tr = 1.73$ and determinant $Det = 0.73$. The discriminant $Tr^2 - 4Det = 0.05$ of the characteristic equation is positive. The eigenvalues are $z_1 = 0.7506$ and $z_2 = 0.9768$, thus E_L is a *stable node*.

The J matrix of the upper equilibrium $E_U(2.51; -0.04)$ is $\begin{bmatrix} 1.08 & 1.26 \\ -0.015 & 0.7 \end{bmatrix}$. It has trace $Tr = 1.78$ and determinant $Det = 0.78$. The discriminant $Tr^2 - 4Det = 0.07$ of the characteristic equation is positive. The eigenvalues are $z_1 = 0.7582$ and $z_2 = 1.0239$, thus E_U is a *saddle*.

In Figure 5a, the J matrix of the lower equilibrium $E_L(0.65; 0.031)$ is $\begin{bmatrix} 0.94 & -0.06 \\ -0.007 & 0.24 \end{bmatrix}$. It has trace $Tr = 1.18$ and determinant $Det = 0.22$. The discriminant $Tr^2 - 4Det = 0.49$ of the characteristic equation is positive. The eigenvalues are $z_1 = 0.2394$ and $_2 = 0.9384$, thus E_L is a *stable node*.

The J matrix of the upper equilibrium $E_U(2.36; 0.015)$ is $\begin{bmatrix} 1.06 & -0.24 \\ -0.007 & 0.24 \end{bmatrix}$. It has trace $Tr = 1.30$ and determinant $Det = 0.25$. The discriminant $Tr^2 - 4Det = 0.68$ of the characteristic equation is positive. The eigenvalues are $z_1 = 0.2380$ and $z_2 = 1.0614$, thus E_U is a *saddle*.

In Figure 5b, the J matrix of the lower equilibrium $E_L(0.49; 0.041)$ is $\begin{bmatrix} 1.10 & -0.33 \\ 0.007 & 0.24 \end{bmatrix}$. It has trace $Tr = 1.34$ and determinant $Det = 0.27$. The discriminant $Tr^2 - 4Det = 0.73$ of the characteristic equation is positive. The eigenvalues are $z_1 = 0.2405$ and $z_2 = 0.9054$, thus E_L is a *stable node*. The J matrix of the upper equilibrium $E_U(3.27; 0.067)$ is $\begin{bmatrix} 1.06 & -0.24 \\ 0.007 & 0.24 \end{bmatrix}$. It has trace $Tr = 1.30$ and determinant $Det = 0.25$. The discriminant $Tr^2 - 4Det = 0.68$ of the characteristic equation is positive. The eigenvalues are $z_1 = 0.2427$ and $z_2 = 1.0948$, thus E_U is a *saddle*.

A.3 Data and code

All graphs used for Figures 4 and 5 are available online:

- Figure 4a: https://www.desmos.com/calculator/lislx1r6ap
- Figure 4b: https://www.desmos.com/calculator/halg55pg3l
- Figure 4c https://www.desmos.com/calculator/hattvq7l1c
- Figure 5a https://www.desmos.com/calculator/quseosbxhj
- Figure 5b https://www.desmos.com/calculator/ajxgir6kvl

Benchmark cases (Sections 4.2.1-4.2.4):

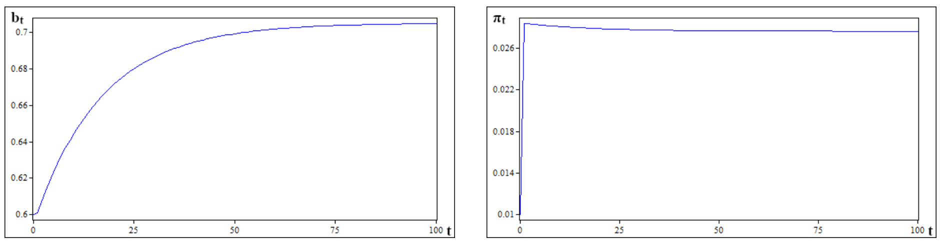
- $\beta = 0$ (4.2.1) https://www.desmos.com/calculator/7qi0skhqnd
- $\eta = 0$ (4.2.2) https://www.desmos.com/calculator/qaxwsxnumz
- $\epsilon = 0$ (4.2.3) https://www.desmos.com/calculator/exbvj8mv0c
- $\mu = 0$ (4.2.4) https://www.desmos.com/calculator/nzhdqwxtcs

The Matlab code used to simulate the model, and calculate the equilibria and stability conditions is available on request.

B Additional Figures and Tables

Figure B1: Time series for the main variables - Case 4a

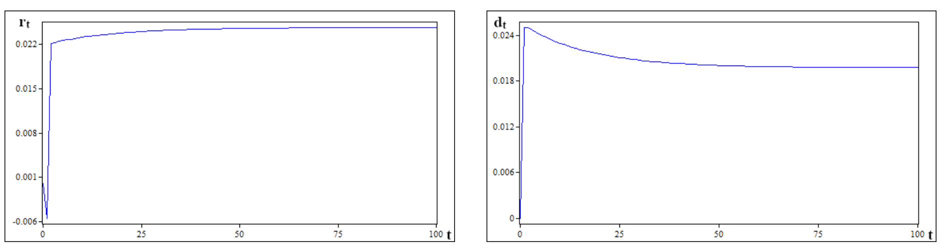
(a) Panel a - Debt ratio b_t (b) Panel b - Inflation rate π_t



Note: Time series for the first 100 periods. Convergence to the long-run value of the variables.

Figure B2: Time series for the auxiliary variables - Case 4a

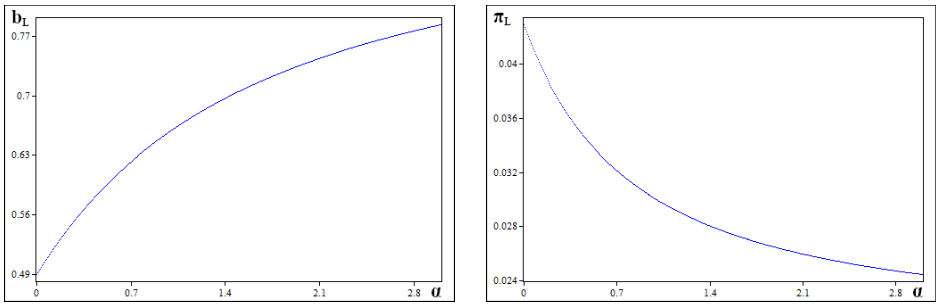
(a) Panel c - Interest rate r_t (b) Panel d - Deficit d_t



Note: Time series for the first 100 periods. Convergence to the long-run value of the variables.

Figure B3: Bifurcation diagram for α

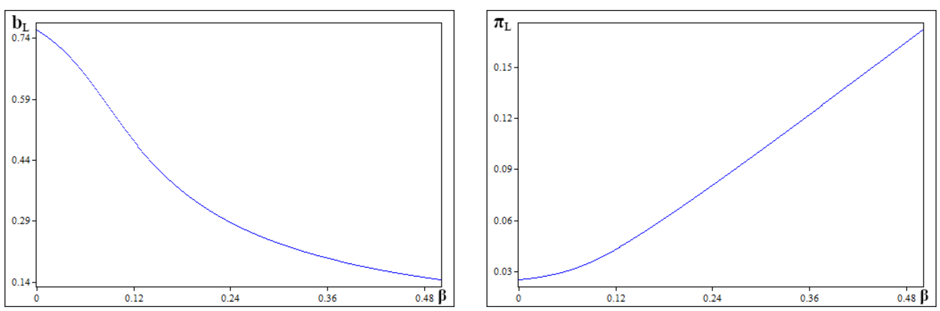
(a) Panel a - Debt ratio (b_L) (b) Panel b - Inflation rate (π_L)



Note: same set of parameters of Figure 4a. Panel a shows the equilibrium value of public debt (b_L) and panel b that of inflation (π_L) as α changes between 0 and 3.

Figure B4: Bifurcation diagram for β

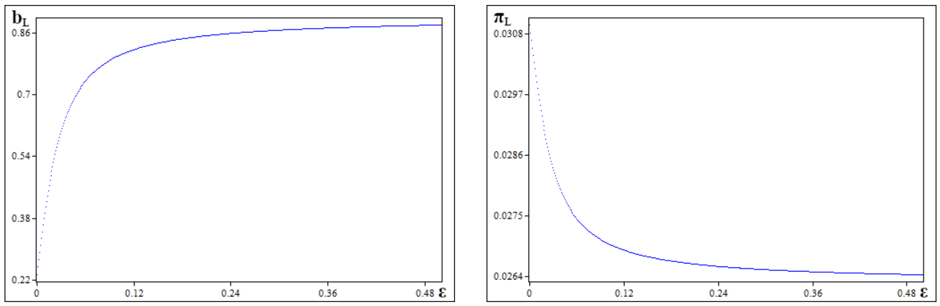
(a) Panel a - Debt ratio (b_L) (b) Panel b - Inflation rate (π_L)



Note: same set of parameters of Figure 4a. Panel a shows the equilibrium value of public debt (b_L) and panel b that of inflation (π_L) as β changes between 0 and 0.5.

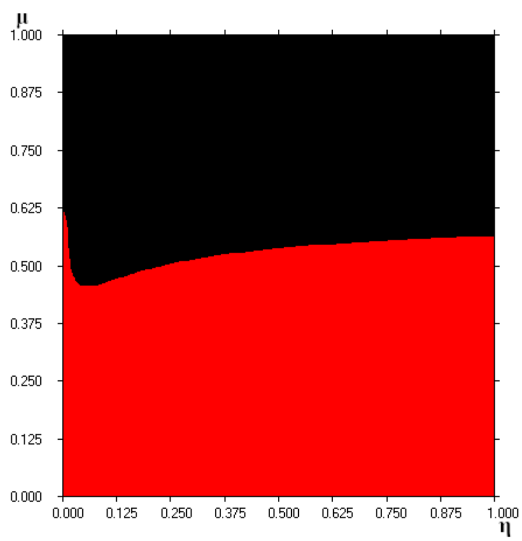
Figure B5: Bifurcation diagram for ϵ

(a) Panel a - Debt ratio (b_L) (b) Panel b - Inflation rate (π_L)



Note: same set of parameters of Figure 4a. Panel a shows the equilibrium value of public debt (b_L) and panel b that of inflation (π_L) as ϵ changes between 0 and 0.5.

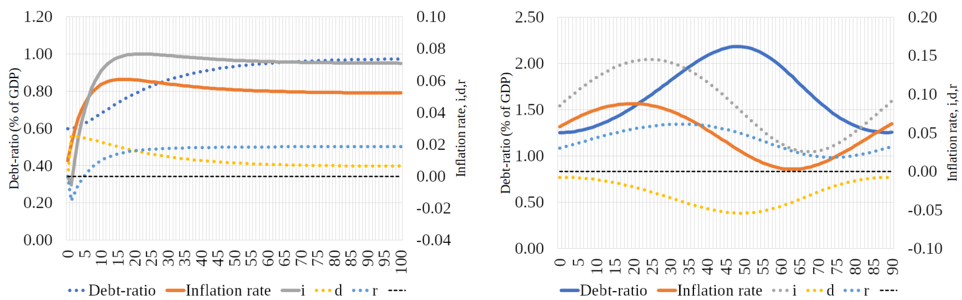
Figure B6: Two-dimensional bifurcation diagram for η and μ



Note: Two-dimensional bifurcation diagram for different values of $0 \leq \eta \leq 1$ and $0 \leq \mu \leq 1$. The red areas represent converging trajectories to E_L, while the black areas represent diverging trajectories.

Figure B7: Time series of all variables (main and auxiliary) for different values of μ

(a) Panel a: $\mu = 0.1$ (b) Panel b: $\mu = 0.5849$



Note: Main (debt ratio, inflation rate) and auxiliary variables (nominal and real interest rate, deficit) in the y-axis, versus time in the x-axis. In Panel a, $\mu = 0.1$ as in Figure 4a (note the convergence of all variables in the long-run to equilibrium E_L). In Panel b, $\mu = 0.5849$ as in Figure 14 (note the Neimark-Sacker bifurcation with the formation of an orbit that prevents the variables from stabilizing).

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Chapter 4: The effects of a green monetary policy on firms financing cost

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Abstract

The monetary policy operations of a Central Bank (CB) involve allocation decisions when purchasing assets and taking collateral. A green monetary policy aims to steer or tilt the allocation of assets and collateral towards low-carbon industries, to reduce the cost of capital for these sectors in comparison to high-carbon ones. Starting from a corporate bonds purchase program (e.g. CSPP) that follows a carbon-neutral monetary policy, we analyze how a shift in the CB portfolio allocation towards bonds issued by low-carbon companies can favor green firms in the market. Relying on optimal portfolio theory, we study how the CB might include the risk related to the environmental sustainability of firms in its balance sheet. In addition, we analyze the interactions between the neutral or green CB re-balancing policy and the evolutionary choice (i.e. by means of exponential replicator dynamics) of a population of firms that can decide to be green or not according to bonds borrowing cost.

Keywords: Monetary Policy; Optimal Portfolio Allocation; Environmental Economics; Interacting Agents; Evolutionary Dynamics.
JEL classification: E52, E58, G11, C61, C73, Q50.

Statement of joint authorship and acknowledgments

I confirm that this Chapter - "The effects of a green monetary policy on firms financing cost" - was jointly co-authored with Germana Giombini and Sebastian Ille. I contributed to 70% of this work.
I confirm that this Chapter is an original work in its entirety and that has not been published in any Journal.
This chapter was presented at the XXII Scientific Conference "(Re)discovering the Drivers of Economic Development" of AISSEC (Italian Association for the Study of Comparative Economic Systems), University of Chieti-Pescara (16-18 June 2022), and at the Workshop on Economic Science with Heterogeneous Interacting Agents 2022 (WEHIA22), University of Catania (22-24 June 2022).
I would like to thank the participants at these workshops for their valuable insights, Edgar J. Sanchez Carrera (University of Urbino) and Elmer Sterken (University of Groningen) for their precious teachings and suggestions.

1 Introduction

The core operations of a Central Bank (CB) include conducting monetary policy operations, managing foreign exchange reserves, and operating large value payment systems. These core operations, for which we use the shorthand of monetary policy operations, involve allocation decisions when purchasing assets and taking collateral, through the so-called 'eligibility criteria'.

The major CBs accept private sector papers (corporate bonds, bank bonds, and bank loans) for asset purchases and collateral, and this credit policy practice has been further intensified under quantitative easing after the global financial crisis. As for the European Central Bank (ECB), the largest items on the Eurosystem balance sheet are securities holdings under the Asset Purchases Program (APP), which was launched in October 2014, and loans to EU credit institutions as part of monetary policy operations. Since then, several Asset Purchase Programs (APPs) have been introduced, allowing the ECB to buy government bonds (PSPP), asset-backed securities (ABSPP) and covered bonds (CBPP3). On March 2016, the ECB announced its intention to start buying corporate bonds directly through the implementation of the corporate sector purchase program (CSPP) as an additional component of the APP (ECB 2016).

Figure 1 shows the ECB net APP purchases, by program.[1] In August 2022, the ECB corporate bond holdings from the CSPP and other collateral monetary policy operations were 344,558 mil. EUR, while the overall APP holdings were 3,262,730 mil. EUR.[2] Thus, around 10.5% of ECB balance sheet is private corporate bonds and, as long as reinvestments in these assets will continue, this amount is expected to remain stable in the next few years (ECB 2022a).

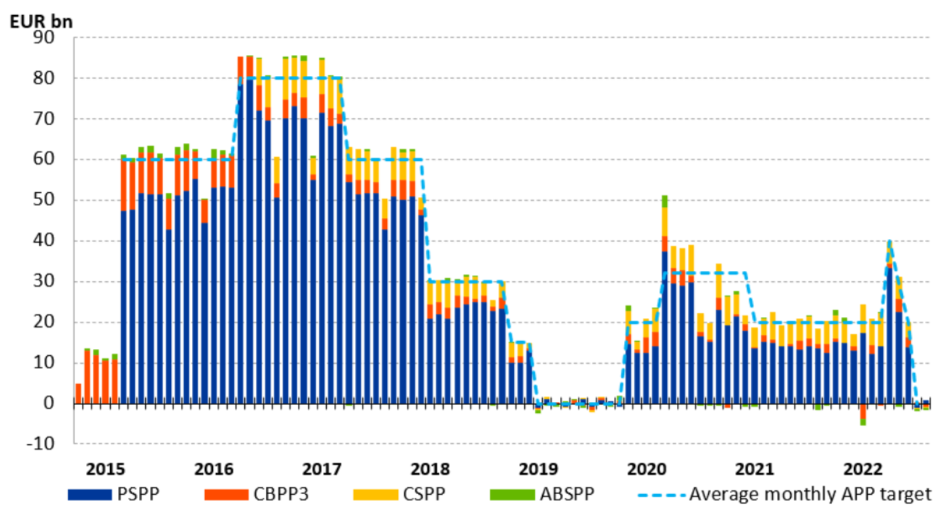
Analogously, the Bank of England (BoE) decided on a number of non-standard monetary policy measures, including the Corporate Bond Purchase Scheme (CBPS or the Scheme), which was launched in August 2016 and further expanded in 2020 (BoE 2021a).

[1] On 9 June 2022 the ECB Governing Council decided to discontinue net asset purchases under the APP as of 1 July 2022. Reinvestments of the principal payments from maturing securities purchased under the programs will continue, in full, for an extended period of time and as long as necessary to maintain ample liquidity conditions and an appropriate monetary policy stance (ECB 2022a).

[2] At amortized cost, in EUR millions, at month-end.

The Federal Reserve (FED), as well, established the Secondary Market Corporate Credit Facility (SMCCF) on March 23, 2020, to support credit to employers by providing liquidity to the market for outstanding corporate bonds (FED 2021).

Figure 1: ECB APP net purchases, by program



Note: The average monthly APP targets were first set by the ECB Governing Council at the beginning of the PSPP in March 2015. The additional envelope of €120 billion decided by the Governing Council on 12 March 2020 has been linearised for illustration in this chart, while it will be implemented in full according to the established principles

Source: ECB 2022a

Following these measures, a consistent part of the securities held in the CB portfolios has become bonds of private companies.
The aim of this chapter is to shed light on the mechanisms through which a CB can implement a green monetary policy to steer or tilt the allocation of assets and collateral towards low-carbon industries, and reduce the cost of capital for these sectors in comparison to high-carbon ones.
Starting from a corporate bonds purchase program that follows a carbon-neutral monetary policy, we analyze how a shift in the CB portfolio allocation towards bonds issued by low-carbon companies can favor green firms in the market. By means of a *'green monetary policy'* the CB internalizes externalities and public failures deriving from climate change through the inclusion of climate-related risks in the portfolio assessment.

The CB operates according to a market efficiency principle so that the optimal portfolio choice encompasses three objectives: obtaining high returns, containing risks, and reducing firms' environmental footprint.
Finally, we analyze the interactions between the neutral or green CB rebalancing policy and the evolutionary choice (i.e. by means of exponential replicator dynamics) of a population of firms that can decide to be green or not according to bonds borrowing cost.
We obtain some main findings.
First, some scenarios are characterized by a strong path dependency in which if a large share of firms employed non-green technology, no investment in green technology occurs in the long run, even if the non-green investment equilibrium is inefficient. We define this equilibrium *technology trap* and show that CSPP monetary policy helps the industry leave the technology trap. Second, green and non-green bond riskiness is a key factor that impacts borrowing costs. The larger the average financial risk of bonds, the lower the share of bonds in the CB portfolio, and the larger the cost. Third, the degree of market competition and of market (im)perfections contribute to amplifying the effects of the green monetary policy by affecting the transmission channel. In the presence of imperfect competition and (or) a high degree of market imperfections the *technology trap* is more likely to happen, and the green monetary policy seems to foster the adoption of green technologies.
The chapter is organized as follows.
Section 2 provides the institutional background and a short literature review on the issue of greening the monetary policy of central banks. Section 3 first analyses a *'neutral monetary policy'* based on modern portfolio theory (3.1), and then a *'green monetary policy'* by introducing a further CB objective based on the carbon intensity of firms (3.2). The section concludes with a numerical example of the results (3.3). Section 4 studies the interactions between the monetary policy strategy undertaken by the CB and the investment decision of a population of firms based on bond borrowing costs. Section 5 concludes.

2 Literature review and institutional background

Market neutrality has generally been the CB guiding principle of asset purchase programs:[3] the monetary authority buys a proportion of the market portfolio of available corporate and bank bonds (usually investment-grade bonds) to reduce price distortions from their eligible asset purchases[4]. However, this strategy might imply a carbon bias because capital-intensive companies and sectors tend to be more carbon-intensive (Papoutsi, Piazzesi, and Schneider 2021).

The existence of climate externalities requires a reconsideration of market neutrality. In the presence of market failures, adhering to the market neutrality principle may reinforce pre-existing inefficiencies that give rise to a suboptimal allocation of resources. If the market misprices the risks associated with climate change underestimating the social costs of investment, adhering to the market neutrality principle may instead support a market structure that hampers an efficient allocation of resources. In view of such market failures, a market efficiency principle would explicitly recognize that a supposedly 'neutral' market allocation may be suboptimal in the presence of externalities. Indeed, market failures may drive a wedge between market prices on the one hand and efficient asset values that internalize the externalities on the other (Schnabel 2021).

Corporate bond holdings expose CBs to different types of financial risk that might be related to climate change: extreme weather events such as wildfires or floods can hit companies' or their customers' premises and destroy their warehouses, manufacturing plants, data centers, and supply chains implying additional "physical risk" (Alogoskoufis et al. 2021). In addition, so-called transaction risks result from societal and economic shifts toward a low-carbon and more climate-friendly production model. Such shifts could mean that some sectors of the economy face significant transformations in asset values or higher costs of doing business that alter the value of investments held by banks and insurance companies (Gourdel et al. 2022).

[3] In the ECB case, the operationalization of this principle entails the monetary authority purchases securities in proportion to their relative market capitalization (Coeure' 2015).

[4] For example, the Bank of England's Corporate Bond Purchase Scheme (CBPS) follows a principle similar to market neutrality. The CBPS is conducted with the objective of minimizing the impact of asset purchases on the relative borrowing costs across sectors. The principle is implemented via sector key targets, with the potential for deviations (BoE 2021b).

For these reasons, some CBs have started to *greener* monetary policy operations to reduce the financial risk related to climate change and to promote a green transition of industries and firms.

On November 5, 2021, the Bank of England considered the climate impact of the issuers of bonds within the framework of the CBPS: "with this approach, we will incentivize firms to take decisive actions that support an orderly transition to net zero. Purchases will then be tilted or skewed within sectors towards the debt of eligible firms that are performing relatively strongly in support of net zero, and responding most to the incentives we are setting, and away from those who are not" (BoE 2021a, BoE 2021b).

As announced in July 2022, also the Eurosystem aims to gradually decarbonize its corporate bond holdings on a path aligned with the goals of the Paris Agreement. To that end, the ECB will tilt its purchases towards issuers with a better climate performance by reinvesting the sizeable redemptions expected over the coming years. The overall volume of corporate bond purchases will, however, continue to be determined solely by monetary policy considerations and the role played by such purchases in achieving the ECB's inflation target (ECB 2022b). The ECB has also announced that when government and corporate bonds come to maturity in the context of its QE program, new bonds will be bought in the market to keep the money stock (money base) unchanged. This creates a 'window of opportunities' for the ECB. It could replace the old bonds with new 'environmental bonds' over time to establish a well-diversified portfolio that also includes the value and the risk profile of climate change and carbon transition effects (Grauwe 2019).

Therefore, the objective of a green monetary policy is to steer or tilt the allocation of assets and collateral toward low-carbon sectors and firms. This could reduce the cost of capital for those companies and sectors in comparison to high-emission industries. The allocation policy must be designed and executed so that it does not interfere with the effective implementation of monetary policy and the transmission mechanism. Price stability is and should remain the top priority for central banks.

In this chapter, we fix the dimension of the corporate bonds purchase program (i.e. the overall CB demand of private bonds), and focus on the composition of the CB balance sheet between two typologies of corporate bonds: green and non-green bonds. We study how steering the CB eligibility criteria towards low-carbon bonds issued by environmentally friendly companies, following the market efficiency principle, can help the financing condition, favoring green companies in the market.

3 The Model

Equation (1) shows the total amount of corporate bonds in an economy, eligible[5] for a CB purchase program (B_T) given by green corporate bonds B_G issued by companies to finance environmentally sustainable projects, and non-green/conventional corporate bonds B_N issued by firms for investment that are not related to emission or pollution abatement technologies:

$$B_T = B_G + B_N \tag{1}$$

We define the share of green bonds $x = \frac{B_G}{B_T}$, and the complementary share of non-green bonds $1 - x = \frac{B_N}{B_T}$ in the economy.
For simplicity, we assume that the CB can identify the type of bond without ambiguity. While the assumption does not alter the conclusions of the chapter, it avoids dealing with various criteria that are often different for each type of institution and/or asset purchase program under consideration, since no international standard has been established yet (OECD 2017 and see for a taxonomy, Commission 2020)[6].
If green and conventional bonds were perfect substitutes for banks, production and investment in both sectors would not be affected (Ferrari and Landi 2021) after the CB tilts the portfolio composition towards green bonds and keeps the total assets constant. However, green and non-green bonds signal two different types of use of the financial resources and hence, are imperfect substitutes both for the issuing firms and for investors (Flammer 2021, Zerbib 2019, Gianfrate and Peri 2019). We, therefore, model both types using two distinct supply functions.
The aggregate supply of corporate green bonds in the market negatively depends on green bond yield: $B_G = f(\mu_G)$. Indeed, when the interest rate on this specific category of bonds (μ_G) increases, the firms' relative supply of bonds decreases because it becomes more costly for companies to finance sustainable-friendly projects through the issuance of green bonds.

[5] The bond and issuer eligibility conditions set forth by the European Central Bank can be found in ECB 2016, Zaghini 2019.

[6] The Eurosystem has developed a climate scoring methodology to assess the climate performance of eligible issuers that is based on three sub-scores: (i) backward-looking climate metrics, in the form of (disclosed) past GHG emissions and emission intensities (normalized by revenue); (ii) forward-looking climate metrics, such as whether the issuer has credible and ambitious decarbonization targets in place; and (iii) the quality of climate disclosures, such as their completeness and their verification by third parties. These metrics are based on publicly available data as well as other relevant information and methodologies, such as science-based targets, etc. (ECB 2022c).

The aggregate supply function is modeled by means of the unitary isoelastic function given by eq. (2a). Similarly, the green bond supply in terms of share $x(\mu_G)$ is given by eq. (2b), the inverse supply function $\mu_G(x)$ is (2c):

$$B_G(\mu_G) = \frac{\alpha}{\mu_G} \Leftrightarrow \tag{2a}$$

$$x(\mu_G) = \frac{\alpha}{\mu_G \, B_T} \Leftrightarrow \tag{2b}$$

$$\mu_G(x) = \frac{\alpha}{x \, B_T} \tag{2c}$$

Analogously, the aggregate supply of corporate non-green bonds in the market negatively depends on non-green bonds yield: $B_N = f(\mu_N)$. This aggregate supply function is unitary isoelastic and given by eq. (3a). The equivalent non-green bonds supply in terms of share $1 - x(\mu_N)$ is eq. (3b), as well as the inverse supply function $\mu_N(1-x)$ is (3c):

$$B_N(\mu_N) = \frac{\beta}{\mu_N} \Leftrightarrow \tag{3a}$$

$$1 - x(\mu_N) = \frac{\beta}{\mu_N \, B_T} \Leftrightarrow \tag{3b}$$

$$\mu_N(1-x) = \frac{\beta}{(1-x) \, B_T} \tag{3c}$$

By definition, the total amount of corporate bonds in the economy as well as the yield on bonds must be positive ($B_T, \mu_G, \mu_N > 0$), it follows from eqs. (2c) and (3c) that also $\alpha, \beta > 0$. The parameters α and β are scaling factors of the aggregate supplies of green and non-green bonds respectively, a proxy of the relative market size of the two types of bonds considered.

3.1 Neutral monetary policy

The total volume of corporate bonds purchased by the CB through a large-scale purchase program is only determined by monetary policy considerations, i.e. inflation targeting (Bacchiocchi and Giombini 2021), thus, we assume that the representative CB is the only corporate bonds investor in the economy and acquires the total amount of eligible bonds in the economy[7]. Therefore, we focus only on the relative composition (i.e green or non-green) of purchase program B_T and study the impact of a CB strategy

[7] This holds without loss of generality when there are no spillovers between the CB and other corporate bonds investors.

that includes environmental considerations (i.e. *green monetary policy*), to study the occurrence of portfolio re-balance and its effect on the cost of bonds for firms.

Based on modern portfolio theory (Bodie, Kane, and Marcus 2021), the CB considers the average expected yields of green μ_G and non-green bonds μ_N, their average volatility (i.e., the standard deviation of their returns), given respectively by $\sigma_G, \sigma_N > 0$, and the covariance between the two types of corporate bonds $\sigma_{G,N}$[8]. The covariance $\sigma_{G,N}$ is related to the correlation coefficient $r_{G,N} = \frac{\sigma_{G,N}}{\sigma_G\,\sigma_N}$, which, to be economically meaningful, must range between -1 (i.e. perfect negative correlation) and $+1$ (i.e. perfect positive correlation). Thus, it holds that:

$$-1 \leq \frac{\sigma_{G,N}}{\sigma_G\,\sigma_N} \leq 1 \tag{4}$$

According to the capital asset pricing model (CAPM), the CB portfolio expected yield $\mu_P(x)$ is a convex combination of the individual yields, where the weights are the share of green bonds $x \in (0,1)$ and non-green bonds $1-x$ (i.e. the complementary part) in the CB portfolio and in the market:

$$\mu_P(x) = x\,\mu_G + (1-x)\,\mu_N \tag{5}$$

Substituting the inverse supply functions of green (2c) and non-green bonds (3c) into eq. (5), and defining the CB portfolio's expected variance $\sigma_P^2(x)$, based on the volatility (i.e. standard deviation) $\sigma_i > 0, i = G, N$, and the covariance $\sigma_{G,N}$ of the individual type of bonds, we obtain:

$$\begin{cases} \mu_P(x) = \frac{\alpha}{B_T} + \frac{\beta}{B_T} \\ \sigma_P^2(x) = x^2\,\sigma_G^2 + (1-x)^2\;\sigma_N^2 + 2\,x\;(1-x)\;\sigma_{G,N} \end{cases} \tag{6}$$

The system of equations in (6) determines a tuple of points, i.e. the expected yield and expected variance of the portfolio, in relation to share x. It describes the mean-variance trade-off that the CB faces for all the possible combinations/allocations of green (x) and non-green $(1-x)$ bonds[9]. Consequently, corporate bonds come in a variety of risk-reward levels depending on the issuing company's creditworthiness. While the CB prefers assets that

[8] To use standard deviations we assume that returns are normally distributed and that the CB, as an investor, has access to sufficient information to evaluate these variables.

[9] The efficient frontier is the set of portfolios which satisfy the condition that no other portfolio exists with a higher expected return but with the same standard deviation of return (i.e., the risk).

have the highest expected return, it also seeks to minimize uncertainty about corporate bonds future returns. We assume that the CB chooses the combination of green and non-green bonds with the optimal risk-reward level, i.e. the portfolio allocation that offers the maximum return-to-risk ratio, i.e. the optimal portfolio x^* in the CAPM. The CB risk-averse preference function in a *neutral monetary policy* setup can be formalized as a capital allocation line defined by the following (7):

$$\mu_P(x) = r_F + S_P \, \sigma_P(x) \tag{7}$$

The CB maximizes the portfolio return $\mu_P(x)$ for a given portfolio risk $\sigma_P(x)$, where S_P is the Sharpe ratio or reward-to-risk ratio (Sharpe 1971), and $r_F \geq 0$ is the equivalent risk-free asset (i.e. the yield associated to a risk-free asset, for example, a short-term U.S. treasury bond). Equation (7) shows the trade-off between the expected portfolio return $\mu_P(x)$ and its volatility $\sigma_P(x)$ and thus defines the risk-aversion preference of the CB. The CB is willing to hold a riskier portfolio if and only if it guarantees a higher average return reflected in S_P. Therefore, the CB maximizes the reward-to-risk ratio S_P given the constraints in (6) by determining the share x that maximizes the Sharpe ratio of a portfolio that is on the envelope of the Markowitz bullet (Markowitz 1952):[10]

$$\max_{x} \quad S_P = \frac{\mu_P(x) - r_F}{\sigma_P(x)} \quad \text{s.t.} \qquad \text{constraints in (6)} \tag{8}$$

Note that $\mu_P(x) \geq r_F$ in (8) requires that:

$$\frac{\alpha + \beta}{B_T} \geq r_F \tag{9}$$

From the Sharpe ratio condition (8), it is also required that $\sigma_P^2(x) > 0$ in (6). It must therefore hold that:

$$\sigma_{G,N} > -\frac{x\sigma_G^2}{2(1-x)} - \frac{(1-x)\sigma_N^2}{2x} \tag{10}$$

[10]Graphically, the slope of the optimal set, the maximum Sharpe ratio, is such that it is tangent to the portfolio efficient frontier (Sharpe 1971).

The problem in (8) can be reduced to solving the unconstrained maximization problem

$$\max_{x} \quad \frac{\frac{\alpha}{B_T} + \frac{\beta}{B_T} - r_F}{\sqrt{x_G^2\, \sigma_G^2 + (1-x)^2\ \sigma_N^2 + 2\, x\ (1-x)\ \sigma_{G,N}}} \tag{11}$$

The solutions to problem (11) return the optimal shares of green and non-green corporate bonds in the CB portfolio and thus in the market, given by:

$$x^* = \frac{\sigma_N^2 - \sigma_{G,N}}{\sigma_G^2 + \sigma_N^2 - 2\,\sigma_{G,N}} \tag{12a}$$

$$1 - x^* = \frac{\sigma_G^2 - \sigma_{G,N}}{\sigma_G^2 + \sigma_N^2 - 2\,\sigma_{G,N}} \tag{12b}$$

From condition (4) and given that (12a), (12b) $\in (0,1)$, it must hold:

$$\sigma_N^2 > \sigma_{G,N} \tag{13a}$$

$$\sigma_G^2 > \sigma_{G,N} \tag{13b}$$

In the following, we define the derivatives of the optimal shares (12a), (12b) with respect to the model parameters:

$$\frac{\partial x^*}{\partial \sigma_N^2} = \frac{\sigma_G^2 - \sigma_{G,N}}{\left(\sigma_G^2 - 2\sigma_{G,N} + \sigma_N^2\right)^2} > 0 \tag{14a}$$

$$\frac{\partial x^*}{\partial \sigma_G^2} = \frac{\sigma_{G,N} - \sigma_N^2}{\left(\sigma_G^2 - 2\sigma_{G,N} + \sigma_N^2\right)^2} < 0 \tag{14b}$$

$$\frac{\partial x^*}{\partial \sigma_{G,N}} = \frac{\sigma_N^2 - \sigma_G^2}{\left(\sigma_G^2 - 2\sigma_{G,N} + \sigma_N^2\right)^2} \gtreqless 0 \tag{14c}$$

$$\frac{\partial^2 x^*}{\partial \sigma_N^2 \partial \sigma_G^2} = \frac{\sigma_N^2 - \sigma_G^2}{\left(\sigma_G^2 - 2\sigma_{G,N} + \sigma_N^2\right)^3} \gtreqless 0 \tag{14d}$$

$$\frac{\partial^2 x^*}{\partial \sigma_G^2 \partial \sigma_{G,N}} = \frac{2\sigma_{G,N} + \sigma_G^2 - 3\sigma_N^2}{\left(\sigma_G^2 - 2\sigma_{G,N} + \sigma_N^2\right)^3} \gtreqless 0 \tag{14e}$$

$$\frac{\partial^2 x^*}{\partial \sigma_N^2 \partial \sigma_{G,N}} = -\frac{2\sigma_{G,N} - 3\sigma_G^2 + \sigma_N^2}{\left(\sigma_G^2 - 2\sigma_{G,N} + \sigma_N^2\right)^3} \gtreqless 0 \tag{14f}$$

As expected, an increase of the variance (i.e. financial risk) reduces the optimal share of the correspondent corporate bond in the CB portfolio, while the effect of the covariance on x^* can be positive, negative, or null, depending on the difference between the two variances.
Given the optimal shares, it is possible to retrieve the optimal amount of green B_G^* and non-green bonds B_N^* in the market:

$$B_G^* = x^* B_T \tag{15a}$$

$$B_N^* = (1-x)^* B_T \tag{15b}$$

Substituting the optimal portfolio amount of green and non-green bonds into the aggregate inverse supply functions (2c) and (3c), provides the equilibrium bonds yields μ_G^* and μ_N^*:

$$\mu_G^* = \frac{\alpha}{B_G^*} = \frac{\alpha\,(\sigma_G^2 + \sigma_N^2 - 2\,\sigma_{G,N})}{B_T\,(\sigma_N^2 - \sigma_{G,N})} \tag{16a}$$

$$\mu_N^* = \frac{\beta}{B_N^*} = \frac{\beta\,(\sigma_G^2 + \sigma_N^2 - 2\,\sigma_{G,N})}{B_T\,(\sigma_G^2 - \sigma_{G,N})} \tag{16b}$$

These bond yields represent the cost of capital for each type of firm issuing the bond. Given eq. (16a), (16b), the monetary authority can reduce the yield/cost of capital for green companies and increase the yield/cost of capital for non-green firms by altering the composition x^* of its balance sheet without modify the latter's total dimension (B_T).

3.2 Green monetary policy

The existence of climate externalities, and physical and transitional risks related to climate change question market neutrality, as it could reinforce pre-existing inefficiencies that give rise to erroneous prices and suboptimal resources allocation. The objective of the *green monetary policy* is to internalize such externalities and risks to obtain an efficient allocation of financial resources that take into consideration climate related issues.
In other words, the CB desires to re-balance its portfolio to reduce the cost of capital for firms that invest in sustainable/green projects, while fixing, at the same time, the overall dimension of the balance sheet B_T.
By increasing the relative share x^* of green bonds, the CB reduces the borrowing cost for environmental sustainable firms and it renders more costly for companies to finance non-green investment projects. This *green monetary policy* should encourage firms to invest and shift to an environmental

sustainable production. We model the *green monetary policy* by introducing a steering or tilting factor (Schoenmaker 2021) that governs the CB's portfolio:

$$p = \frac{C_N}{C_G} \tag{17}$$

where $C_i, i = G, N$ is a synthetic indicator of the environmental footprint of the i-type issuer, e.g. the average carbon emissions and/or other environmental measures. Note that the average environmental footprint indicator of non-green issuers C_N is greater than the same indicator for green issuers C_G. This is consistent with studies such as Fatica, Panzica, and Rancan 2021, where green bond issued by non-financial corporations are associated with a reduction in firm-level carbon emissions induced by climate friendly investment projects.

Since the tilting factor p in eq. (17) is the ratio between the two footprint indicators, it always exceeds 1. Moreover, this ratio defines the extent of the greening monetary policy and accounts for the additional risks (physical, transitional) related to the carbon footprint of firms that issue corporate bonds to finance non-sustainable investment. Since these projects (linked to conventional bonds) are not green, they: (1) are more exposed to adverse climatic events and natural disasters that bring direct and indirect physical assets damages (e.g. business disruption, system failures, disruption of transportation facilities and telecommunications infrastructure, etc.), (2) are more vulnerable to an increasing legal and regulatory environmental-friendly framework where compliance risk as well as litigation and liability costs associated with climate-sensitive investments undermine business profitability, (3) become target of economic policy that demand a reduction in the use of fossil fuels and carbon emission (e.g. carbon tax) (Alogoskoufis et al. 2021, ECB/ESRB 2021).

The climate-related risks become relevant and are internalized via the CB corporate bond purchase program. As they affect the variance of the corresponding bonds (σ_N^2), we define a modified variance $\hat{\sigma_N}^2$ that considers beside the financial risk, these climate-related risks:

$$\hat{\sigma_N}^2 = p\ \sigma_N^2 \tag{18}$$

given that the tilting/steering factor $p > 1$, the overall risk of non-green corporate bonds increases[11]. In this way, the CB internalizes the externalities and public failures through the inclusion of climate-related risks in the

[11]Note that the case of *neutral monetary policy*, is obviously the special case in which $p = 1$.

portfolio assessment. Therefore, following the market efficiency principle, the optimal portfolio choice in a *green monetary policy* setting encompasses three objectives: obtaining high returns, containing risk/volatility, and reducing firms' environmental footprint, defined by:

$$\begin{gathered}\max_x s_P = \frac{\mu_P(x) - r_F}{\sigma_P(x)} \quad \text{s.t.} \\ \begin{cases} \mu_P(x) & = \frac{\alpha}{B_T} + \frac{\beta}{B_T} \\ \sigma_P^2(x) & = x^2\, \sigma_G^2 + (1-x)^2 \;\, \hat{\sigma_N}^2 + 2\, x\;\,(1-x)\;\, \sigma_{G,N} \end{cases}\end{gathered} \tag{19}$$

and the corresponding solutions in (12a) and (12b) with the substitution of $\hat{\sigma_N}^2$ in eq. (18).
Since

$$\frac{\partial x^*}{\partial p} = \frac{\sigma_N^2\left(\sigma_G^2 - \sigma_{G,N}\right)}{\left(\sigma_G^2 + p\,\sigma_N^2 - 2\,\sigma_{G,N}\right)^2} > 0 \tag{20}$$

from condition (13b), the CB optimal portfolio contains a higher share of green bonds x^* and a lower share of non-green bonds $1 - x^*$. The optimal amount of the two types of bonds B_G^* and B_N^* is given by eqs. (15a) and (15b), the bonds yields μ_G^* and μ_N^* are given by (16a) and (16b) after substituting $\hat{\sigma_N}^2$ in (18):

$$\mu_G^* = \frac{\alpha}{B_G^*} = \frac{\alpha\,(\sigma_G^2 + \hat{\sigma_N}^2 - 2\,\sigma_{G,N})}{B_T\,(\hat{\sigma_N}^2 - \sigma_{G,N})} \tag{21a}$$

$$\mu_N^* = \frac{\beta}{B_N^*} = \frac{\beta\,(\sigma_G^2 + \hat{\sigma_N}^2 - 2\,\sigma_{G,N})}{B_T\,(\sigma_G^2 - \sigma_{G,N})} \tag{21b}$$

The CB lowers the financing costs for environmentally sustainable firms and tightens the financing conditions of non-green companies, i.e. increasing the so-called green premium or *greenium* (E. Agliardi and R. Agliardi 2021, Caramichael and Rapp 2022), as

$$\begin{aligned}\frac{\partial \mu_G^*}{\partial p} &= \frac{\alpha\,\sigma_N^2\left(\sigma_{G,N} - \sigma_G^2\right)}{B_T\left(\sigma_{G,N} - p\,\sigma_N^2\right)^2} < 0 \\ \frac{\partial \mu_N^*}{\partial p} &= -\frac{\beta\,\sigma_N^2}{B_T\left(\sigma_{G,N} - \sigma_G^2\right)} > 0\end{aligned} \tag{22}$$

A short numerical example shows the impact of a *green monetary policy* CSPP undertaken by a representative CB. In the economy, a volume of eligible corporate bonds equal to $B_T = 140,000$ millions EUR or USD is acquired by the Central Bank through the CSPP. The scaling factors of the aggregate bonds supply are $\alpha = 2300$ for green bonds, and $\beta = 4000$ for non-green bonds. Furthermore, the CB can observe the yields trend to assess the financial risk related to these assets. The volatility, given by the standard deviation, of green bonds $\sigma_G = 0.20$ is higher than that of non-green bonds $\sigma_N = 0.15$, [12] and covariance between the two types of bonds is $\sigma_{G,N} = -0.002$, corresponding to a moderate negative correlation coefficient $r_{G,N} = -0.067$. The risk-free asset has a yield of $r_F = 0.02$. The assumptions satisfy conditions (4), (9), (10), (13), and Table 1 compares the optimal shares, amounts and yields of green and non-green bonds for a *neutral monetary policy* ($p = 1$) and for a *green monetary policy* ($p = 1.1$).

Table 1: Comparison between neutral and green monetary policy

Type of mon. pol. (p)	x^*	$1 - x^*$	B_G^*	B_N^*	μ_G^*	μ_N^*
Neutral ($p = 1$)	36.8%	63.2%	51,579	88,421	4.46%	4.52%
Green ($p = 1.1$)	40.9%	59.1%	54,473	85,527	4.22%	4.68%

Table 1 shows that if the tilting factor $p > 1$, that is, as long as the CB accounts for the additional risks related to the carbon footprint of firms that issue corporate bonds to finance non-sustainable investment, the financing conditions of green firms improve, *ceteris paribus*.
Figures 2, shows the impact of green σ_G, non-green bond σ_N variances, tilting factor p and covariance $\sigma_{G,N}$ on the optimal bond share x^* in the CB portfolio.
Figures 3 and 4 show the effect of the same variables on the equilibrium values of green μ_G^* and non-green μ_N^* bond yields.
These 3-dimensional graphs highlight how the variances and covariance of these assets affect non-linearly the impact of the *green monetary policy* on bond shares and rates.

[12] In this example, the environmental-friendly investment related to green bonds is on average riskier or more volatile than the conventional investment linked to non-green bonds because it makes use of relatively new and less consolidated technology.

Figure 2: Optimal share x^* changes on relevant parameters

(a) x^* changes on σ_G, σ_N (b) x^* changes on p, σ_G (c) x^* changes on p, $\sigma_{G,N}$

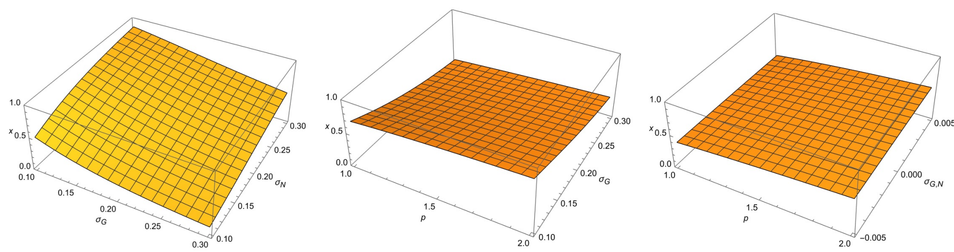


Figure 3: Optimal green bonds rate μ_G^* changes on relevant parameters

(a) μ_G^* changes on σ_G, σ_N (b) μ_G^* changes on p, σ_G (c) μ_G^* changes on p, $\sigma_{G,N}$

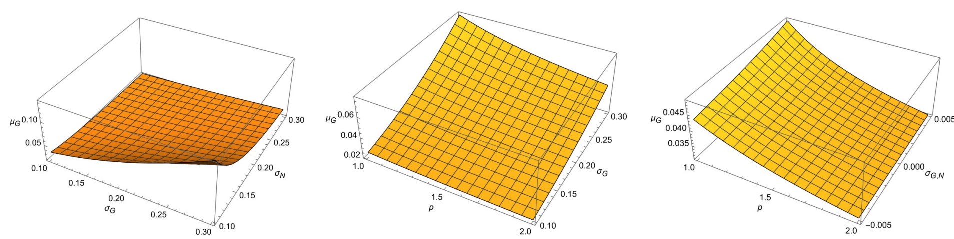
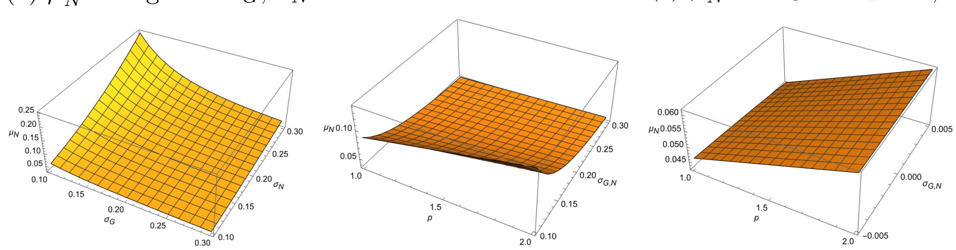


Figure 4: Optimal green bonds rate μ_N^* changes on relevant parameters

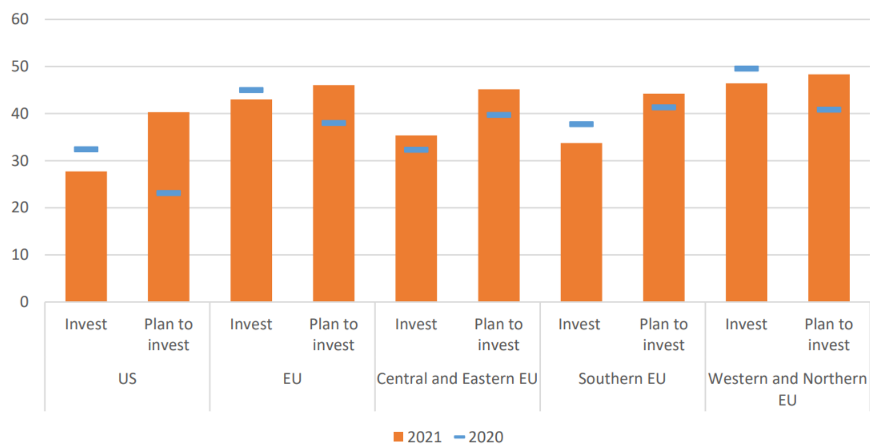
(a) μ_N^* changes on σ_G, σ_N (b) μ_N^* changes on p, σ_G (c) μ_N^* changes on p, $\sigma_{G,N}$



4 Green monetary policy and firm investment choice

In this section, we consider the interaction between monetary policies, i.e. neutral or green, and the investment choice of firms in a given sector.
The investment survey of the European Investment Bank (EIB) in Figure 5 shows that an increasing number of firms is investing in green/climate-related measures (EIB 2022)[13].

Figure 5: Firms (in %) investing or planning to invest in climate-related measures



Source: EIBIS 2021, EIBIS 2020
Note: The base is all firms (data not shown for those who said do not know/refused to answer)
Question: Has your company already invested to tackle the impacts of weather events and reduce carbon emissions?

Furthermore, Europe has also become a world leader in the issuance of green bonds. In late 2021, the volumes issued by companies and national and sub-national governments in the EU-27 reached €497 bn compared to a bond volume of non-European issuers at around €558 bn (Fatica and Panzica 2021).

[13]The share of firms investing in climate measures in 2021 is marginally below the share in 2020, which is likely the result of the repercussions the COVID-19 pandemic had on firms' investment plans. Overall, the share of EU firms investing in climate-related measures is significantly higher than in the United States, with companies in Western and Northern Europe leading the trend (EIB 2022).

Building on this evidence and similar to Pindyck 1988, 1991, we model the potential impact of a CSPP program on a population of firms that invests capital $C(t)$ in each period t. The population of firms belongs to an industry with two technologies of production: a green technology G and a non-green technology N. Consequently, the firms in the sector can invest capital $C(t)$ at every period t (e.g. every year) in either green/climate-related technology $G(t)$ (i.e. 'green investment') or in non-green technology $N(t)$ (i.e. 'non-green investment'). The share of green investment in the industry is $0 \leq y(t) = \frac{G(t)}{C(t)} \leq 1$ and the complementary share of non-green investment is $1 - y(t) = \frac{N(t)}{C(t)}$, assuming that the background growth rate of bond capital $r(t)$ is independent of the technology investment choice $i = G(t), N(t)$ at each time t.
We assume that firms make investment choices under limited information: firms do not know exactly what the return on investment of each technology will be and/or are not able to compute the optimal alternative following traditional profit maximization rules. In this case, the decision cannot be based on the expected return on investment as in a perfect information setting. Instead, firms imitate the investment behavior of other firms. More specifically, each company in the industry simply observes a small subset of other firms and replicates the investment strategy of the most successful ones.
Similar to Shaffer 1991, and Calcagnini, Gardini, et al. 2022, we assume that the firm investment on technology $i(t)$ earns a marginal return $MR_i(t)$:

$$MR_G(t) = a_G - b_G\, y(t) \tag{23a}$$
$$MR_N(t) = a_N - b_N\, [1 - y(t)] \tag{23b}$$

where the parameters $a_G, a_N, b_G, b_N > 0$ depend on the characteristics of the manufacturing technology i of the sector and are assumed to be constant in time[14]. The total earnings $E_i(t)$ from a given technology investment/adoption $i(t)$ are the integral of (23a),(23b) with respect to the correspondent investment, i.e.:

$$E_G(t) = a_G\, y(t) - \frac{b_G}{2}\, y(t)^2 \tag{24a}$$
$$E_N(t) = a_N\, [1 - y(t)] - \frac{b_N}{2}\, [1 - y(t)]^2 \tag{24b}$$

[14]For this reason we can refer to them as *structural parameters.*

Given a relatively small firm size, firms are price-taker in the bonds market. At each time t, firms can issue either a green bond at a constant interest rate μ_G^* to finance the investment in the green technology G, or they can issue non-green bonds at a constant interest rate μ_N^* to finance the investment in non-green technology N[15]. The cost of the two alternative types of bonds is determined by the portfolio optimization problem of the monetary authority in relation to its policy and defined by (21). For the sake of simplicity, both types of bonds have the same maturity. As a result, the borrowing cost of a firm is given by the principal amount to be reimbursed at maturity, which coincides with the value of the investment, and the (fixed) interest rate μ_G^* or μ_N^* on this debt,[16]

$$C_G(t) = y(t)\,(\mu_G^* + 1) \tag{25a}$$

$$C_N(t) = [1 - y(t)]\,(\mu_N^* + 1) \tag{25b}$$

Considering both the total earnings from the investment (24a), (24b) and the corporate bond cost (25a), (25b), we define the firms return on green investment $\pi_G(y)$ as a function of green investment in the industry at time t, and the firms return on non-green investment $\pi_N(1-y)$ as a function of non-green investment at time t[17],

$$\pi_G\,(y) = a_G\,y - \frac{b_G}{2}\,y^2 - (\mu_G^* + 1)\,y \tag{26a}$$

$$\pi_N\,(1-y) = a_N\,\,(1-y) - \frac{b_N}{2}\,\,(1-y)^2 - (\mu_N^* + 1)\,\,(1-y) \tag{26b}$$

The CB corporate bonds purchase program can follow the *neutral monetary policy* or the *green monetary policy* framework. The type of program affects the relative bonds' cost μ_G^* and μ_N^* (in eqs. (21a),(21b)), and therefore, the firms' decisions to invest in environmental-friendly technology.

The decision of the firms to invest in the green technology $y \in [0, 1]$ is assumed to evolve in discrete time, according to an exponential replicator dynamics R, as in Cabrales and Sobel 1992, Bacchiocchi and Bischi 2022:

$$y\,(t+1) = f\,(y\,(t)) = (1-\eta)\,\,y\,(t) + \eta\,\frac{y\,(t)}{y\,(t) + (1 - y\,(t))\,\,e^{-\gamma\,g(y(t))}} \tag{27}$$

[15] Here we do not consider the phenomenon of green-washing, in which some firms issue green bonds to bear a lower financing cost employing the proceeds in non-green investment.

[16] Since the maturity of green and non-green corporate bonds is the same, it is sufficient to compare firm' borrowing cost in only one period of time.

[17] For sake of brevity we omit t in eqs. (26a), (26b).

The dynamic model (27) describes the time evolution of the green investment by introducing adaptive adjustments based on a direct comparison of the expected firm's return on investment:

$$g\left(y\left(t\right)\right)=\pi_{G}\left(y\left(t\right)\right)-\pi_{N}\left(1-y\left(t\right)\right) \tag{28}$$

According to (27) and (28), at each discrete time t, the share of green investment y increases (decreases) in $t+1$ when a firm's return in green investments is expected to be higher (lower) than the return on non-green investments. The parameter $\gamma>0$ represents the speed of technology adoption and expresses the firms' ability and propensity to switch to the alternative manufacturing technology as a profit gain is observed in the current time period. The velocity of technology adoption is strictly related to adjustment costs and the irreversibility of investment[18], and a low value of γ indicates a slow speed of adoption. Equation (27) also captures the level of inertia as a consequence of the degree of competitiveness between firms, measured by the parameter $0\leq\eta\leq1$. For $\eta\to0$ the firms of the industry have the highest degree of inertia. In this case, investment choices do not change over time, since $y(t+1)=y(t)=y(0)$; while for $\eta\to1$, no anchoring exits since a firm's survival critically depends on quickly adopting the most profitable technology of production, i.e. $y(t)\to1$ if $g(y)>0$ and $y(t)\to0$ if $g(y)<0$.

4.1 Analysis

Since $y(0)\in[0;1]$ then $y(t)\in[0;1]$ for each $t\geq0$, as it follows from the inequality $0\leq\frac{y}{y+(1-y)\,e^{-\gamma\,g(y)}}\leq1$. Additionally, it is straightforward to see that two pure fixed points exist at $y^{*}=0$ and $y^{*}=1$ (i.e. *pure equilibria*), where "all firms invest in non-green technology N" and "all firms invest in green technology G", respectively. The interior fixed points (i.e. *mixed equilibria*) are then given by the solution to $g(y^{*})=0$ in (28).

[18]It is determined by whether once installed capital has little or no value unless used in production (Bertola 1998), its industry or firm-specificity (Pindyck 1991), and as a consequence its intangibility, the difficulty of re-employment, market imperfections (Calcagnini, Giombini, and Travaglini 2019).

Solving for $\pi_G = \pi_N$ with respect to y, we obtain the position of the interior fixed points:[19]

$$y^*_{1,2} = \frac{c \pm \sqrt{c^2 - 4d\left(1 - a_N + \frac{b_N}{2} + \mu^*_N\right)}}{2d} \tag{29a}$$

$$\text{where } c = 2 - a_G - a_N + b_N + \mu^*_G + \mu^*_N \tag{29b}$$

$$d = \frac{1}{2}\left(b_N - b_G\right) \tag{29c}$$

where μ^*_G and μ^*_N are given by (21a) and (21b) respectively.

Two interior fixed points exist if and only if $0 < y^*_{1,2} < 1$ and the discriminant $\Delta = c^2 - 4d\left(1 - a_N + \frac{b_N}{2} + \mu^*_N\right) > 0$.
The asymptotic stability of the fixed points in discrete time is given by the following condition: $-1 < R'(y^*) < 1$, where $R'(y^*)$ is the derivative of (27) at fixed point y^* [20]. The derivatives $R'(y^*)$ at each of the four fixed points are:

$$R'(0) = 1 - \eta\left(1 - e^{\gamma\left(1 - a_N + \frac{b_N}{2} + \mu^*_N\right)}\right) \tag{30}$$

$$R'(1) = 1 - \eta\left(1 - e^{\gamma\left(1 - a_G + \frac{b_G}{2} + \mu^*_G\right)}\right) \tag{31}$$

$$R'(y^*_1) = 1 - \frac{\gamma\eta r(r - c)(c - 2d - r)}{4d^2} \tag{32a}$$

$$\text{with } r = \sqrt{\left(b_G - b_N\right)\left(2 - 2a_N + b_N + \mu_N\right) + c^2} \tag{32b}$$

$$R'(y^*_2) = 1 - \frac{\gamma\eta r(r + c)(c - 2d + r)}{4d^2} \tag{33a}$$

[19]Since (26a) and (26b) are second-degree polynomials, only none, one or two interior fixed points exist.

[20]The stability condition includes both an upper and a lower threshold for the slope of the non-linear function R at the equilibrium point, and the two limiting values -1 and $+1$ constitute two different conditions of non-hyperbolicity of the fixed point. When the condition of non-hyperbolicity $R'(y^*) = 1$ is crossed, as parameters vary, potentially three bifurcations can occur: fold, transcritical (or stability exchange), and pitchfork bifurcation. The bifurcation occurring at $R'(y^*) = -1$ is denoted as flip, at which the fixed point changes its oscillatory stability (i.e. convergence through damped oscillations) into oscillatory instability (i.e. trajectories starting close to y^* exhibit oscillatory expansion).

where μ_G^* and μ_N^* are given by (21a) and (21b) respectively.

Given the complexity of the derivatives, we cannot derive analytical conditions in terms of the model parameters. We, therefore, explore numerically the dynamical proprieties of the system (27) when parameters change to infer relevant economic implications.
In particular, we will define four scenarios with at least one internal fixed point [21] for different values of the *structural parameters* that define the characteristic of the manufacturing technology $i = G, N$ of the industry: a_G, a_N, b_G, b_N. We take the parameter values in Table 1 as a benchmark case for a *neutral monetary policy* setting and investigate how a change in p influences the share of green and non-green investment in the industry.

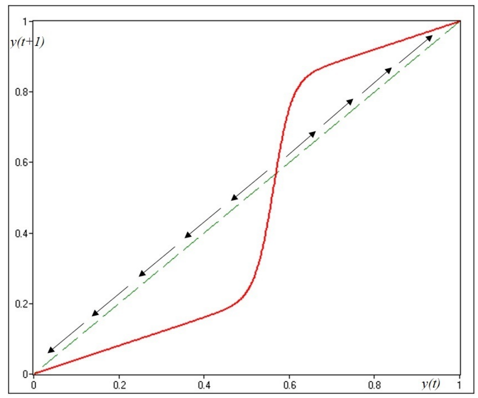
4.2 Unstable internal equilibrium and path dependency

We start with the easiest scenario in which one internal unstable fixed point exists at $y_1^* = 0.569$ ($R'(y_1^*) = 7.30$). The pure equilibria at $y^* = 0$, ($R'(0) = 0.40$) and $y^* = 1$ ($R'(1) = 0.40$) are stable. The time series plot in Figure 6a shows that the interior equilibrium is a separatrix and defines the basins of attraction of the two attracting pure equilibria. Starting from the initial condition (*i.c.*) 0.56 at which 56% of the investment in the industry is in green technology and the remainder of 44% are in the conventional non-green technology, the time series in red, given by (27), converges to $y^* = 0$, i.e. all the firms of the sector eventually invest in non-green technology in the long-run. This holds for all $i.c. < y_1^*$ as highlighted by the arrows in the phase plot of Figure 6b. For all $i.c. > y_1^*$ (such as $i.c. = 0.57$ of the blue time series in Fig. 6a), R converges to $y^* = 1$, i.e. all the companies invest in green technology after a certain period of time t.

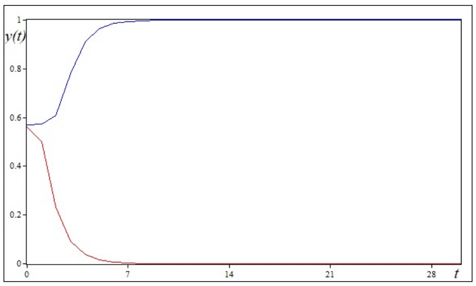
[21] We will ignore those scenarios in which only *pure equilibria* exist, i.e. firms invest fully in green or non-green technology independently from the starting conditions.

Figure 6: Scenario of unstable internal equilibrium $y_1^* = 0.569$

(b) Panel B: phase plot



(a) Panel A: time series

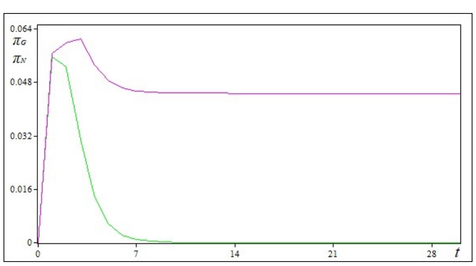


Parameters: $a_G = 1.2, a_N = 1.24, b_G = 0.2, b_N = 0.3, \eta = 0.6, \gamma = 400, \alpha = 2300, \beta = 4000, BT = 140000, \sigma_G = 0.2, \sigma_N = 0.15, \sigma_{G,N} = -0.002, p = 1, r_F = 0.02$. In panel A, for the red time series the initial condition ($i.c.$) is 0.56, and for the blue time series is $i.c. = 0.57$.

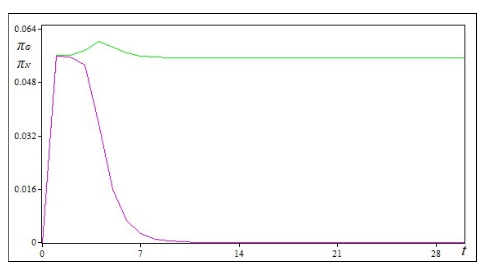
In the former case, in Fig. 7a, profits from green investment $\pi_G = 0$, while the non-green investment generates an equilibrium profit $\pi_N = 0.046$. Fig. 7b shows the latter case in which green investment leads to a profit $\pi_G = 0.055$, and non-green profits are $\pi_N = 0$ in the long-run.

Figure 7: Profits' evolution

(a) Panel A: Convergence to $y^* = 0$



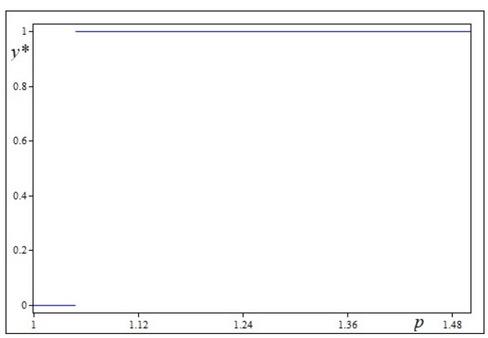
(b) Panel B: Convergence to $y^* = 1$



Parameters: same parameters of Figure 6. In panel A $i.c. = 0.56$, in panel B $i.c. = 0.57$. The green curve represents green profit π_G, the pink curve non-green profit π_N.

This scenario is characterized by a strong path dependency: if a large share of firms employed non-green technology, no investment in green technology occurs in the long-run, while if a critical share of the firms invests in green technology, eventually the entire firm population will adopt the latter technology. Furthermore, note that the all non-green investment equilibrium is Pareto sub-optimal in terms of profits compared to the all green investment equilibrium (i.e. $0.046 < 0.055$). This constitutes a *technology trap*, where all the firms in the sector are stuck with a sub-optimal choice.
CSPP monetary policy can be used to help the industry leave *technology trap*. This is demonstrated by the bifurcation diagram[22] for parameter p of Figure 8. In the previous scenario, the CB ran a neutral monetary policy (i.e. $p = 1$). By increasing p, the monetary authority moves towards a green monetary policy reducing the cost of corporate green bonds. Consequently, increasing p shifts the internal equilibrium and increases the basin of attraction of the full green investment. At $i.c. = 0.56$, a value of $p = 1.04$ leads to a convergence towards the all green investment equilibrium. For higher p values, lower initial conditions converge to the same equilibrium. In other words, the parameter p acts on the value of the unstable internal equilibrium y_1^*, which represents a threshold/separator between the two asymptotically stable pure equilibria: increasing p reduces y_1^*.

Figure 8: Bifurcation diagram for p



Parameters: same parameters of Figure 6. The $i.c. = 0.56$.

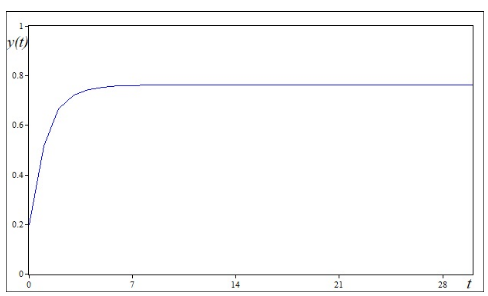
[22] In dynamical systems, a bifurcation diagram shows the values visited or approached asymptotically (fixed points, periodic orbits, or chaotic attractors) of a system as a function of a bifurcation parameter in the system.

4.3 Stable internal equilibrium and transition to deterministic chaos

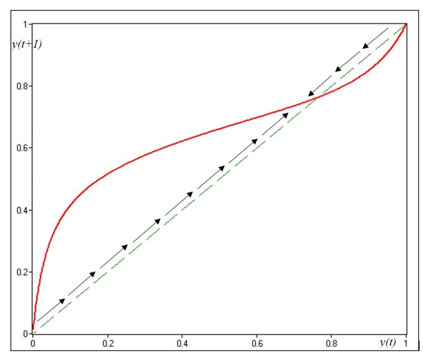
We consider the case with only one internal equilibrium $y_1^* = 0.763$, which is stable ($R'(y_1^*) = 0.47$). The two pure equilibria are unstable ($R'(0) = 12.60, R'(1) = 2.45$). Figure 9 highlights the evolution in time (9a) of the green investment share starting from $i.c. = 0.2$. The firm population converges to $y_1^* = 0.76$ (i.e. 76% green technology, 24% non-green technology adoption in the sector). In this case, y_1^* is the unique global attractor of the system and is reached for every $0 < i.c. < 1$ (Fig. 9b).

Figure 9: Scenario of unique stable equilibrium $y_1^* = 0.76$

(a) Panel A: time series



(b) Panel B: phase plot



Parameters: $a_G = 1.22, a_N = 1.16, b_G = 0.4, b_N = 0.35, \eta = 0.6, \gamma = 50, \alpha = 2300, \beta = 4000, BT = 140000, \sigma_G = 0.2, \sigma_N = 0.15, \sigma_{G,N} = -0.002, p = 1, r_F = 0.02$. The $i.c.$ is 0.2.

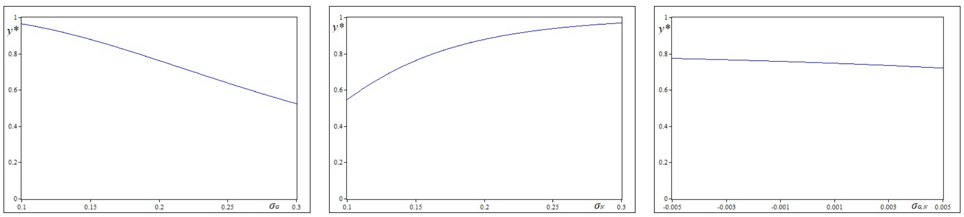
Figure 10 presents the bifurcation diagrams for the standard deviations (i.e. proxy of the financial risk) of green bonds (Fig. 10a), non-green bonds (Fig. 10b), and the covariance between the two typologies of bonds (Fig. 10c)[23]. An increase in the average financial risk of green bonds σ_G translates into a lower share of these assets in the CB portfolio, and it leads to a rise in the cost of borrowing for these firms. Consequently, the share of green investment gradually falls at the equilibrium (Fig. 10a). The opposite holds for an increase of average financial risk of non-green bonds σ_N as shown in Fig. 10b. The share of green investment rises and the share of non-green in-

[23]The range of variation of the parameters in this Figure and in all the subsequent bifurcation diagrams is subject to conditions in eqs. (4), (9), (10), (13).

vestment falls. Lastly, increasing the covariance $\sigma_{G,N}$ from a negative value (correlation) to a positive (correlation) mildly decreases the share of green investment at the equilibrium (Fig. 10c).

Figure 10: Bifurcation diagrams for variances

(a) Panel A: bif. diag. σ_G (b) Panel B: bif. diag. σ_N (c) Panel C: bif. diag. $\sigma_{G,N}$



Parameters: same parameters and *i.c.* of Figure 9.

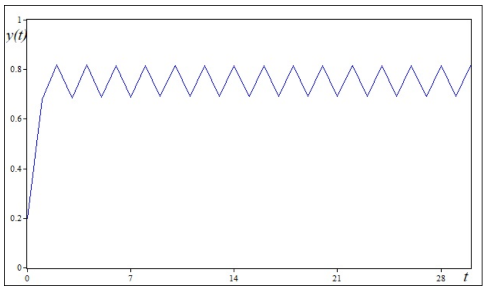
Note that Figure 9 is obtained given a low speed of technology adoption in the industry: $\gamma = 50$. Starting with the same parameter values and initial condition, Figure 11 demonstrates that an increase in the speed of technology adoption ($\gamma = 200$) causes systemic instability. The firm population periodically shifts between $y = 0.69$ and $y = 0.82$ as shown in Figure 11a, and the phase plot[24] of Figure 11b. In economic terms, the population of firms adopts a technology more rapidly than in the earlier scenario, which creates a periodic adaptation of the other technology as firms choose another technology in each period[25].

[24] The phase plot shows that the point where the system (in red) intercepts the bisector is the same. However, the increase of γ warps R, lowering the point derivative at the previous equilibrium to less than -1. The system undergoes a flip bifurcation.

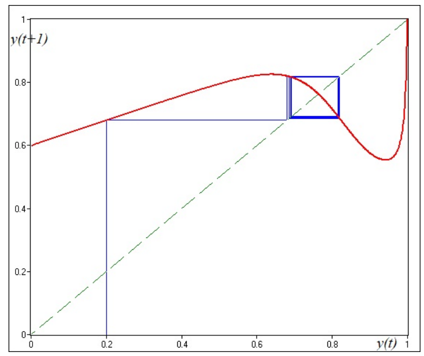
[25] This is caused by a periodic shift in the profits associated with each technology. While not shown here, we demonstrate this for the following scenario.

Figure 11: Convergence towards a cycle-2 period

(a) Panel A: time series



(b) Panel B: phase plot



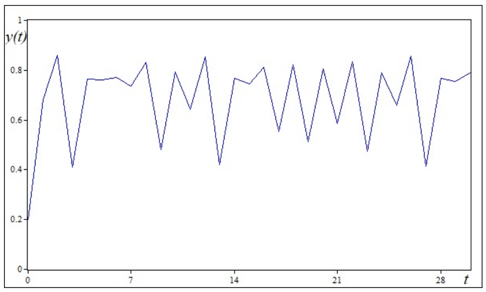
Parameters: same parameters and *i.c.* of Figure 9, except for $\gamma = 200$.

Further increasing γ to 400 leads to the creation of a region of deterministic chaos[26] (Fig. 12). In this specific case, the time evolution of the green investment share is erratic (Fig. 12a).
The economic consequence of such erratic motion is a low level of predictability regarding the manufacturing technology adopted in the industry. The only information at disposal of policymakers is that trajectories are bounded into a finite trapping region. However, the trapping region where the chaotic motion is confined may cover a large area of the phase space, as it occurs e.g. in the situation shown in Figure 12b from around $y = 0.4.$ to $y = 0.85$. Such uncertainty may disappear acting on the primary cause of the chaotic behavior, namely the excessive speed of technology adoption. From this point of view, a certain value of inertia in switching investment decisions, even if it is sub-optimal in terms of firms' decisions due to the delay toward more efficient inputs reallocation, could be beneficial at a collective level and for policymakers as it avoids such erratic time patterns (Fig. 12a).

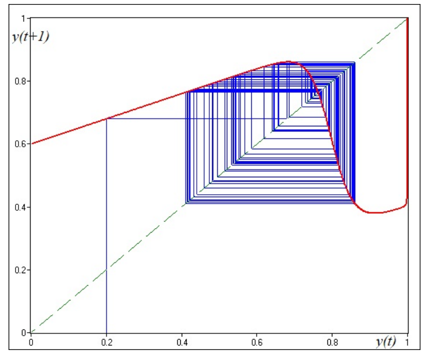
[26] The chaotic attractor characterizes a system that is sensitive dependent on initial conditions (see e.g. Devaney 1986, Lorenz 1989, Medio and Lines 2001).

Figure 12: Convergence towards a deterministic chaos region

(a) Panel A: time series



(b) Panel B: phase plot



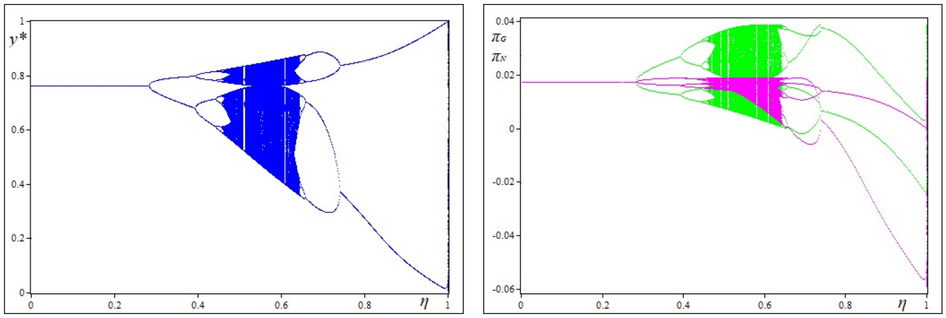
Parameters: same parameters and *i.c.* of Figure 9, except for $\gamma = 400$.

The population dynamics should be similarly affected by the degree of market competition. Figure 13a plots the bifurcation diagram for various values of η. For low values of η, and thus low market competition, the firm population converges to a single interior equilibrium. Bifurcations occur at higher values eventually leading to chaotic behavior for values of η exceeding 0.5. Interestingly, in highly competitive markets we observe non-chaotic but periodic behavior, which is defined by a periodic shift between two equilibria. Consequently, investment in green technology is only chaotic in imperfectly competitive markets. Figure 13b shows the corresponding average profits for both technologies. We can see that the periodic shifts and the chaotic behavior at higher η are caused by initially periodic and then chaotic shifts in the firm profits associated with each technology. At very high levels of competition, profits periodically shift between two values for each technology, rendering green investment more profitable in the current period and non-green investment more profitable in the next.

Figure 13: Bifurcation diagrams for η

(a) Panel A: bif. diagram η

(b) Panel B: bif. diagram of profits for η

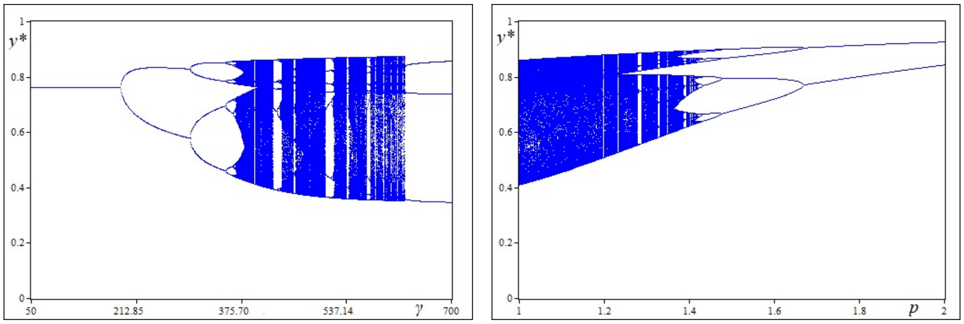


Parameters: same parameters and *i.c.* of Figure 9, except for $\gamma = 400$. In panel B, the green curve represents green profit π_G, the pink curve non-green profit π_N.

Figure 14a shows the bifurcation diagram for different rates of technology adoption γ. Here, we observe an effect similar to higher levels of competition. The system bifurcates as adoption rates increase (for $\gamma \approx 200$), eventually leading to chaotic behavior at $\gamma = 400$ as demonstrated in Fig. 12. Very high rates of technology adoption eventually also lead to periodic behavior, but here the firm population periodically shifts between three equilibria.
Similar to the previous scenario, a green monetary policy can stabilize investment decisions. Figure 13b shows the impact of p given the neutral monetary scenario in Figure 12. Values of p exceeding 1.4 stabilize technology adoption and eventually lead to periodic shifts.

Figure 14: Bifurcation diagrams for relevant parameters

(a) Panel A: bif. diagram γ (b) Panel B: bif. diagram p



Parameters: same parameters and $i.c.$ of Figure 9, except for $\gamma = 400$.

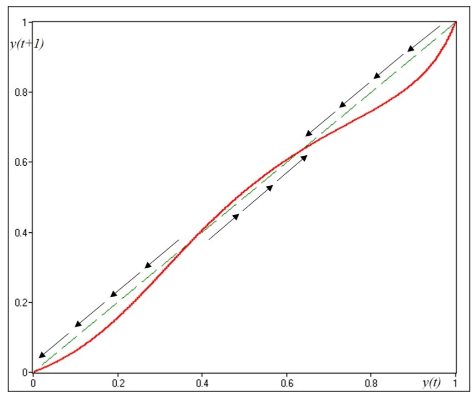
4.4 Two internal equilibria (unstable and stable)

Figure 15 shows the case of two internal equilibria: the first $y_1^* = 0.377$ is unstable ($R'(y_1^*) = 1.25$), the second $y_2^* = 0.631$ is stable ($R'(y_2^*) = 0.75$), and correspondingly equilibrium $y^* = 0$ is stable ($R'(0) = 0.47$) and $y^* = 1$ is unstable ($R'(1) = 3.08$). Figure 15a shows two time series: the red starts from $i.c. = 0.2$ and converges quite rapidly to the equilibrium of full non-green investment $y^* = 0$, whereas the blue starts from $i.c. = 0.5$ and converges after a relatively longer period of time to the mixed (or internal) stable equilibrium $y_2^* = 0.63$ where 63% of the firms in the industry employ green technology. The corresponding phase plot is given in Figure 15b, showing the path dependency of the system. A critical share of at least 37.7% of firms adopting green technology is needed to converge to the upper equilibrium. Any initial condition with fewer firms will remain trapped at the lower equilibrium at which no firm adopts a green technology. Figures 15c and 15d illustrate the firm profits if the population converges to the low or high stable equilibrium, respectively. The low equilibrium of full non-green investment is characterized by null green profits ($\pi_G = 0$) and non-green profits π_N equal to 0.006. The high equilibrium is an interior fixed point and thus, as illustrated in section 4.1, characterized by the same profits' value.

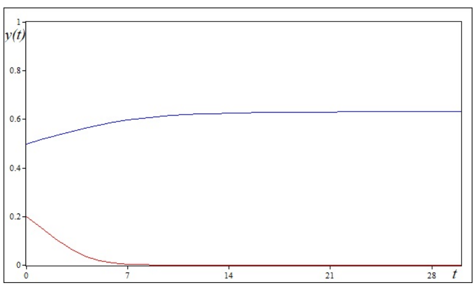
In particular, both green and non-green investments earn 0.047, an amount significantly superior to the panel C scenario. Consequently, the low equilibrium is Pareto inefficient and constitutes a *technology trap*.

Figure 15: Scenario of two stable equilibria $y^* = 0$, $y_2^* = 0.63$

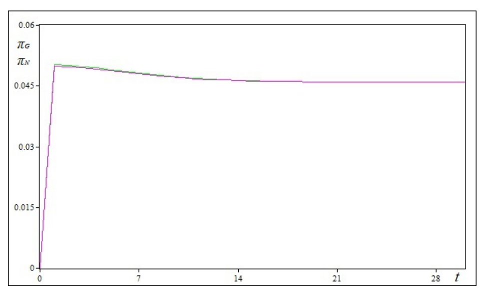
(b) Panel B: phase plot



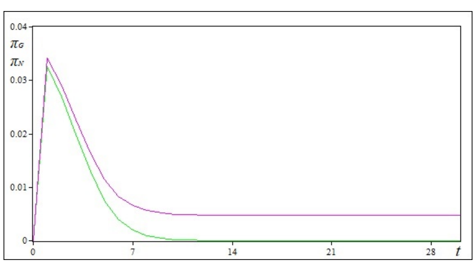
(a) Panel A: time series



(d) Panel D: profits high equilibrium



(c) Panel C: profits low equilibrium

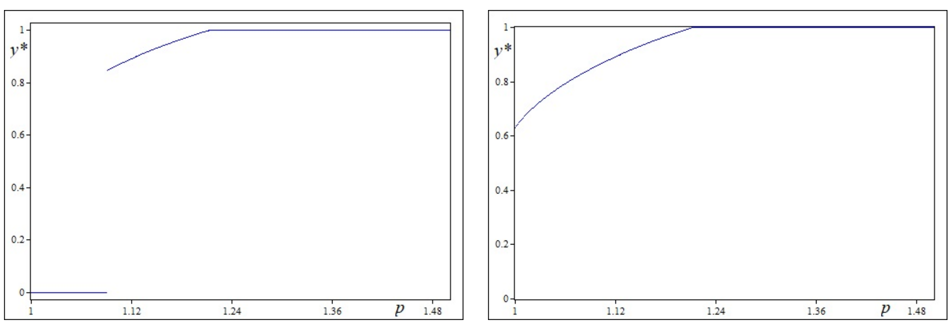


Parameters: $a_G = 1.25, a_N = 1.24, b_G = 0.42, b_N = 0.38, \eta = 0.7, \gamma = 300, \alpha = 2300, \beta = 4000, BT = 140000, \sigma_G = 0.2, \sigma_N = 0.15, \sigma_{G,N} = -0.002, p = 1, r_F = 0.02$. In panel A, for the red time series the initial condition (*i.c.*) is 0.2, and for the blue time series is *i.c.* $= 0.5$. In panel C *i.c.* $= 0.2$, in panel D *i.c.* $= 0.5$. The green curve represents green profit π_G, the pink curve non-green profit π_N.

A green policy by the CB can then help escape this trap as highlighted in Figure 16. The bifurcation diagram of Fig. 16a corresponds to the case of the red time series in Fig. 15a. Indeed, for a *neutral monetary policy* ($p = 1$) the equilibrium value is $y^* = 0$. A *green monetary policy* that progressively augments p causes the firm population to escape the trap. At $p \approx 1.10$, the population shifts from the low to the high equilibrium. To a lesser extent, the beneficial effect can also be observed if the firm population has a critical number of firms, which initially adopt green technology. However, increasing p does not lead to a shift between the equilibria, but a higher equilibrium value of the higher fixed point. Fig. 16b shows the situation for an initial condition of 0.5, where the industry is already on the socially optimal equilibrium. Here, moving to a *green monetary policy* ($1 < p < 1.21$) increases the initial mixed equilibrium value from $y_2^* = 0.63$ to $y^* = 1$ for $p > 1.21$.

Figure 16: Bifurcation diagram for p

(a) Panel A: $i.c. = 0.2$ (b) Panel B: $i.c. = 0.5$



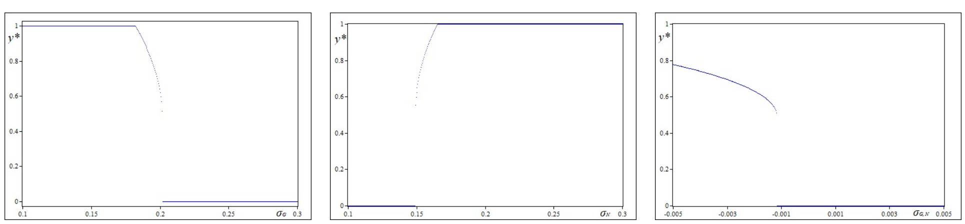
Parameters: same parameters of Figure 15.

Finally, let us take a look at the bifurcation diagrams for the variances in Figure 17 by referring to the situation of Fig. 15a with $i.c. = 0.5$.
In Fig. 17a for low values of green bonds standard deviation σ_G (i.e. proxy of the financial risk), firms face low borrowing cost. This makes relatively more profitable green technology and the entire industry opts for this investment strategy $y^* = 1$. Increasing green bonds risk rises interest rates and financing costs for firms, so we assist to a small window of σ_G where the mixed equilibrium is reached (both technologies are used), while for $\sigma_G > 0.21$ green bonds interest rate becomes too high and the entire sector chooses to

implement the conventional/non-green technology $y^* = 0$. The exact same qualitative behavior, but in an opposite direction, happens in Fig. 17b for σ_N. A strong increase in the financial risk of non-green bonds boosts their cost and pushes all firms in the sector to embrace the more profitable green technology $y^* = 1$. The last panel 17c depicts the bifurcation diagram for the covariance $\sigma_{G,N}$. For negative values from -0.005 to around -0.001 (i.e. negative correlation), we see a mild reduction of the mixed equilibrium. For $\sigma_{G,N} > -0.001$ a sharp jump occurs that leads the system to reach the full non-green investment equilibrium $y^* = 0$. When green bonds risk is higher than its non-green counterpart (i.e. in this case $\sigma_G = 0.2 > \sigma_N = 0.15$), an uncorrelated or positively correlated relationship between bonds reduces for the CB the benefits of portfolio diversification. Since it is no longer possible to mitigate the system risk (i.e. portfolio variance σ_P^2) acquiring both types of bonds, the CB gradually reduces the demand for the riskier green bonds. As a result, the higher share of non-green bonds in the CB portfolio decreases the cost of the latter and rises the yield of the green assets. At a certain point (e.g. in this example for $\sigma_{G,N} = -0.001$), non-green bonds become cheaper for all the firms in the industry, and the final choice is to employ the conventional technology.

Figure 17: Bifurcation diagrams for variances

(a) Panel A: bif. diag. σ_G (b) Panel B: bif. diag. σ_N (c) Panel C: bif. diag. $\sigma_{G,N}$



Parameters: same parameters of Figure 15. The *i.c.* is 0.5.

4.5 Two internal equilibria (stable and unstable)

The last relevant scenario is characterized again by two internal equilibria, but with opposite stability properties: the lower interior equilibrium $y_1^* = 0.181$ is stable ($R'(y_1^*) = 0.44$), the second interior equilibrium $y_2^* = 0.480$ is unstable ($R'(y_2^*) = 1.95$), while $y^* = 0$ is unstable ($R'(0) = 3.67$) and $y^* = 1$ is stable ($R'(1) = 0.30$). The scenario is depicted in Figure 18.

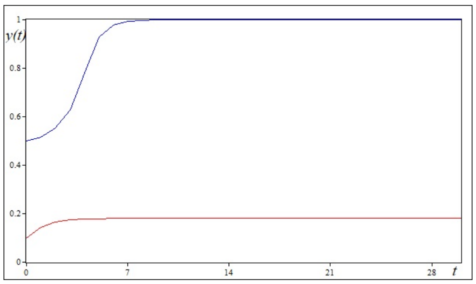
In Figure 18a the red time series starts from $i.c. = 0.1$ and converges quite rapidly to the lower mixed equilibrium $y_1^* = 0.18$, whereas the blue starts from $i.c. = 0.5$ and approaches, after a relatively long period of time, to the equilibrium of full green investment $y^* = 1$. The internal unstable equilibrium $y_2^* = 0.48$, the threshold between the two basins of attractions, is shown in Figure 18b. As previously stressed, the path dependence phenomenon can be better visualized from the phase plot, where for all $i.c. < y_2^*$ the mixed eq. $y_1^* = 0.18$ is reached, while for all $i.c. > y_2^*$ the pure eq. $y^* = 1$ is attained in the long run.

The possibility of having two fixed points depending on the initial state of the industry translates into different profit evolution. In Fig. 18c, the firm population converges to the lower mixed equilibrium. Profits for both technologies are equal at 0.027. In Fig. 18d, the population eventually only adopts green technology [27]. Green profits π_G converge to the same value at 0.027. In this particular scenario, no Pareto inefficient allocation occurs and the green monetary policy of the CB is ineffective from the efficiency point of view.

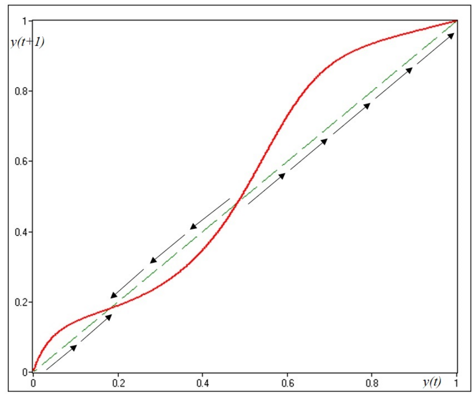
[27] It is irrelevant that non-green investment makes zero profits since no firm adopts this technology.

Figure 18: Scenario of two stable equilibria $y^* = 0.18$, $y^* = 1$

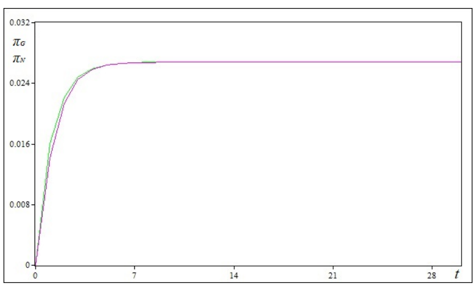
(a) Panel A: time series



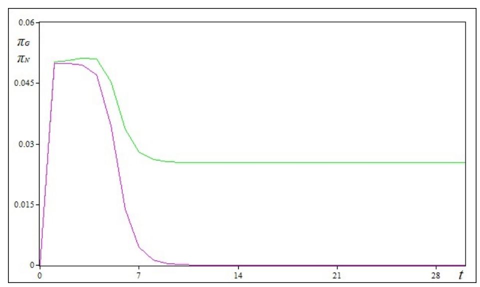
(b) Panel B: phase plot



(c) Panel C: profits low equilibrium



(d) Panel D: profits high equilibrium



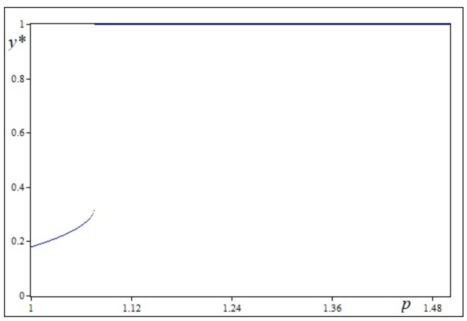
Parameters: $a_G = 1.22, a_N = 1.25, b_G = 0.3, b_N = 0.42, \eta = 0.7, \gamma = 300, \alpha = 2300, \beta = 4000, BT = 140000, \sigma_G = 0.2, \sigma_N = 0.15, \sigma_{G,N} = -0.002, p = 1, r_F = 0.02$. In panel A, for the red time series the initial condition ($i.c.$) is 0.1, for the blue time series $i.c. = 0.5$. In panel C $i.c. = 0.1$, in panel D $i.c. = 0.5$. The green curve represents green profit π_G, the pink curve non-green profit π_N.

In this scenario, however, *green monetary policy* CSPP could still be useful to encourage the adoption of green technology. The bifurcation diagrams in Figure 19 demonstrate the impact of p in both scenarios, respectively. While the policy is ineffective in the high equilibrium scenario (Figure 19b), increasing p beyond 1.10 helps the firm population to move from the low equilibrium to the high equilibrium in Figure 19a.

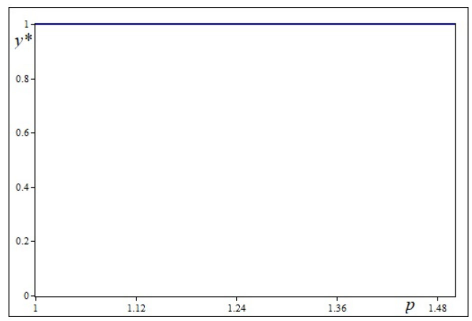
So, despite being ineffective in increasing firms' returns (i.e. moving from the low equilibrium to the high one does not increase the efficiency of the industry), the *green monetary policy* is still useful in encouraging companies to switch to a more sustainable technology of production, reducing the total carbon emission for the same amount of firms' profitability.

Figure 19: Bifurcation diagram for p

(a) Panel A: $i.c. = 0.1$



(b) Panel B: $i.c. = 0.5$



Parameters: same parameters of Figure 18.

5 Conclusion

In recent years, it has become increasingly evident that climate change is one of the main sources of structural change impacting the financial system. Indeed, it affects all agents in the economy, in all sectors and geographic areas with potentially nonlinear dynamics. Moreover, while the quantification of impact, time horizon, and future pathway are uncertain, there is a high degree of certainty that some combination of physical and transitional risks will materialize in the near future, affecting negatively the stability of the financial systems, and the economic systems as a whole. Therefore, CB monetary policies have been starting to consider risks related to climate change with the aim to strengthen the role of the financial system to manage risk and mobilize capital for green and low-carbon investments in the broader context of environmentally sustainable development.

In this chapter, we developed a model of CSPP that internalized climate-related externalities by means of a tilting factor of the environmental footprint of green and non-green firms. We showed that a shift in the CB portfolio allocation toward bonds issued by low-carbon companies can favor green firms in the market. We modeled firm investment choices with exponential replicator dynamics and explored numerically the dynamical proprieties of the system.

We obtained some main findings.

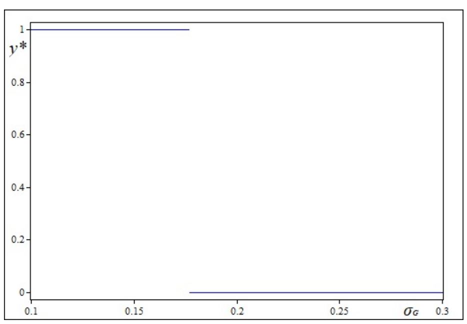
First, some scenarios are characterized by a strong path dependency in which if a large share of firms employed non-green technology, no investment in green technology occurs in the long run, even if the non-green investment equilibrium is inefficient. We define this equilibrium *technology trap* and show that CSPP monetary policy helps the industry leave the technology trap. Second, green and non-green bond riskiness is a key factor that impacts borrowing costs. The larger the average financial risk of bonds, the lower the share of bonds in the CB portfolio, and the larger the firms' borrowing cost. Third, the degree of market competition and of market (im)perfections contribute to amplifying the effects of the green monetary policy by affecting the transmission channel. In the presence of imperfect competition and (or) a high degree of market imperfections the *technology trap* is more likely to happen, the green monetary policy seems to foster the adoption of green technology and to stabilize investment decisions.

Our future research agenda aims at studying two possible extensions. Firstly, we plan to study a model that incorporates the risk of green-washing. A second extension takes into account the interaction between the green monetary CSPP and fiscal policies.

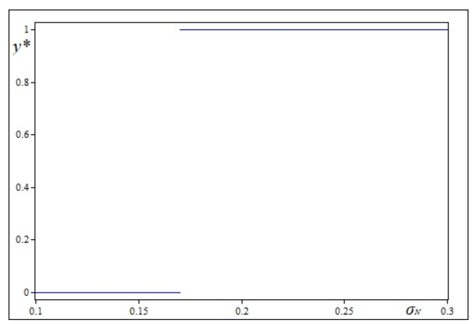
A Additional Figures

Figure 20: Bif. diagrams for variances in the unstable equilibrium scenario

(a) Panel A: bif. diag. σ_G

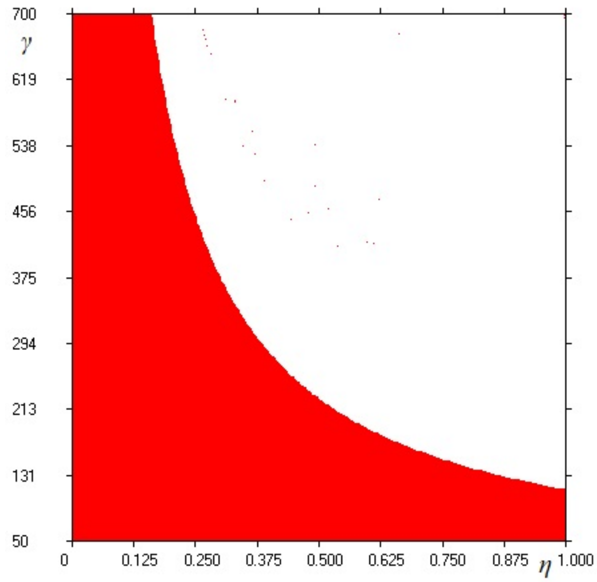


(b) Panel A: bif. diag. σ_N



Parameters: same parameters of Figure 6. The *i.c.* is 0.52.

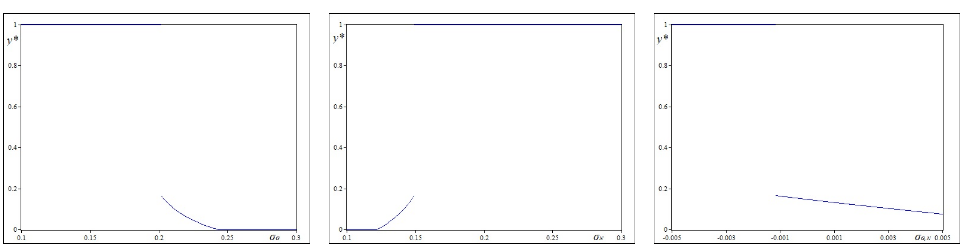
Figure 21: Two-dimensional bifurcation diagram for η and γ in the stable internal equilibrium scenario



Parameters: same parameters of Figure 9. The *i.c.* is 0.2. In red the area of stability/convergence to the unique internal stable equilibrium $y_1^* = 0.763$, in white the region of instability characterized by period cycles of different frequency or deterministic chaotic attractors.

Figure 22: Bif. diagrams for variances in the two internal equilibria (stable and unstable) scenario

(a) Panel A: bif. diag. σ_G (b) Panel B: bif. diag. σ_N (c) Panel C: bif. diag. $\sigma_{G,N}$



Parameters: same parameters of Figure 18. The *i.c.* is 0.5.

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Chapter 5: Non-performing loans, expectations and banking stability: a dynamic model

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Abstract

This chapter proposes dynamic oligopolistic models to describe heterogenous banks that compete in the loan market. Two boundedly rational banks adopt an adaptive behavior to increase their profits under different assumptions of limited information and bounded computational ability, in the presence of a share of credits that might not be reimbursed (i.e. non-performing loans). Each Nash equilibrium is an equilibrium point of the dynamic adjustments as well. Thus, the repeated strategic interactions between banks may converge to a rational equilibrium according to the parameters' values and the initial conditions. As a case study, we assume an isoelastic nonlinear demand and linear costs as in Puu (1991), and we analyze the influence of the economic parameters on the local stability of the unique equilibrium, as well as the kinds of attractors that characterize the long-run behavior of the banks. Moreover, we study the global structure of the basins of attraction and the degrees of stability of the Nash equilibrium under two different dynamic adjustments: adaptive best reply and gradient dynamics. We obtain interesting policy insights on how different risk factors interact to generate banking stress and fragility. Finally, we show that different monetary policies set by the central bank may produce a variety of lending behaviors affecting banking stability.

Keywords: Oligopolistic banking model, Non-performing loans, Discrete dynamical system, Bounded rationality, Non-linearity.
JEL Classification: C61, C62, D43, E51.

Statement of joint authorship and acknowledgments

I confirm that this Chapter - “Non-performing loans, expectations and banking stability: a dynamic model” - was jointly co-authored with Germana Giombini and Gian Italo Bischi. I contributed to 70% of this work.
This entire chapter was published as an article:
Bacchiocchi, A., Bischi, G. I., Giombini, G. (2022). ”Non-performing loans, expectations and banking stability: A dynamic model”. In: *Chaos, Solitons & Fractals*, 157, 111906. DOI: 10.1016/j.chaos.2022.111906.
This chapter was presented at the 12th International Conference Nonlinear Economic Dynamics – NED, University Cattolica del Sacro Cuore, Milan (12-15 Sept. 2021).
I would like to thank the participants at the 12th NED Conference, two anonymous reviewers of ”Chaos, Solitons & Fractals” for their constructive comments, as well as Fabio Tramontana, Giorgio Calcagnini (University of Urbino), Giacomo Rondina (University of San Diego) and Sebastian Ille (Northeastern University London) for their useful insights and suggestions.

1 Introduction

The shock of the COVID-19 pandemic has created an unprecedented challenge for the economic and financial markets. While the decline of GDP has been symmetric or comparable across member countries due to the necessary limitations posed by governments to contain the spread of the disease, asymmetric effects are appearing across sectors of the economy. The differentiated impacts involve not only productivity and consumption but also the functioning of financial and banking systems.

Prompted by regulators, especially after the unsuccessful experience of the Great Financial Crisis, banks have built capital and liquidity buffers, improved risk management practices, and internalized part of the social cost of risk-taking (Schivardi et al., 2017). Thanks to these efforts, they were certainly better prepared to cope with a shock in 2020 than they were in 2008. However, questions are still open about whether this updated EU macroprudential framework, along with the contingency measures implemented in the aftermath of the Covid-19 crisis, will be sufficient to prevent the evolution of an initial liquidity crisis into a worrisome solvency one (Coeuré, 2020).

The implications of the coronavirus pandemic for the financial and banking sectors are still to be fully determined. However, economic growth and forward-looking indicators of default risk suggest that bankruptcies and corporate insolvencies will rise significantly by the end of 2021 (Banerjee et al., 2020), and the share of credits that will not be reimbursed (i.e. non-performing loans, hereinafter NPLs) is expected to increase as the impact of the COVID-19 crisis on the real economy intensifies (OECD, 2021). Indeed, EU data show that the NPL ratio for all EU banks experienced a first uptick, rising to 2.9 percent in Q1-2020, up from 2.6 percent in Q4-2019, even though with a diversified situation between member states (European Commission, 2020).

The literature shows that the relative amount of NPLs on overall credits is a function of both internal factors, such as the banks' management lending strategy, and on external factors like the exogenous business cycle stage of the economy.

For this reason, the first goal of this chapter is to study, theoretically, the characteristics that could expose credit institutions to financial suffer, especially in a context characterized by an economic slowdown (Calcagnini et al., 2018). We propose a dynamic oligopoly model to describe a system of heterogenous banks that compete in the loan market.

Several theoretical and empirical studies have focused on the issue (Ari et al.,

2020b; Couppey-Soubeyran et al., 2020; Goodell, 2020; Zhang et al., 2020). The majority of these works rely on traditional analyses of banks' yearly balance sheets that are used to assess bank performances in terms of their capacity to generate profitability in the short term. However, they are only partially informative of the bank and market financial sustainability. In this vein, ECB (2010) states that desirable features for banks' performance measures should encompass more aspects of the long-run performance than just profitability embedded in market-oriented indicators, and other elements such as the quality and riskiness of assets, should be taken into account. Furthermore, performance measurements do not consider the systemic relevance of a bank, which is one of the key factors behind instability, thus neglecting relevant vulnerabilities of the system as a whole.
A dynamic model represents a novelty in the investigation of the role of NPL on market stability and provides a better setup to study the complex structures of relationships and equilibria that characterize the banking system over time. It allows us to study the evolution of NPLs over a long time horizon, analyzing how the quality and riskiness of assets in banks' portfolios may endanger the capacity of each credit institution to generate profits and thus be competitive in the banking sector (i.e. financial sustainability). The peculiarities of the banking sector are well represented by an oligopoly system, which allows us to capture both the cooperation (i.e. the interbank market) and competition (in the loan market) relationships between credit institutions.
Besides, we model the banking system by means of an oligopolistic model for several other reasons (Leon, 2015). First, one of the key conditions of perfect competition is missing: i.e. the absence of entry and exit barriers, as financial regulation is one of the major constraints to free entry in this industry. The financial sector is among the most regulated sectors in many countries, and different regulations set entry requirements for domestic and foreign banks, capital requirements, and other regulations that affect bank activities. Moreover, the degree of contestability in banking is also influenced by non-legal barriers, such as technical ones. The presence of scale and scope economies may create obstacles to outside banks; network economies may also create an additional barrier in the measure that incumbents choose to share or extend their network to exclude rivals from the market and limit competition.
Second, the banking industry is typically affected by asymmetric information problems. Private information may limit effective competition from uninformed outside banks due to the adverse selection problem, as potential entrant banks stem from their inability to distinguish new (good) borrowers

from old (bad) borrowers who have been rejected by their previous bank.
Third, the banking industry is characterized by market concentration, that is the aggregate share of banking assets held by the largest banks is relatively high, with some degree of variability among countries (Calice and Leonida, 2018).
Therefore, we focus on a duopoly, where two boundedly rational banks adopt an adaptive behavior to increase their profits under different assumptions of limited information and computational ability, in the presence of NPLs. Each equilibrium point of the dynamic adjustments proposed is a Nash equilibrium, i.e. coincides with the corresponding oligopolistic competition outcome obtained under assumptions of complete information and rational (profit maximizing) banks. Thus, in the dynamic framework employed, the repeated strategic interactions between banks may converge to a rational equilibrium according to the parameters' values and the initial conditions considered.
As a case study, we consider an isoelastic loan demand and linear costs, as proposed by Puu (1991) in a general oligopoly setting, and we study the influence of the economic parameters on the local stability of the unique Nash equilibrium as well as the kinds of attractors that characterize the long-run behavior of the banks when the Nash equilibrium is unstable. The first dynamic adjustment proposed, in discrete time, is the classical best reply approach with naïve expectations, i.e. the two banks are assumed to know the demand function and solve the profit maximization problem, thus computing the correct reaction functions, but they are not informed about competitor's choices. Consequently, to compute the best reply they assume the currently observed competitor's loan decision as the expected next choice, generating the well-known Cournot tâtonnement, as in Puu (1991, 1995).
The second dynamic adjustment proposed is the discrete-time gradient dynamics (see Bischi and Naimzada, 2000, and references therein) where the two banks do not solve any profit maximization problem as they simply adjust their next period choices according to the estimated profit derivative (i.e. the marginal profit). In this dynamic adjustment the loans supplied by the banking system move along the direction of the expected profit gradient. So, banks have not only limited information but also limited computation ability because they do not know the best reply strategy, and adjust their decisions according to local profit-increasing arguments. However, even in this instance any positive fixed point of the dynamic duopoly model is a Nash equilibrium, the same of the duopoly system with fully rational agents (i.e. an intersection of the reaction functions).

In both the dynamic models with bounded rationality, we study under what conditions (i.e. for which sets of parameters and which initial conditions) the dynamic adjustment will converge to a Nash equilibrium. Moreover, in the case of isoelastic demand considered, a unique Nash equilibrium exists, but different kinds of attractors, periodic or chaotic, can be obtained when it is unstable, giving rise to long-run evolutions that never settle to a rational equilibrium.

Finally, we analyze the global structure of the basins of attraction and we compare the degrees of stability of the Nash equilibrium under the two different dynamic adjustments proposed. We obtain interesting policy insights on how different risk factors and activities interact in producing concrete situations of banking stress and fragility, which is a highly important issue to the goal of increasing banks' resilience to adverse shocks. Different monetary policies set by the Central Bank produce a variety of behaviors affecting banking stability.

The chapter is organized as follows. Section 2 describes the banking activities in an oligopolistic model, while Section 3 focuses on the duopoly case with isoelastic demand. In Section 4 we study the case of an adaptive best reply, whereas we focus on gradient dynamics in Section 5. Section 6 provides an economic interpretation of the findings and Section 7 concludes.

2 The banking activities in an oligopolistic model

The core banking activities are generally the production of deposit and loan services. The typical balance sheet of a representative credit institution i is defined on one side by the sum of three macro-categories of assets or uses, namely reserves R, loans L, and financial investment B (which include mainly bonds, securities, investment funds, and, to a lesser extent, riskier assets). These three categories correspond to the core activities of each banking institution and they are financed by the resources or liability on the other side of the balance sheet, given by the bank's overall amount of deposits collected from customers, D (see Figure 1).

Figure 1: Balance sheet of a bank i

Assets	Liabilities
Reserves R_i Loans L_i Bonds/Securities B_i	Deposits D_i
Net position on the interbank market M_i	

In this theoretical setup a representative commercial bank i that operates in a market of N heterogeneous banks decides, in each time period t, to provide credits in the form of loans $L_i(t)$ and to invest in financial assets $B_i(t)$.
According to international banking standards that aim at increasing the stability of the financial markets (i.e. Basel Accords), each credit institution needs to satisfy a reserve requirement R_i, which includes e.g. cash holding for clients, provisions, deposit protection funds, etc., and is assumed to be a fraction (q) of the bank liabilities:

$$R_i = qD_i \tag{1}$$

Finally, the interbank market allows banks to borrow (lend) an amount of money M_i to (from) other banks at the rate r.[1]
The position on the interbank market M_i for the single banking institution i is given by the difference between its overall liabilities and assets:

$$M_i = D_i - R_i - L_i - B_i$$

so that $M_i < 0$ is the case of a bank that borrows at rate r from the interbank market to finance its activities; vice versa, $M_i > 0$ represents a situation of a bank that invests its surplus of resources (net deposit) by lending at rate r to the other banks in the industry.

[1] As the prefix suggests, the interbank market is a market where each trade represents an agreement between the banks to exchange amounts of money at a rate r. This rate is set by the Central Bank and holds for all the credit institutions in the interbank market.

By substituting (1) into the above expression, we obtain:

$$M_i = D_i(1-q) - L_i - B_i \tag{2}$$

A market clearing condition holds so that the aggregate or the sum of every position in the interbank market (M) should always be equal to zero:

$$M = \sum_{i=1}^{N} M_i = 0.$$

We consider an oligopolistic Monti-Klein model (Klein (1971), Monti (1972)) where N price-makers banks compete ($i = 1, 2, ..., N$) influencing the loan rate r_L, the deposit rate r_D and the bond rate r_B.[2]

The main research question of this chapter is to analyze how NPLs affect the banking stability and the volume of lending for all credit institutions in the industry. To this aim, the riskiness of the lending activity of each bank is captured by a parameter ρ_i, which measures the bank's expected share of loans that will not be reimbursed at maturity. That is, each bank bears losses due to NPLs, defined as $NPLs_i = \rho_i * L_i$, namely the amount of credits that the bank i foresees will not be reimbursed at maturity. Thus, the share of loans for which the bank receives a positive return $r_L(L)$ is only a fraction of the overall amount of credits, and it is defined by the expression $L_i - \rho_i \, L_i$.

Furthermore, alongside the loss in the NPLs yield, also the principal amount of these bad loans will become an irrecoverable part for the bank. Indeed, we will take account of it by including the capital loss ($-\rho_i L_i$) in the expected profit equation (3).

Thus, the parameter ρ_i captures the riskiness of the lending activity, which each bank considers when it comes to foreseeing the expected profit.

Finally, bank's costs are related to the size of loans and deposits provided, as well as to the amount of financial investment. Consequently, we define $C(L_i, D_i, B_i)$ for each bank i in the market as the cost of managing a volume D of deposits, a volume L of loans and a volume B of bonds.

The expected profit for a representative bank i in an oligopolistic market $i = 1, 2, ..., N$, is:

$$\pi_i = r_L(L)(L_i - \rho_i L_i) - \rho_i L_i + r_B(B)B_i + rM_i - r_D(D)D_i - C(L_i, D_i, B_i) \tag{3}$$

[2] r_B can be thought as an average rate of a bank's financial investment in securities, funds, and riskier assets.

where $L = \sum_{i=1}^{N} L_i$, $D = \sum_{i=1}^{N} D_i$ and $B = \sum_{i=1}^{N} B_i$; r_L, r_D, r_B are functions of the overall quantity of loans L, deposits D and bonds B.

Substituting equation (2) - the net position of the bank on the interbank market - in equation (3), we get:

$$\pi_i = [r_L(L)(1-\rho_i)-\rho_i-r]L_i+[r_B(B)-r]B_i+[r(1-q)-r_D(D)]D_i-C(L_i,D_i,B_i) \tag{4}$$

The interest rate on the interbank market r represents one of the principal instruments at disposal of the central banks to influence the monetary condition in the secondary market.[3]
Accordingly, the bank's expected profit (4) can be seen as the sum of the intermediation margins/spreads on loans, deposits, and bonds (taking also into account the riskiness of these activities), net of management costs.
Indeed, the three main variables L, D, B may vary in volume depending on the spread between their rate and the main refinancing rate r.
We also notice that the refinancing rate r settled by the Central Bank through open market operations (OMOs), which consist of a large-scale acquisition or sale of bonds in the primary market (i.e. the policy monetary course), influences the profitability of all the three main banking operations.
The profit-maximizing behavior for each bank i in the industry is obtained by solving the following first-order conditions, for $i = 1, ..., N$:

$$\frac{\partial \pi_i}{\partial L_i} = r_L(L)(1-\rho_i) - \rho_i - r + r'_L(L)(1-\rho_i)L_i - \frac{\partial C_i}{\partial L_i} = 0 \tag{5}$$

$$\frac{\partial \pi_i}{\partial D_i} = r(1-q) - r_D(D) - r'_D(D)D_i - \frac{\partial C_i}{\partial D_i} = 0 \tag{6}$$

$$\frac{\partial \pi_i}{\partial B_i} = r_B(B) - r + r'_B(B)B_i - \frac{\partial C_i}{\partial B_i} = 0 \tag{7}$$

If the N banks in the market are homogeneous, namely the cost function and the expected share of NPLs is the same for every institution i, then a Cournot equilibrium of the banking industry exists, defined as an N-tuple of triples (L_i^*, D_i^*, B_i^*) that for every i maximizes the profit of bank i, taking the volume of loans, deposits, and bonds of other banks as given.

[3] In this simple model we do not include the interest rate on the main refinancing operations (MRO), that are the most important instrument for the ECB to provide the bulk of liquidity to the banking system in the primary market. Generally, the two aforementioned interest rates are very similar, thus, in the following, we just consider r as the main refinancing interest rate.

This Cournot equilibrium is characterized by the same volume of services offered by each bank: $L_i^* = L^*/N$, $D_i^* = D^*/N$, $B_i^* = B^*/N$.
As stated above, one of the main research questions of the paper is to analyze how an increase of ρ_i affects the banking stability and the volume of lending for all credit institutions in the industry. To this aim, and given the first-order conditions described above, we can exclusively focus on the loan market to answer how different economic parameters, degree of bank rationality, and level of information affect the equilibrium and its stability. Thus, the bank expected profits (4) can be simplified as follows:

$$\pi_i = [r_L(L)(1-\rho_i) - \rho_i - r]L_i - C(L_i).$$

3 The loan market with isoelastic demand in a duopoly

In this Section, we focus on the loan market and consider the case of a duopoly, where the two competing banks are characterized by different linear costs $C(L_i) = c_{Li}\, L_i$, $i = 1, 2$, and diverse risk in the lending activity ρ_i.
As to the factors influencing loan demand, most studies include an economic activity variable (such as real GDP or industrial production) and financing costs (i.e. interest rates or bank lending rates) as its main determinants. Another determinant is the opportunity cost of bank loans (i.e. the cost of alternative sources of finance). Finally, we assume an isoelastic demand function, so that consumers' loan total expenditure is constant, as proposed by Puu (1991). The assumption of constant elasticity is consistent with the absence of liquidity constraints and market frictions. Indeed, empirical analyses suggest that loan demand price elasticity might increase with income, while might become highly price sensitive at higher-than-normal rates. Moreover, evidence suggests that loan size is far more responsive to changes in loan maturity than to changes in interest rate (Karlan and Zinman, 2019). Thus, we model the demand function as follows:

$$r_L(L) = \frac{\alpha y + \beta r_B}{L}\ , \quad L = L_1 + L_2 \tag{8}$$

where we shall assume $L \neq 0$ in the following, that is $L_i \geq 0$, $i = 1, 2$, with at least one of them strictly positive. The numerator is the sum of two components, which we take as given. The term αy captures the transactions demand for credit (indeed, the strongest is the local aggregate production and income y from households and firms, the higher will be the demand for credits), and $\alpha > 0$ measures the sensitivity of the loan rate to the inverse of the credit-to-GDP ratio L/y; the second term considers that loan interest rates may correlate with other investment opportunities or financing, so that the behavioral parameter $\beta > 0$ captures the degree of substitutability for borrowers of the two alternative way of financing, loans and bonds (Bernanke and Blinder, 1988).

Following Puu (1991), by inserting the nonlinear demand function (8) and its derivative $r'_L(L)$ in the first order condition (5), we get a closed form of the unique solution of the expected profit maximization problem that bank i faces at time t in order to choose the loan strategy:

$$L_i(t+1) = \arg\max_{L_i} \pi_i^e(t+1) \tag{9}$$

given by:

$$L_i(t+1) = R_i\left(L_j^e(t+1)\right) = -L_j^e(t+1) + \sqrt{\frac{(\alpha y + \beta r_B)(1-\rho_i)}{r + \rho_i + c_{L_i}} L_j^e(t+1)} \tag{10}$$
$$\text{with} \quad i, j = 1, 2 \quad j \neq i$$

where $R_i(\cdot)$ are the *reaction functions* and $L_j^e(t+1)$ is the expected decision of the competitor.

A Nash equilibrium is located at the intersections of the reaction curves. If players (banks) correctly forecast the competitors' decisions (rational expectations), $L_j^e(t+1) = L_j(t+1)$, then the Nash equilibria can be directly computed (one-shot game). However, in a bounded rationality setting, banks may not know beforehand the competitors' choices, and, consequently, they formulate some reasonable forecasts, on the basis of their information set. The simplest assumption, proposed by Cournot (1838), is that of *naïve expectations*, $L_j^e(t+1) = L_j(t)$, i.e. each bank expects that the decision of the other one will remain the same as in the previous period.

The *naïve expectations* assumption introduces a form of information asymmetry in the market because the bank i can only observe the volume of loans granted by its competitor j at the actual period of time t and, on the basis of this, it settles the optimal volume of loans. This hypothesis could be more realistic than rational expectations because, usually, it is particularly complex to forecast the amount of loans provided by other banks (especially competitors) at time $t+1$, while it is more plausible referring to balance sheets or to other public observable information (i.e. public disclosure). Under this assumption[4] equation (10) generates a discrete-time dynamical system, called *best reply dynamics*:

$$L_i(t+1) = R_i\left(L_j(t)\right) = -L_j(t) + \sqrt{\frac{L_j(t)}{k_i}} \qquad i,j=1,2 \quad j \neq i \tag{11}$$

with

$$k_i = \frac{r+\rho_i+c_{Li}}{\left(\alpha y+\beta r_B\right)\left(1-\rho_i\right)} \qquad i=1,2 \tag{12}$$

Notice that every Nash equilibrium is also an equilibrium of the best reply dynamics, because the intersections of the reaction curves are the fixed points of the difference equation (11). However, such equilibria are not reached in one shot, they may be reached asymptotically, in the long run, if they are stable under the best reply dynamics. This may be seen as an evolutionary explanation of the outcome of a Nash equilibrium. Moreover, the dynamical system (11) may not converge to a Nash equilibrium, as it may exhibit asymptotic convergence to periodic or chaotic attractors (see e.g. Rand 1978, Dana and Montrucchio 1986, Puu 1991, 1998, Bischi et al., 2000, 2010). The unique Nash equilibrium can be expressed by using the aggregate parameters (12):

$$\mathbf{L}^* = (L_1^*, L_2^*) = \left(\frac{k_2}{(k_1+k_2)^2}; \frac{k_1}{(k_1+k_2)^2}\right) \ , \tag{13}$$

and its stability properties, following Puu (1991), can be given in terms of the ratio:

$$k_1/k_2 = \frac{r+\rho_1+c_{L_1}}{r+\rho_2+c_{L_2}}. \tag{14}$$

[4]Other kinds of expectations mechanisms can be used, such as adaptive expectations, see for example Szidarovszky and Okuguchi (1988), Bischi and Kopel (2001).

4 Adaptive best reply

Following Puu (1991), see also Agliari et al. (2005) or the book Bischi et al. (2010), in this Section we consider an adaptive adjustment that implies inertia (or anchoring). Indeed, as the banks realize that their best reply is not reliable enough, due to imperfect information on competitor's choice, they do not immediately jump to the computed "optimal" solution, but they prefer to settle on a weighted average (i.e. a convex combination) between the computed (sub-optimal) best reply R_i and their previous choice, according to the adaptive scheme in (15).

The discrete dynamical system (15) assumes the form $(L_1(t+1), L_2(t+1)) = B(L_1(t), L_2(t))$, and the map B is given by:

$$B : \begin{cases} L_1(t+1) = & (1-\lambda_1) L_1(t) + \lambda_1 R_1(L_2(t)) \\ L_2(t+1) = & (1-\lambda_2) L_2(t) + \lambda_2 R_2(L_1(t)) \end{cases} \tag{15}$$

where the reaction functions are defined by equation (11) and the parameters $\lambda_i \in [0,1]$ capture how much the banks consider reliable the computed best reply based on imperfect information. Thus, best reply is obtained for $\lambda_i \to 1$, whereas complete inertia (i.e. no change at all) occurs as $\lambda_i \to 0$. Notice that each dynamic equation now includes two dynamic variables on the right-hand side, as the loans decided by banks at time t are a weighted average between the previous volume of loans and the reaction to competitor's choice arising from the solution of the profit maximization problem (with naïve expectations).

Generally, smaller values of the parameters λ_i, i.e. larger degree of inertia, enhance stability, as both the region of stability in the space of parameters and the basin of attraction of the stable Nash equilibrium, widen.

Concerning the study of the local stability of the Nash equilibrium[5] under the adaptive adjustment (15), let us consider the Jacobian matrix of the map (15) computed in (13):

$$J(L_1^*, L_2^*) = \begin{bmatrix} 1-\lambda_1 & \frac{\lambda_1}{2}\left(\frac{k_2}{k_1}-1\right) \\ \frac{\lambda_2}{2}\left(\frac{k_1}{k_2}-1\right) & 1-\lambda_2 \end{bmatrix}$$

[5] Notice that even if $(0,0)$ is an equilibrium for the map B, it will not be considered in the following because the demand function (8) is not defined in it, i.e. it is an economically unfeasible point.

The stability conditions, in terms of trace $Tr^* = 2 - \lambda_1 - \lambda_2$ and determinant $\Delta^* = (1-\lambda_1)(1-\lambda_2) + \lambda_1\lambda_2 \frac{(k_2-k_1)^2}{4k_1k_2}$ become (see e.g. Elaydi, 2008, Medio and Lines, 2001):

$$\begin{cases} 1 - Tr^* + \Delta^* = \lambda_1\lambda_2\left(1 + \frac{(k_2-k_1)^2}{4k_1k_2}\right) > 0 \\ \\ 1 + Tr^* + \Delta^* = 4 - 2\lambda_1 - 2\lambda_2 + \lambda_1\lambda_2 + \lambda_1\lambda_2\frac{(k_2-k_1)^2}{4k_1k_2} > 0 \\ \\ 1 - \Delta^* = \lambda_1 + \lambda_2 - \lambda_1\lambda_2 - \lambda_1\lambda_2\frac{(k_2-k_1)^2}{4k_1k_2} \;\; > 0 \end{cases}$$

where k_i are given by (12). The first two conditions are always satisfied, whereas the third stability condition defines a region of stability of the Nash equilibrium in the space of parameters. In the best reply dynamics ($\lambda_1 = 1, \lambda_2 = 1$) feasible (i.e. bounded and non-negative) trajectories are obtained provided that $k_1/k_2 \in [4/25, 25/4] = [0.16, 6.25]$.
Moreover, the Nash equilibrium (13) is stable if and only if $k_1/k_2 \in (3 - 2\sqrt{2}, 3 + 2\sqrt{2}) \simeq (0.17, 5.83)$.
In Figure 2, obtained with $\rho_1 = 0.054$ and $\rho_2 = 0.005$, the stable equilibrium (13) is shown, together with its basin of attraction, represented by the yellow region, whereas the grey region represents the set of initial conditions that generate unfeasible trajectories, i.e. diverging or involving negative values of L_i. [6]

[6] The parameter values resemble economic and financial values that occur in the real world.

Starting from ρ_i, if we take into account the value of NPLs to total gross loans over the last 20 years in the Eurozone (and other OCSE nations), it has ranged, on average, from a minimum of 0.005 for the banking sector of virtuous member countries, up to 0.20 in less virtuous ones (data retrieved from https://data.worldbank.org/indicator/FB.AST.NPER.ZS?locations=XC-IT-DE).

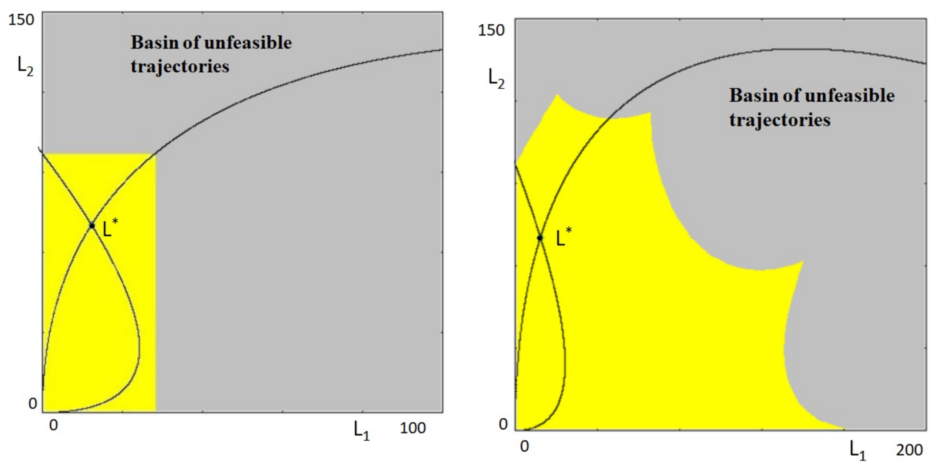
The value of parameter α is related to the inverse of the credit-to-GDP ratio L/y. Data of domestic credit to the private sector, again from World Bank, shows that for many developed countries this ratio can vary between 40% and 130%. To meet this condition, in our simulations, the parameter α should range from 0.02 to 0.07.

The return from corporate and treasury bonds r_B shows large variability. It depends on the specific country, as well as on the maturity of the securities. However, on average, it can go from 0.005 to 0.06 in recent years (https://fred.stlouisfed.org/graph/?id=IRLTLT01EZA156N).

Variable loan costs c_{Li} are specific to each financial institution and are also related to the other parameter values, in particular to interest rates r_L and r. Plausible values of c_{Li} are between 0.001 and 0.01.

A comparison between the pure best reply model (2a) and the adaptive one with inertia (2b) is also shown. In the latter case, the basin of attraction of the unique stable equilibrium is larger and jagged for lower lambdas ($\lambda_1 = 0.6, \lambda_2 = 0.7$), ceteris paribus.

Figure 2: Comparison of the stable Nash equilibrium without and with inertia



(a) $\lambda_1 = 1, \lambda_2 = 1, \alpha = 0.03, y = 200, \beta = 15, r_B = 0.01, r = 0.001, \rho_1 = 0.054, \rho_2 = 0.005, c_{L1} = 0.005, c_{L2} = 0.005$.

(b) Same parameters with $\lambda_1 = 0.6$, $\lambda_2 = 0.7$.

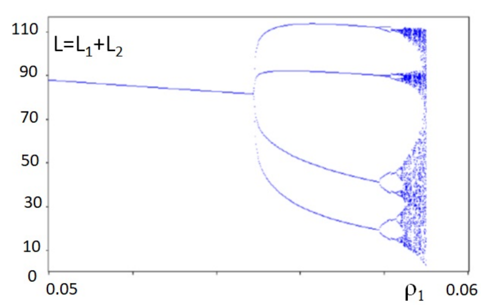
If the two marginal costs and the expected share of NPLs are very different, so that k_1/k_2 exits the interval of stability, the Nash equilibrium becomes unstable.

Indeed, in the particular case without inertia, i.e. $\lambda_1 = \lambda_2 = 1$, if k_1/k_2 falls outside the interval $(3 - 2\sqrt{2}, 3 + 2\sqrt{2})$ the asymptotic dynamics of the best reply model undergoes a degenerate subcritical Neimark-Sacker bifurcation leading to a 1:4 resonance case (see e.g. Kuznetsov, 1998, ch. 9, or Mira, 1987, ch. 5) because the trace of the Jacobian matrix vanishes in the particular case without inertia, and the iterated map (15) becomes

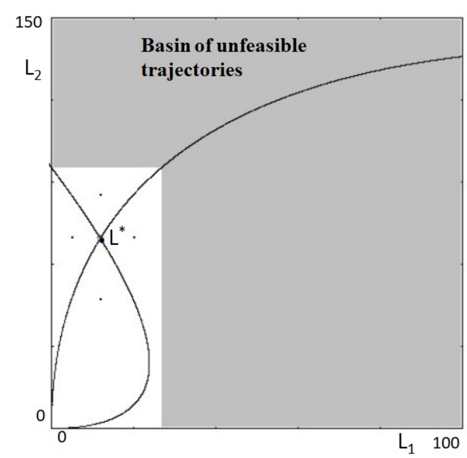
Finally, the official interest rate r set by the major Central Banks (i.e. ECB, Federal Reserve, and Bank of England) exhibits a similar trend over time (from 0 to 0.1). In particular, the last years have witnessed a strong reduction of the interest rates (i.e. expansive monetary policy) towards values very close to zero to ease financial conditions in the aftermath of the Great Financial Crisis and to sustain the economy after the shock of the Covid-19 pandemic.

decoupled after two iterations, being $L_i(t+2) = R_i(R_j(L_i(t)))$.
This kind of dynamical systems have some particular local and global properties, as shown in Bischi et al., 2000, Agliari et al., 2002, due to the fact that it behaves essentially as a one-dimensional map. For example, the degenerate N-S bifurcation of the map B corresponds to a flip bifurcation of the second iterate. This is highlighted in the bifurcation diagram of Figure 3a where ρ_1 is increased (all other parameters being the same) and $L = L_1 + L_2$ is represented in the vertical axis. After the N-S bifurcation the system converges to a 4-period cycle (Figure 3b) and could even exhibit chaotic motions around the Nash equilibrium, as shown in Figure 4 (left panel) in the phase portrait, and in the right panel of Figure 4 with the time series counterpart, the chaotic attractor with 4-periodicity obtained with $\rho_1 = 0.059$ and $\rho_2 = 0.005$, assumes the form of a cross. A further increase of ρ_1 creates a contact with the basin boundary (i.e. boundary crisis) that destroys the attractor, making the system unstable.

Figure 3: The degenerate Neimark-Sacker bifurcation in the best reply dynamics



(a) $\lambda_1 = 1, \lambda_2 = 1, \alpha = 0.03,$
$y = 200, \beta = 15, r_B = 0.01, r = 0.001,$
$\rho_2 = 0.005, c_{L1} = 0.005, c_{L2} = 0.005.$



(b) Same parameters with fixed $\rho_1 = 0.055$.

Figure 4: The cross chaotic attractor in the best reply dynamics

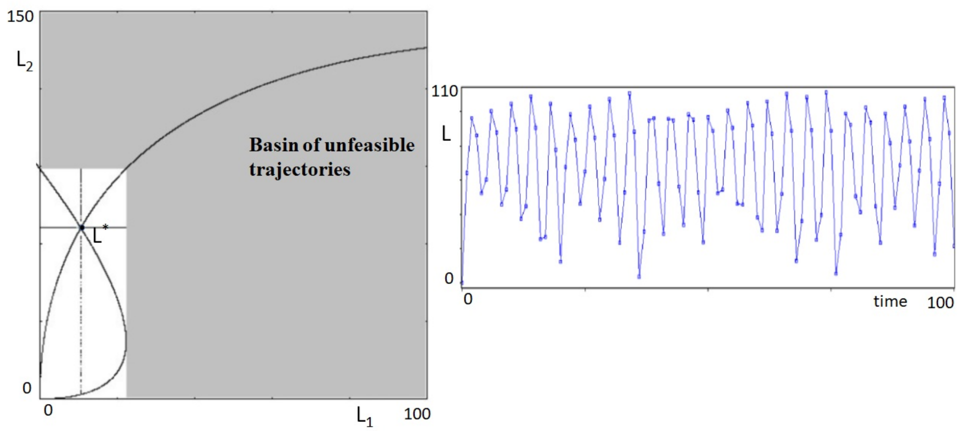


Figure 4: $\lambda_1 = 1, \lambda_2 = 1, \alpha = 0.03, y = 200, \beta = 15, r_B = 0.01, r = 0.001, \rho_1 = 0.059, \rho_2 = 0.005, c_{L1} = 0.005, c_{L2} = 0.005$.

Likewise, also in the case of adaptive best reply with inertia, the feasibility and the stability conditions can be expressed in terms of the ratio between k_1 and k_2, i.e. the stability depends on the heterogeneity of firms and on the inertia, or prudence, parameters λ_1 and λ_2. For example, for the adaptive best reply model with $\lambda_1 = 0.6$ and $\lambda_2 = 0.7$ an excessive bank heterogeneity translates into a subcritical Neimark-Sacker with a hard stability loss, i.e. just after the bifurcation unfeasible trajectories are obtained for any initial condition different from the Nash equilibrium (see Agliari et al., 2005, for a detailed study of the local and global dynamics connected with this subcritical N-S bifurcation).

Figure 5: Towards subcritical N-S bifurcation in the adaptive best reply dynamics

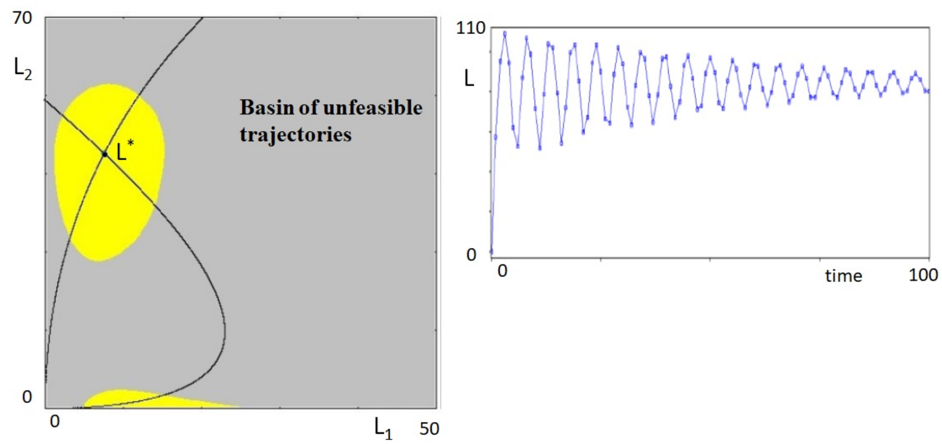


Figure 5: $\lambda_1 = 0.6, \lambda_2 = 0.7, \alpha = 0.03, y = 200, \beta = 15, r_B = 0.01, r = 0.001, \rho_1 = 0.095, \rho_2 = 0.005, c_{L1} = 0.005, c_{L2} = 0.005$

Figure 5 shows the scenario before the N-S bifurcation with a small non-connected basin of attraction as ρ_1 increases, while in the right panel, it is possible to appreciate the damped oscillation toward the Nash equilibrium. Moreover, we can compare Figure 5 with Figure 3 to draw an important remark on the degree of stability of the aforementioned dynamic adjustments. In the first case of pure best reply the system loses stability via a degenerate N-S bifurcation for $\rho_1 = 0.055$, while with lower lambdas in the adaptive case, all other things being equal, the N-S bifurcation happens for ρ_1 greater than 0.095. Thus, introducing inertia increases the overall stability of the banking system allowing for a greater diversity between the two banks in the system. In all these figures, in order to simulate an increasing heterogeneity between banks we moved ρ_1 to be far larger than ρ_2.
These findings underline, under the assumption of best reply adjustment, the crucial role on stability of the heterogeneity of banks, represented by different marginal costs $c_{L1} \neq c_{L2}$ and different expected share of NPLs $\rho_1 \neq \rho_2$. The latter parameters reflect a diverse risk-taking in the lending activity. For example, if $\rho_1 > \rho_2$ bank 1 is lending a higher share of subprime or risky loans or is doing a worse screening activity of borrowers, than bank 2. Consequently, bank 1 will take into account a higher share of NPLs in defining the next optimal loan amount. However, it is possible

to add another degree of diversity when the efficiency in managing a given volume of loans is different for the two credit institution, e.g. $c_{L1} > c_{L2}$. The reasons can be several: the banks have different dimensions that imply a diverse degree of economy of scale, and/or more efficient internal practices in handling the credits, and/or different monitoring expenses, etc.

Furthermore, in the adaptive case a different level of anchoring, or prudence, in choosing the next move (i.e. loan supply) can also cause instability, or the failure to reach the Nash equilibrium. In the latter case, the heterogeneity is due to the λ parameters, e.g. $\lambda_1 \neq \lambda_2$. They represent relevant behavioral parameters and are a proxy of the level of risk-aversion of each bank (recall the discourse on the bank sub-optimal best reply based on imperfect information, in Section 3). This means that the level of information asymmetry perceived in the market on competitors' decisions can influence the banks' behavior (more cautious or less), and this in turn may potentially endanger the stability of the banking sector. To sum up, Figure 6 shows a two-dimensional bifurcation diagram in the parameters' plane ρ_1, ρ_2 for the adaptive best reply case, where all the three levers of bank diversity are present (e.g. $\lambda_1 \neq \lambda_2$, $c_{L1} \neq c_{L2}$, and $\rho_1 \neq \rho_2$). The stability region of the Nash equilibrium is represented by a yellow color, whereas black represents unfeasible trajectories, even starting from initial conditions taken in a very small neighborhood of the equilibrium.

Figure 6: The stability region of the adaptive best reply dynamics in the plane ρ_1, ρ_2

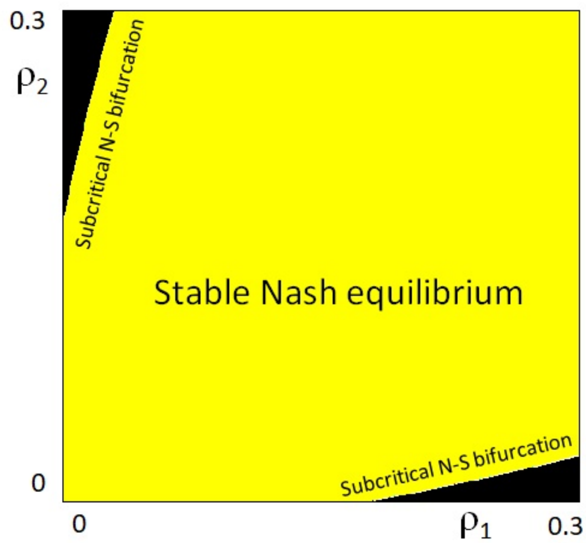


Figure 6: $\lambda_1 = 0.8$, $\lambda_2 = 0.9$, $\alpha = 0.05$, $y = 200$, $\beta = 15$, $r_B = 0.05$, $r = 0.03$, $c_{L1} = 0.004$, $c_{L2} = 0.005$.

5 Gradient dynamics

An alternative dynamic adjustment mechanism, involving a lower degree of rationality, is obtained by considering the so-called gradient dynamics, based on the assumption that the banks adjust their loans supply over time proportionally to their marginal profits (see e.g. Dixit, 1986, Flam, 1993, Bischi and Naimzada, 2000).

In essence, we assume that each bank does not have complete knowledge of the demand function or is not able to solve the optimization problem, hence it tries to infer how the market will respond to small changes in loan supply by an empirical estimate of the marginal profit, that may be obtained by market research or by brief experiments performed at the beginning of period t.

We assume that the banks are able to obtain a correct empirical estimate of the marginal profits $\left(\frac{\partial \pi_i}{\partial L_i}\right)^{(e)} = \frac{\partial \pi_i}{\partial L_i}$ $i = 1,2$. This local estimate of expected marginal profits is generally easier to obtain than a solution of the optimization problem that requires computational skills as well as a global knowledge of the demand function (involving values of L that may be very different from the current ones).

In this case, the banks behave as local profit maximizers, the local adjustment process being one where a bank increases its loan supply if it perceives a positive marginal profit and decreases the loan amount if marginal profit is negative:

$$L_i(t+1) = L_i(t) + \alpha_i(L_i)\frac{\partial \pi_i}{\partial L_i}(L_1, L_2) \ ; \qquad i = 1,2 \tag{16}$$

where $\alpha_i(L_i)$ is a positive function, which gives the extent of loan supply variation of bank i following a given profit signal.

It is easy to verify that, being $\alpha_i(L_i) > 0$, from the equilibrium conditions $L_i(t+1) = L_i(t)$, $i = 1,2$, one gets the first order conditions (5), i.e. $\frac{\partial \pi_i}{\partial L_i}(L_1, L_2) = 0$, $i = 1,2$, hence any positive equilibrium point of (16) is a Nash equilibrium (the trivial equilibrium $L_i = 0$, $i = 1,2$, is not considered feasible).

An adjustment mechanism similar to (16) has been proposed by some authors with constant α_i (see e.g. Dixit, 1986, Flam, 1993). Instead, following Bischi and Naimzada (2000a), we assume α_i proportional to L_i, i.e. $\alpha_i(L_i) = v_i L_i$, $i = 1,2$, where v_i is a positive *speed of adjustment*, equivalent to the assumption that the "relative change" is proportional to the marginal profit:

$$\frac{L_i(t+1) - L_i(t)}{L_i(t)} = v_i \frac{\partial \pi_i}{\partial L_i}(L_1, L_2).$$

If we insert into (16) the marginal profit in the right-hand side of (5) with the isoelastic inverse loan demand (8) and its derivative $r_L'(L) = -(\alpha y + \beta r_B)/(L_1 + L_2)^2$, the discrete dynamical system (16) assumes the form $(L_1(t+1), L_2(t+1)) = T(L_1(t), L_2(t))$, and the map T is given by:

$$T : \begin{cases} L_1(t+1) = & L_1(t) + v_1(\alpha y + \beta r_B)(1-\rho_1) L_1(t) \left(\frac{L_2(t)}{(L_1(t)+L_2(t))^2} - k_1\right) \\ \\ L_2(t+1) = & L_2(t) + v_2(\alpha y + \beta r_B)(1-\rho_2) L_2(t) \left(\frac{L_1(t)}{(L_1(t)+L_2(t))^2} - k_2\right) \end{cases} \tag{17}$$

where the aggregate parameters k_i, $i = 1,2$, are defined by (12).

It is evident, as expected, that the unique feasible equilibrium is the Nash equilibrium (13), i.e. the same obtained under the assumption of full rationality or under the best reply dynamic adjustment. However, its local stability properties, as well as the global structure of its basin of attraction when it is stable, are different. Thus, a comparison between the two kinds of dynamic adjustments may give an interesting insight.
It is worth stressing that the map (17) is not defined along the line $L_1+L_2=0$ (*line of non-definition* δ_s). Since the state variables L_1, L_2 represent the loans offered by the banks, we are only interested in the dynamics of (17) in the region $\mathbb{R}^2_+ = \{L_1, L_2 \,|\, L_1 \geq 0,\ L_2 \geq 0\}$, and the only point of $\mathbb{R}^2_+$ belonging to the line δ_s is $(0,0)$. However, the presence of this point may have a crucial influence on the structure of the basins in $\mathbb{R}^2_+$.
Following the standard local stability analysis based on the computation of the Jacobian matrix:

$$DT(L_1,L_2) = \begin{bmatrix} 1+v_1\left(\alpha y+\beta r_B\right)(1-\rho_1)\left(\frac{L_2(L_2-L_1)}{(L_1+L_2)^3}-k_1\right) & v_1\left(\alpha y+\beta r_B\right)(1-\rho_1)\frac{L_1(L_1-L_2)}{(L_1+L_2)^3} \\ v_2\left(\alpha y+\beta r_B\right)(1-\rho_2)\frac{L_2(L_2-L_1)}{(L_1+L_2)^3} & 1+v_2\left(\alpha y+\beta r_B\right)(1-\rho_2)\left(\frac{L_1(L_1-L_2)}{(L_1+L_2)^3}-k_2\right) \end{bmatrix}$$

at the equilibrium point L_1^*, L_2^*

$$DT(L_1^*,L_2^*) = \begin{bmatrix} 1+v_1\left(\alpha y+\beta r_B\right)(1-\rho_1)k_1\left(\frac{k_1-k_2}{k_1+k_2}-1\right) & v_1\left(\alpha y+\beta r_B\right)(1-\rho_1)\frac{k_1-k_2}{k_1+k_2} \\ v_2\left(\alpha y+\beta r_B\right)(1-\rho_2)\frac{k_2-k_2}{k_1+k_2} & 1+v_2\left(\alpha y+\beta r_B\right)(1-\rho_2)k_2\left(\frac{k_2-k_1}{k_1+k_2}-1\right) \end{bmatrix}$$

the sufficient condition for the stability of L^* is that the eigenvalues of the Jacobian matrix $DT(L_1^*, L_2^*)$ are inside the unit circle of the complex plane. As mentioned in the previous Section, this is true if and only if the following conditions in terms of the trace Tr^* and the determinant Δ^* hold:

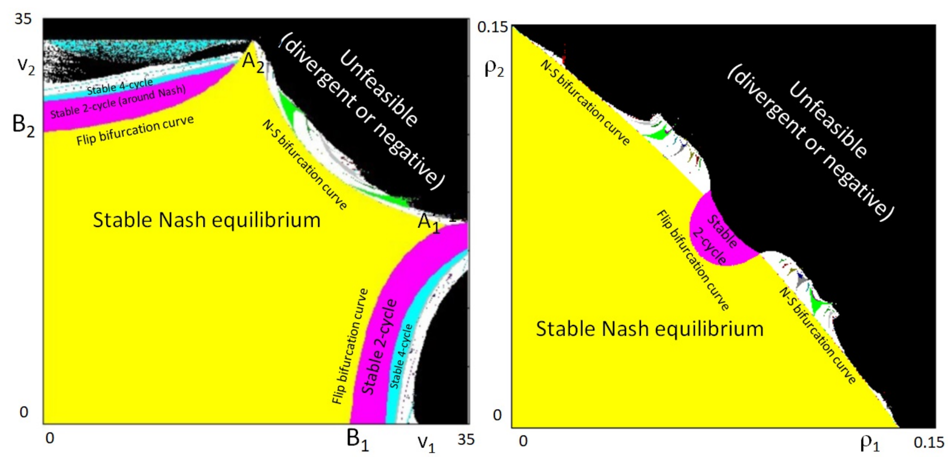
$$\begin{aligned} &1-Tr^*+\Delta^* = \left(\alpha y+\beta r_B\right)^2(1-\rho_1)(1-\rho_2)k_1k_2v_1v_2 > 0 \\ &1+Tr^*+\Delta^* = \left(\alpha y+\beta r_B\right)^2(1-\rho_1)(1-\rho_2)k_1k_2v_1v_2 \\ &\qquad -4\tfrac{k_1k_2}{k_1+k_2}\left(v_1\left(\alpha y+\beta r_B\right)(1-\rho_1)+v_2\left(\alpha y+\beta r_B\right)(1-\rho_2)\right)+4>0 \\ &1-\Delta^* = 2\tfrac{k_1k_2}{k_1+k_2}\left(v_1\left(\alpha y+\beta r_B\right)(1-\rho_1)+v_2\left(\alpha y+\beta r_B\right)(1-\rho_2)\right) \\ &\qquad -\left(\alpha y+\beta r_B\right)^2(1-\rho_1)(1-\rho_2)k_1k_2v_1v_2>0 \end{aligned} \tag{18}$$

where k_i, $i=1,2$, are defined in (12). The first condition is always satisfied, whereas the other two define a bounded region of stability in the parameter space. The second condition defines the condition of flip (or period-doubling) bifurcation, the third condition the N-S bifurcation.

Given the unitary costs c_{L_1}, c_{L_2} and the expected loan default ρ_1, ρ_2, the stability region can be represented in the plane $\mathcal{V} = \{v_1, v_2 \,|\, v_1 \geq 0, v_2 \geq 0\}$, as shown in yellow in Figure 7a.
This region is symmetric with respect to the diagonal $v_1 = v_2$ and bounded by the positive branches of two equilateral hyperbolas (see Bischi et al., 1999b) whose equations are obtained from the second and the third condition of (18). The coordinates of the vertices of this region are:

$$A_1 = \left(\frac{2}{(r+\rho_1+c_{L_1})}, \frac{2}{(r+\rho_2+c_{L_2})}\right) \quad A_2 = \left(\frac{2}{(r+\rho_2+c_{L_2})}, \frac{2}{(r+\rho_1+c_{L_1})}\right)$$
$$B_1 = \left(\frac{k_1+k_2}{(r+\rho_1+c_{L_1})\,(r+\rho_2+c_{L_2})}, 0\right) \quad B_2 = \left(0, \frac{k_1+k_2}{(r+\rho_1+c_{L_1})\,(r+\rho_2+c_{L_2})}\right).$$

Figure 7: The stability region of gradient dynamics in the plane



(a) $\alpha = 0.03, y = 200, \beta = 15,$
$r_B = 0.01, r = 0.05, \rho_1 = 0.055,$
$\rho_2 = 0.005, c_{L1} = 0.005, c_{L2} = 0.005.$

(b) $v_1 = 15, v_2 = 20, \alpha = 0.03,$
$y = 200, \beta = 15, r_B = 0.05,$
$r = 0.03, c_{L1} = 0.004, c_{L2} = 0.005.$

In Figure 7a the region of stability of the Nash equilibrium is represented by yellow color in the parameters' plane (v_1, v_2), whereas in 7b it is represented in the parameters' plane (ρ_1, ρ_2). The other colors represent periodic cycles of different periods, such as pink for period 2, light blue for period 4, etc., whereas the white area is a region of bounded attractors that may be periodic (with periods greater than 15), quasiperiodic, or chaotic, the black region represents unfeasible trajectories, i.e. diverging or involving negative values. In the left panel (7a), along the boundary of the yellow region connecting the points A_1 and A_2 a supercritical N-S bifurcation occurs, whereas along the arc of hyperbola connecting B_1 with A_1, as well as along the one between B_2 and A_2, a supercritical flip bifurcation occurs. Analogously, in the right panel (7b) a supercritical flip bifurcation occurs along the line separating yellow and pink colors, a supercritical N-S bifurcation along the other boundaries of the yellow region.

The arguments given so far are based on local stability results. However, such insights may lead to misleading conclusions if they are not supported by an analysis of the basins of attraction, because it may occur that an equilibrium, even if it is locally stable, may be so close to a boundary of its basin that any practical stability is lost because a small perturbation may lead the system to evolve to another region of the phase space (even at infinite distance, along a diverging trajectory). For example, in Figure 8 stable Nash equilibrium is shown with its own basin of attraction represented by the yellow region. However, the topological structure of this basin is quite irregular, being multiply connected and quite intermingles with portions of the basin of diverging trajectories represented by the grey shaded region.

Figure 8: The Nash equilibrium and its stability region in the Gradient Dynamics

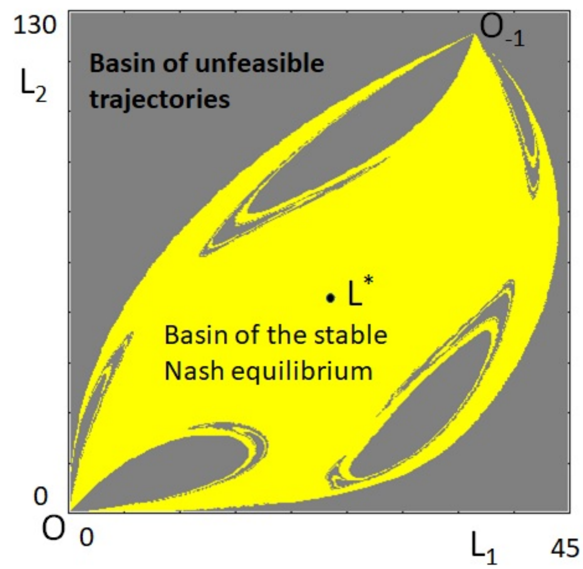
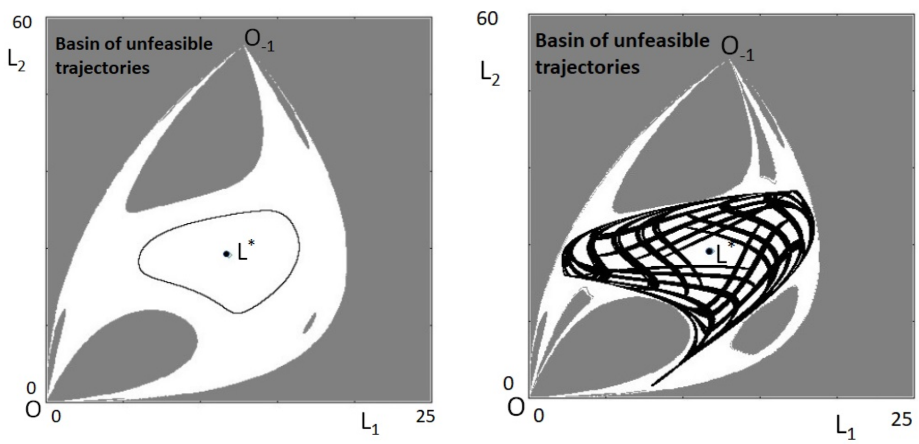


Figure 8: $v_1 = 23, v_2 = 40, \alpha = 0.05, y = 200, \beta = 15, r_B = 0.05, r = 0.03, \rho_1 = 0.055, \rho_2 = 0.005, c_{L1} = 0.005, c_{L2} = 0.005$.

The global structure of the boundaries that separate these basins is strongly influenced by the following two global features of the map (17): (i) it is a *noninvertible map* of the plane, so its global geometric properties can be characterized by the method of *critical curves* (see Mira et al., 1996); (ii) the map T has denominators which vanish along a one-dimensional subset of the plane, on which a *focal point* exists, located at the singular point $(0, 0)$, where the map assumes the form $0/0$ (see Bischi et al., 1999a). For an analytical and numerical analysis of these global dynamical properties of the map (17), and how these are related to (i) and (ii), we refer to Bischi et al. (2001) where a map with the same mathematical structure is analyzed and applied to a different economic context.
Here we are interested in some numerical simulations concerning the kind of bounded non-equilibrium dynamics observed outside the stability region. Indeed, when one of the two flip bifurcation curves is crossed as some parameters are varied, then a stable cycle of period two located around the unstable Nash equilibrium is observed, and further parameters' changes may

lead to the well known period doubling route to chaos. When the N-S bifurcation curve is crossed an attracting closed invariant curve is obtained, along which quasi-periodic or periodic motion occurs (the case of periodic windows is related to the existence of Arnold tongues, the green regions clearly visible in the upper parts of both Figures 7a and 7b). In fact, with speeds of adjustment $v_1 = 23$ and $v_2 = 23$, a stable limit cycle is observed around the unstable Nash equilibrium (just after the supercritical N-S bifurcation), as shown in Figure 9a together with its basin of attraction (the white region), whereas Figure 9b displays a chaotic attractor obtained with $v_1 = 24$ and $v_2 = 23.8$.

Figure 9: The supercritical Neimark-Sacker bifurcation in the Gradient Dynamics



(a) $v_1 = 23, v_2 = 23, \alpha = 0.03,$
$y = 200, \beta = 15, r_B = 0.01,$
$r = 0.05, \rho_1 = 0.055, \rho_2 = 0.005,$
$c_{L1} = 0.005, c_{L2} = 0.005.$

(b) $v_1 = 24, v_2 = 23.8, \alpha = 0.03,$
$y = 200, \beta = 15, r_B = 0.01,$
$r = 0.05, \rho_1 = 0.055, \rho_2 = 0.005,$
$c_{L1} = 0.005, c_{L2} = 0.005.$

In both cases, the grey region represents the set of initial conditions generating unbounded trajectories. Moreover, in the case of the chaotic attractor shown in Figure 9b, it is very close to the basin boundary, hence the system is very vulnerable as a small parameter change may cause a contact between the bounded attractor and its basin boundary, thus giving rise to a global bifurcation at which the chaotic attractor is transformed into a chaotic repellor, after which almost all the initial conditions will generate diverging trajectories. In an economic interpretation, divergence means that the two banks, and so the loan market, cannot find a suitable adjustment around the Nash equilibrium, thus the duopoly system collapses. This situation happens for very large values of the ρ_i parameter, thus in the presence of severe bank financial distress. We may also assist to a transitory phase, whose length depends on the parameter values, where the two banks try to find a compromise in their loan supply along a transient motion around the Nash equilibrium. The transitory phase characterized by system instability, may stabilize in the long-run. In this respect, Figure 10 shows the loan quantity and profit evolution for both banks in this particular situation. Bank 2, characterized by slightly larger costs but a far lower probability of default than bank 1, offers a greater amount of loans in equilibrium (i.e. in the long-run, after about 10 periods). Figure 10 shows that despite damped oscillations of quantities and profits [7], both banks reach the Nash equilibrium.

Figure 10: The banks' profit evolution

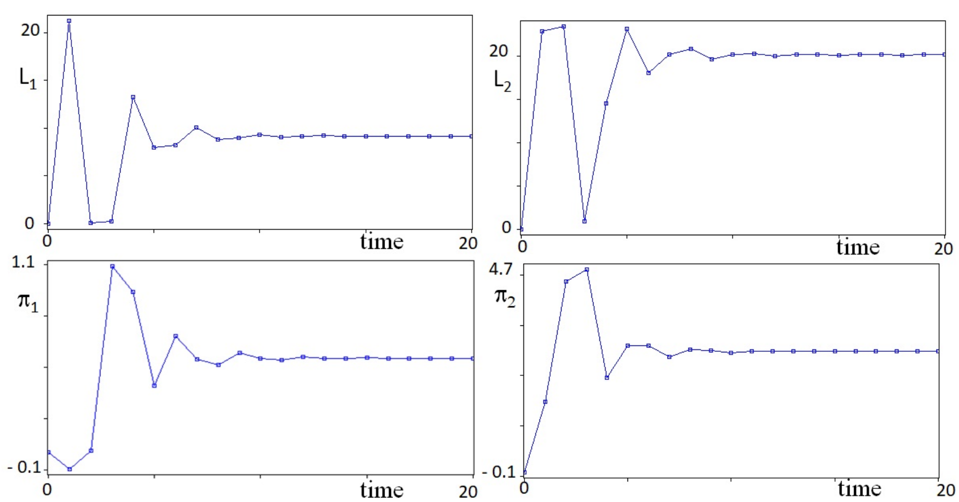


Figure 10: $v_1 = 15$, $v_2 = 15$, $\alpha = 0.05$, $y = 200$, $\beta = 15$, $r_B = 0.01$, $r = 0.05$, $\rho_1 = 0.08$, $\rho_2 = 0.01$, $c_{L1} = 0.003$, $c_{L2} = 0.005$.

[7]Although bank 1 exhibits negative profit in period 1, it remains in the market and its loan supply stabilizes to a positive value.

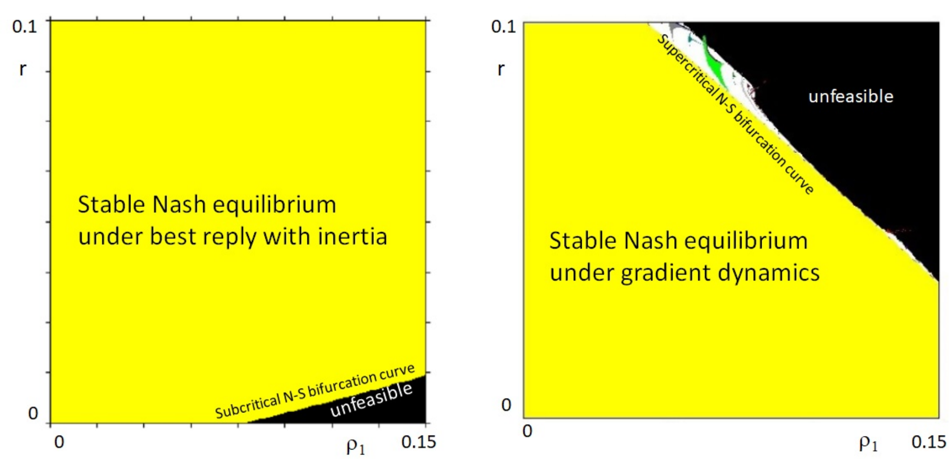
6 Economic implications

In this Section, we focus on the economic implications of the dynamic adjustments proposed, showing the similarities and differences between the two models. First of all, our analyses suggest that the impact of interest rate on banking stability depends on the level of rationality that characterizes bank competition.

In the model, the parameter r is exogenously determined by the Central Bank according to its inflation and output targets (Bacchiocchi and Giombini, 2021), so that different values of the parameter r correspond to expansive or restrictive monetary policies for low and high r values, respectively. We start by investigating the impact of monetary policies, as captured by different r values, on the volume of loans provided, NPLs, and market stability, in the two models.

In Figures 11a and 11b we compare the dynamics of the two models (adaptive best reply and gradient dynamics) to the same range of interest rates. We take as reference bank 1, so the x-axis measures the amount of expected NPLs ρ_1 (i.e. proxy of the expected financial riskiness of its lending activity), while the y-axis shows different monetary policies r.

Figure 11: A stability comparison for different monetary policies



(a) $\lambda_1 = 0.7, \lambda_2 = 0.7, \alpha = 0.03,$ $y = 200, \beta = 15, r_B = 0.01, \rho_2 = 0.005,$ $c_{L1} = 0.005, c_{L2} = 0.005.$

(b) $v_1 = 15, v_2 = 15, \alpha = 0.03, y = 200,$ $\beta = 15, r_B = 0.01, \rho_2 = 0.005,$ $c_{L1} = 0.005, c_{L2} = 0.005.$

Figure 11a shows the impact of monetary policies assuming the adaptive best reply dynamics. We obtain that divergence occurs (for high levels of ρ_1) in the presence of expansive monetary measure, i.e. for very low levels of the interbank rate r.[8]

Overall the system is stable for a wide range of r and ρ_1 values, suggesting that the transmission mechanisms of monetary policies work properly in the presence of inertia and banks that compete by an (adaptive) best reply strategy.

Figure 11b shows the impact of monetary policies on the share of NPLs in the presence of banks that compete by means of gradient dynamics. We obtain that the yellow stability region shrinks, and the system moves to instability (in white) or divergence (in black) as long as the interbank rate r increases, for a larger set of ρ_1 values than those of the previous Figure 11a.

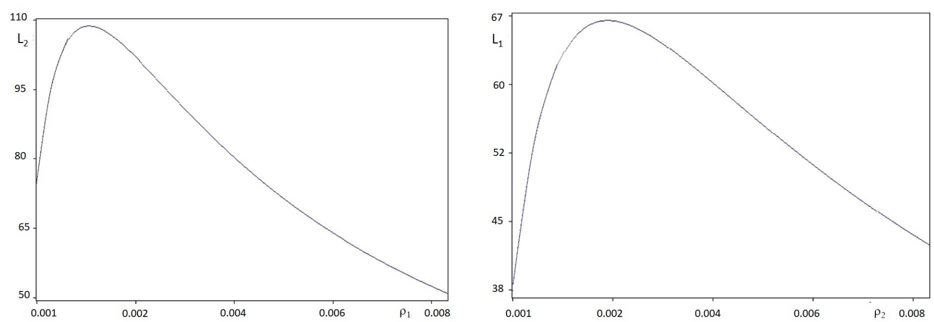
This finding suggests that in the presence of a lower degree of banks rationality (i.e: gradient dynamics), the monetary policy set by the Central Bank performs worsen than in the presence of more rational agents (i.e.: adaptive best reply). Likely, in the former case the transmission mechanisms (that works through the price or quantity channels) encounters obstacles related to the bounded rationality of banks. These obstacles refer to the capacity of banks to modify their loan supply, potentially affecting the potency of forward guidance and leading to powerful mitigation of the effects of monetary policy. As discussed in Farhi and Werning (2019), under forward guidance the intended interest rate path is directly and exhaustively communicated by the central bank to the economic actors, which usually pay close attention to these announcements (i.e. signalling channel).

However, expectations for other endogenous macroeconomic variables, such as output or inflation, are not under the direct control of the central bank, nor directly announced and, thus, agents can only form beliefs about them indirectly. If banks do not promptly anticipate the changes in monetary policy course and/or if the latter are not adequately communicated by the central bank, they can end up reacting late to restrictive monetary policies (i.e. high r values) that would aim at reducing loan quantities by increasing loan costs. Therefore, as long as banks do not react rapidly to restrictive monetary policies by reducing loan supply, a larger amount of loans fails to be reimbursed because of the increased cost (high r) leading the system to

[8] In Figure 11a we used λ_1 and λ_2 equal to 0.7, but the same qualitative result occurs in the best reply case with lambdas equal to 1. The only difference is given by the fact that the yellow basin of attraction is smaller, thus lower values of ρ_1 can cause divergence, coherently with the stability argument seen in the previous Section 4.

instability and divergence.
Secondly, our analyses show that the market dynamics depend on bank interdependence for both models considered. That is, the bank i financial stress has a non-linear impact on the credit supply of the competitor j. For low ρ_i values the competitor j increases its market share at the expense of bank i, but when the expected NPLs share ρ_i exceeds a certain threshold, the difficulty of the bank i becomes detrimental also for the competitor j, which reduces its loan supply.

Figure 12: The banks' interdependence



(a) $v_1 = 15, v_2 = 15, \alpha = 0.03, y = 200,$ $\beta = 15, r_B = 0.01, r = 0.001, \rho_2 = 0.01,$ $c_{L1} = 0.002, c_{L2} = 0.003.$

(b) $v_1 = 15, v_2 = 15, \alpha = 0.03, y = 200,$ $\beta = 15, r_B = 0.01, r = 0.001, \rho_1 = 0.02,$ $c_{L1} = 0.002, c_{L2} = 0.003.$

Focusing on the gradient dynamics, Figure 12a shows the impact of ρ_1 on the credit supply of the competitor, L_2; while Figure 12b highlights the effect of ρ_2 on the credit supply of the bank 1, L_1. The threshold or apex of the curve could be different depending on the value of the other parameters, especially the banks' variable cost c_{L1} and c_{L2}. This finding confirms that whatever is the bank taken as a reference, the financial stress of a credit institution translates into a potential suffering situation for all the other banks in the market, leading to a credit crunch (i.e. reduction of the overall loan supply).
The effect works through two channels: the interbank channel and the signalling channel via loan interest rate r_L. As modeled in the bank's profit function (3), the banks lend each other in the interbank market. If one bank i perceives a high risk on its credit recovery, it will likely reduce not only the volume of loans provided to households and firms L_i, but also to the banks'

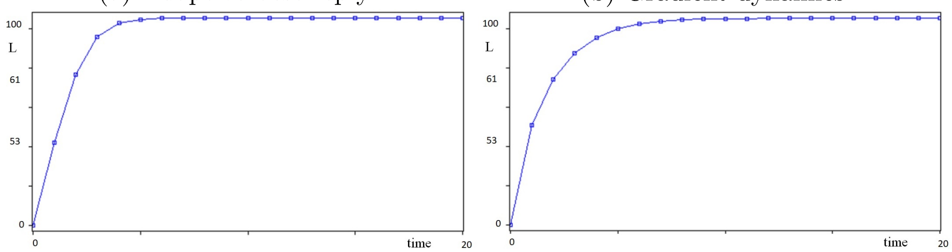
counterpart j, M_i.
In this situation, the other bank(s) counterpart j would react by shrinking their loan supply too.
The other channel works through the cost of loans. In fact, a higher expected loan default means a greater insolvency risk for the bank i. It can be interpreted as an additional cost to bear in the lending activity. For this reason, the struggling credit institution i will shrink its loan supply re-balancing its portfolio towards the other type of assets modeled (i.e. financial investment B). [9]
The reduced loan supply of one of the two credit institutions in our duopoly banking market pushes up the loan rate r_L, which, in such cases, acts as a signal of expected financial difficulties, and brings down the overall loan demand from private, see equation (8). The result is that, in equilibrium, all the banks in the industry, including the financially healthier competitor(s) j, face a smaller market, thus being forced to provide a lower loan quantity.
These considerations imply that the higher the concentration of institutions in the banking sector, the larger the aforementioned credit crunch effect. In this vein, in a duopoly, the financial tensions of one of the two banks have a strong impact on the unique competitor. If the market is more fragmented or competitive (i.e. with a greater number of banks) the impacts on the other competitors, by means of the interbank and signalling channels, are relatively lower.
A final remark comes from the speed required to reach the unique Nash equilibrium.
Figures 13a and 13b show the time series of L (i.e. the overall supply of loans in this market) assuming the adaptive best reply, and the gradient dynamics, respectively. We obtain that the speed of convergence is faster in the presence of the (sub-optimal) optimizing behavior[10] of Figure 13a, than in the presence of the local adjustment that characterizes Figure 13b.
The result is consistent with the following underlying assumption.
A greater degree of agents' rationality (i.e. adaptive best reply) leads to a faster banks' reaction to changes in competitors' loan offers and banking market conditions, while in the case of more limited knowledge and computational ability (i.e. gradient dynamics), the adjustment is relatively slower.

[9] In this simple model we focus just on the loan market without considering the possible effect of re-balancing the banks' portfolio or changing its dimension (i.e. quantitative changes). This requires a deeper analysis of the interdependence between loan, deposit, and bond markets.

[10] *Ceteris paribus*, in Figure 13a, with λ_1 and λ_2 equal to 1 (i.e. the "pure" Best Reply), the equilibrium is reached after 3 periods of time only.

Figure 13: The speed to the Nash equilibrium

(a) Adaptive best reply (b) Gradient dynamics



(a) $\lambda_1 = 0.7, \lambda_2 = 0.7, \alpha = 0.03,$
$y = 200, \beta = 15, r_B = 0.01, r = 0.001,$
$\rho_1 = 0.03, \rho_2 = 0.02, c_{L1} = 0.002,$
$c_{L2} = 0.003.$

(b) $v_1 = 15, v_2 = 15, \alpha = 0.03,$
$y = 200, \beta = 15, r_B = 0.01, r = 0.001,$
$\rho_1 = 0.03, \rho_2 = 0.02, c_{L1} = 0.002,$
$c_{L2} = 0.003.$

7 Conclusions and future research

This chapter analyzed two duopoly models to describe banks that compete in a loan market described by an isoelastic demand function as in Puu (1991). The two dynamic models are characterized by different kinds of boundedly rational adjustments to increase their profits under different assumptions on limited information and computational ability, as well as in the presence of NPLs. We first discussed on the adaptive best reply mechanism where each bank reacts with inertia to competitor's decision. Then, we focused on a dynamic adjustment mechanism that involves a lower degree of rationality, by considering the gradient dynamics, based on the assumption that the banks adjust their loans supply over time proportionally to their marginal profits. The main mathematical properties of similar discrete-time dynamic models have already been studied in the literature, see e.g. Puu (1991) and Agliari et al. (2005) for the former model, Bischi et al. (1999b) and Bischi et al. (2001) for the latter one. However, the meaning of the dynamic variables as well as the structure of the parameters' space is quite different. In particular, the marginal costs are replaced by the aggregate parameters k_1 and k_2, that include the share of loans expected by the banks ρ_i, and the interest rate on interbank market r.

Moreover, a comparison between the two models has not been analyzed in the literature, and such a comparison concerning local and global stability properties of the two models, is particularly interesting when competition between banks is considered. In both cases, we obtain that bank heterogeneity, which derives from either different cost structures, different shares of NPLs, or both, affects the stability of the equilibrium.

In terms of economic implications, the models suggest that in the presence of a larger degree of bounded rationality of banks (i.e.: gradient dynamics), the monetary policy set by the Central Bank performs worsen than in the presence of more rational agents (i.e.: adaptive best reply). Likely, in the former case, the transmission mechanisms (that work through the price or quantity channels) encounter obstacles related to too limited bank rationality. The latter leads the system to divergence or instability for relatively high levels of interest rates and share of expected NPLs. Secondly, our analyses showed that bank interdependence affects the market dynamics so that the financial stress of a credit institution could translate into a suffering situation for all the other banks in the market, leading to a credit crunch.

In terms of future research agenda, some additional elements are worth exploring.

First of all, the identification of the combinations of banking operations harbingers of financial distress or corporate insolvencies is of paramount relevance. Secondly, attention could be deserved to the analysis of the impact of a reserve requirement change, or the effect of monetary policy on financial asset yields, deposits, and loan interest rates. Moreover, the future agenda could focus on the possible interactions among the different markets in which the bank operates. Indeed, the cost function $C(L, D, B)$ could be modeled so that costs would be not perfectly separable, and interaction effects among markets could be explored. In this latter case, both different kinds of boundedly rational adjustments, and NPLs would affect not only the equilibrium of the loan market but also the financial system as a whole. Additionally, costs may be nonlinear to capture the effects of economies or dis-economies of scale.

Last, but not least, the two models analyzed in this chapter also provide arguments about a question often addressed in the literature on dynamic games, concerning the possibility that a repeated game will eventually lead to a Nash equilibrium despite the fact that players are boundedly rational in the short run. This is an evolutionary interpretation of the Nash equilibrium, and traditionally, answers to this question have been given in terms of the local stability of Nash equilibria.

However, even if only a way to behave rationally exists (represented by immediate convergence, in one shot, to a Nash equilibrium), several kinds of boundedly rational adaptive adjustment mechanisms may be observed, characterized by different stability properties. Thus, a comparison between different adjustment mechanisms, related with different information sets or computational abilities, or other features, is always interesting in this context.

Moreover, in a nonlinear model, a study of local stability only may not be sufficient to perform such a comparison. For example, a study of the extension of the basin of attraction of a stable equilibrium can give information about its robustness with respect to exogenous perturbations, but this requires a global analysis of the dynamical system, i.e. a study not based on linear approximations. Since for general higher-dimensional systems such results are hard to come by, we limited ourselves to the case of two banks.

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