# ASYMPTOTIC AVERAGE SOLUTIONS TO LINEAR SECOND ORDER SEMI-ELLIPTIC PDES: A PIZZETTI-TYPE THEOREM

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ABSTRACT. By exploiting an old idea first used by Pizzetti for the classical Laplacian, we introduce a notion of asymptotic average solutions making pointwise solvable every Poisson equation  $\mathcal{L}u(x) = -f(x)$  with continuous data f, where  $\mathcal{L}$  is a hypoelliptic linear partial differential operator with positive semi-definite characteristic form.

## 1. INTRODUCTION

The Poisson-type equations related to hypoelliptic linear second order PDE's with nonnegative characteristic form cannot be studied in  $L^p$  spaces due to the lack of a suitable Calderon-Zygmund theory for the relevant singular integrals. Our paper presents a result allowing to satisfactory study such equations in spaces of continuous functions. We follow a procedure introduced by Pizzetti in his 1909's paper [14] based on the asymptotic average solutions for the classical Poisson-Laplace equation.

1.1. Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^n$ , and let  $f : \Omega \longrightarrow \mathbb{R}$  be a continuous bounded function. Let us denote by  $u_f$  the Newtonian potential of f, i.e.,

$$u_f : \mathbb{R}^n \longrightarrow \mathbb{R}, \qquad u_f(x) := \int_{\Omega} \Gamma(y - x) f(y) \, dy.$$

Here  $\Gamma$  denotes the fundamental solution of the Laplace equation, i.e.,

$$\Gamma(x) = c_n |x|^{2-n}, x \in \mathbb{R}^n \smallsetminus \{0\},\$$

 $\omega_n$  being the volume of the unit ball in  $\mathbb{R}^n$  and  $c_n := \frac{1}{n(n-2)\omega_n}$ .

It is well known that  $u_f \in C^1(\mathbb{R}^n, \mathbb{R})$ , while, in general,  $u_f|_{\Omega} \notin C^2(\Omega, \mathbb{R})$ . However, in the weak sense of distributions,

(1.1) 
$$\Delta u_f = -f \text{ in } \Omega.$$

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As a consequence, if the continuous function f is such that

(1.2) 
$$u_f \notin C^2(\Omega, \mathbb{R}),$$

then the Poisson equation

$$(1.3)\qquad \qquad \Delta v = -f$$

has no classical solutions, i.e., there does not exist a function  $v \in C^2(\Omega, \mathbb{R})$  satisfying

$$\Delta v(x) = -f(x)$$
 for every  $x \in \Omega$ .

Indeed, assume by contradiction that such a function exists. Then, by (1.1),

$$\Delta(u_f - v) = 0 \text{ in } \Omega$$

in the weak sense of distributions, so that, by Caccioppoli–Weyl's Lemma, there exists a function h, harmonic in  $\Omega$ , such that

$$u_f(x) - v(x) = h(x)$$

a.e. in  $\Omega$ . Therefore,  $u_f - v$  being continuous in  $\Omega$ ,

$$u_f = v + h \in C^2(\Omega, \mathbb{R}),$$

in contradiction with (1.2). This proves the existence of continuous functions f such that the Poisson equation (1.3) is not *pointwise* solvable. In his paper [14], Pizzetti introduced a notion of *pointwise weak Laplacian*, making pointwise solvable every Poisson equation with continuous data. Pizzetti started from the following remark. Given a function u of class  $C^2$  in  $\Omega$  one has

(1.4) 
$$\lim_{r \to 0} \frac{M_r(u)(x) - u(x)}{r^2} = \frac{1}{2(n+2)} \Delta u(x)$$

for every  $x \in \Omega$ . Here  $M_r$  denotes the Gauss average

$$M_r(u)(x) := \frac{1}{|B(x,r)|} \int_{\partial B(x,r)} u(y) \, dy,$$

|B(x,r)| being the volume of B(x,r), the Euclidean ball centered at x with radius r. Then, if  $u \in C(\Omega, \mathbb{R})$  is such that the limit at the left hand side of (1.4) exists at a point  $x \in \Omega$ , Pizzetti defines

$$\Delta_a u(x) := 2(n+2) \lim_{r \to 0} \frac{M_r(u)(x) - u(x)}{r^2}.$$

We call  $\Delta_a u(x)$  the asymptotic average Laplacian of u at x. Keeping in mind (1.4), if  $u \in C^2(\Omega, \mathbb{R})$ , then

$$\Delta_a u(x) = \Delta u(x)$$
 for every  $x \in \Omega$ .

We denote by

$$\mathcal{A}(\Omega, \Delta)$$

the class of functions  $u \in C(\Omega, \mathbb{R})$ , such that  $\Delta_a u(x)$  exists at any point  $x \in \Omega$ . Obviously,  $\mathcal{A}(\Omega, \Delta)$  is a (linear) sub-space of  $C(\Omega, \mathbb{R})$ . Moreover, by the previous remark,

$$C^2(\Omega, \mathbb{R}) \subseteq \mathcal{A}(\Omega, \Delta).$$

Pizzetti proved that the Newtonian potentials of continuous bounded functions are contained in  $\mathcal{A}(\Omega, \Delta)$ . Precisely he proved the following theorem.

**Theorem A** (Pizzetti Theorem). Let  $\Omega \subseteq \mathbb{R}^n$ ,  $n \geq 3$ , be a bounded open subset of  $\mathbb{R}^n$  and let  $f: \Omega \longrightarrow \mathbb{R}$  be a bounded continuous function. Then

$$u_f \in \mathcal{A}(\Omega, \Delta)$$

and

$$\Delta_a u_f = -f \ in \ \Omega.$$

The aim of this paper is to extend the notion of asymptotic average solution and Pizzetti's Theorem to the class of linear second order semi-elliptic partial differential operators that we will introduce in the next subsection.

1.2. We will deal with partial differential operators of the type

(1.5) 
$$\mathcal{L} = \sum_{i,j=1}^{n} \partial_{x_i}(\partial_{x_j} a_{ij}(x)), \ x \in \mathbb{R}^n,$$

where  $A(x) := (a_{ij} = a_{ji})_{i,j=1,...,n}$  is a symmetric nonnegative definite matrix,

$$x \mapsto a_{ji}(x), \qquad i, j = 1, \dots, n$$

are smooth functions in  $\mathbb{R}^n$  and

$$\sum_{i=1}^{n} a_{ii}(x) > 0 \text{ for every } x \in \mathbb{R}^{n}.$$

Together with these qualitative properties we assume that  $\mathcal{L}$  is hypoelliptic in  $\mathbb{R}^n$  and endowed with a smooth fundamental solution

$$\Gamma : \{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid x \neq y \} \longrightarrow \mathbb{R},$$

such that

- (i)  $\Gamma(x, y) = \Gamma(y, x) > 0$ , for every  $x \neq y$ ;
- (ii)  $\lim_{x\to y} \Gamma(x,y) = \infty$ , for every  $y \in \mathbb{R}^n$ ;
- (iii)  $\lim_{x\to\infty} \left( \sup_{y\in K} \Gamma(x,y) \right) = 0$ , for every compact set  $K \subseteq \mathbb{R}^n$ ;

(iv)  $\Gamma(x, \cdot)$  belongs to  $L^1_{\text{loc}}(\mathbb{R}^n)$ , for every  $x \in \mathbb{R}^n$ .

We recall that when we say that  $\Gamma$  is a fundamental solution of  $\mathcal{L}$  we mean that, for every  $\varphi \in C_0^{\infty}(\mathbb{R}^n, \mathbb{R})$  and  $x \in \mathbb{R}^n$ :

$$\int_{\mathbb{R}^n} \Gamma(x, y) \mathcal{L}\varphi(y) \, dy = -\varphi(x).$$

1.3. Important examples of operators satisfying our assumptions are the "sum of squares" of homogeneous Hörmander vector fields. Precisely: let

$$X = \{X_1, \dots, X_m\}$$

be a family of linearly independent smooth vector fields such that

(H1)  $X_1, \ldots, X_m$  satisfy the Hörmander rank condition at x = 0, that is,

 $\dim\{Y(0) \mid Y \in \operatorname{Lie}\{X_1, \dots, X_m\}\} = n;$ 

(H2)  $X_1, \ldots, X_m$  are homogeneous of degree 1 with respect to a group of dilations  $(\delta_{\lambda})_{\lambda>0}$  of the following type

$$\delta_{\lambda}: \mathbb{R}^n \longrightarrow \mathbb{R}^n,$$

$$\delta_{\lambda}(x) = \delta_{\lambda}(x_1, \dots, x_n) = (\lambda^{\sigma} x_1, \dots, \lambda^{\sigma_n} x_n),$$

where the  $\sigma_j$ 's are natural numbers such that  $1 \leq \sigma_1 \leq \ldots \leq \sigma_n$ .

Then,

(1.6) 
$$\mathcal{L} = \sum_{j=1}^{m} X_j^2$$

satisfies all the assumptions listed in subsection 1.2 (see [1], [2]).

We stress that the sub-Laplacians on stratified Lie groups in  $\mathbb{R}^n$  are particular cases of the operator  $\mathcal{L}$  in (1.6).

1.4. The extension of Pizzetti's Theorem to the operator  $\mathcal{L}$  in (1.5) rests on some representation formulas on the superlevel set of  $\Gamma$ . If  $x \in \mathbb{R}$  and r > 0, define

$$\Omega_r(x) := \left\{ y \in \mathbb{R}^n : \Gamma(x, y) > \frac{1}{r} \right\}.$$

We will call  $\Omega_r(x)$  the  $\mathcal{L}$ -ball centered at x and with radius r. It is easy to recognize that  $\Omega_r(x)$  is a nonempty bounded open set of  $\mathbb{R}^n$ . Moreover

(1.7) 
$$\bigcap_{r>0} \Omega_r(x) = \{x\}$$

and

$$\frac{|\Omega_r(x)|}{r} \longrightarrow 0 \text{ as } r \longrightarrow 0.^1$$

Remark 1.1. If  $\mathcal{L} = \Delta$ , then

$$\Omega_r(x) = B(x,\rho), \text{ with } \rho = (c_n r)^{\frac{1}{n-2}}.$$

Let  $\Omega \subseteq \mathbb{R}^n$  be open and let  $u \in C^2(\Omega, \mathbb{R})$ . Then, for every  $\mathcal{L}$ -ball,  $\Omega_r(x)$  such that  $\overline{\Omega_r(x)} \subseteq \Omega$  and for every  $\alpha > -1$  we have

(1.8) 
$$u(x) = M_r(u)(x) - N_r(\mathcal{L}u)(x),$$

where  $M_r$  and  $N_r$  are the following average operators:

(1.9) 
$$M_r(u)(x) := \frac{\alpha + 1}{r^{\alpha + 1}} \int_{\Omega_r(x)} u(y) K(x, y) \, dy,$$

where

$$K(x,y) := \frac{\langle A(y) \nabla_y \Gamma(x,y), \nabla_y \Gamma(x,y) \rangle}{(\Gamma(x,y))^{\alpha+2}};$$

(1.10) 
$$N_r(w)(x) := \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha \left( \int_{\Omega_r(x)} \left( \Gamma(x,y) - \frac{1}{\rho} \right) w(y) \, dy \right) \, d\rho.$$

The proof of the representation formula (1.8) can be found in [4].

*Remark* 1.2. If  $\mathcal{L} = \Delta$  and  $\alpha = \frac{2}{n-2}$ , then the kernel K is constant and  $M_r$  becomes the Gauss average on the Euclidean ball  $B(x, \rho)$ , with  $\rho = (c_n r)^{\frac{1}{n-2}}$ .

Letting

(1.11) 
$$Q_r(x) := N_r(1) = \frac{\alpha + 1}{r^{\alpha + 1}} \int_0^r \rho^\alpha \left( \int_{\Omega_r(x)} (\Gamma(x, y) - \frac{1}{\rho} \right) dy \, d\rho,$$

an easy computation shows that

$$Q_r(x) = \int_0^r \frac{\Omega_{\rho}(x)}{\rho^2} \left(1 - \left(\frac{\rho}{r}\right)^{\alpha+1}\right) d\rho.$$

*Remark* 1.3. If  $\mathcal{L} = \Delta$  and  $\alpha = \frac{2}{n-2}$ , then, letting  $\rho = (c_n r)^{\frac{1}{n-2}}$ , we get

$$\frac{M_r(u)(x) - u(x)}{Q_r(x)} = 2(n+2) \frac{\frac{1}{|B(x,\rho)|} \int_{B(x,\rho)} u(y) \, dy - u(0)}{\rho^2},$$

so that, by (1.4),

<sup>&</sup>lt;sup>1</sup>If E is a measurable set of  $\mathbb{R}^n$ , |E| denotes its Lebesgue measure.

(1.12) 
$$\lim_{r \to 0} \frac{M_r(u)(x) - u(x)}{Q_r(x)} = \Delta u(x).$$

The limit in (1.12) extends to all the operators  $\mathcal{L}$  in (1.5). Indeed, if u is a  $C^2$  function in an open set  $\Omega \subseteq \mathbb{R}^n$ , from the representation formula (1.8) and the identity (1.7), using Corollary 2.5 in Section 2, one immediately gets

$$\lim_{r \to 0} \frac{M_r(u)(x) - u(x)}{Q_r(x)} = \mathcal{L}u(x).$$

Then, in analogy with the case  $\mathcal{L} = \Delta$ , we introduce the following definition.

**Definition 1.4.** Let  $\mathcal{L}$  be a partial differential operator satisfying the assumptions of subsection 1.2 and let u be a continuous function in an open set  $\Omega \subseteq \mathbb{R}^n$ . We say that

$$u \in \mathcal{A}(\Omega, \mathcal{L}),$$

 $\mathbf{i}\mathbf{f}$ 

Then,  $u_f \in \mathcal{A}(\mathbb{R}^n,$ 

$$\lim_{r \longrightarrow 0} \frac{M_r(u)(x) - u(x)}{Q_r(x)}$$

exists in  $\mathbb{R}$  at every point  $x \in \Omega$ . In this case we define

$$(\mathcal{L}_a(u))(x) := \lim_{r \to 0} \frac{M_r(u)(x) - u(x)}{Q_r(x)}.$$

Furthermore, if  $f \in C(\Omega, \mathbb{R})$  and there exists  $u \in \mathcal{A}(\Omega, \mathcal{L})$  such that

$$(\mathcal{L}_a u)(x) = f(x)$$
 for every  $x \in \Omega$ ,

we say that u is an *asymptotic average solution to* 

$$\mathcal{L}_a u = f \text{ in } \Omega.$$

In the case f = 0 this definition was first introduced in the paper [6].

The main result of our paper is the following theorem which extends Pizzetti's Theorem to the operators (1.5).

**Theorem 1.5.** Let  $f : \mathbb{R}^n \longrightarrow \mathbb{R}$  be a compactly supported continuous function. Define

$$u_f(x) := \int_{\mathbb{R}^n} \Gamma(x, y) f(y) \, dy, \qquad x \in \mathbb{R}^n.$$
  
$$\mathcal{L}) \text{ and}$$
  
$$\mathcal{L}_a u_f = -f \text{ in } \mathbb{R}^n.$$

We will prove this theorem in the next section. Here, by using a result in [6], we show a consequence of Theorem 1.5.

**Theorem 1.6.** Let  $f, u : \mathbb{R}^n \longrightarrow \mathbb{R}$  be compactly supported continuous functions. Then,

$$\mathcal{L}_a u = -f \ in \ \mathbb{R}^n$$

if and only if

$$\mathcal{L}u = -f \text{ in } \mathcal{D}'(\mathbb{R}^n).$$

*Proof.* By the previous Theorem 1.5,

$$\mathcal{L}_a u = -f \text{ in } \mathbb{R}^n$$

if and only if

$$\mathcal{L}_a(u-u_f)=0 \text{ in } \mathbb{R}^n.$$

Then, by Corollary 3.4 in [6],  $u - u_f \in C^{\infty}(\mathbb{R}^n, \mathbb{R})$  and

$$\mathcal{L}(u - u_f) = 0$$

in the classical sense (and vice versa). Since  $\mathcal{L}$  is hypoelliptic, this is equivalent to say that

$$\mathcal{L}(u-u_f)=0$$
 in  $\mathcal{D}'(\mathbb{R}^n),$ 

or that

(1.13) 
$$\mathcal{L}(u) = \mathcal{L}(u_f) \text{ in } \mathcal{D}'(\mathbb{R}^n).$$

On the other hand,  $\Gamma$  being a fundamental solution of  $\mathcal{L}$ ,  $\mathcal{L}(u_f) = -f$  in  $\mathcal{D}'(\mathbb{R}^n)$ . Then, (1.13) can be written as follows:

$$\mathcal{L}u = -f \text{ in } \mathcal{D}'(\mathbb{R}^n).$$

This completes the proof.

1.5. **Bibliographical note.** In recent years asymptotic mean value formulas characterizing classical or viscosity solutions to linear and nonlinear second order Partial Differential Equations have been proved by many authors; we refer to [6, 12, 11, 8, 5, 10, 7, 13, 3]. In those papers one can find quite exhaustive bibliography on this subject.

We would also like to quote the papers [4] and [9] where the notion of asymptotic sub-harmonic function is introduced in sub-Riemannian settings to extend classical results by Blaschke, Privaloff, Reade and Saks.

## 2. Proof of Theorem 1.5

For the readers' convenience, we split this section in two subsections.

2.1. Let G be a compact subset of  $\mathbb{R}^n$  and let r > 0. Define

(2.1) 
$$G_r := \bigcup_{x \in G} \Omega_r(x).$$

Then, we have the following lemma.

**Lemma 2.1.** For every compact set  $G \subseteq \mathbb{R}^n$  and for every r > 0, the set  $\overline{G}_r$  is compact.

*Proof.* It is enough to prove that  $G_r$  is bounded. We argue by contradiction and assume that  $G_r$  is not bounded. Then, there exists a sequence  $(z_n)$  in  $G_r$  such that

$$|z_n| \longrightarrow \infty.$$

By the very definition of  $G_r$ , for every  $n \in \mathbb{N}$ , there exists  $x_n \in G$  such that  $z_n \in \Omega_r(x_n)$ . This means that

$$\Gamma(x_n, z_n) > \frac{1}{r}.$$

As a consequence,

$$\frac{1}{r} < \Gamma(x_n, z_n) \le \sup_{x \in G} \Gamma(x, z_n),$$

so that, by the assumption (iii) related to  $\Gamma$ 

$$0 < \frac{1}{r} \le \lim_{n \to \infty} \left( \sup_{x \in G} \Gamma(x, z_n) \right) = 0.$$

This contradiction shows that  $G_r$  is bounded.

# 2.2. In this subsection we prove the following lemma.

**Lemma 2.2.** Let G be a compact subset of  $\mathbb{R}^n$  and let r > 0. Then, there exists a positive constant  $C_r(G)$  such that

(2.2) 
$$\sup_{x \in G} Q_r(x) \le C_r(G).$$

*Proof.* Keeping in mind the definition of  $Q_r(x)$  (see (1.11)) for every  $x \in G$  we get

(2.3) 
$$Q_{r}(x) \leq \frac{\alpha+1}{r^{\alpha+1}} \int_{0}^{r} \rho^{\alpha} \left( \int_{\Omega_{r}(x)} \Gamma(x,y) \, dy \right) \, d\rho$$
$$\leq (\text{by (2.1)}) \quad \frac{\alpha+1}{r^{\alpha+1}} \int_{0}^{r} \rho^{\alpha} \left( \int_{G_{r}} \Gamma(x,y) \, dy \right) \, d\rho.$$

On the other hand, if  $\varphi \in C_0^{\infty}(\mathbb{R}^n, \mathbb{R})$  is such that  $\varphi = 1$  on  $G_r, \varphi \ge 0$  (such a function exists thanks to Lemma 2.1), we have

$$\begin{split} \int_{G_r} \Gamma(x,y) \, dy &\leq \int_{\mathbb{R}^n} \varphi(y) \Gamma(x,y) \, dy \\ &\leq \sup_{x \in G} \int_{\mathbb{R}^n} \varphi(y) \Gamma(x,y) \, dy \\ &= C_{\varphi}(G). \end{split}$$

Using this estimate in (2.3) we obtain

$$\sup_{x \in G} Q_r(x) \leq C_{\varphi}(G) \frac{\alpha + 1}{r^{\alpha + 1}} \int_0^r \rho^{\alpha} d\rho$$
$$= C_{\varphi}(G) := C_r(G).$$

Remark 2.3. Since  $Q_{\rho}(x) \subseteq Q_r(x)$  for every  $\rho \in ]0, r[$ , we can assume

$$C_{\rho}(G) \leq C_r(G)$$

for every  $0 < \rho < r$ .

2.3. Now, we show a kind of continuity property of the  $\Omega_r(x)$  balls with respect to the Euclidean topology. Precisely, we prove the following lemma.

**Lemma 2.4.** For every  $x \in \mathbb{R}^n$  and for every R > 0 there exists r > 0 such that

$$\Omega_r(x) \subseteq B(x, R).$$

*Proof.* We still argue by contradiction and assume the existence of R > 0 such that  $\Omega_r(x) \notin B(x, R)$  for every r > 0. Then, if  $(r_n)$  is a sequence of real positive numbers such that  $r_n \searrow 0$ , for every  $n \in \mathbb{N}$  there exists  $y_n \in \Omega_{r_n}(x)$  such that

$$y_n \notin B(x, R).$$

This means

$$y_n \notin B(x, R)$$
 and  $\Gamma(x, y_n) > \frac{1}{r_n}$ .

Since  $\Gamma(x, y) \longrightarrow 0$  as  $y \longrightarrow \infty$  and  $\frac{1}{r_n} \longrightarrow \infty$ , the sequence  $(y_n)$  is bounded. As a consequence, we may assume

$$\lim_{n \to \infty} y_n = y^*$$

for a suitable  $y^* \in \mathbb{R}^n$ . Then  $y^* \notin B(x, R)$ . In particular  $y \neq x$  so that  $\Gamma(x, y) < \infty$ . On the other hand,

$$\Gamma(x, y^*) = \lim_{n \longrightarrow \infty} \Gamma(x, y_n) \ge \lim_{n \longrightarrow \infty} \frac{1}{r_n} = \infty.$$

This contradiction proves the lemma.

From the previous lemma we obtain the following corollary.

**Corollary 2.5.** Let  $f : \mathbb{R}^n \longrightarrow \mathbb{R}$  be a continuous function. Then, for every  $x \in \mathbb{R}^n$ ,

$$\sup_{y \in \Omega_r(x)} |f(y) - f(x)| \longrightarrow 0 \text{ as } r \longrightarrow 0.$$

*Proof.* Since f is continuous at x, for every  $\varepsilon > 0$  there exists R > 0 such that

$$\sup_{y \in B(x,r)} |f(y) - f(x)| < \varepsilon.$$

By the previous lemma, there exists  $r_0 > 0$  such that  $\Omega_{r_0}(x) \subseteq B(x, r)$ . Then, for every  $r < r_0$ ,

$$\sup_{y\in\Omega_r(x)}|f(y)-f(x)|\leq \sup_{y\in\Omega_{r_0}(x)}|f(y)-f(x)|\leq \sup_{y\in B(x,r)}|f(y)-f(x)|<\varepsilon.$$

We have so proved that for every  $\varepsilon > 0$  there exists  $r_0 > 0$  such that

$$\sup_{y \in \Omega_r(x)} |f(y) - f(x)| < \varepsilon$$

for every  $r < r_0$ . Hence,

$$\lim_{r \to 0} \left( \sup_{y \in \Omega_r(x)} |f(y) - f(x)| \right) = 0.$$

2.4. Let f as in Theorem 1.5 and, to simplify the notation, let us denote  $u_f$  by u. The aim of this subsection is to prove the following identity:

(2.4) 
$$u(x) = M_r(u)(x) + N_r(f)(x) \quad \forall x \in \mathbb{R}^n.$$

To this end we choose a sequence  $(f_p)$  in  $C_0^{\infty}(\mathbb{R}^n, \mathbb{R})$  with the following properties:

- (i) there exists a compact set  $K \subseteq \mathbb{R}^n$  such that  $\operatorname{supp} f \subseteq K$  and  $\operatorname{supp} f_p \subseteq K$  for every  $p \in \mathbb{N}$ ;
- (ii)  $\sup_K |f_p f| \longrightarrow 0 \text{ as } p \to \infty.$

For simplicity reasons, let us put  $u_p = u_{f_p}$ , i.e.,

$$u_p(x) = \int_{\mathbb{R}^n} \Gamma(x, y) f_p(y) \ dy = \int_K \Gamma(x, y) f_p(y) \ dy.$$

Then, by Lebesgue's dominated convergence Theorem,

$$u(x) = \lim_{p \to \infty} u_p(x) = \int_K \Gamma(x, y) \lim_{p \to \infty} f_p(y) \, dy,$$

10

for every  $x \in \mathbb{R}^n$ . Actually, we have a stronger result. For every compact set  $G \subseteq \mathbb{R}^n$ ,

$$\begin{aligned} \sup_{G} |u_p - u| &\leq \sup_{x \in G} \left| \int_{K} \Gamma(x, y) (f_p(y) - f(y)) \, dy \right| \\ &\leq \sup_{K} |f_p - f| \sup_{x \in G} \int_{K} \Gamma(x, y) \, dy \\ &= C(G, K) \sup_{K} |f_p - f|. \end{aligned}$$

We explicitly observe that C(G, K) is a strictly positive finite constant. Hence,

(2.5) 
$$\sup_{G} |u_p - u| \longrightarrow 0 \text{ as } p \longrightarrow \infty.$$

Moreover, for every  $p \in \mathbb{N}$ ,

$$u_p \in C^{\infty}(\mathbb{R}^n, \mathbb{R})$$
 and  $\mathcal{L}u_p = -f_p$ .

Then, by identity (1.8),

$$u_p(x) = M_r(u_p)(x) - N_r(\mathcal{L}u_p)(x)$$
$$= M_r(u_p)(x) + N_r(f_p)(x)$$

for every  $p \in \mathbb{N}$ .

We have already noticed that  $u_p(x) \longrightarrow u(x)$  as  $p \longrightarrow \infty$ . To prove (2.4) we now show that

(2.6) 
$$\lim_{p \to \infty} M_r(u_p)(x) = M_r(u)(x)$$

and

(2.7) 
$$\lim_{p \to \infty} N_r(f_p)(x) = N_r(f)(x).$$

For every  $x \in \mathbb{R}^n$  we have

$$|M_{r}(u_{p})(x) - M_{r}(u)(x)| = |M_{r}(u_{p} - u)(x)|$$

$$\leq \sup_{\Omega_{r}(x)} |u_{p} - u|M_{1}(1)(x)|$$

$$= \sup_{\Omega_{r}(x)} |u_{p} - u|.$$

Since  $\overline{\Omega_r(x)}$  is compact (see Lemma 2.1), and keeping in mind (2.5), the last right hand side goes to zero as  $p \longrightarrow \infty$ . Then,

$$|M_r(u_p)(x) - M_r(u)(x)| \longrightarrow 0 \text{ as } p \longrightarrow \infty,$$

proving (2.6).

Let us now prove (2.7). For every  $x \in \mathbb{R}^n$ , we have

$$|N_r(f_p)(x) - N_r(f)(x)| \le |N_r(|f_p - f|)(x)|$$
  
 $\le \sup_{K} |f_p - f|Q_r(x)|$ 

Then, for every compact set  $G \subseteq \mathbb{R}^n$ ,

$$\sup_{G} |N_{r}(f_{p}) - N_{r}(f)| \leq \sup_{K} |f_{p} - f| \sup_{x \in G} |Q_{r}(x)| \\
\leq (\text{by (2.2)}) \quad C_{r}(G) \sup_{K} |f_{p} - f|.$$

So we have proved that  $(N_r(f_p))$  is uniformly convergent to  $N_r(f)$  on every compact subset of  $\mathbb{R}^n$ . This, in particular, implies (2.7).

2.5. In this subsection we complete the proof of Theorem 1.5. To this end we first remark that, thanks to (2.4), for every  $x \in \mathbb{R}^n$ , we have

$$\frac{M_r(u)(x) - u(x)}{Q_r(x)} = -\frac{N_r(f)(x)}{Q_r(x)},$$

so that, as f(x) is constant with respect to  $y \in \Omega_r(x)$ ,

$$\left|\frac{M_r(u)(x) - u(x)}{Q_r(x)} + f(x)\right| = \frac{1}{Q_r(x)} |N_r(f(x) - f)(x)| \\ \leq \sup_{y \in \Omega_r(x)} |f(u) - f(y)|Q_r(x).$$

By Corollary 2.5 and Remark 2.3, the left hand side of the previous inequality goes to zero as  $r \longrightarrow 0$ . Hence,

$$\lim_{r \to 0} \frac{M_r(u)(x) - u(x)}{Q_r(x)} = -f(x)$$

for every  $x \in \mathbb{R}^n$ . This completes the proof of Theorem 1.5.

# Declarations

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