Decisions in Economics and Finance
https://doi.org/10.1007/s10203-023-00408-y

Recycled and non-recycled exhaustible resource: an optimal control strategy for input allocation

Silvia Bertarelli[1] · **Chiara Lodi**[2,3] · **Stefania Ragni**[1]

Received: 14 April 2022 / Accepted: 7 July 2023

Abstract
The United Nations aim to perform a transition toward a sustainable environment where people can live by decoupling economic growth from resource use. Through the definition of the Agenda 2030 and the corresponding sustainable development goals, this transition asks for a lower dependence on non-renewable resources and for the use of recycled materials in a finite term perspective. In this respect, we provide an optimal control model which searches for an efficient allocation of labor between non-recycling and recycling sectors exploiting a given non-renewable resource. The optimization process is carried out over a finite time horizon in accordance with the need of rapidly achieving the targets imposed by the ecological transition. By employing the classical tools of optimal control theory, a complete theoretical analysis of the model well-posedness is developed under the assumption of linear production in both sectors. The approach is applied in order to simulate a hypothetical test case.

Keywords Recycling · Non-renewable resource · Optimal control · Input allocation · Waste disposal

JEL Classification Q61 · Q3 · Q53

✉ Stefania Ragni
stefania.ragni@unife.it

Silvia Bertarelli
silvia.bertarelli@unife.it

Chiara Lodi
chiara.lodi@uniurb.it
http://www.sustainability-seeds.org

1 Department of Economics and Management, University of Ferrara, Via Voltapaletto 11, 44121 Ferrara, Italy

2 Department of Economics, Society, Politics, University of Urbino Carlo Bo, Via Saffi 42, 61029 Urbino, PU, Italy

3 SEEDS, Ferrara, Italy

Published online: 12 August 2023

1 Introduction

Since 2016, through the definition of the Agenda 2030 and the corresponding Sustainable Development Goals, the United Nations (UNs) have drawn a particular attention to a necessary transition toward a sustainable scenario where people and the Earth could coexist and flourish (see United Nations 2015). Today, this transition has already begun, but the urgency of an action is higher than the actual velocity of the transition itself. European Union (EU) has concretized this issue as an urgent deadline in the EU Green Deal with the purpose of reinforcing existing agreements and managing the “just and inclusive transition” transformation (see European Commission 2023). It is required economic growth to be decoupled from resource use; this means that a lower dependence on non-renewable (or depletable) resources and a better waste management are encouraged by the adoption of closed-loop approach including the use of recycled materials. Non-renewables include a large variety of resource stocks, such as oil, coal, gas, uranium, and gold, which are generated through natural processes with very slow formation rate that is timescales relevant to humans; therefore, their endowment is fixed and finite in the Earth’s crust and environmental systems (see Perman et al. 2003; Sweeney 1993; Withagen 1999; Hotelling 1931). Depletable resources have a great importance in the current socio-economic scenario. First of all, most of them still represent important inputs for industrial production; second, their natural stock is limited and geographically concentrated, so that supply shortages may negatively affect global markets. Finally, their waste disposal efficiency and recycling involve crucial issues for the ecological transition. In 2014, the EU Waste Framework Directive outlined the waste management hierarchy which underlines the importance of waste production prevention, re-use, recycling, recovery and ending with disposal. Within this hierarchy, recycling plays a pivotal role since it allows to use generated waste as a new resource for production and it also allows to scale down the demand for extraction of resources from ores (European Environmental Agency 2014). Recycling process is an important part of circular economy approach as well. It minimizes the amount of waste sent to landfill and helps in preventing and reducing the negative socio-economic-environmental effects of using raw natural resources (European Environmental Agency 2021). Given this scenario, this paper deals with the problem of both preserving a non-renewable natural resource, whose shortage could lead to economic and environmental depletion, and reducing the waste sent to landfill by recycling. In the framework of optimal control theory, we search for an optimal allocation of inputs in order to maximize the social welfare determined by consumption patterns and environmental damages, which are generated by the waste disposal and by the usage of the non-renewable depletable input.

The economic literature studied the waste management under a municipality perspective at first. In this context, several papers have dealt with optimal control models for analyzing different waste management strategies, such as recycling procedure and landfilling (see Highfill and McAsey 1997; Huhtala 1997). More recently, a slightly different perspective has been adopted to understand which is the interaction among recycling, employment of natural resource and consumption (see Highfill and McAsey 2001; Huhtala 1999; André and Cerdá 2006). In this approach, both the waste management sector and the conventional production, requiring the resource extraction, are involved in the optimal control process. Inspired and motivated by the current scenario and by the above-mentioned literature, we focus our attention on a specific optimal strategy to control the use of both non-recycled and recycled depletable resources when the time horizon is set as finite. Our approach may be considered as the finite counterpart of the analysis developed in Huhtala (1999), where the study of the economy steady state is carried out over the infinite time horizon.

Our assumption of a fixed and finite temporal threshold is original and crucial. On one hand, it is motivated by the UNs and EU policies which impose that some specific environmental targets must be achieved up to a given year in order to complete the ecological transition. On the other hand, accounting for a finite time horizon also allows to better understand the short-term dynamics of the non-renewable resource management and the corresponding recyclability. Fast responses can potentially support the ecological transition and reduce environmental damages for the forthcoming future. Our model emphasizes the effects of technological characteristics of recycling and extraction programs, as well as the social pressure about environmental issues on policymakers.

Precisely, our model assumes that the production of a consumption good requires the employment of a given amount of input, represented by a non-renewable resource, in its non-recycled and recycled form. The economy is endowed with a fixed amount of labor which is employed between the non-recycling and the recycling sectors. The first sector refers to the production of the raw non-recycled material, while the second one relates to the recovery of the recycled resource from waste disposal. The social planner efficiently allocates labor so that the social welfare is maximized by accounting for both current and future damages. Thus, the main issues faced by the whole economic system are: the environmental damages caused by both the use of non-recycled resource and the accumulation of waste stock, the optimal allocation of labor between non-recycling and recycling sectors and the management of non-recycled depletable resource. From a mathematical viewpoint, the problem is faced by classical techniques of optimal control theory (see Léonard and Van Long 1992; Weber 2011; Halkos and Papageorgiou 2015). The objective function includes a suitable damage function along the whole time horizon together with a scrap value function, which involve both current and future damages due to the employment of non-recycled depletable resource and to the waste accumulation. We provide necessary optimality conditions under the assumption of linear production in both non-recycling and recycling sectors. A complete theoretical analysis is carried out in order to prove the model well-posedness. The study is completed by a simulation of a hypothetical case of a non-renewable resource. By considering different scenarios regarding the technological framework and future damages, our paper contributes to the existing literature as a support for social planners to choose suitable strategies for non-renewable resource conservation and waste recycling management. In this respect, we would like to notice that the optimal allocation of labor together with the related state variables is not available in closed form, and then, they need to be approximated. In the literature, a partitioned symplectic Runge–Kutta scheme is determined as the correct numerical tool for handling the Lagrangian's first-order conditions on the discrete formulation (see Bonnans and Varin 2006; Hager 2000; Ragni et al. 2010). In this respect, we adopt the numerical procedure proposed in Diele et al. (2011), where the nearly Hamiltonian system arising from the necessary optimality conditions is discretized by exploiting the exponential Lawson integration in the framework of a partitioned symplectic Runge–Kutta method.

Concerning the outline of this paper, the starting point for defining the economic model is described in Sect. 2, where the assumptions on consumption and production are stated. Then, the optimal control plan is defined in Sect. 3. We characterize the optimal trajectories for the production and we study their properties in Sect. 4. In this respect, a characterization of the optimal control variable is provided in Sect. 5 under the assumption of linear production. The model is applied for simulating a hypothetical non-renewable resource's employment in a mixed conventional and recycling production process; the results are described in Sect. 6. Finally, conclusions are drawn in Sect. 7.

2 Assumptions on consumption and production

We account for an economy where a final consumption good is produced through a non-renewable resource. This input can be purchased from two sectors: one relates to a conventional production of the resource itself (non-recycling sector) and the other one exploits the existing waste stock obtained by a recycling process (recycling sector). Both inputs are employed for the production of the final good C over a time horizon $[0, T]$, with given length $T > 0$. Therefore, at each time t, $C(t)$ is made up of two different inputs related to the same exhaustible resource: the non-recycled material denoted by $V(t)$ and the recycled material $R(t)$, which is recovered from a waste disposal site. Therefore, production depends on these two factors, respectively; more precisely, it is evaluated by the CES function

$$C = \left[\theta V^{\rho} + (1-\theta) R^{\rho}\right]^{1/\rho}, \tag{1}$$

where $\theta \in (0, 1)$ is a time-invariant parameter with $\frac{\theta}{1-\theta}\left[\frac{R}{V}\right]^{1-\rho}$ representing the technical rate of substitution (in absolute value) and ρ is related to the elasticity of substitution between non-recycled and recycled materials. In the case when the two inputs are qualitatively homogeneous (i.e., they are perfect substitutes), then $\rho = 1$ and the CES function is linear. In our framework, we assume that $0 < \rho < 1$, meaning that the two inputs are imperfect substitutes. This means that the quality of recycled and non-recycled resources is different, and in our context, we may suppose that the quality of the recycled input does not exceed the quality of the non-recycled one.

Concerning the non-recycled stock resource, we denote by $S(t)$ the stock of resource at time t; then, the following equation describes the time evolution of the stock depletion process

$$\begin{aligned} \dot{S}(t) &= -V(t), \\ S(0) &= S_0, \end{aligned} \tag{2}$$

where $S_0 > 0$ represents the initial amount of the available non-recycled input. According to Eq. (2), the evolution over time of the non-recycled input corresponds to the amount of resource implemented in the production process.

From a technical perspective, the consumption process generates waste which can be partially saved and recycled for the production. For the sake of simplicity, both waste production and recycling process are assumed to be instantaneous and occur at the same time. In this framework, let $W(t)$ denote the cumulative amount of waste at time t. As the flow of waste $R(t)$ is used by the recycling sector and thus, removed from the accumulated stock $W(t)$, then the process of waste accumulation is modeled by the following equation

$$\begin{aligned} \dot{W}(t) &= \gamma_1 V(t) - (1-\gamma_2)R(t), \\ W(0) &= W_0, \end{aligned} \tag{3}$$

where $W_0 \geq 0$ represents the initial stock of recyclable waste inherited at the beginning of the process from the past, γ_1 and $(1-\gamma_2)$ denote the waste generation rate of the non-recycled and the recycled materials, respectively. Assume that the constant parameters γ_1 and γ_2 are exogenous such that $\gamma_1, \gamma_2 \in (0, 1)$. In the sequel, we suppose that the waste content rate γ_1 of non-recycled resource exceeds the waste content rate γ_2 of recycled input so that $\gamma_1 > \gamma_2$.

Moreover, we also assume that there is no waste degradation process.

The considered economy is endowed with a fixed amount of labor $L > 0$, which is devoted to get both inputs and is taken as exogenous. Let $l_v(t)$ and $l_r(t)$ be the labor demand used in conventional sector and recycling at every time t, respectively, so that the labor market equilibrium is given by the following condition

$$l_v(t) + l_r(t) = L. \tag{4}$$

The marginal productivity of labor is positive and decreasing in both sectors, then production functions $V(t) = h(l_v)$ and $R(t) = g(l_r)$ satisfy the following features: $h(0) = g(0) = 0$, $h'(l_v) > 0$, $g'(l_r) > 0$, $h''(l_v) \leq 0$ and $g''(l_r) \leq 0$. For the sake of notation, we denote $l(t) = l_v(t)$ for each $t \in [0, T]$. From a mathematical viewpoint, function $l(t)$ lies in the following space of bounded functions:

$$\mathcal{A} = \{l : [0, T] \to \mathbb{R} \;\; l \text{ is Lebesgue measurable and } 0 \leq l(t) \leq L, \;\forall t \in [0, T]\},$$

which represents the admissible set for labor. Due to condition (4), we have $l_r(t) = L - l(t)$ at any time and notice that $l_r \in \mathcal{A}$. In this respect, we set $V(t)$ and $R(t)$ as follows

$$V(t) = h(l(t)), \quad R(t) = g(L - l(t)), \tag{5}$$

for any t. In the sequel, (5) is exploited in (2) and (3), where forcing terms are bounded by the following relationships

$$-h(L) \leq -V(t) \leq 0, \qquad -(1-\gamma_2)g(L) \leq \gamma_1 V(t) - (1-\gamma_2)R(t) \leq \gamma_1 h(L), \tag{6}$$

at any time t. We notice that the previous inequalities hold due to the production function monotonicity. From an economic point of view, the first bound in (6) means that the employment of the non-recycled input cannot exceed its maximum admissible level when the total amount of labor is completely devoted to the non-recycling sector. On the other hand, the second bound completes the first one, but it refers to the recycling sector. More precisely, the waste accumulation of non-renewable resource cannot exceed neither the maximum admissible level of resource when the total amount of labor is used in the non-recycling sector nor the maximum admissible level of resource when labor is totally employed in the recycling sector.

As a further remark, the assumption that labor $l(t)$ is Lebesgue measurable is crucial in the analysis of well-posedness developed in the next Sect. 5. Actually, it implies that right-hand sides can be integrated in (2) and (3); thus, the differential system admits a unique solution. We notice that, by integrating equations (2) and (3) in time and exploiting bounds (6), we get

$$S_0 - h(L)T \leq S(t) \leq S_0,$$

and

$$W_0 + (\gamma_2 - 1)g(L)T \leq W(t) \leq W_0 + \gamma_1 h(L)T,$$

for any time t. In the sequel, we assume that the time horizon length T does not exceed an established temporal threshold so that

$$T \leq \min\left\{\frac{S_0}{h(L)}, \frac{W_0}{(1-\gamma_2)g(L)}\right\}. \tag{7}$$

Condition (7) assures that the following inequalities hold

$$S_0 - h(L)T \geq 0, \qquad W_0 + (\gamma_2 - 1)g(L)T \geq 0. \tag{8}$$

As a consequence, $S(t)$ and $W(t)$ get non-negative values, i.e., $S(t) \geq 0$ and $W(t) \geq 0$, and they are bounded functions of time. From an economic viewpoint, we remark that T does not exceed the minimum value between two different temporal levels. The first one is the time necessary for completely depleting the original non-recycled resource in the case when labor is all devoted to the non-recycling sector. The second level represents the time necessary for letting the recyclable part of waste disposal be empty in the case when labor is totally employed in the recycling sector.

We will see that inequalities in (8) are crucial in order to prove the results in the next Sect. 5.

3 Optimal production program

A crucial issue in the circular economy approach is represented by guaranteeing the maximum welfare benefit from economic activities provided that non-renewable resource stock is maintained over time and the waste recycling capability increases.

In this framework, we analyze the social planner program which consists in efficiently allocating a restricted amount of labor, $l(t)$, in order to maximize the social welfare that accounts for different contributions. The first one consists of the consumers' utility $U(C)$ deriving from the consumption of the final produced good. In order to provide an example of optimal trajectories, we consider a standard isoelastic utility function of parameter $\sigma \in (0, 1)$ given by

$$U(C) = \frac{C^{1-\sigma}}{1-\sigma};$$

therefore, due to (1) and (5), the utility depends on labor l according to the following relationship

$$U(C(l)) = \frac{[\theta\,[h(l)]^\rho + (1-\theta)\,[g(L-l)]^\rho]^{(1-\sigma)/\rho}}{1-\sigma}. \tag{9}$$

We also assume that the social welfare takes into account the environmental benefit and damage related to recycled and non-recycled stocks both in the transient horizon $[0, T]$ and in the future. Actually, social welfare grows according to the benefit $B(S)$ which comes from preserving the non-renewable resource stock $S(t)$ over the whole time horizon. The term $B(S)$ is assumed to be strictly increasing with $B(0) = 0$ and strictly concave in S, then $B'(S) > 0$, $B(S) \geq 0$ and $B''(S) < 0$. We also suppose that social welfare diminishes according to the environmental damage $D(W)$ caused by accumulating waste stock $W(t)$ in the transient at any time over the whole interval $[0, T]$. This damage function is strictly increasing with $D(0) = 0$ and strictly convex in W, then $D'(W) > 0$, $D(W) \geq 0$ and $D''(W) > 0$.

Furthermore, in a future perspective, we add another term $\Psi(S_T, W_T)$ which depends on final values $S(T) = S_T$ and $W(T) = W_T$ and can be interpreted as the value of an integral of future utility flow related to the future damage associated with the resource implementation. Indeed, this term is split into two contributions as $\Psi(S_T, W_T) = \Psi_1(S_T) + \Psi_2(W_T)$; in particular, according to some examples in the literature (see for instance Léonard and Van Long (1992)), we set

$$\Psi_1(S_T) = \nu\, S_T\, e^{-\delta T}, \qquad \Psi_2(W_T) = -\mu\, W_T\, e^{-\delta T},$$

where $\nu > 0$ and $\mu > 0$ represent two constant weights which are exogenously chosen. The first term $\Psi_1(S_T)$ represents the value of an integral of future utility flow related to the future damage associated with the exhaustible original resource implementation, starting from time T with a non-renewable resource stock S_T. We notice that $\Psi_1(S_T)$ is increasing with the value S_T, which means that the future utility flow is higher as S_T increases. In this respect, the damage is as lower as the future utility flow $\Psi_1(S_T)$ gets a higher value, which corresponds to a higher non-renewable input stock S_T. On the other hand, the second term $\Psi_2(W_T)$ can be interpreted as the value of an integral of future utility flow related to the future damage from waste disposal site starting from time T with a waste stock W_T. As $\Psi_2(W_T)$ is decreasing with the value W_T, then the future utility flow is higher as W_T decreases. More precisely, the environmental damage due to the cumulative amount of non-renewable resource in waste disposal is as lower as the future utility flow $\Psi_2(W_T)$ gets a higher value, which corresponds to a smaller value for waste stock W_T.

Under the previous assumptions, the objective function is defined as

$$J(l(\cdot)) = \int_0^T e^{-\delta t}[U(C(l(t))) + B(S(t)) - D(W(t))]\,dt + \Psi(S_T, W_T), \quad (10)$$

where $\delta > 0$ is the constant discount rate over time. Our goal consists of searching for a control $l^* \in \mathcal{A}$ such that

$$J(l^*) = \max_{l \in \mathcal{A}} J(l), \quad (11)$$

subject to conditions (5) and state equations (2) and (3).

4 Optimal trajectory of the economy

With the aim of characterizing an optimal strategy for allocating the labor devoted to non-recycling and recycling sectors, we first define the current value Hamiltonian $H = H(l, S, W, \varphi_1, \varphi_2)$ as

$$H = \frac{[C(l)]^{1-\sigma}}{1-\sigma} + B(S) - D(W) + \varphi_1\,[-h(l)] + \varphi_2\,[\gamma_1 h(l) - (1-\gamma_2) g(L-l)],$$

where the costate variables φ_1 and φ_2 are the shadow prices of the non-recycled resource and recyclable stock, respectively. According to the well-known Pontryagin's Maximum Principle, we notice that an optimal solution $l^*(t)$ together with state variables $S(t)$, $W(t)$ and costate ones $\varphi_1(t)$, $\varphi_2(t)$ must satisfy the following optimality necessary conditions:

(i) $l^*(t)$ maximizes $H(l, S(t), W(t), \varphi_1(t), \varphi_2(t))$, provided that $l^*(t)$ belongs to the admissible set $[0, L]$;

(ii) state dynamics is described by (2)-(3), where the role of $l(t)$ is played by $l^*(t)$, and costate variables are continuous functions of t with piecewise continuous derivatives which satisfy the following equations

$$\dot{\varphi}_1(t) = \delta\varphi_1(t) - B'(S(t)), \qquad \varphi_1(T) = \nu\, e^{-\delta T},$$

and

$$\dot{\varphi}_2(t) = \delta\varphi_2(t) + D'(W(t)), \qquad \varphi_2(T) = -\mu\, e^{-\delta T}.$$

These equations can be integrated in order to obtain

$$\varphi_1(t) = \nu e^{-\delta(2T-t)} + \int_t^T e^{-\delta(s-t)} B'(S(s))ds, \tag{12}$$

$$\varphi_2(t) = -\mu e^{-\delta(2T-t)} - \int_t^T e^{-\delta(s-t)} D'(W(s))ds, \tag{13}$$

for all $t \in [0, T]$. It is worthwhile to notice that φ_1 gets non-negative values since the non-recycled resource stock is a "good" for the society. This means that the larger the stock of non-recycled resource, the better is for the society, since we are indirectly preserving Earth resources. Furthermore, φ_2 gets negative values, as expected, since the waste stock may represent a "bad" for the society. Specifically, the larger is the waste stock, the worse is social welfare since waste harms the environment; on the other hand, a larger waste stock implies a larger stock for recycling. The condition $\varphi_2 \leq 0$ yields that the negative effect of harming the environment prevails over the positive effect of increasing the endowment of recyclable resource.

As we search for l^* maximizing the Hamiltonian function H so that the necessary optimality condition stated in item (i) is satisfied, then we need to evaluate the following derivative

$$\frac{\partial H}{\partial l} = C(l)^{1-\sigma-\rho} \cdot \Upsilon(l) - \Phi(l, \varphi_1, \varphi_2),$$

where we set

$$\Phi(l, \varphi_1, \varphi_2) = \varphi_1 h'(l) - \varphi_2 \left[\gamma_1 h'(l) + (1 - \gamma_2) g'(L - l)\right],$$

and

$$\Upsilon(l) = \theta \frac{h'(l)}{h(l)^{1-\rho}} - (1 - \theta) \frac{g'(L - l)}{g(L - l)^{1-\rho}}.$$

We notice that

$$\frac{dC}{dl} = [C(l)]^{1-\rho} \Upsilon(l),$$

then $\Phi(l, \varphi_1, \varphi_2)$ is the marginal benefit related to the recycled input and the marginal consumption arising from the employment of a marginal unit of labor in the non-recycling sector depends on $\Upsilon(l)$. We also remark that, due to the assumptions on functions $h(l)$ and $g(l)$, the following Inada conditions hold:

$$\lim_{l \to 0^+} \Upsilon(l) = +\infty, \qquad \lim_{l \to L^-} \Upsilon(l) = -\infty. \tag{14}$$

They are crucial in order to prove the next result providing a necessary condition for optimal control.

Proposition 1 *Let $l^* \in \mathcal{A}$ be an optimal control. Then, there exists a constant value $\widehat{L} \leq L$ such that $\Upsilon(\widehat{L}) = 0$ and the following relationships hold for all $t \in [0, T]$:*

$$0 \leq l^*(t) \leq \widehat{L}, \tag{15}$$

and

$$C(l^*(t))^{1-\sigma-\rho} \cdot \Upsilon(l^*(t)) = \Phi(l^*(t), \varphi_1(t), \varphi_2(t)). \tag{16}$$

Furthermore, the optimal labor satisfies the following condition

$$\frac{\theta}{1-\theta}\left[\frac{g(L-l^*(t))}{h(l^*(t))}\right]^{1-\rho} \geq \frac{g'(L-l^*(t))}{h'(l^*(t))}, \tag{17}$$

at any time $t \in [0, T]$.

Proof We assume that l^* represents an optimal control and exploit the necessary condition stated in the previous item (i). In this respect, according to the evaluation of the shadow prices in (12) and (13), we have $\varphi_1 > 0$, $\varphi_2 < 0$; moreover, $h'(l) > 0$, $g'(L-l) > 0$ for all l. Then, Φ gets finite values such that $\Phi(l, \varphi_1, \varphi_2) > 0$ for any l, φ_1 and φ_2. Furthermore, as already mentioned, we have $\Upsilon(l) \to +\infty$ as $l \to 0^+$ and $\Upsilon(l) \to -\infty$ as $l \to L^-$. In addition, we get

$$\begin{aligned}\Upsilon'(l) = (\rho-1)\left(\theta[h(l)]^{\rho-2}[h'(l)]^2 + (1-\theta)[g(L-l)]^{\rho-2}[g'(L-l)]^2\right)\\ +\theta[h(l)]^{\rho-1}h''(l) + (1-\theta)[g(L-l)]^{\rho-1}g''(L-l) < 0,\end{aligned}$$

for any $0 < l < L$. Therefore, $\Upsilon(l)$ is decreasing with respect to l; thus, the existence of a constant threshold $\widehat{L}$, $0 < \widehat{L} < L$, can be established such that $\Upsilon(\widehat{L}) = 0$ and $\Upsilon(l) > 0$ for $l \in (0, \widehat{L})$.

In this framework, we recall that $0 < \rho < 1$ and notice that C gets finite values such that $C(l) > 0$ at each $l \in \mathcal{A}$. Then, for any φ_1 and φ_2, Inada conditions (14) yield $[C(l)]^{1-\sigma-\rho} \cdot \Upsilon(l) - \Phi(l, \varphi_1, \varphi_2) \to +\infty$ as $l \to 0^+$ and $[C(l)]^{1-\sigma-\rho} \cdot \Upsilon(l) - \Phi(l, \varphi_1, \varphi_2) \to -\infty$ as $l \to L^-$.

As a conclusion, $\partial H/\partial l \to +\infty$ when $l \to 0^+$ and $\partial H/\partial l \to -\infty$ when $l \to L^-$. Then, since function $\partial H/\partial l$ is continuous with respect to l, it follows that $\partial H/\partial l$ nullifies at $l^*(t)$ which lies inside interval $(0, L)$. This assertion yields (16).

In addition, due to condition $\Phi > 0$, l^* has to satisfy relationship $\Upsilon(l^*(t)) > 0$ which yields conditions (15) and (17) to be satisfied. In this way, the proof is completed. □

In the previous proof, we get $\Upsilon(l^*) > 0$: this condition means that the optimal path l^* is reached in the case when the consumption is increasing with respect to the labor employed in the non-recycling sector.

Moreover, equation (16), which represents the so-called Euler condition, states that the optimal labor allocation occurs when the marginal benefit of employing one more unit of labor in the non-recycling sector equals the marginal cost related to both the non-recyclable input depletion and the waste stock accumulation.

5 Optimal labor under linear production and special benefit and damage functions

As already mentioned, Proposition 1 states optimal control features arising from the Pontryagin's Maximum Principle. Anyway, the arguments developed in its proof cannot be exploited in order to state the existence of an optimal control solution under general assumptions on functions $h(l)$ and $g(l)$. Actually, convexity requirements for both production functions $h(l)$ and $g(l)$ are not enough to assure that the same convexity feature holds for the Hamiltonian function too. However, the further assumption of linear production in both non-recycling and recycling sectors simplifies the problem in a support for a complete theoretical analysis of well-posedness. For this reason, we suppose that

$$V(t) = h(l(t)) = m_1 l(t), \qquad R(t) = g(L - l(t)) = m_2(L - l(t)), \tag{18}$$

at any time, where the labor productivities $m_1 > 0$ and $m_2 > 0$ are fixed and exogenous. According to equations (2) and (3), state dynamics is described by

$$\dot{S}(t) = -m_1\, l(t),$$

and

$$\dot{W}(t) = \gamma_1\, m_1\, l(t) - (1 - \gamma_2)\, m_2\, (L - l(t)),$$

in time horizon $[0, T]$. First of all, we notice that the length T is assumed to be under the threshold established in (7), then conditions (8) hold and the state variables are bounded functions according to the following relationships:

$$0 \le \underline{S} \le S(t) \le S_0, \qquad 0 \le \underline{W} \le W(t) \le \overline{W}, \qquad t \in [0, T], \tag{19}$$

where $\underline{S} = S_0 - m_1 L T$, $\underline{W} = W_0 + (\gamma_2 - 1) m_2 L T$ and $\overline{W} = W_0 + \gamma_1 m_1 L T$.

Furthermore, the benefit $B(S)$ and the damage $D(W)$ are defined as the following functions:

$$B(S) = 2 c_S \sqrt{S}, \qquad D(W) = c_W \frac{W^2}{2}, \tag{20}$$

where $c_S > 0$ and $c_W > 0$ represent the weights of the benefit and damage, respectively, in the optimization process during the whole time horizon. It is evident that, under this choice, we have $B'(S) = c_S/\sqrt{S}$ and $D'(W) = c_W\, W$.

In the framework described so far, the following result provides a proof of existence and uniqueness for the optimal solution.

Proposition 2 *Under assumptions (18) and (20), there exists a labor function $l^* \in \mathcal{A}$ which is the unique optimal solution of problem (11).*

Proof As a first step, we notice that the existence of an optimal control l^* can be proved by employing the results outlined for a general minimum Bolza problem in Theorem 4.1 and Corollary 4.1, page 68, in Fleming and Rishel (1975). With this aim, we write the problem in equivalent form with similar notation as in Fleming and Rishel (1975): the state variables are arranged in vector form such that $x(t) = (S(t), W(t))^T$ and the corresponding differential system is revisited as

$$\dot{x}(t) = f(l(t)),$$

where $x(0) = x_0 := (S_0, W_0)^T$ and the right-hand side is defined as $f(u) = \alpha + \beta l$ with vector coefficients given by $\alpha = (0, (\gamma_2 - 1)m_2 L)^T$ and $\beta = (-m_1, \gamma_1 m_1 + (1 - \gamma_2)m_2)^T$. In addition, by interpreting the definition of feasible class $\mathcal{F}'$ given in Fleming and Rishel (1975), here $\mathcal{F}'$ merely corresponds to the class of all pairs (x_0, l) such that $l \in \mathcal{A}$ and the state variable can be integrated as

$$x(t) = x(0) + \int_0^t f(l(s))\, ds, \qquad 0 \le t \le T, \tag{21}$$

by prescribing the initial condition $x(0) = x_0$ which is given and fixed in our problem. Furthermore, using definitions in (20) of benefit and damage functions, we consider the functional

$$\mathcal{L}(t, x(t), l(t)) = e^{-\delta t}\left(c_W \frac{[x^{(2)}(t)]^2}{2} - 2c_S\sqrt{x^{(1)}(t)} - \frac{[C(l(t))]^{1-\sigma}}{1-\sigma}\right),$$

with $x^{(1)}(t)$ and $x^{(2)}(t)$ corresponding to the entries of vector $x(t)$ for any time t. In this framework, the so-called performance index can be defined as

$$\widetilde{J}(x_0, l) = \int_0^T \mathcal{L}(t, x(t), l(t))\, dt - \Psi(x^{(1)}(T), x^{(2)}(T), T).$$

We remark that our functional $J(l)$ in (11) takes on the same value as $\widetilde{J}(x_0, l)$ but opposite in sign (i.e., $\widetilde{J}(x_0, l) = -J(l)$). As a consequence, our model (11) is equivalent to the problem which consists of finding a control $l^* \in \mathcal{A}$ such that the corresponding performance index $\widetilde{J}(x_0, l^*)$ is minimized in the class $\mathcal{F}'$. The result provided by Theorem 4.1 and Corollary 4.1 in Fleming and Rishel (1975) represents a sufficient condition for the existence of a solution, and it is employed for the specific problem we face. Actually, in our case, it can be applied since $f(u)$ is continuous and there exists a positive constant $C_1 = \max\{|\alpha|, |\beta|\}$ such that $|f(l)| \le C_1(1 + |l|)$ for all $l \in \mathcal{A}$. In addition, Theorem 4.1 in Fleming and Rishel (1975) relies on further assumptions (a), (b), (c), (d), (e) which hold in our case according to the following items:

- The assumption (a) stated in Theorem 4.1 of Fleming and Rishel (1975) is merely satisfied; indeed, the feasible class $\mathcal{F}'$ is not empty due to the fact that it is possible to pick any $l \in \mathcal{A}$ which is Lebesgue integrable, then the state system admits a unique solution which is obtained by the integration in (21).

- The subset of $\mathbb{R}$ involved in the assumption (b) stated in Theorem 4.1 of Fleming and Rishel (1975) corresponds to the interval $[0, L]$ where each labor function $l(t)$ has its value. The same assumption is verified as $[0, L]$ is a closed set.
- Due to (19), $x(t)$ is bounded for any t; therefore, each labor $l \in \mathcal{A}$ steers $x(T)$ to a compact set where the scrap function Ψ is continuous with respect to its inputs. Then, the assumption (c) stated in Theorem 4.1 of Fleming and Rishel (1975) is satisfied.
- The assumption (d) in Theorem 4.1 is replaced by the corresponding (d') stated in Corollary 4.1 of Fleming and Rishel (1975); it is satisfied since $\mathcal{L}(t, x, \cdot)$ is convex in $[0, L]$. Indeed, it is not so difficult to verify that

$$\frac{\partial \mathcal{L}}{\partial l} = e^{-\delta t}[C(l)]^{-\sigma}\left(\sigma[C(l)]^{-1}\left(\frac{dC}{dl}\right)^2 - \frac{d^2C}{dl^2}\right) > 0,$$

since

$$\frac{d^2C}{dl^2} = (\rho-1)[C(l)]^{1-2\rho}\,\theta(1-\theta)L^2[m_1 m_2]^\rho \cdot [l(L-l)]^{\rho-2} < 0,$$

under the assumption $0 < \rho < 1$.
- Also, the assumption (e) in Theorem 4.1 is replaced by the corresponding (e') stated in Corollary 4.1 of Fleming and Rishel (1975). In this respect, due to the bounds in (19), for any $t \in [0, T]$ we have $0 < \sqrt{x^{(1)}(t)} \leq \sqrt{S_0}$ and

$$[x^{(2)}(t)]^2 \geq \underline{W}^2 \geq (1-\gamma_2)^2\,(m_2 T)^2\,[l(t)]^2 - 2W_0(1-\gamma_2)m_2 LT.$$

Furthermore, due to the fact that the consumption function $C(\cdot)$ is continuous, we may consider its maximum value $\overline{C} = \max_{0\leq z\leq L} C(z)$. It follows that there exist two constants $c_1 = e^{-\delta T}(c_W/2)(1-\gamma_2)^2\,(m_2 T)^2 > 0$ and $c_2 = e^{-\delta T}(c_W W_0(1-\gamma_2)m_2 LT + 2c_S\sqrt{S_0} + \overline{C}^{1-\sigma}/(1-\sigma)) > 0$ such that

$$\mathcal{L}(t, x(t), l(t)) \geq c_1|l(t)|^2 - c_2.$$

Thus, assumption (e') stated in Corollary 4.1 of Fleming and Rishel (1975) is satisfied.

Under the previous argument, all the assumptions stated in Theorem 4.1 of Fleming and Rishel (1975) hold; as a consequence, there exists $l^* \in \mathcal{A}$ minimizing the performance index $\widetilde{J}(x_0, l)$ on the class $\mathcal{F}'$. Due to the equivalence between the two problems, l^* represents an optimal control which solves also our model (11). Therefore, the existence of a solution is proved for the problem at hand.

As the next step, we focus on the uniqueness proof. We notice that, in correspondence with any optimal solution l^* of (11), the Hamiltonian function $H(l, S, W, \varphi_1, \varphi_2)$ defined in the previous Sect. 4 is maximized with respect to l, according to the Maximum Principle statement (i). In this respect, we remark that it is not so difficult to get the second derivative:

$$\frac{\partial^2 H}{\partial l^2} = [C(l)]^{1-\sigma-2\rho} \left(-\sigma\,[\Upsilon(l)]^2 + (\rho-1)\theta(1-\theta)L^2[m_1 m_2]^\rho[l(L-l)]^{\rho-2}\right).$$

Since $\partial^2 H/\partial l^2 < 0$, then the Hamiltonian function is strictly concave in l and it can admit no more than one maximum value. As a conclusion, problem (11) can admit no more than one solution. □

In this framework, under the assumption of linear production, condition (15) is satisfied by setting

$$\widehat{L} = \frac{L}{1+\left[\frac{1-\theta}{\theta}\right]^{1/(1-\rho)}\left[\frac{m_2}{m_1}\right]^{\rho/(1-\rho)}}.$$

Remark 1 We notice that the choice of the temporal threshold T assures that relationships (19) hold: this is crucial to bound both costate variables $\varphi_1(t)$ and $\varphi_2(t)$. Actually, due to the previous assumptions, the derivative $B'(S) = c_S/\sqrt{S}$ is decreasing with respect to S so that $0 < B'(S_0) \leq B'(S) \leq B'(\underline{S})$ for each S, with $\underline{S} \leq S \leq S_0$. Then, from (12) we get

$$\nu e^{-\delta(2T-t)} < \varphi_1(t) \leq \nu e^{-\delta(2T-t)} + \frac{B'(\underline{S})}{\delta}\left(1-e^{-\delta(T-t)}\right),$$

which yields the following bounds

$$\nu e^{-\delta(2T-t)} < \varphi_1(t) \leq \nu e^{-\delta(2T-t)} + \frac{B'(\underline{S})}{\delta}, \tag{22}$$

that holds for any $t \in [0, T]$. By a similar argument, starting from $D'(W) = c_W\,W$, we get $0 < D'(\underline{W}) \leq D'(W) \leq D'(\overline{W})$ for each W, with $\underline{W} \leq W \leq \overline{W}$; these inequalities yield the bounds

$$-\mu e^{-\delta(2T-t)} - \frac{D'(\overline{W})}{\delta} \leq \varphi_2(t) < -\mu e^{-\delta(2T-t)}, \tag{23}$$

for any $t \in [0, T]$. Then, we set

$$\begin{aligned} \overline{\chi} &= (m_1\nu + (\gamma_1 m_1 + (1-\gamma_2)m_2)\,\mu)\,e^{-\delta(T-t)}, \\ \overline{\overline{\chi}} &= \overline{\chi} + \frac{m_1 B'(\underline{S}) + (\gamma_1 m_1 + (1-\gamma_2)m_2)\,D'(\overline{W})}{\delta} \end{aligned} \tag{24}$$

and notice that, in correspondence with any labor strategy $l \in \mathcal{A}$, it is possible to obtain the following relationship

$$\underline{\chi} \leq \Phi(l, \varphi_1, \varphi_2) \leq \overline{\overline{\chi}}, \tag{25}$$

which holds for any $t \in [0, T]$. These bounds for the term Φ are exploited in order to prove the results in the next Subsections 5.1 and 5.2.

5.1 Waste stock accumulation dynamics

A crucial issue of the current analysis deals with the way of accumulating/reducing waste over time. Actually, we are interested in investigating the rate of accumulation in order to understand whether waste stock increases or not. It is easy to verify that the inequality $\dot{W}(t) \geq 0$ holds in the case when the optimal labor allocation exceeds the level $\overline{L}$ defined by

$$\overline{L} = \frac{(1-\gamma_2)m_2}{\gamma_1 m_1 + (1-\gamma_2)m_2} L. \tag{26}$$

It follows that $W(t)$ increases over a given time interval in the case when the following relationship holds

$$l^*(t) \geq \overline{L}; \tag{27}$$

on the other hand, $W(t)$ is decreasing in time in the opposite case. Therefore, the comparison between the optimal labor l^* and the threshold $\overline{L}$ is crucial in order to establish waste accumulation dynamics over time. This comparison is carried out in the next Proposition which starts from some assumptions on the value

$$\Upsilon(\overline{L}) = \theta m_1^\rho \overline{L}^{\rho-1} - (1-\theta)m_2^\rho (L - \overline{L})^{\rho-1}.$$

We recall that $\frac{dC}{dl}(\overline{L}) = [C(\overline{L})]^{1-\rho}\Upsilon(\overline{L})$, then $\Upsilon(\overline{L})$ is related to the rate of variation of the consumption good in correspondence with a marginal increase in labor employed in the non-recycling sector at $l = \overline{L}$. The following result provides some sufficient conditions for predicting the monotonic behavior of waste stock.

Proposition 3 *We consider the following two different situations.*

(a) When we assume that

$$\Upsilon(\overline{L}) = \theta m_1^\rho \overline{L}^{\rho-1} - (1-\theta)m_2^\rho (L - \overline{L})^{\rho-1} > 0, \tag{28}$$

we set

$$\overline{\tau} = \frac{[C(\overline{L})]^{1-\sigma-\rho}\,\Upsilon(\overline{L})}{\nu m_1 + \mu(\gamma_1 m_1 + (1-\gamma_2)m_2)},$$

and

$$\overline{\overline{\tau}} = \overline{\tau} - \frac{m_1 B'(\underline{S}) + (\gamma_1 m_1 + (1-\gamma_2)m_2)D'(\overline{W})}{\delta \cdot (\nu m_1 + \mu(\gamma_1 m_1 + (1-\gamma_2)m_2))};$$

then we prove the following statements:

(a_1) if $\overline{\tau} < e^{-\delta 2T}$, *then the waste stock reduces over the whole time horizon* $[0, T]$*;*
(a_2) if $e^{-\delta 2T} < \overline{\overline{\tau}} < \overline{\tau} < e^{-\delta T}$, *then there exist two temporal thresholds*

$$\bar{t} = 2T + \frac{\ln(\overline{\tau})}{\delta}, \quad \bar{\bar{t}} = 2T + \frac{\ln(\overline{\overline{\tau}})}{\delta}, \qquad 0 < \bar{\bar{t}} < \bar{t} < T, \tag{29}$$

such that the waste stock $W(t)$ *accumulates in the initial part of the time horizon for* $t \in [0, \bar{\bar{t}}]$, *after that it reduces for* $t \in [\bar{t}, T]$*;*
(a_3) if $e^{-\delta T} < \overline{\overline{\tau}}$, *then the waste stock accumulates over the whole time horizon* $[0, T]$.

(b) In the opposite situation, under the assumption that

$$\Upsilon(\overline{L}) = \theta m_1^{\rho} \overline{L}^{\rho-1} - (1-\theta) m_2^{\rho} (L - \overline{L})^{\rho-1} \leq 0, \tag{30}$$

the waste stock is decreasing in time over the whole horizon $[0, T]$.

Proof As already mentioned, if condition (27) holds, then waste stock increases; in the opposite case, $W(t)$ is decreasing. Due to the results in Proposition 1, condition (27) can be satisfied for any time t such that

$$\frac{\partial H}{\partial l}(\overline{L}, S(t), W(t), \varphi_1(t), \varphi_2(t)) > 0, \tag{31}$$

which means that $\frac{\partial H}{\partial l}$ gets non-negative values at $l = \overline{L}$.

As what concerns the first situation (a), we assume that (28) holds. We notice that, due to a priori estimate (25) where $\overline{\chi}$ and $\overline{\overline{\chi}}$ are defined as in (24), if the following inequality holds

$$\overline{\overline{\chi}} < C(\overline{L})^{1-\sigma-\rho}\Upsilon(\overline{L}), \tag{32}$$

then we get

$$\Phi(\overline{L}, \varphi_1(t), \varphi_2(t)) < C(\overline{L})^{1-\sigma-\rho}\Upsilon(\overline{L}).$$

As a consequence, (32) represents a sufficient condition for satisfying (31) and (27). In this respect, we notice that inequality (32) holds for any t such that $e^{\delta t} < e^{\delta 2T}\overline{\overline{\tau}}$.

On the other hand, it is possible to verify that the other inequality

$$C(\overline{L})^{1-\sigma-\rho}\Upsilon(\overline{L}) < \overline{\chi}, \tag{33}$$

represents a sufficient condition for having

$$\frac{\partial H}{\partial l}(\overline{L}, S(t), W(t), \varphi_1(t), \varphi_2(t)) < 0, \tag{34}$$

so that $l^*(t) < \overline{L}$. In particular, (33) holds for any t such that $\overline{\tau}e^{\delta 2T} < e^{\delta t}$.

Starting from this argument, we may prove the three different statements. The first assumption $\overline{\tau} < e^{-\delta 2T}$ yields $\overline{\tau}e^{\delta 2T} \leq 1 < e^{\delta t}$ at each time $t \in [0, T]$; it follows that (33) is satisfied, and it implies (31) for any time. Therefore, we get $l^*(t) < \overline{L}$ so that $\dot{W}(t) \leq 0$ over the whole time horizon. It follows that waste stock reduces at every time.

In the second case when $e^{-\delta 2T} < \overline{\overline{\tau}} < \overline{\tau} < e^{-\delta T}$, we get $0 < \overline{\overline{t}} < \overline{t} < T$. Under this assumption, starting from $0 < t < \overline{\overline{t}}$, we exploit the exponential function and we get that inequality $e^{\delta t} < e^{\delta 2T}\overline{\overline{\tau}}$ holds; thus, condition (32) is satisfied for all $t \in [0, \overline{\overline{t}}]$. Due to the previous argument, we obtain $\dot{W}(t) > 0$ so that the waste stock is increasing in the same interval $[0, \overline{\overline{t}}]$. Moreover, when we start from $\overline{t} < t < T$ and apply the exponential function, then inequality $\overline{\tau}e^{\delta 2T} < e^{\delta t}$ holds; thus, condition (33) is satisfied for all $t \in [\overline{t}, T]$. Hence, $\dot{W}(t) < 0$ so that the waste stock is decreasing in the same interval $[\overline{t}, T]$.

Under the third assumption $e^{-\delta T} < \overline{\overline{\tau}}$, we have $e^{\delta t} \leq e^{\delta T} < e^{\delta 2T}\overline{\overline{\tau}}$ for any $0 \leq t \leq T$, which yields (33) at each time in $[0, T]$. It follows that waste reduces over the whole time horizon.

Concerning the second situation (b), we notice that assumption (30) yields

$$\frac{\partial H}{\partial l}(\overline{L}, S(t), W(t), \varphi_1(t), \varphi_2(t)) < 0;$$

then we get $l^*(t) \leq \widehat{L} \leq \overline{L}$ for all $t \in [0, T]$ due to the results in the proof of Proposition 1. It follows that $\dot{W}(t) \leq 0$ over the whole time horizon; therefore, the waste stock reduces for all $t \in [0, T]$. □

According to the previous results, waste accumulation/reduction is strictly related to the values of weights ν and μ. Actually, a possible condition to assure global reduction in waste consists of prescribing weights ν and μ which are sufficiently large in comparison with an economic system characterized by patient agents (i.e., small value for δ) in a relatively short time horizon (i.e., short final time T): in this case, the quantity $\nu m_1 + \mu(\gamma_1 m_1 + (1-\gamma_2)m_2)$ may be large enough to let condition $\overline{\tau} \leq e^{-\delta 2T}$ hold. In the opposite case, an accumulation of waste happens either during an initial period of time or, in the worst case, in the whole time horizon.

5.2 Even distribution of labor between sectors

Another interesting issue consists of arguing about the case when the same amount of labor is employed both in the non-recycling sector and in the recycling one, i.e., $l^*(t) = L/2$ at a certain $\widehat{t} \in [0, T]$. In this respect, we provide the following result.

Proposition 4 *Suppose that* $\theta m_1^{\rho} - (1-\theta)m_2^{\rho} > 0$*; then, set*

$$\widehat{\tau} = \frac{[C(L/2)]^{1-\sigma-\rho}\,\Upsilon(L/2)}{\nu m_1 + \mu(\gamma_1 m_1 + (1-\gamma_2)m_2)},$$

and

$$\widehat{\widehat{\tau}} = \widehat{\tau} - \frac{m_1 B'(\underline{S}) + (\gamma_1 m_1 + (1-\gamma_2) m_2) D'(\overline{W})}{\delta \cdot (\nu m_1 + \mu(\gamma_1 m_1 + (1-\gamma_2) m_2))}.$$

Under the further assumption

$$e^{-\delta 2T} < \widehat{\widehat{\tau}} < \widehat{\tau} < e^{-\delta T}, \tag{35}$$

define $\widehat{t_1} = 2T + \ln(\widehat{\widehat{\tau}})/\delta$ *and* $\widehat{t_2} = 2T + \ln(\widehat{\tau})/\delta$. *It is possible to prove that*

(a) for every $t \in [0, \widehat{t_1}]$ *the optimal choice* $l^*(t)$ *is over* $L/2$, *i.e.,* $l^*(t) > L/2$;
(b) on the other hand, for every $t \in [\widehat{t_2}, T]$ *the optimal choice* $l^*(t)$ *is under* $L/2$, *i.e.,* $l^*(t) < L/2$.

Proof Condition $\theta m_1^\rho - (1-\theta) m_2^\rho > 0$ is sufficient to assure that $\Upsilon(L/2) > 0$. Moreover, under the assumption that (35) is verified, then we obtain $0 < \widehat{t_1} < \widehat{t_2} < T$. By exploiting the same approach developed in the proof of the previous Proposition 3, it is possible to verify that the following different situations occur:

(a) for any $t \in [0, \widehat{t_1}]$ we get $e^{\delta t} < e^{\delta 2T} \widehat{\widehat{\tau}}$, which yields

$$\frac{\partial H}{\partial l}(L/2, S(t), W(t), \varphi_1(t), \varphi_2(t)) > 0,$$

therefore, the optimal choice at t satisfies the condition $l^*(t) > L/2$;
(b) for any $t \in [\widehat{t_2}, T]$, we get $e^{\delta 2T} \widehat{\tau} < e^{\delta t}$, which implies

$$\frac{\partial H}{\partial l}(L/2, S(t), W(t), \varphi_1(t), \varphi_2(t)) < 0,$$

therefore the optimal choice at t satisfies the condition $l^*(t) < L/2$.

Thus, the proof is completed. □

Remark 2 As a consequence of the previous Proposition 4, we may argue that any switching time $\widehat{t}$ such that $l^*(\widehat{t}) = L/2$ may be located between $\widehat{t_1}$ and $\widehat{t_2}$, i.e.,

$$0 < \widehat{t_1} < \widehat{t} < \widehat{t_2} < T.$$

The role of this temporal threshold deserves attention: actually $\widehat{t}$ represents a switching time where the social planner moves on preferring to employ more labor in a resource sector rather than in the other one.

Remark 3 In the benchmark case, when both sectors are supposed to be equally productive (i.e., $m_1 = m_2$), condition $\theta m_1^\rho - (1-\theta) m_2^\rho > 0$ yields $2\theta - 1 > 0$; then we obtain

$$\widehat{\tau} = \frac{[m_1 L/2]^{-\sigma} (2\theta - 1)}{\nu + \mu(\gamma_1 + 1 - \gamma_2)}.$$

It follows that both parameters $\widehat{\tau}$ and $\widehat{\widehat{\tau}}$ are independent of ρ, thus switching time $\widehat{t}$ lies in the temporal interval $[\widehat{t_1}, \widehat{t_2}]$ which does not vary according to the elasticity of substitution between the resources.

Remark 4 As a final remark related to the previous results in Propositions 3 and 4, we notice that the temporal thresholds $\bar{t}$, $\bar{\bar{t}}$, $\widehat{t_1}$ and $\widehat{t_2}$ decrease with respect to the values chosen for ν and μ with rates

$$\frac{\partial \bar{t}}{\partial \nu} = \frac{\partial \bar{\bar{t}}}{\partial \nu} = \frac{\partial \widehat{t_1}}{\partial \nu} = \frac{\partial \widehat{t_2}}{\partial \nu} = -\frac{m_1}{\delta(\nu m_1 + \mu(\gamma_1 m_1 + (1-\gamma_2)m_2))},$$

and

$$\frac{\partial \bar{t}}{\partial \mu} = \frac{\partial \bar{\bar{t}}}{\partial \mu} = \frac{\partial \widehat{t_1}}{\partial \mu} = \frac{\partial \widehat{t_2}}{\partial \mu} = -\frac{\gamma_1 m_1 + (1-\gamma_2)m_2}{\delta(\nu m_1 + \mu(\gamma_1 m_1 + (1-\gamma_2)m_2))}.$$

As a consequence, when the social planner chooses a certain policy by fixing given values of damage weights ν and μ, then the intertemporal dynamics of the variables at hand updates according to non-recycling and recycling technologies through the quantities m_1 and $\gamma_1 m_1 + (1-\gamma_2)m_2$. Actually, we get

$$\frac{\partial \bar{t}}{\partial \nu} = \frac{m_1}{\gamma_1 m_1 + (1-\gamma_2)m_2} \cdot \frac{\partial \bar{t}}{\partial \mu}.$$

Thus, threshold $\bar{t}$ decreases with the same rate with respect to both ν and μ under the assumption that

$$\frac{m_1}{m_2} = \frac{1-\gamma_2}{1-\gamma_1}.$$

On the other hand, the rate of reduction $\frac{\partial \bar{t}}{\partial \nu}$ overcomes the other one $\frac{\partial \bar{t}}{\partial \mu}$ in the case when

$$\frac{m_1}{m_2} < \frac{1-\gamma_2}{1-\gamma_1}.$$

In this respect, we may get similar dynamics by changing the ratio m_1/m_2 rather than $(1-\gamma_2)/(1-\gamma_1)$. Similar argument can be developed as what concerns the rates of reduction for $\bar{\bar{t}}, \widehat{t_1}, \widehat{t_2}$.

6 Discussion of some results from simulating a hypothetical test case

In order to evaluate the optimal allocation of labor between non-recycling and recycling sectors, we perform some simulations related to a hypothetical case of non-renewable resource. We provide a counterfactual analysis of the model by considering both consumption and production decisions, and the corresponding external effects in terms of future damages. We account for different interesting scenarios, where all the variables are dimensionless.

Starting from the parameters of the consumption function, we set $\theta = 0.6$, which represents the contribution of the non-recycled input on total production (and consumption) of the final good. Setting this parameter higher than 0.5 means that the market share of non-recycled resource producers is higher than the recycling firms' one; this situation occurs when the recycling market is less developed than the conventional one and reflects the current state of the recycling sector all around the world. Furthermore, we consider different values of the elasticity of substitution between non-recycled and recycled materials: $\rho = 0.005$, $\rho = 0.5$ and $\rho = 0.955$ such that $0 < \rho < 1$.

The starting value of non-recycled resource stock, S_0, is normalized and fixed at 100, while the initial stock of recycled resource, W_0, is set at 10.

Concerning the waste accumulation process, as already pointed out in Sect. 2, we set $\gamma_1 = 0.6$ and $\gamma_2 = 0.5$. According to this condition, the waste content rate γ_1 of non-recycled resource exceeds the waste content rate γ_2 of recycled input; it is consistent with the assumption that the content of non-recycled resource exceeds the content of recycled input in making consumption good, due to the previous setting $\theta = 0.6 > 0.5$.

Finally, we assume the discount rate δ is 0.01 and the utility parameter σ is 0.2. In addition, concerning the benefit $B(S)$ and the damage $D(W)$ defined as the functions in (20), we suppose $c_S \ll 1$ and $c_W \ll 1$: this assumption means that, in the optimization process, utility $U(C)$ is much more weighted with respect to the environmental damages related to $B(S)$ and $D(W)$, due to the fact that consumption has usually more relevance to the community wellness than the environment protection in the short temporal term. In particular, we set $c_S = c_W = 0.001$.

In the following sections, two kinds of simulation are described in order to investigate the role of labor productivity and weights of future damages on the optimal allocation of labor, between recycling and non-recycling sectors.[1]

In this respect, due to the fact that the optimal strategy cannot be evaluated in closed form, then it is necessary to approximate the solution of the optimal control model by a suitable numerical algorithm. Precisely, the optimal control $l^*(t)$, $t \in [0, T]$, is evaluated by employing the classical bisection method to approximate the root of (16) in the interval $[0, \widehat{L}]$ at each time t. The precision required for the bisection approximation is set at 10^{-12}. In addition, by following the approach proposed in Diele et al. (2011), the differential system related to the optimality necessary conditions is discretized by a partitioned symplectic Runge–Kutta method: the state equations in $S(t)$ and $W(t)$ are integrated by the classical Euler scheme, while the costate variables $\varphi_1(t)$ and $\varphi_2(t)$ are approximated by the exponential Lawson algorithm (see Lawson 1967) related to the symplectic counterpart Radau IA 1-stage (see Lambert 1991). Therefore, in the framework of the forward-backward integration, the resulting numerical scheme consists of equations with opposite orientations, where the discrete state

[1] For the sake of brevity, we do not provide any simulation about the role of non-recycling and recycling waste generation process related to parameters γ_1 and γ_2, which are considered as given and fixed. Actually, due to the remarks in Sect. 5, intertemporal waste dynamics and even distribution of labor between the two sectors depend on damage weights, together with ratios m_1/m_2 and $(1 - \gamma_2)/(1 - \gamma_1)$.

variables start from an initial temporal condition and a final condition is imposed on the discrete costate variables. This drawback is overcome by constructing successive approximations in a sweep approach converging to the required solution. We notice that, in general, the numerical convergence of any iterative approach depends on the length T of the temporal interval and the procedure is often faster as time horizon length decreases (see for instance McAsey et al. 2012; Ragni 2020, 2022). Thus, we set the temporal length at $T = 80$. On one hand, all the previous settings assure that the bounds in (7) are satisfied; on the other hand, the time horizon length is chosen so small that the iterative algorithm converges to the required exact solution. The time step length is set at $\Delta t = 0.005$ in the process of numerical integration.

Finally, our numerical experiments are carried out in Matlab environment.

6.1 Labor productivity

A first set of simulations has been carried out in order to understand the role played by non-recycling and recycling technologies on labor allocation, waste accumulation and non-recycled resource depletion. More precisely, labor productivities m_1 and m_2 vary according to the following different scenarios on recycling technology:

1. $m_1 = m_2$, so that the recycling productivity equals the non-recycling one;
2. $m_1 > m_2$, meaning that the recycling productivity is lower than the non-recycling one;
3. $m_1 < m_2$, thus the recycling productivity is greater than the non-recycling one.

We assume the productivity of non-recycling activity is fixed at $m_1 = 0.1$. On the other hand, we let m_2 vary from 0.1, to 0.095 and 0.12. By combining all the possible values of ρ and m_2, we may carry out some dynamic simulations to perform a counterfactual analysis of the model. The weights of future damages related to non-recycled input depletion and waste accumulation are $\nu = 1$ and $\mu = 1$; this allows us to describe a situation where the policymaker is accounting for all environmental aspects, by giving the same importance to the final value of both recycled and non-recycled material stocks.

The results are shown in Figs. 1, 2 and 3, respectively. Each figure consists of three plots corresponding to all the three different levels of ρ. Precisely, the first row is always related to $\rho = 0.005$, the second row corresponds to $\rho = 0.5$ and the third row is related to $\rho = 0.955$. On the left hand side column, the optimal trajectories of labor allocated between non-recycling and recycling sectors are shown: the solid line represents the optimal labor $l^*(t)$ devoted to non-recycled input over the whole time horizon, the dashed line describes the labor devoted to recycling $L - l^*(t)$. On the right column, the dynamics of state variables (waste stock and non-recycled resource stock) are provided.

In Fig. 1, which refers to the baseline situation where the two different technologies are identically productive (i.e., $m_1 = m_2$), we focus on non-recycled and recycled resources substitution since ρ is the only varying parameter. It is evident that, for any value of ρ, the amount of labor employed in the non-recycling sector is higher than the other one starting from the beginning up to the switching time $\widehat{t}$ discussed in Remark 2 for an even allocation of labor between sectors. After that, an opposite trend

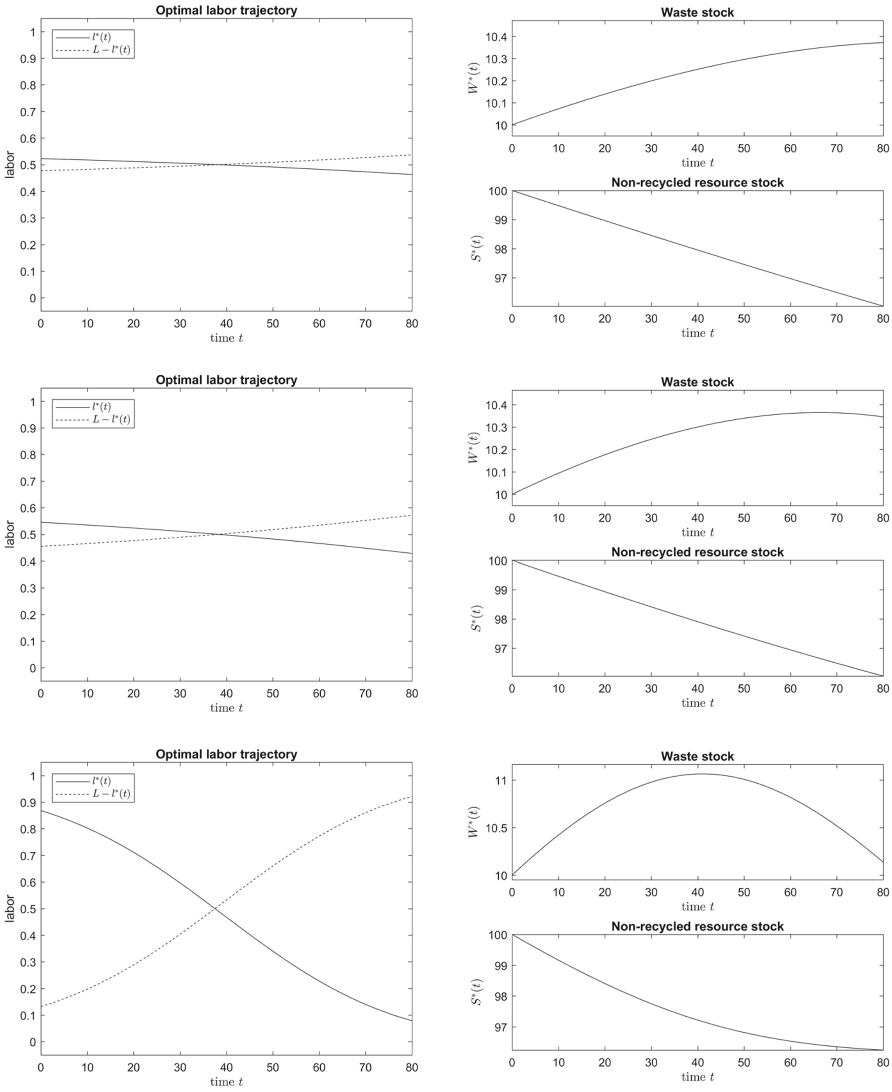

Fig. 1 The results are obtained in correspondence with $m_2 = 0.1$. We set $\rho = 0.005$ on the first row, $\rho = 0.5$ on the second row, $\rho = 0.955$ on the third row

is observed. The time switching point $\widehat{t}$ is around 38, and its location is not affected by the elasticity of substitution between the inputs, as expected due to Proposition 4 and its consequence in Remark 3. Moreover, non-recycled resource stock is decreasing over time.

In addition, in the case when the elasticity of substitution is sufficiently low (i.e., $\rho = 0.005$ in the first row of Fig. 1), the waste stock accumulates over the whole time horizon; this behavior corresponds to the property (a_1) in Proposition 3. Concerning

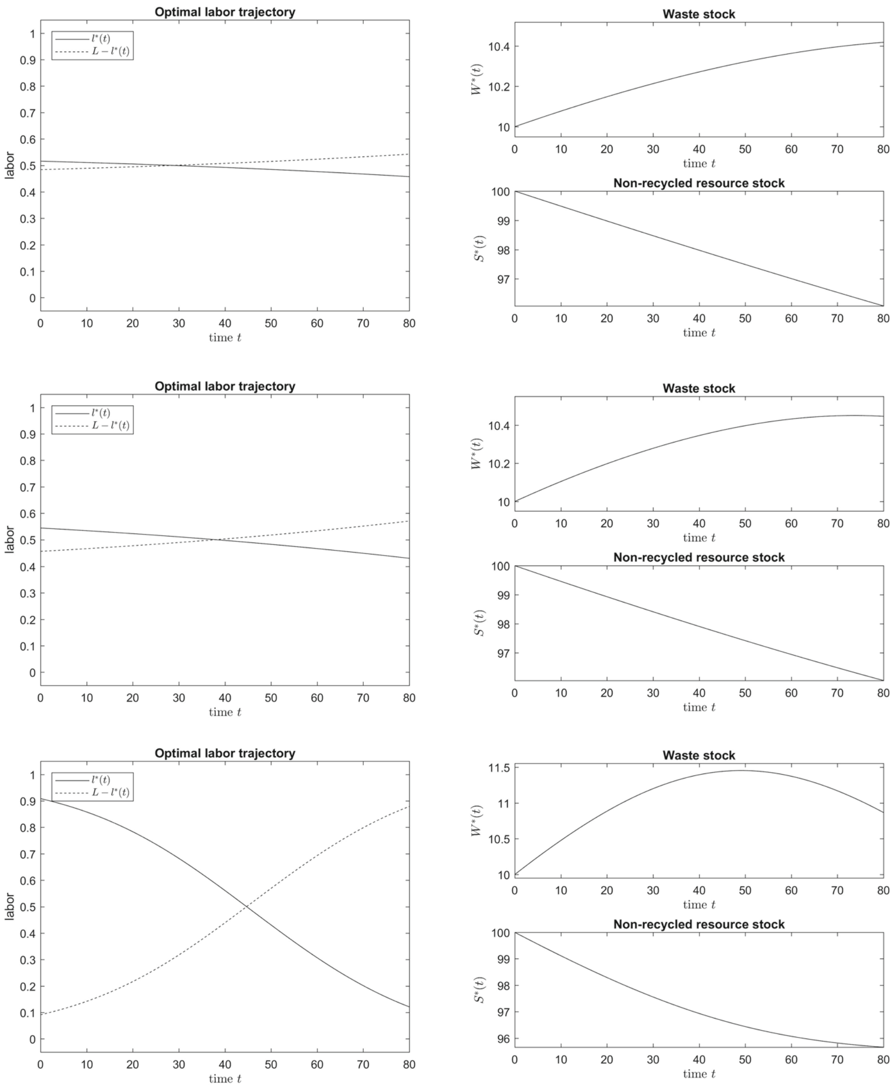

Fig. 2 The results are obtained in correspondence with $m_2 = 0.095$. We set $\rho = 0.005$ on the first row, $\rho = 0.5$ on the second row, $\rho = 0.955$ on the third row

higher values of ρ (in the second and third rows of Fig. 1), the waste stock is accumulated in the first part of the time horizon and, after that, it steadily decreases in the remaining of the temporal interval; this behavior is consistent with the result in the property (a_2) of Proposition 3. It is evident that the time when the waste begins to decrease depends on the chosen value of ρ; indeed, it is as closer to the time origin as ρ is increasing, thus the decline of waste stock is as faster as the recycled input is a closer substitute for the non-recycled one.

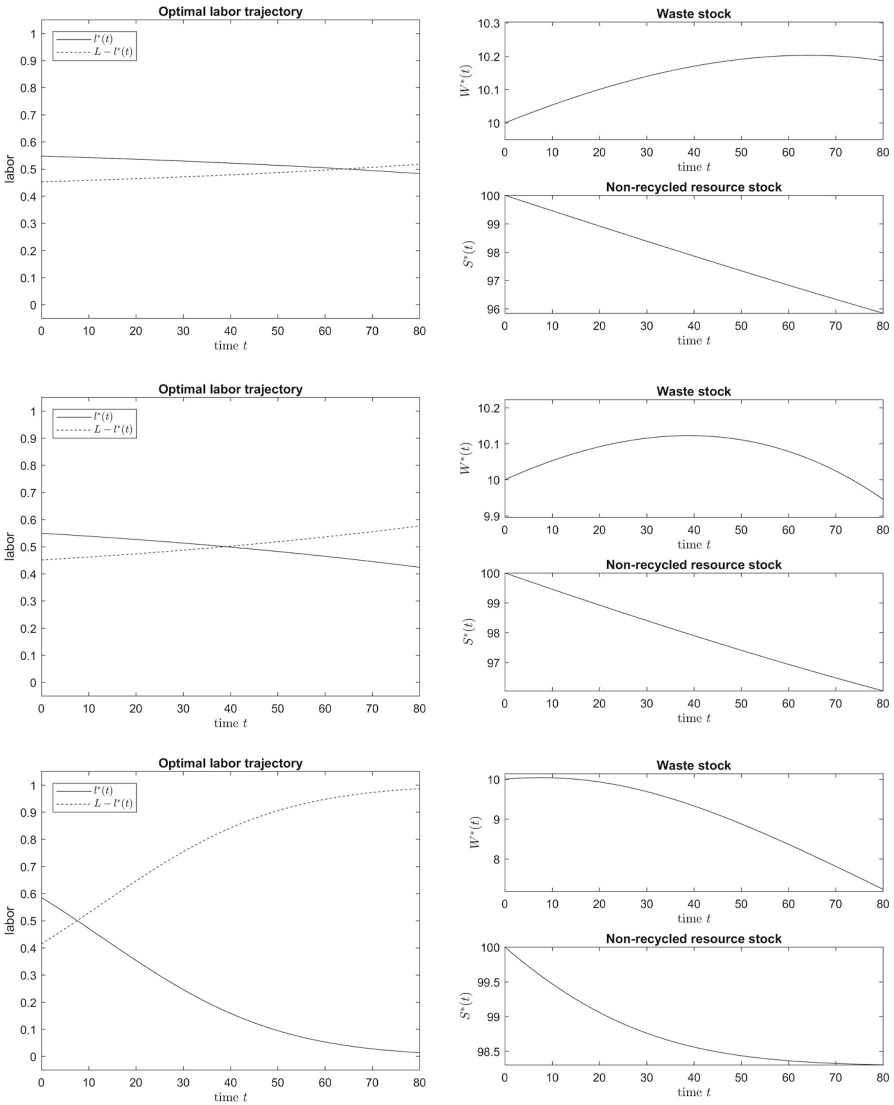

Fig. 3 The results are obtained in correspondence with $m_2 = 0.12$. We set $\rho = 0.005$ on the first row, $\rho = 0.5$ on the second row, $\rho = 0.955$ on the third row

The second scenario is shown in Fig. 2, where we assume that $m_1 > m_2$ corresponding to the case where the recycling technology is less efficient than the non-recycling one (i.e., the current state of the available technology for recycling). Labor trajectories have similar trends as described in the first scenario $m_1 = m_2$, but the switching time $\widehat{t}$ depends on the chosen value of ρ. The position of $\widehat{t}$ is shifted forward as closer to T as the elasticity of substitution ρ is larger. This result points out that, in the optimal transition, the social planner employs a large share of workers in the conventional sector when recycling is less productive than non-recycling and inputs are close substitutes. This implies that, despite inputs are qualitatively similar, recycling remains unexploited for a long time period. Indeed, labor is transferred from the non-recycling to the recycling sector only when approaching the deadline T.

As a consequence, less workers are employed in recycling. Waste accumulation increases over the whole time horizon for low and middle quality inputs. In the case when technology for recycling is able to produce high-quality (and close substitute) inputs, then waste accumulation increases in the first path of the time horizon, after that it reduces. This result is consistent with the situation (a_3) stated in Proposition 3. A low productive technology for recycling, even though for high-quality inputs, is not able to anticipate and boost the transition.

The last scenario assumes $m_1 < m_2$. This assumption reflects the current objectives of EU policies that are designed to foster recycling technologies. The results are shown in Fig. 3. Similar behaviors arise for all values of ρ with respect to the previous scenarios in terms of labor trajectories. The switching point $\widehat{t}$ such that $l^*(\widehat{t}) = L/2$ (Remark 2) reduces as the value ρ increases. Concerning waste stock dynamics, a decreasing trend is shown only for all high-quality inputs along the entire time path. On the other hand, as expected, a depletion of the non-recycled input stock occurs. Crucially, the reduction in non-recycled resource is slower for high-quality inputs. It means that, as expected, fostering recycling technology together with creating a high-quality substitute for the non-recycled input represents the right option to slow down non-renewable resource depletion. Actually, in the last case when recycled and non-recycled inputs are close substitutes (bottom of Fig. 3), the non-recycled input stock converges to a stationary level and the labor is completely employed in the recycling sector in the last part of time interval.

6.2 Future damages

As a further analysis, other simulations focus on the role of future damages caused by non-recycled resource employment and waste accumulation. We fix identical labor productivity for recycling and non-recycling activities such that $m_1 = m_2 = 0.1$. Again, we account for different values of the elasticity of substitution; we start from $\rho = 0.005$, then we assume $\rho = 0.5$ and $\rho = 0.955$. A counterfactual analysis is carried out by assuming different values of weights ν and μ, as these parameters affect the relative shadow prices of both non-recycled and recycled resources. Precisely, we account for three different scenarios:

1. a low and identical importance of both types of damage, i.e., $\nu = \mu = 0.1$;
2. a high and identical importance of both damages, i.e., $\nu = \mu = 2$;
3. a higher importance of final waste accumulation damage with $\nu = 1$ and $\mu = 2$.

As what concerns the first scenario in Fig. 4, the amount of labor employed in the non-recycling sector is always higher than the one used for recycling activities. Moreover, there exists no switching time $\widehat{t} \leq T$ such that $l^*\left(\widehat{t}\right) = L/2$; this is due to the fact that the inequalities in (35) are not satisfied, but in this case it holds that $e^{-\delta T} < \widehat{\widehat{\tau}}$. Then, the labor trajectory plots show a constant positive trend over time and an increasing number of workers employed in non-recycling sector. In accordance

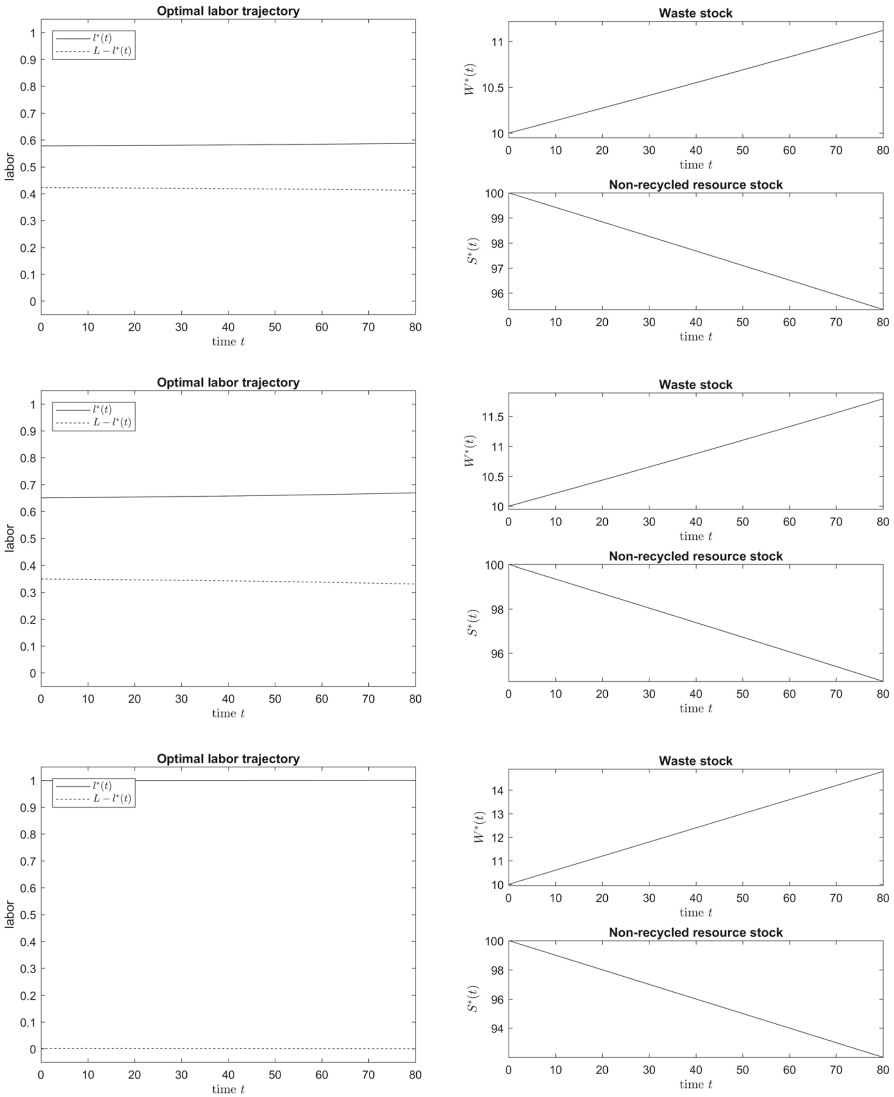

Fig. 4 The results are obtained in correspondence with $\nu = \mu = 0.1$. We set $\rho = 0.005$ on the first row, $\rho = 0.5$ on the second row, $\rho = 0.955$ on the third row. Moreover, $m_1 = m_2 = 0.1$ for all the different cases

with the case of situation (a_3) in Proposition 3, this result is associated with a waste stock accumulation and a non-recycled resource depletion.

Figure 5 shows the dynamics related to the second scenario. The labor trajectory related to the recycling sector is always above the non-recycling labor trajectory. The former has an increasing trend in the whole time horizon, the latter globally decreases. It is due to the fact that the inequalities in (35) are not satisfied, but we have $\widehat{\tau} < e^{-\delta 2T}$ in this case. Again, creating a close substitute for the non-recycled input represents

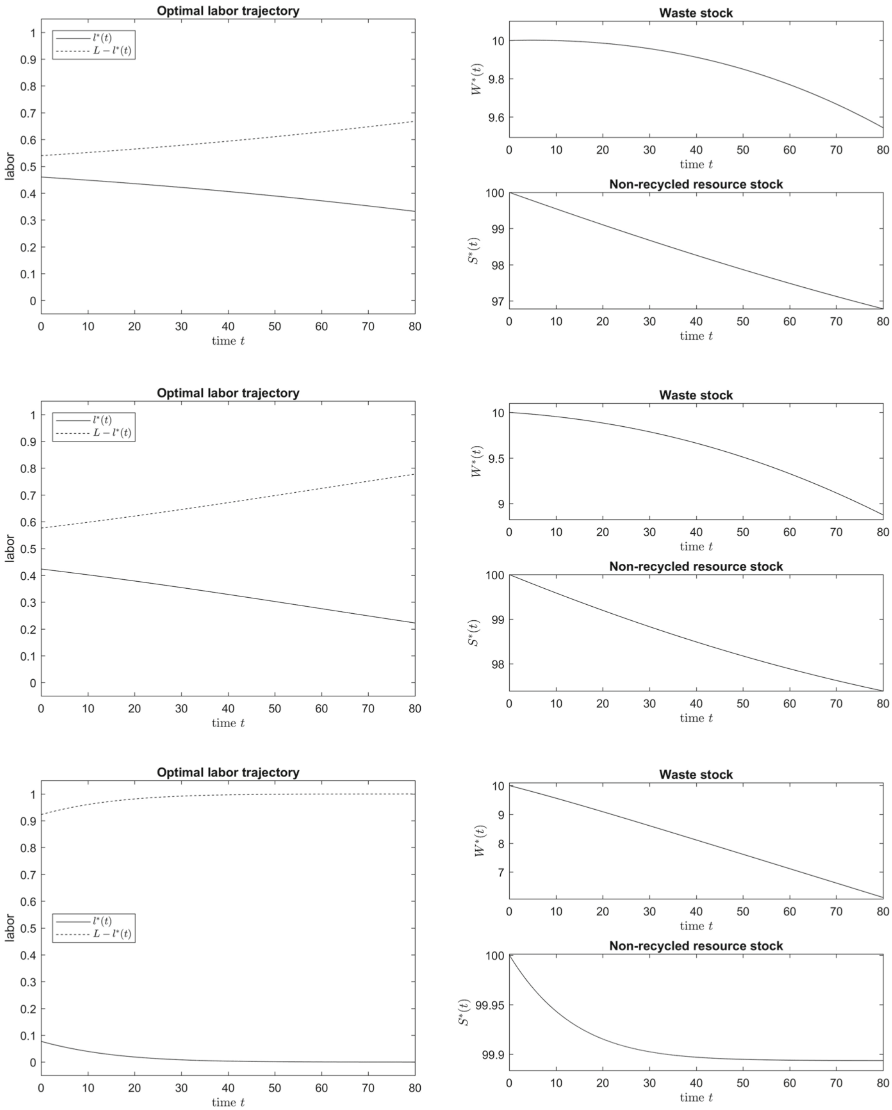

Fig. 5 The results are obtained in correspondence with $\nu = \mu = 2$. We set $\rho = 0.005$ on the first row, $\rho = 0.5$ on the second row, $\rho = 0.955$ on the third row. Moreover, $m_1 = m_2 = 0.1$ for all the different cases

the best option to preserve non-renewable resources: for the highest elasticity of substitution (bottom of Fig. 5), non-recycled input stock remains stationary and labor is totally employed in the recycling sector. Concerning recycled waste disposal, waste stock progressively reduces, especially when inputs are close substitutes, due to the wider share of workers allocated in recycling. Non-recycled resource availability is also depleting; anyway, when the level of substitution is 0.955, it initially declines up

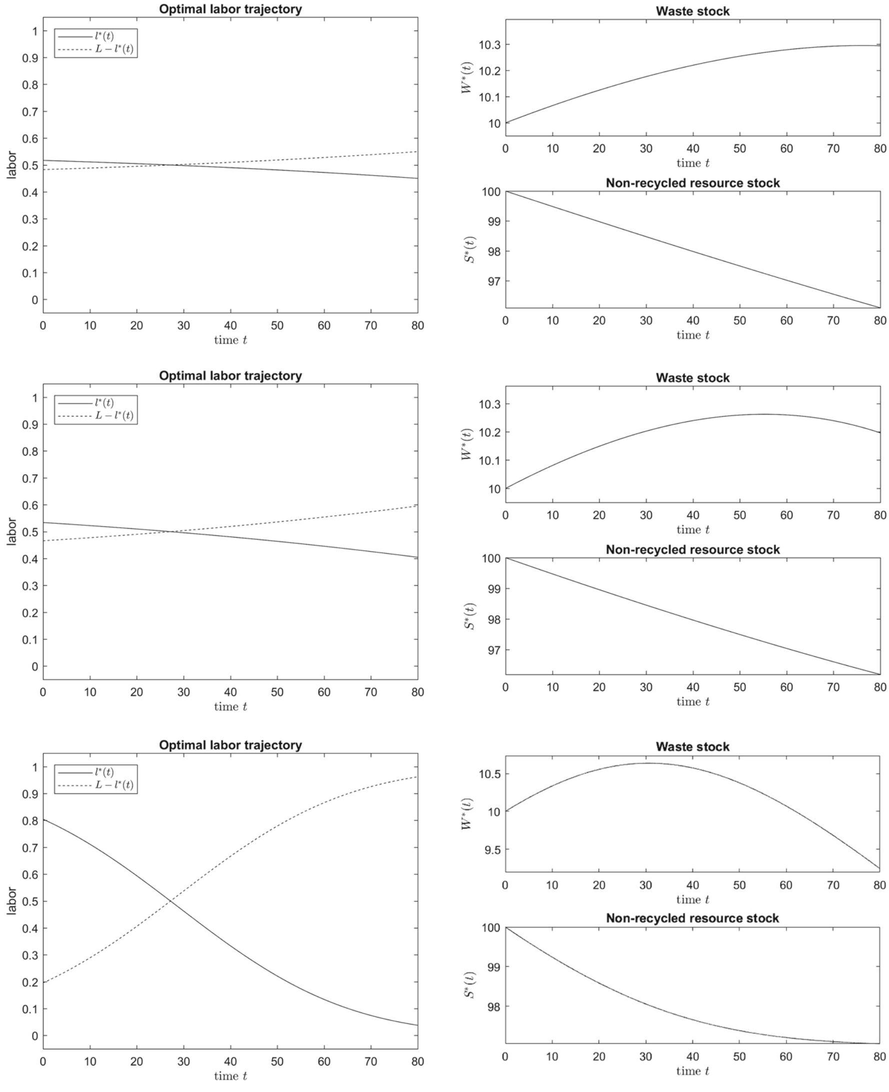

Fig. 6 The results are obtained in correspondence with $\nu = 1$ and $\mu = 3$. We set $\rho = 0.005$ on the first row, $\rho = 0.5$ on the second row, $\rho = 0.955$ on the third row. Moreover, $m_1 = m_2 = 0.1$ for all the different cases

to a given time, after that depletion ceases and $S(t)$ lies around a given lower bound ($S_{min} \approx 99.89$) whose value is very close to the initial stock.

The dynamics related to the third scenario is shown in Fig. 6. The amount of labor employed in the non-recycling sector is higher than the one employed in the recycling sector starting from the beginning of the time horizon up to the switching time $\widehat{t}$, after that an opposite trend is observed. For any value of ρ, the location of $\widehat{t}$ is around 27. In correspondence with a low value of the elasticity of substitution (i.e., $\rho = 0.005$

in the first row of Fig. 6), the waste stock accumulates over the whole time interval. Concerning the other values of ρ, the waste stock is accumulated in the first part of the time horizon and, after that, it is decreasing in the final part of the temporal interval; this behavior corresponds to the result (a_2) in Proposition 3. The decline of waste stock is as faster as the recycled input is a closer substitute for the non-recycled one. Again non-recycled resource stock always reduces over time.

Finally, for the sake of brevity, the results obtained for $\nu = 2 > 1 = \mu$ are omitted as the dynamics does not qualitatively change in a substantial way with respect to the second scenario shown in Fig. 5.

7 Concluding remarks

Predictive modeling of an optimal strategy to control non-recycling and recycling activities may represent an efficient tool to help policymakers in their decision-making process for both environmental conservation and social welfare maximization. According to Huhtala (1999), the necessary condition to guarantee the existence of a long run equilibrium is recycling. In this respect, an optimal control model has been described with the aim of finding an efficient allocation of labor between non-recycling and recycling sectors to provide a non-renewable resource used in the production of consumption goods. In this framework, the desired final level of both natural resource and waste stocks can be interpreted as target to be achieved for future sustainability. Differently from Huhtala (1999), the optimization process has been carried out over a finite time horizon in accordance with the need of rapidly achieving the Sustainable Development Goals adopted by UNs Member States and promoted by EU Green Deal. By employing classical tools of optimal control theory, a complete theoretical analysis of the model has been developed under the assumption of linear production in both sectors: the existence of a solution and its uniqueness have been proved by applying the Pontryagin's Maximum Principle and its consequences. The optimal production-consumption paths result when both current and future environmental damages are taken into account.

The model has been employed in order to determine the optimal strategy of labor allocation in a hypothetical test case. Numerical simulations highlight that the existence of a recycled input to substitute for the depletable resource and social pressures on political mechanisms for current and future damages are crucial in order to preserve it by the finite deadline coming from the urgency of environmental issues. However, this entails a strong effort to enhance the recycling technology and a growing social awareness to environmental themes to translate into large damage weights in the optimization process. Waste reduction occurs over the whole time horizon and the non-recycled input stock remains stationary as a large share of labor is employed in the recycling sector over a certain time interval. Producing high-quality recycled inputs speeds the whole transition process up. An increase in recycling efficiency unambiguously decreases the level of waste stock as in Huhtala (1999). Differently from Huhtala (1999), whose main focus is on tax-subsidy schemes, this model concentrates the attention on structural characteristics relating to technology and social preferences in the optimal allocation of inputs. Political pressure on policymakers and fast responses in terms of more advanced recycling technologies or, more generally, strong incentives to attract labor in the recycling sector may support the ecological transition, reduce environmental damages and assure intergenerational equity for the forthcoming future.

As a future purpose, we aim to employ our optimal control model as a tool for studying the management of non-exhaustible resources, such as timber materials and biofuels, in sustainable transition.

Acknowledgements The authors would like to thank three anonymous Referees for careful reading of the manuscript and making useful comments which helped to improve the paper. The work of Chiara Lodi was supported by the project "Innovation and global challenges in an interconnected world: the role of local resources and socio-economic conditions" funded by the Ministry of University and Scientific Research within the context of the Research Project of Relevant National Interest (PRIN) 2017.

Funding Open access funding provided by Università degli Studi di Ferrara within the CRUI-CARE Agreement.

Declarations

Conflict of interest The authors have no competing interests to declare that are relevant to the content of this article.

References

André, F.J., Cerdá, E.: On the dynamics of recycling and natural resources. Environ. Resource Econ. **33**, 199–221 (2006)

Bonnans, J.F., Varin, J.L.: Computation of order conditions for symplectic partitioned Runge–Kutta schemes with application to optimal control. Numer. Math. **103**(1), 1–10 (2006)

Diele, F., Marangi, C., Ragni, S.: Exponential Lawson integration for nearly Hamiltonian systems arising in optimal control. Math. Comput. Simul. **81**, 1057–1067 (2011)

European Commission, The European Green Deal, COM/2019/640 final, Document 52019DC0640 (2023). https://eur-lex.europa.eu/legal-content/EN/TXT/?qid=1588580774040&uri=CELEX%3A52019DC0640 (Access 30th June)

European Environmental Agency, EEA Signals 2014—Well-being and the environment. Waste: A problem or a resource?, (2014). https://eea.europa.eu/publications/signals-2014/articles/waste-a-problem-or-a-resource (Access 19th Nov 2022)

European Environmental Agency, Waste recycling in Europe (2021). https://eea.europa.eu/ims/waste-recycling-in-europe (Access 19th Nov 2022)

Fleming, H.F., Rishel, R.W.: Deterministic and Stochastic Optimal Control. Springer, Berlin (1975)

Hager, W.W.: Runge–Kutta methods in optimal control and the transformed adjoint system. Numer. Math. **87**(2), 247–282 (2000)

Halkos, G., Papageorgiou, G.: Dynamical Methods in Environmental and Resource Economics, MPRA Paper 67845. University Library of Munich, Germany (2015)

Highfill, J., McAsey, M.: An application of optimal control to the economics of recycling. SIAM Rev. **43**(4), 679–693 (2001)
Highfill, J., McAsey, M.: Municipal waste management: recycling and landfill space constraint. J. Urban Econ. **41**, 118–136 (1997)
Highfill, J., McAsey, M.: Landfilling versus backstop recycling when income is growing. Environ. Resource Econ. **19**, 37–52 (2001)
Hotelling, H.: The economics of exhaustible resources. J. Polit. Econ. **39**, 137–175 (1931)
Huhtala, A.: A post-consumer waste management model for determining optimal levels of recycling and landfilling. Environ. Resource Econ. **10**, 301–314 (1997)
Huhtala, A.: Optimizing production technology choices: conventional production vs. recycling. Resour. Energy Econ. **21**(1), 1–18 (1999)
Lambert, J.D.: Numerical Methods for Ordinary Differential Systems. Wiley, Hoboken (1991)
Lawson, J.D.: Generalized Runge–Kutta processes for stable systems with large Lipschitz constants. SIAM J. Numer. Anal. **4**, 372–380 (1967)
Léonard, D., Van Long, N.: Optimal Control Theory and Static Optimization in Economics. Cambridge University Press, New York (1992)
McAsey, M., Mou, L., Han, W.: Convergence of the forward-backward sweep method in optimal control. Comput. Optim. Appl. **53**, 207–226 (2012)
Perman, R., Ma, Y., McGilvray, J., Common, M.: Natural Resource and Environmental Economics. Pearson Education Limited, Edinburgh Gate (2003)
Ragni, S., Diele, F., Marangi, C.: Steady-state invariance in high-order Runge–Kutta discretization of optimal growth models. J. Econ. Dynam. Control **34**, 1248–1259 (2010)
Ragni, S.: A fixed-point iteration method for optimal control problems governed by parabolic equations. AIP Conf. Proc. **2293**, 420054 (2020)
Ragni, S.: A constructive method for solving parabolic equations with opposite orientations arising in optimal control. J. Math. Anal. Appl. **512**(1), 126092 (2022)
Sweeney, J.L.: Economic theory of depletable resources: An introduction, Chapter 17. In: Kneese, A.V., Sweeney, J.L. (eds.) Handbook of Natural Resource and Energy Economics, vol. 3, pp. 759–854. Elsevier, Amsterdam (1993)
United Nations—Transforming our World: The 2030 Agenda for Sustainable Development, A/RES/70/1 (2015). https://sdgs.un.org/2030agenda (Access 30th June 2023)
Weber, T.A.: Optimal Control Theory With Applications in Economics. The MIT Press, Cambridge (2011)
Withagen, C.: Optimal Extraction of Non-Renewable Resources, Chapter 3 in Handbook of Environmental and Resource Economics. Edward Elgar Publishing, Cheltenham (1999)