Non-performing loans, expectations and banking stability: a dynamic model

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Abstract

This paper proposes dynamic oligopolistic models to describe heterogenous banks that compete in the loan market. Two boundedly rational banks adopt an adaptive behavior to increase their profits under different assumptions of limited information and bounded computational ability, in the presence of a share of credits that might not be reimbursed (i.e. non-performing loans). Each Nash equilibrium is an equilibrium point of the dynamic adjustments as well. Thus, the repeated strategic interactions between banks may converge to a rational equilibrium according to the parameters' values and the initial conditions.

As a case study, we assume an isoelastic nonlinear demand and linear costs as in Puu (1991), and we analyze the influence of the economic parameters on the local stability of the unique equilibrium, as well as the kinds of attractors that characterize the long-run behavior of the banks. Moreover, we study the global structure of the basins of attraction and the degrees of stability of the Nash equilibrium under two different dynamic adjustments: adaptive best reply and gradient dynamics. We obtain interesting policy insights on how different risk factors interact to generate banking stress and fragility. Finally, we show that different monetary policies set by the Central Bank may produce a variety of lending behaviors affecting banking stability.

Keywords: Oligopolistic banking model, Non-performing loans, Discrete dynamical system, Bounded rationality, Non-linearity.

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1 Introduction

The shock of the COVID-19 pandemic has created an unprecedented challenge for the economic and financial markets. While the decline of GDP has been symmetric or comparable across member countries due to the necessary limitations posed by governments to contain the spread of the disease, asymmetric effects are appearing across sectors of the economy. The differentiated impacts involve not only productivity and consumption but also the functioning of financial and banking systems.

Prompted by regulators especially after the unsuccessful experience of the Great Financial Crisis, banks have built capital and liquidity buffers, improved risk management practices, and internalized part of the social cost of risk-taking (Schivardi et al., 2017). Thanks to these efforts, they were certainly better prepared to cope with a shock in 2020 than they were in 2008. However, questions are still open whether this updated EU macroprudential framework along with the contingency measures implemented in the aftermath of the Covid-19 crisis will be sufficient to prevent the evolution of an initial liquidity crisis into a worrisome solvency one (Coeuré, 2020).

The implications of the coronavirus pandemic for the financial and banking sectors are still to be fully determined. However, economic growth and forward-looking indicators of default risk suggest that bankruptcies and corporate insolvencies will rise significantly by the end of 2021 (Banerjee et al., 2020), and the share of credits that will not be reimbursed (i.e. non-performing loans, hereinafter NPLs) is expected to increase as the impact of the COVID-19 crisis on the real economy intensifies (OECD, 2021). Indeed, EU data show that the NPL ratio for all EU banks experienced a first uptick, rising to 2.9 percent in Q1-2020, up from 2.6 percent in Q4-2019, even though with a diversified situation between member states (European Commission, 2020).

The literature shows that the relative amount of NPLs on overall credits is a function of both internal factors, such as the banks' management lending strategy, and on external factors like the exogenous business cycle stage of the economy.

For this reason, the first goal of this paper is to study, theoretically, the characteristics that could expose credit institutions to financial suffer, especially in a context characterized by an economic slowdown (Calcagnini et al., 2018). We propose a dynamic oligopoly model to describe a system of heterogenous banks that compete in the loan market.

Several theoretical and empirical studies have focused on the issue (Ari et al., 2020b; Couppey-Soubeyran et al., 2020; Goodell, 2020; Zhang et al., 2020). The majority of these works rely on traditional analyses of banks' yearly balance sheets that are used to assess bank performances in terms of their capacity to generate profitability in the short term. However, they are only partially informative of the bank and market financial sustainability. In this vein, ECB (2010) states that desirable features for banks' performance measures should encompass more aspects of the long-run performance than just profitability embedded in market-oriented indicators, and other elements such as the quality and riskiness of assets, should be taken into account. Furthermore, performance measurements do not consider the systemic relevance of a bank, which is one of the key factors behind instability, thus neglecting relevant vulnerabilities of the system as a whole.

A dynamic model represents a novelty in the investigation of the role of NPL on market stability and provides a better setup to study the complex structures of relationships and equilibria that characterize the banking system over time. It allows to study the evolution of NPLs over a long time horizon, analyzing how the quality and riskiness of assets in banks' portfolio may endanger the capacity of each credit institution to generate profits and thus being competitive in the banking sector (i.e. financial sustainability).

The peculiarities of the banking sector are well represented by an oligopoly system, which allows us to capture both the cooperation (i.e. the interbank market) and competition (in the loan market) relationships between credit institutions.

Besides, we model the banking system by means of an oligopolistic model for several other reasons (Leon, 2015). First, one of the key conditions of perfect competition is missing: i.e. the absence of entry and exit barriers, as financial regulation is one of the major constraints to free entry in this industry. The financial sector is among the most regulated sectors in many countries, and different regulations set entry requirements for domestic and foreign banks, capital requirements and other regulations that affect bank activities. Moreover, the degree of contestability in banking is also influenced by non-legal barriers, such as technical ones. The presence of scale and scope economies may create obstacles to outside banks; network economies may also create an additional barrier in the measure that incumbents choose to share or extend their network to exclude rivals from the market and limit competition.

Second, the banking industry is typically affected by asymmetric information problems. Private information may limit effective competition from uninformed outside banks due to the adverse selection problem, as potential entrant banks stems from their inability to distinguish new (good) borrowers from old (bad) borrowers who have been rejected by their previous bank.

Third, the banking industry is characterized by market concentration, that is the aggregate share of banking assets held by the largest banks is relatively high, with some degree of variability among countries (Calice and Leonida, 2018).

Therefore, we focus on a duopoly, where two boundedly rational banks adopt an adaptive behavior to increase their profits under different assumptions of limited information and computational ability, in the presence of NPLs. Each equilibrium point of the dynamic adjustments proposed is a Nash equilibrium, i.e. coincides with the corresponding oligopolistic competition outcome obtained under assumptions of complete information and rational (profit maximizing) banks. Thus, in the dynamic framework employed, the repeated strategic interactions between banks may converge to a rational equilibrium according to the parameters' values and the initial conditions considered.

As a case study, we consider an isoelastic loan demand and linear costs, as proposed by Puu (1991) in a general oligopoly setting, and we study the influence of the economic parameters on the local stability of the unique Nash equilibrium as well as the kinds of attractors that characterize the long-run behavior of the banks when the Nash equilibrium is unstable. The first dynamic adjustment proposed, in discrete time, is the classical best reply approach with naïve expectations, i.e. the two banks are assumed to know the demand function and solve the profit maximization problem, thus computing the correct reaction functions, but they are not informed about competitor's choices. Consequently, to compute the best reply they assume the currently observed competitor's loan decision as the expected next choice, generating the well known Cournot tâtonnement, as in Puu (1991, 1995).

The second dynamic adjustment proposed is the discrete time gradient dynamics (see Bischi and Naimzada, 2000, and references therein) where the two banks do not solve any profit maximization problem as they simply adjust their next period choices according to estimated profit derivative (i.e. the marginal profit). In this dynamic adjustment the loans supplied by the banking system move along the direction of expected profit gradient. So, banks have not only limited information but also limited computation ability because they do not know the best reply strategy, and adjust their decisions according to local profit increasing arguments. However, even in this instance any positive fixed point of the dynamic duopoly model is a Nash equilibrium, the same of the duopoly system with fully rational agents (i.e. an intersection of the reaction functions).

In both the dynamic models with bounded rationality we study under what conditions (i.e. for which sets of parameters and which initial conditions) the dynamic adjustment will converge to a Nash equilibrium. Moreover, in the case of isoelastic demand considered, a unique Nash equilibrium exists, but different kinds of attractors, periodic or chaotic, can be obtained when it is unstable, giving rise to long-run evolutions that never settle to a rational equilibrium.

Finally, we analyze the global structure of the basins of attraction and we compare the degrees of stability of the Nash equilibrium under the two different dynamic adjustments proposed. We obtain interesting policy insights on how different risk factors and activities interact in producing concrete situations of banking stress and fragility, which is a highly important issue to the goal of increase banks' resilience to adverse shocks. Different monetary policies set by the Central Bank produce a variety of behaviors affecting banking stability.

The paper is organized as follows. Section 2 describes the banking activities in an oligopolistic model, while Section 3 focuses on the duopoly case with isoelastic demand. In Section 4 we study the case of an adaptive best reply, whereas we focus on gradient dynamics in Section 5. Section 6 provides an economic interpretation of the findings and Section 7 concludes.

2 The banking activities in an oligopolistic model

The core banking activities are generally the production of deposit and loan services. The typical balance sheet of a representative credit institution i is defined on one side by the sum of three macro-categories of assets or uses, namely reserves R, loans L and financial investment (which include mainly bonds and securities) B. These three categories correspond to the core activities of each banking institution and they are financed by the resources or liability in the other side of the balance sheet, given by the bank's overall amount of deposits collected from customers, D.

In this theoretical setup a representative commercial bank i that operates in a market of N heterogeneous banks decides, in each time period t, to provide credits in the form of loans $L_i(t)$ and to invest in financial assets $B_i(t)$.

According to international banking standards that aim at increasing the stability of the financial markets (i.e. Basel Accords), each credit institution needs to satisfy a reserve requirement R_i that is assumed to be a fraction (q) of the bank liabilities:

$$R_i = qD_i \tag{1}$$

Finally, the interbank market allows banks to borrow (lend) an amount of money M_i to (from) other banks at the rate r.[1]

The position on the interbank market M_i for the single banking institution i is given by the difference between its overall liabilities and assets:

$$M_i = D_i - R_i - L_i - B_i$$

so that $M_i < 0$ is the case of a bank that borrows at rate r from the interbank market to finance its activities; vice versa, $M_i > 0$ represents a situation of a bank that invests its surplus of resources (net deposit) by lending at rate r to the other banks in the industry.

By substituting (1) into the above expression, we obtain:

$$M_i = D_i(1 - q) - L_i - B_i \tag{2}$$

A market clearing condition holds so that the aggregate or the sum of every position in the interbank market (M) should always be equal to zero:

$$M = \sum_{i=1}^{N} M_i = 0.$$

We consider an oligopolistic Monti-Klein model (Klein (1971), Monti (1972)) where N price-makers banks compete ($i = 1, 2, ..., N$) influencing the loan rate r_L, the deposit rate r_D and the bond rate r_B.[2]

The main research question of this paper is to analyze how NPLs affect the banking stability and the volume of lending for all credit institutions in the industry. To this aim, the riskiness of the lending activity of

[1] As the prefix suggests, the interbank market is a market where each trade represents an agreement between the banks to exchange amounts of money at a rate r. This rate is set by the Central Bank and holds for all the credit institutions in the interbank market.

[2] r_B can be thought as an average rate of bank's financial investment in securities, funds, and riskier assets.

each bank is captured by a parameter ρ_i, which measures the bank expected share of loans that will not be reimbursed at maturity. That is, each bank bears losses due to NPLs, defined as $NPLs_i = \rho_i * L_i$, namely the amount of credits that the bank i foresees will not be reimbursed at maturity. Thus, the share of loans for which the bank receives a positive return $r_L(L)$ is only a fraction of the overall amount of credits, and it is defined by the expression $L_i - \rho_i\, L_i$.

Furthermore, alongside the loss in the NPLs yield, also the principal amount of these bad loans will become an irrecoverable part for the bank. Indeed, we will take account of it by including the capital loss $(-\rho_i L_i)$ in the expected profit equation (3).

Thus, the parameter ρ_i captures the riskiness of the lending activity, which each bank considers when it comes to foresee the expected profit.

Finally, bank's costs are related to the size of loans and deposits provided, as well as to the amount of financial investment. Consequently, we define $C(L_i, D_i, B_i)$ for each bank i in the market as the cost of managing a volume D of deposits, a volume L of loans and a volume B of bonds. The expected profit for a representative bank i in an oligopolistic market $i = 1, 2, ..., N$, is:

$$\pi_i = r_L(L)(L_i - \rho_i L_i) - \rho_i L_i + r_B(B)B_i + rM_i - r_D(D)D_i - C(L_i, D_i, B_i) \tag{3}$$

where $L = \sum_{i=1}^{N} L_i$, $D = \sum_{i=1}^{N} D_i$ and $B = \sum_{i=1}^{N} B_i$; r_L, r_D, r_B are functions of the overall quantity of loans L, deposits D and bonds B.

Substituting equation (2) - the net position of the bank on the interbank market - in equation (3), we get:

$$\pi_i = [r_L(L)(1 - \rho_i) - \rho_i - r]L_i + [r_B(B) - r]B_i + [r(1 - q) - r_D(D)]D_i - C(L_i, D_i, B_i) \tag{4}$$

The interest rate on interbank market r represents one of the principal instruments at disposal of the central banks to influence the monetary condition in the secondary market.[3]

Accordingly, the bank's expected profit (4) can be seen as the sum of the intermediation margins/spreads on loans, deposits and bonds (taking also into account the riskiness of these activities), net of management costs.

[3] In this simple model we do not include the interest rate on the main refinancing operations (MRO), that are the most important instrument for the ECB to provide the bulk of liquidity to the banking system in the primary market. Generally, the two aforementioned interest rates are very similar, thus in the following we just consider r as the main refinancing interest rate.

Indeed, the three main variables L, D, B may vary in volume depending on the spread between their rate and the main refinancing rate r.

We also notice that the refinancing rate r settled by the Central Bank through open market operations (OMOs), which consist of a large scale acquisition or sale of bonds in the primary market (i.e. the policy monetary course), influences the profitability of all the three main banking operations.

The profit-maximizing behavior for each bank i in the industry is obtained by solving the following first-order conditions, for $i = 1, ..., N$:

$$\frac{\partial \pi_i}{\partial L_i} = r_L(L)(1-\rho_i) - \rho_i - r + r'_L(L)(1-\rho_i)L_i - \frac{\partial C_i}{\partial L_i} = 0 \tag{5}$$

$$\frac{\partial \pi_i}{\partial D_i} = r(1-q) - r_D(D) - r'_D(D)D_i - \frac{\partial C_i}{\partial D_i} = 0 \tag{6}$$

$$\frac{\partial \pi_i}{\partial B_i} = r_B(B) - r + r'_B(B)B_i - \frac{\partial C_i}{\partial B_i} = 0 \tag{7}$$

If the N banks in the market are homogeneous, namely the cost function and the expected share of NPLs is the same for every institution i, then a Cournot equilibrium of the banking industry exists, defined as an N-tuple of triples (L_i^*, D_i^*, B_i^*) that for every i maximizes the profit of bank i, taking the volume of loans, deposits and bonds of other banks as given. This Cournot equilibrium is characterized by the same volume of services offered by each bank: $L_i^* = L^*/N$, $D_i^* = D^*/N$, $B_i^* = B^*/N$.

As stated above, one of the main research questions of the paper is to analyze how an increase of ρ_i affects the banking stability and the volume of lending for all credit institutions in the industry. To this aim, and given the first order conditions described above, we can exclusively focus on the loan market to answer how different economic parameters, degree of bank rationality and level of information affect the equilibrium and its stability. Thus, the bank expected profits (4) can be simplified as follows:

$$\pi_i = [r_L(L)(1-\rho_i) - \rho_i - r]L_i - C(L_i).$$

3 The loan market with isoelastic demand in a duopoly

In this Section, we focus on the loan market and consider the case of a duopoly, where the two competing banks are characterized by different linear costs $C(L_i) = c_{Li}\, L_i$, $i = 1, 2$, and diverse risk in the lending activity ρ_i.

As to the factors influencing loan demand, most studies include an economic activity variable (such as real GDP or industrial production) and financing costs (i.e. interest rates or bank lending rates) as its main determinants. Another determinant is the opportunity cost of bank loans (i.e. the cost of alternative sources of finance). Finally, we assume an isoelastic demand function, so that consumers' loan total expenditure is constant, as proposed by Puu (1991). The assumption of constant elasticity is consistent with the absence of liquidity constraints and market frictions. Indeed, empirical analyses suggest that loan demand price elasticity might increase with income, while might become highly price sensitive at higher-than-normal rates. Moreover, evidence suggests that loan size is far more responsive to changes in loan maturity than to changes in interest rate (Karlan and Zinman, 2019). Thus, we model the demand function as follows:

$$r_L(L) = \frac{\alpha y + \beta r_B}{L} \ , \quad L = L_1 + L_2 \tag{8}$$

where we shall assume $L \neq 0$ in the following, that is $L_i \geq 0$, $i = 1, 2$, with at least one of them strictly positive. The numerator is the sum of two components, which we take as given. The term αy captures the transactions demand for credit (indeed, the strongest is the local aggregate production and income y from households and firms, the higher will be the demand for credits), and $\alpha > 0$ measures the sensitivity of the loan rate to the inverse of the credit-to-GDP ratio L/y; the second term considers that loan interest rates may correlate with other investment opportunities or financing, so that the behavioral parameter $\beta > 0$ captures the degree of substitutability for borrowers of the two alternative way of financing, loans and bonds (Bernanke and Blinder, 1988).

Following Puu (1991), by inserting the nonlinear demand function (8) and its derivative $r'_L(L)$ in the first order condition (5), we get a closed form of the unique solution of the expected profit maximization problem that bank i faces at time t in order to choose the loan strategy:

$$L_i(t+1) = \arg\max_{L_i} \pi_i^e(t+1) \tag{9}$$

given by:

$$L_i(t+1) = R_i\left(L_j^e(t+1)\right) = -L_j^e(t+1) + \sqrt{\frac{(\alpha y + \beta r_B)(1-\rho_i)}{r + \rho_i + c_{L_i}} L_j^e(t+1)} \qquad i,j = 1,2 \quad j \neq i \tag{10}$$

where $R_i(\cdot)$ are the *reaction functions* and $L_j^e(t+1)$ is the expected decision of the competitor.

A Nash equilibrium is located at the intersections of the reaction curves. If players (banks) correctly forecast the competitors' decisions (rational expectations), $L_j^e(t+1) = L_j(t+1)$, then the Nash equilibria can be directly computed (one-shot game). However, in a bounded rationality setting, banks may not know beforehand the competitors' choices, and consequently they formulate some reasonable forecast, on the basis of their information set. The simplest assumption, proposed by Cournot (1838), is that of *naïve expectations*, $L_j^e(t+1) = L_j(t)$, i.e. each bank expects that the decision of the other one will remain the same as in the previous period.

The *naïve expectations* assumption introduces a form of information asymmetry in the market because the bank i can only observe the volume of loans granted by its competitor j at the actual period of time t and, on the basis of this, it settles the optimal volume of loans. This hypothesis could be more realistic than rational expectations because, usually, it is particularly complex to forecast the amount of loans provided by other banks (especially competitors) at time $t+1$, while it is more plausible referring to balance sheets or to other public observable information (i.e. public disclosure).

Under this assumption[4] equation (10) generates a discrete-time dynamical system, called *best reply dynamics*:

$$L_i(t+1) = R_i(L_j(t)) = -L_j(t) + \sqrt{\frac{L_j(t)}{k_i}} \qquad i,j = 1,2 \quad j \neq i \tag{11}$$

with

$$k_i = \frac{r + \rho_i + c_{Li}}{(\alpha y + \beta r_B)(1-\rho_i)} \qquad i = 1,2 \tag{12}$$

[4]Other kinds of expectations mechanisms can be used, such as adaptive expectations, see for example Szidarovszky and Okuguchi (1988), Bischi and Kopel (2001).

Notice that every Nash equilibrium is also an equilibrium of the best reply dynamics, because the intersections of the reaction curves are the fixed points of the difference equation (11). However, such equilibria are not reached in one shot, they may be reached asymptotically, in the long run, if they are stable under the best reply dynamics. This may be seen as an evolutionary explanation of the outcome of a Nash equilibrium. Moreover, the dynamical system (11) may not converge to a Nash equilibrium, as it may exhibit asymptotic convergence to periodic or chaotic attractors (see e.g. Rand 1978, Dana and Montrucchio 1986, Puu 1991, 1998, Bischi et al., 2000, 2010). The unique Nash equilibrium can be expressed by using the aggregate parameters (12):

$$\mathbf{L}^* = (L_1^*, L_2^*) = \left(\frac{k_2}{(k_1+k_2)^2}; \frac{k_1}{(k_1+k_2)^2} \right) \ , \tag{13}$$

and its stability properties, following Puu (1991), can be given in terms of the ratio:

$$k_1/k_2 = \frac{r+\rho_1+c_{L_1}}{r+\rho_2+c_{L_2}} \tag{14}$$

4 Adaptive best reply

Following Puu (1991), see also Agliari et al. (2005) or the book Bischi et al. (2010), in this Section we consider an adaptive adjustment that implies inertia (or anchoring). Indeed, as the banks realize that their best reply is not reliable enough, due to imperfect information on competitor's choice, they do not immediately jump to the computed "optimal" solution, but they prefer to settle on a weighted average (i.e. a convex combination) between the computed (sub-optimal) best reply R_i and their previous choice, according to the adaptive scheme in (15).

The discrete dynamical system (15) assumes the form $(L_1(t+1), L_2(t+1)) = B\left(L_1(t), L_2(t)\right)$, and the map B is given by:

$$B : \begin{cases} L_1\,(t+1) = & (1-\lambda_1)\,L_1\,(t) + \lambda_1\,R_1\,(L_2(t)) \\ \\ L_2\,(t+1) = & (1-\lambda_2)\,L_2\,(t) + \lambda_2\,R_2\,(L_1(t)) \end{cases} \tag{15}$$

where the reaction functions are defined by equation (11) and the parameters $\lambda_i \in [0,1]$ capture how much the banks consider reliable the computed best reply based on imperfect information. Thus, best reply is obtained for $\lambda_i \to 1$, whereas complete inertia (i.e. no change at all) occurs as $\lambda_i \to 0$. Notice that each dynamic equation now includes two dynamic variables on the right hand side, as the loans decided by banks at time t are a weighted average between the previous volume of loans and the reaction to competitor's choice arising from the solution of the profit maximization problem (with naïve expectations).

Generally, smaller values of the parameters λ_i, i.e. larger degree of inertia, enhance stability, as both the region of stability in the space of parameters and the basin of attraction of the stable Nash equilibrium, widen.

Concerning the study of the local stability of the Nash equilibrium[5] under the adaptive adjustment (15), let us consider the Jacobian matrix of the map (15) computed in (13):

$$J(L_1^*, L_2^*) = \begin{bmatrix} 1-\lambda_1 & \frac{\lambda_1}{2}\left(\frac{k_2}{k_1}-1\right) \\ \frac{\lambda_2}{2}\left(\frac{k_1}{k_2}-1\right) & 1-\lambda_2 \end{bmatrix}$$

The stability conditions, in terms of trace $Tr^* = 2 - \lambda_1 - \lambda_2$ and determinant $\Delta^* = (1-\lambda_1)(1-\lambda_2) + \lambda_1\lambda_2\frac{(k_2-k_1)^2}{4k_1k_2}$ become (see e.g. Elaydi, 2008, Medio and Lines, 2001):

$$\begin{cases} 1 - Tr^* + \Delta^* = \lambda_1\lambda_2\left(1 + \frac{(k_2-k_1)^2}{4k_1k_2}\right) > 0 \\ \\ 1 + Tr^* + \Delta^* = 4 - 2\lambda_1 - 2\lambda_2 + \lambda_1\lambda_2 + \lambda_1\lambda_2\frac{(k_2-k_1)^2}{4k_1k_2} > 0 \\ \\ 1 - \Delta^* = \lambda_1 + \lambda_2 - \lambda_1\lambda_2 - \lambda_1\lambda_2\frac{(k_2-k_1)^2}{4k_1k_2} > 0 \end{cases}$$

[5]Notice that even if $(0,0)$ is an equilibrium for the map B, it will not be considered in the following because the demand function (8) is not defined in it, i.e. it is an economically unfeasible point.

where k_i are given by (12). The first two conditions are always satisfied, whereas the third stability condition defines a region of stability of the Nash equilibrium in the space of parameters. In the best reply dynamics ($\lambda_1 = 1, \lambda_2 = 1$) feasible (i.e. bounded and non negative) trajectories are obtained provided that $k_1/k_2 \in [4/25, 25/4] = [0.16, 6.25]$.

Moreover, the Nash equilibrium (13) is stable if and only if $k_1/k_2 \in (3 - 2\sqrt{2}, 3 + 2\sqrt{2}) \simeq (0.17, 5.83)$.

In Figure 1, obtained with $\rho_1 = 0.054$ and $\rho_2 = 0.005$, the stable equilibrium (13) is shown, together with its basin of attraction, represented by the yellow region, whereas the grey region represents the set of initial conditions that generate unfeasible trajectories, i.e. diverging or involving negative values of L_i. [6]

A comparison between the pure best reply model (1a) and the adaptive one with inertia (1b) is also shown. In the latter case, the basin of attraction of the unique stable equilibrium is larger and jagged for lower lambdas ($\lambda_1 = 0.6, \lambda_2 = 0.7$), ceteris paribus.

[6]The parameter values resemble economic and financial values that occur in real world.

Starting from ρ_i, if we take into account the value of NPLs to total gross loans over the last 20 years in the Eurozone (and other OCSE nations), it has ranged, on average, from a minimum of 0.005 for the banking sector of virtuous member countries, up to 0.20 in less virtuous ones (data retrived from https://data.worldbank.org/indicator/FB.AST.NPER.ZS?locations=XC-IT-DE).

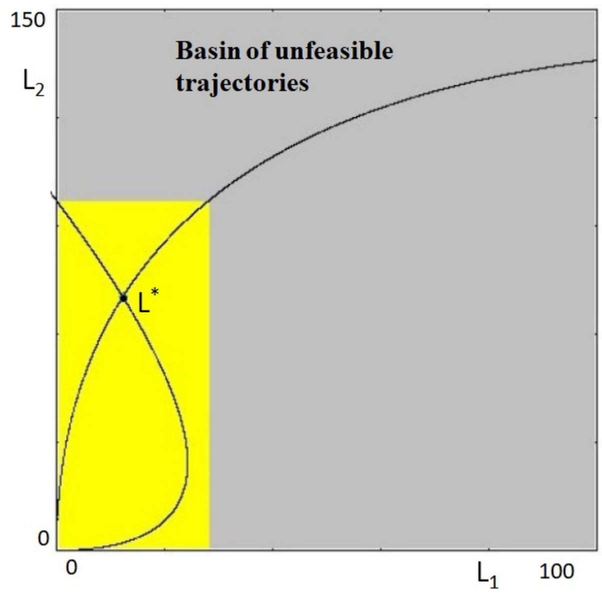
The value of parameter α is related to the inverse of the credit-to-GDP ratio L/y. Data of domestic credit to private sector, again from World Bank, shows that for many developed countries this ratio can vary between 40% and 130%. To met this condition, in our simulations, the parameter α should range from 0.02 to 0.07.

The return from corporate and treasury bonds r_B shows large variability. It depends on the specific country, as well as on the maturity of the securities. However, on average, it can go from 0.005 to 0.06 in recent years (https://fred.stlouisfed.org/graph/?id=IRLTLT01EZA156N).

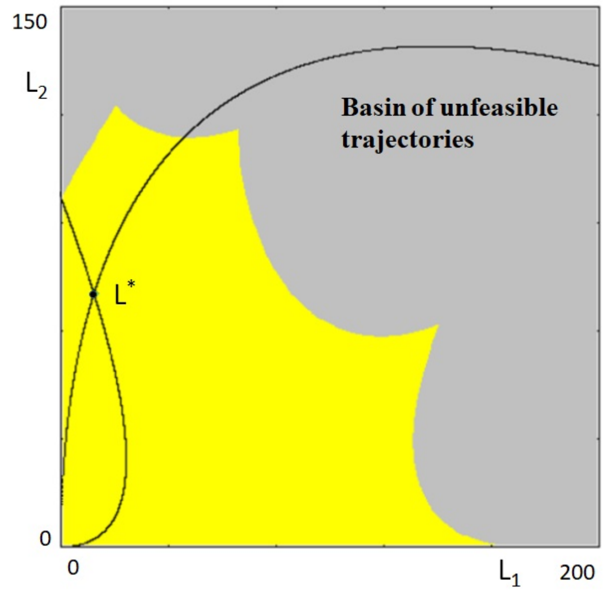
Variable loan costs c_{Li} are specific of each financial institution and are also related to the others parameter values, in particular to interest rates r_L and r. Plausible values of c_{Li} are between 0.001 and 0.01.

Finally, the official interest rate r set by the major Central Banks (i.e. ECB, Federal Reserve, and Bank of England) exhibits a similar trend over time (from 0 to 0.1). In particular, the last years have witnessed a strong reduction of the interest rates (i.e. expansive monetary policy) towards values very close to zero to ease financial conditions in the aftermath of the Great Financial Crisis, and to sustain the economy after the shock of Covid-19 pandemic.

Figure 1: Comparison of the stable Nash equilibrium without and with inertia



(a) $\lambda_1 = 1, \lambda_2 = 1, \alpha = 0.03, y = 200,$
$\beta = 15, r_B = 0.01, r = 0.001, \rho_1 = 0.054,$
$\rho_2 = 0.005, c_{L1} = 0.005, c_{L2} = 0.005$



(b) Same parameters with $\lambda_1 = 0.6$, $\lambda_2 = 0.7$

If the two marginal costs and the expected share of NPLs are very different, so that k_1/k_2 exits the interval of stability, the Nash equilibrium becomes unstable.

Indeed, in the particular case without inertia, i.e. $\lambda_1 = \lambda_2 = 1$, if k_1/k_2 falls outside the interval $(3 - 2\sqrt{2}, 3 + 2\sqrt{2})$ the asymptotic dynamics of the best reply model undergoes a degenerate subcritical Neimark-Sacker bifurcation leading to a 1:4 resonance case (see e.g. Kuznetsov, 1998, ch.9, or Mira, 1987, ch. 5) because the trace of the Jacobian matrix vanishes in the particular case without inertia, and the iterated map (15) becomes decoupled after two iterations, being $L_i(t+2) = R_i\left(R_j(L_i(t))\right)$. This kind of dynamical systems have some particular local and global properties, as shown in Bischi et al., 2000, Agliari et al., 2002, due to the fact that it behaves essentially as a one-dimensional map. For example, the degenerate N-S bifurcation of the map B corresponds to a flip bifurcation of the second iterate. This is highlighted in the bifurcation diagram of Figure 2a where ρ_1 is increased (all other parameters being the same) and $L = L_1 + L_2$ is represented in the vertical axis. After the N-S bifurcation the system converges to a 4-period cycle (Figure 2b) and could even exhibit chaotic motions around the Nash equilibrium, as shown in Figure 3 (left panel) in the phase portrait, and in the right panel of Figure 3 with the time series counterpart, the chaotic attractor with 4-periodicity obtained with $\rho_1 = 0.059$ and $\rho_2 = 0.005$, assumes the form of a cross. A further increase of ρ_1 creates a contact with the basin boundary (i.e. boundary crisis) that destroys the attractor, making the system unstable.

Figure 2: The degenerate Neimark-Sacker bifurcation in the best reply dynamics

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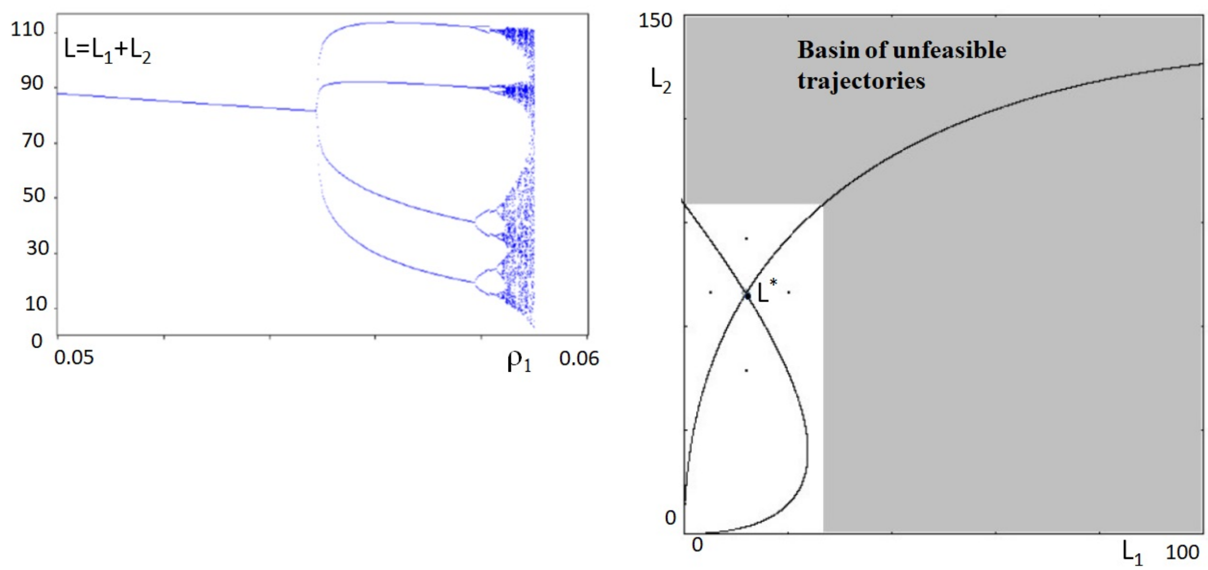


(a) $\lambda_1 = 1, \lambda_2 = 1, \alpha = 0.03, y = 200, \beta = 15, r_B = 0.01, r = 0.001, \rho_2 = 0.005, c_{L1} = 0.005, c_{L2} = 0.005$

(b) Same parameters with fixed $\rho_1 = 0.055$

Figure 3: The cross chaotic attractor in the best reply dynamics

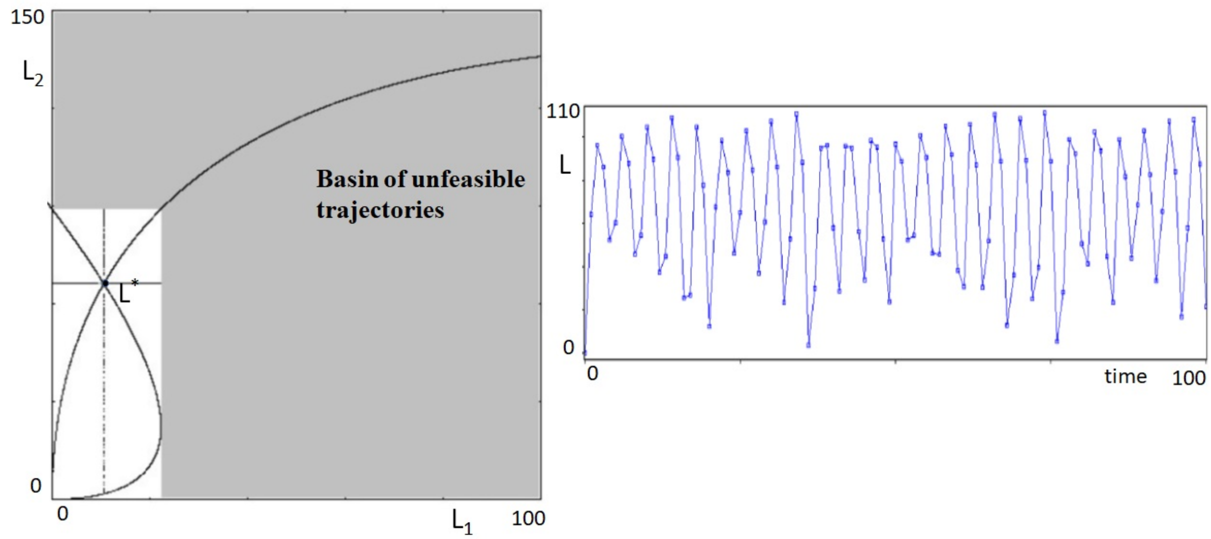


Figure 3: $\lambda_1 = 1, \lambda_2 = 1, \alpha = 0.03, y = 200, \beta = 15, r_B = 0.01, r = 0.001, \rho_1 = 0.059, \rho_2 = 0.005, c_{L1} = 0.005, c_{L2} = 0.005$

Likewise, also in the case of adaptive best reply with inertia the feasibility and the stability conditions can be expressed in terms of the ratio between k_1 and k_2, i.e. the stability depends on the heterogeneity of firms and on the inertia, or prudence, parameters λ_1 and λ_2. For example, for the adaptive best reply model with $\lambda_1 = 0.6$ and $\lambda_2 = 0.7$ an excessive bank heterogeneity translates into a subcritical Neimark-Sacker with an hard stability loss, i.e. just after the bifurcation unfeasible trajectories are obtained for any initial condition different from the Nash equilibrium (see Agliari et al., 2005, for a detailed study of the local and global dynamics connected with this subcritical N-S bifurcation).

Figure 4: Towards subcritical N-S bifurcation in the adaptive best reply dynamics

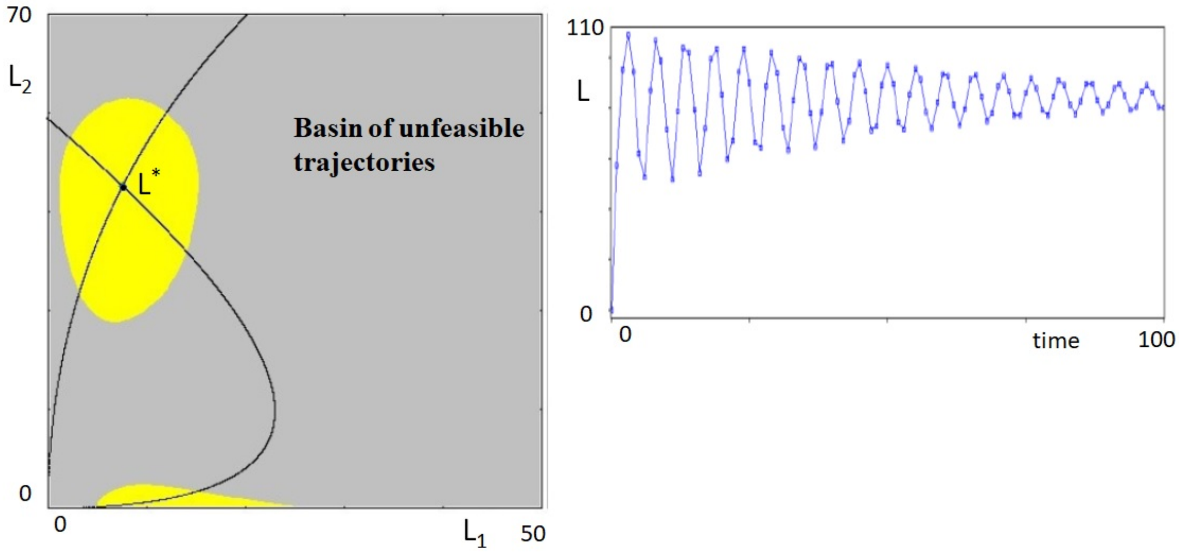


Figure 4: $\lambda_1 = 0.6, \lambda_2 = 0.7, \alpha = 0.03, y = 200, \beta = 15, r_B = 0.01, r = 0.001, \rho_1 = 0.095, \rho_2 = 0.005, c_{L1} = 0.005, c_{L2} = 0.005$

Figure 4 shows the scenario before the N-S bifurcation with a small non-connected basin of attraction as ρ_1 increases, while in the right panel it is possible to appreciate the damped oscillation toward the Nash equilibrium. Moreover, we can compare Figure 4 with Figure 2 to draw an important remark on the degree of stability of the aforementioned dynamic adjustments. In the first case of pure best reply the system loses stability via a degenerate N-S bifurcation for $\rho_1 = 0.055$, while with lower lambdas in the adaptive case, all other things being equal, the N-S bifurcation happens for ρ_1 greater than 0.095. Thus, introducing inertia increases the overall stability of the banking system allowing for a greater diversity between the two banks in the system. In all these figures, in order to simulate an increasing heterogeneity between banks we moved ρ_1 to be far larger than ρ_2.

These findings underline, under the assumption of best reply adjustment, the crucial role on stability of the heterogeneity of banks, represented by different marginal costs $c_{L1} \neq c_{L2}$ and different expected share of NPLs $\rho_1 \neq \rho_2$. The latter parameters reflect a diverse risk-taking in the lending activity. For example, if $\rho_1 > \rho_2$ bank 1 is lending a higher share of sub-prime or risky loans, or is doing a worse screening activity of borrowers, than bank 2. Consequently, bank 1 will take into account an higher share of NPLs in defining the next optimal loan amount. However, it is possible to add another degree of diversity when the efficiency in managing a given volume of loans is different for the two credit institution, e.g. $c_{L1} > c_{L2}$. The reasons can be several: the banks have different dimensions that imply a diverse degree of economy of scale, and/or more efficient internal practices in handling the credits, and/or different monitoring expenses, etc. Furthermore, in the adaptive case a different level of anchoring, or prudence, in choosing the next move (i.e. loan supply) can also cause instability, or the failure to reach the Nash equilibrium. In the latter case, the heterogeneity is due to the λ parameters, e.g. $\lambda_1 \neq \lambda_2$. They represent relevant behavioral parameters, and are a proxy of the level of risk-adversion of each bank (recall the discourse on the bank sub-optimal best reply based on imperfect information, in Section 3). This means that the level of information asymmetry perceived in the market on competitors' decisions can influence the banks' behavior (more cautious or less), and this in turn may potentially endanger the stability of the banking sector. To sum up, Figure 5 shows a two-dimensional bifurcation diagram in the parameters' plane ρ_1, ρ_2 for the adaptive best reply case, where all the three levers of bank diversity are present (e.g. $\lambda_1 \neq \lambda_2$, $c_{L1} \neq c_{L2}$, and $\rho_1 \neq \rho_2$). The stability region of the Nash equilibrium is represented by yellow color, whereas black represents unfeasible trajectories, even starting from initial conditions taken in a very small neighborhood of the equilibrium.

Figure 5: The stability region of the adaptive best reply dynamics in the plane ρ_1, ρ_2

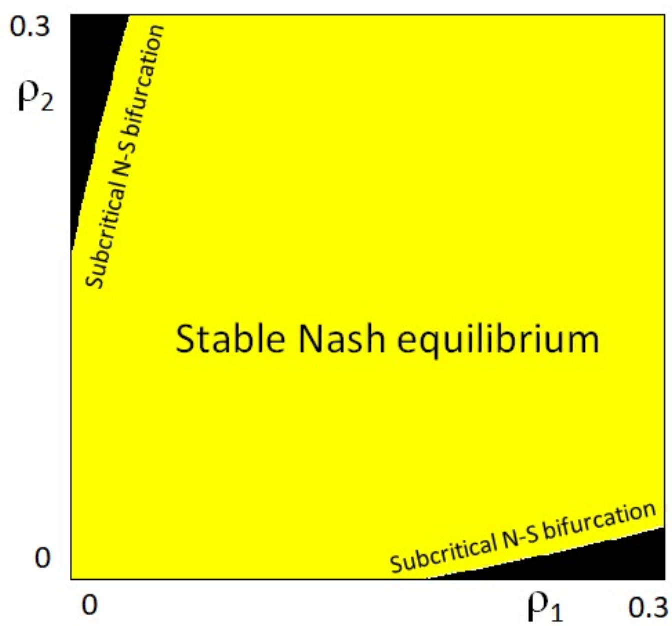


Figure 5: $\lambda_1 = 0.8$, $\lambda_2 = 0.9$, $\alpha = 0.05$, $y = 200$, $\beta = 15$, $r_B = 0.05$, $r = 0.03$, $c_{L1} = 0.004$, $c_{L2} = 0.005$

5 Gradient dynamics

An alternative dynamic adjustment mechanism, involving a lower degree of rationality, is obtained by considering the so-called gradient dynamics, based on the assumption that the banks adjust their loans supply over time proportionally to their marginal profits (see e.g. Dixit, 1986, Flam, 1993, Bischi and Naimzada, 2000).

In essence, we assume that each bank does not have a complete knowledge of the demand function or is not able to solve the optimization problem, hence it tries to infer how the market will respond to small changes in loan supply by an empirical estimate of the marginal profit, that may be obtained by a market research or by brief experiments performed at the beginning of period t. We assume that the banks are able to obtain a correct empirical estimate of the marginal profits $\left(\frac{\partial \pi_i}{\partial L_i}\right)^{(e)} = \frac{\partial \pi_i}{\partial L_i}\ \ i = 1, 2\,.$ This local estimate of expected marginal profits is generally easier to obtain than a solution of the optimization problem that requires computational skill as well as a global knowledge of the demand function (involving values of L that may be very different from the current ones).

In this case, the banks behave as local profit maximizers, the local adjustment process being one where a bank increases its loan supply if it perceives a positive marginal profit and decreases the loan amount if marginal profit is negative:

$$L_i(t+1) = L_i(t) + \alpha_i(L_i)\,\frac{\partial \pi_i}{\partial L_i}(L_1, L_2)\ ; \qquad i = 1, 2 \tag{16}$$

where $\alpha_i(L_i)$ is a positive function, which gives the extent of loan supply variation of bank i following a given profit signal.

It is easy to verify that, being $\alpha_i(L_i) > 0$, from the equilibrium conditions $L_i\,(t+1) = L_i\,(t)$, $i = 1, 2$, one gets the first order conditions (5), i.e. $\frac{\partial \pi_i}{\partial L_i}(L_1, L_2) = 0$, $i = 1, 2$, hence any positive equilibrium point of (16) is a Nash equilibrium (the trivial equilibrium $L_i = 0,\ i = 1, 2$, is not considered feasible).

An adjustment mechanism similar to (16) has been proposed by some authors with constant α_i (see e.g. Dixit, 1986, Flam, 1993). Instead, following Bischi and Naimzada (2000a), we assume α_i proportional to L_i, i.e. $\alpha_i(L_i) = v_i L_i\,, i = 1, 2$, where v_i is a positive *speed of adjustment*, equivalent to the assumption that the "relative change" is proportional to the marginal profit:

$$\frac{L_i(t+1)-L_i(t)}{L_i(t)} = v_i \frac{\partial \pi_i}{\partial L_i}(L_1, L_2).$$

If we insert into (16) the marginal profit in the right hand side of (5) with the isoelastic inverse loan demand (8) and its derivative $r_L^{'}(L) = -(\alpha y + \beta r_B)/(L_1+L_2)^2$, the discrete dynamical system (16) assumes the form $(L_1(t+1), L_2(t+1)) = T(L_1(t), L_2(t))$, and the map T is given by:

$$T : \begin{cases} L_1(t+1) = & L_1(t) + v_1(\alpha y + \beta r_B)(1-\rho_1) L_1(t) \left(\frac{L_2(t)}{(L_1(t)+L_2(t))^2} - k_1 \right) \\ \\ L_2(t+1) = & L_2(t) + v_2(\alpha y + \beta r_B)(1-\rho_2) L_2(t) \left(\frac{L_1(t)}{(L_1(t)+L_2(t))^2} - k_2 \right) \end{cases} \tag{17}$$

where the aggregate parameters k_i, $i = 1, 2$, are defined by (12). It is evident, as expected, that the unique feasible equilibrium is the Nash equilibrium (13), i.e. the same obtained under the assumption of full rationality or under the best reply dynamic adjustment. However, its local stability properties, as well as the global structure of its basin of attraction when it is stable, are different. Thus, a comparison between the two kinds of dynamic adjustments may give an interesting insight.

It is worth to stress that the map (17) is not defined along the line $L_1 + L_2 = 0$ (*line of non-definition* δ_s). Since the state variables L_1, L_2 represent the loans offered by the banks, we are only interested in the dynamics of (17) in the region $\mathbb{R}^2_+ = \{L_1, L_2 \,|\, L_1 \geq 0,\ L_2 \geq 0\}$, and the only point of $\mathbb{R}^2_+$ belonging to the line δ_s is $(0, 0)$. However, the presence of this point may have a crucial influence on the structure of the basins in $\mathbb{R}^2_+$.

Following the standard local stability analysis based on the computation of the Jacobian matrix:

$$DT(L_1, L_2) = \begin{bmatrix} 1 + v_1(\alpha y + \beta r_B)(1-\rho_1)\left(\frac{L_2(L_2-L_1)}{(L_1+L_2)^3} - k_1\right) & v_1(\alpha y + \beta r_B)(1-\rho_1)\frac{L_1(L_1-L_2)}{(L_1+L_2)^3} \\ v_2(\alpha y + \beta r_B)(1-\rho_2)\frac{L_2(L_2-L_1)}{(L_1+L_2)^3} & 1 + v_2(\alpha y + \beta r_B)(1-\rho_2)\left(\frac{L_1(L_1-L_2)}{(L_1+L_2)^3} - k_2\right) \end{bmatrix}$$

at the equilibrium point

$$DT(L_1^*, L_2^*) = \begin{bmatrix} 1 + v_1(\alpha y + \beta r_B)(1-\rho_1)k_1\left(\frac{k_1-k_2}{k_1+k_2} - 1\right) & v_1(\alpha y + \beta r_B)(1-\rho_1)\frac{k_1-k_2}{k_1+k_2} \\ v_2(\alpha y + \beta r_B)(1-\rho_2)\frac{k_2-k_2}{k_1+k_2} & 1 + v_2(\alpha y + \beta r_B)(1-\rho_2)k_2\left(\frac{k_2-k_1}{k_1+k_2} - 1\right) \end{bmatrix}$$

the sufficient condition for the stability of L^* is that the eigenvalues of the Jacobian matrix $DT(L_1^*, L_2^*)$ are inside the unit circle of the complex plane. As mentioned in the previous Section, this is true if and only if the following conditions in terms of the trace Tr^* and the determinant Δ^* hold:

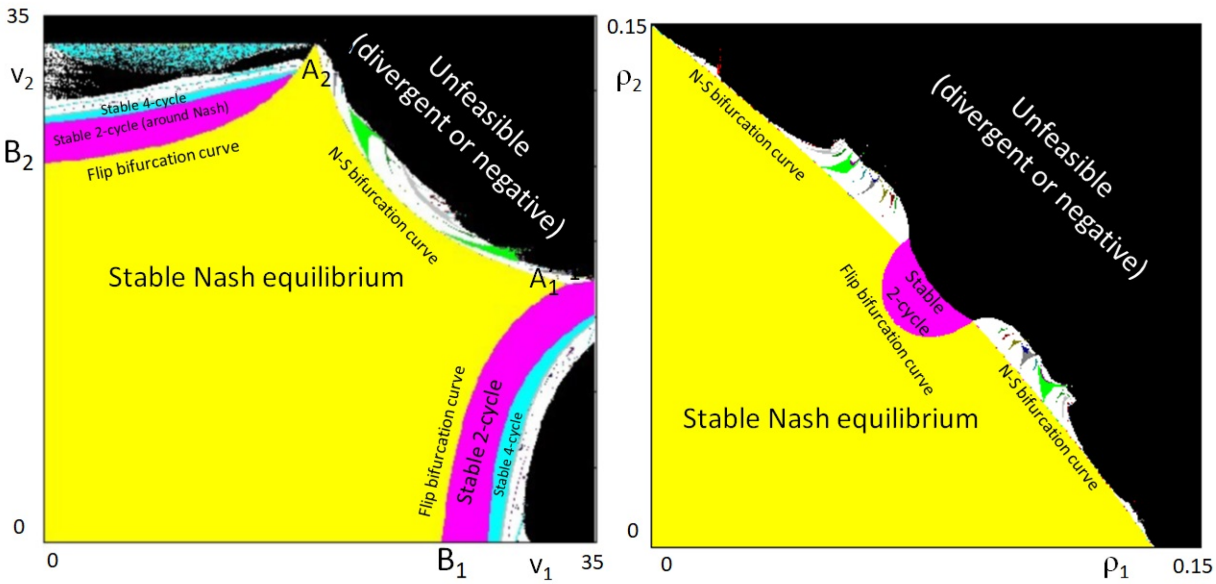
$$\begin{aligned}
&1 - Tr^* + \Delta^* = (\alpha y + \beta r_B)^2 (1-\rho_1)(1-\rho_2) k_1 k_2 v_1 v_2 > 0 \\
&1 + Tr^* + \Delta^* = (\alpha y + \beta r_B)^2 (1-\rho_1)(1-\rho_2) k_1 k_2 v_1 v_2 - 4\tfrac{k_1 k_2}{k_1+k_2} \left(v_1 (\alpha y + \beta r_B)(1-\rho_1) + v_2 (\alpha y + \beta r_B)(1-\rho_2)\right) + 4 > 0 \\
&1 - \Delta^* = 2\tfrac{k_1 k_2}{k_1+k_2} \left(v_1 (\alpha y + \beta r_B)(1-\rho_1) + v_2 (\alpha y + \beta r_B)(1-\rho_2)\right) - (\alpha y + \beta r_B)^2 (1-\rho_1)(1-\rho_2) k_1 k_2 v_1 v_2 > 0
\end{aligned} \tag{18}$$

where $k_i,\ i = 1, 2$, are defined in (12). The first condition is always satisfied, whereas the other two define a bounded region of stability in the parameter space. The second condition defines the condition of flip (or period doubling) bifurcation, the third condition the N-S bifurcation. Given the unitary costs $c_{L_1},\ c_{L_2}$ and the expected loan default $\rho_1,\ \rho_2$, the stability region can be represented in the plane $\mathcal{V} = \{v_1, v_2 \,|\, v_1 \geq 0, v_2 \geq 0\}$, as shown in yellow in Figure 6a.

This region is symmetric with respect to the diagonal $v_1 = v_2$ and bounded by the positive branches of two equilateral hyperbolas (see Bischi et al., 1999b) whose equations are obtained from the second and the third condition of (18). The coordinates of the vertices of this region are:

$$\begin{aligned}
A_1 &= \left(\frac{2}{(r+\rho_1+c_{L_1})}, \frac{2}{(r+\rho_2+c_{L_2})}\right) \quad A_2 = \left(\frac{2}{(r+\rho_2+c_{L_2})}, \frac{2}{(r+\rho_1+c_{L_1})}\right) \\
B_1 &= \left(\frac{k_1+k_2}{(r+\rho_1+c_{L_1})(r+\rho_2+c_{L_2})}, 0\right) \quad B_2 = \left(0, \frac{k_1+k_2}{(r+\rho_1+c_{L_1})(r+\rho_2+c_{L_2})}\right).
\end{aligned}$$

Figure 6: The stability region of gradient dynamics in the plane



(a) $\alpha = 0.03, y = 200, \beta = 15, r_B = 0.01, r = 0.05,$ $\rho_1 = 0.055, \rho_2 = 0.005, c_{L1} = 0.005, c_{L2} = 0.005$

(b) $v_1 = 15, v_2 = 20, \alpha = 0.03, y = 200, \beta = 15,$ $r_B = 0.05, r = 0.03, c_{L1} = 0.004, c_{L2} = 0.005$

In Figure 6a the region of stability of the Nash equilibrium is represented by yellow color in the parameters' plane (v_1, v_2), whereas in 6b it is represented in the parameters' plane (ρ_1, ρ_2). The other colors represent periodic cycles of different periods, such as pink for period 2, light-blue for period 4 etc., whereas the white area is a region of bounded attractors that may be periodic (with period greater than 15), quasiperiodic, or chaotic, the black region represents unfeasible trajectories, i.e. diverging or involving negative values. In the left panel (6a), along the boundary of the yellow region connecting the points A_1 and A_2 a supercritical N-S bifurcation occurs, whereas along the arc of hyperbola connecting B_1 with A_1, as well as along the one between B_2 and A_2, a supercritical flip bifurcation occurs. Analogously, in the right panel (6b) a supercritical flip bifurcation occurs along the line separating yellow and pink colors, a supercritical N-S bifurcation along the other boundaries of the yellow region.

The arguments given so far are based on local stability results. However, such insights may lead to misleading conclusions if they are not supported by an analysis of the basins of attraction, because it may occur that an equilibrium, even if it is locally stable, may be so close to a boundary of its basin that any practical stability is lost because a small perturbation may lead the system to evolve to another region of the phase space (even at infinite distance, along a diverging trajectory). For example, in Figure 7 stable Nash equilibrium is shown with its own basin of attraction represented by the yellow region. However, the topological structure of this basin is quite irregular, being multiply connected and quite intermingles with portions of the basin of diverging trajectories represented by the grey shaded region.

Figure 7: The Nash equilibrium and its stability region in the Gradient Dynamics

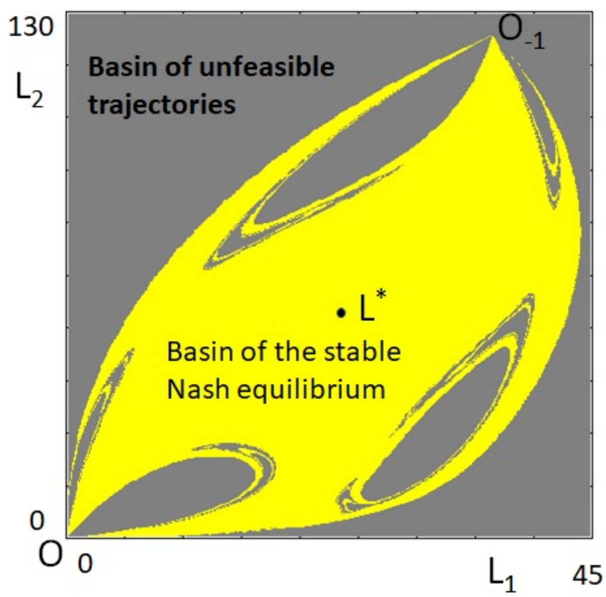
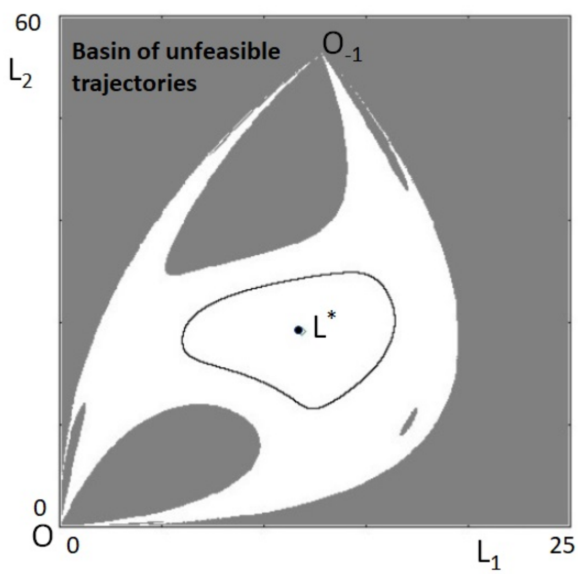


Figure 7: $v_1 = 23, v_2 = 40, \alpha = 0.05, y = 200, \beta = 15, r_B = 0.05, r = 0.03, \rho_1 = 0.055, \rho_2 = 0.005, c_{L1} = 0.005, c_{L2} = 0.005$

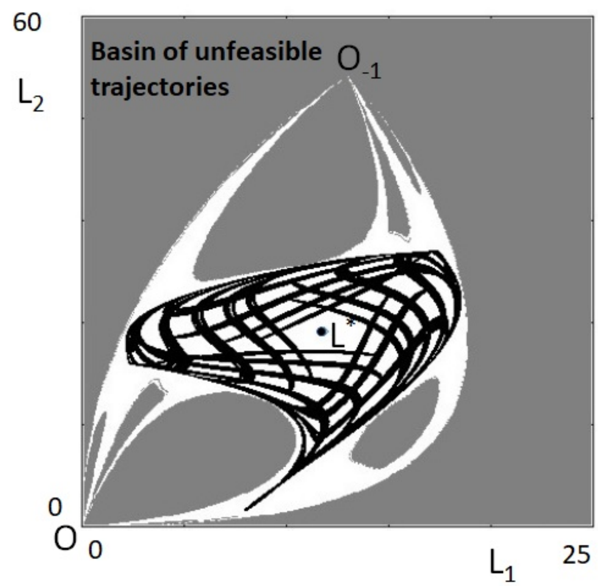
The global structure of the boundaries that separate these basins is strongly influenced by the following two global features of the map (17): (i) it is a *noninvertible map* of the plane, so its global geometric properties can be characterized by the method of *critical curves* (see Mira et al., 1996); (ii) the map T has denominators which vanish along a one-dimensional subset of the plane, on which a *focal point* exists, located at the singular point $(0, 0)$, where the map assumes the form $0/0$ (see Bischi et al., 1999a). For an analytical and numerical analysis of these global dynamical properties of the map (17), and how these are related to (i) and (ii), we refer to Bischi et al. (2001) where a map with the same mathematical structure is analyzed and applied to a different economic context.

Here we are interested in some numerical simulations concerning the kind of bounded nonequilibrium dynamics observed outside the stability region. Indeed, when one of the two flip bifurcation curves are crossed as some parameters are varied, then a stable cycle of period two located around the unstable Nash equilibrium is observed, and further parameters' changes may lead to the well known period doubling route to chaos. When the N-S bifurcation curve is crossed an attracting closed invariant curve is obtained, along which quasiperiodic or periodic motion occurs (the case of periodic windows is related to the existence of Arnold tongues, the green regions clearly visible in the upper parts of both Figures 6a and 6b). In fact, with speeds of adjustment $v_1 = 23$ and $v_2 = 23$, a stable limit cycle is observed around the unstable Nash equilibrium (just after the supercritical N-S bifurcation), as shown in Figure 8a together with its basin of attraction (the white region), whereas Figure 8b displays a chaotic attractor obtained with $v_1 = 24$ and $v_2 = 23.8$.

Figure 8: The supercritical Neimark-Sacker bifurcation in the Gradient Dynamics



(a) $v_1 = 23, v_2 = 23, \alpha = 0.03, y = 200, \beta = 15,$
$r_B = 0.01, r = 0.05, \rho_1 = 0.055, \rho_2 = 0.005,$
$c_{L1} = 0.005, c_{L2} = 0.005$



(b) $v_1 = 24, v_2 = 23.8, \alpha = 0.03, y = 200, \beta = 15,$
$r_B = 0.01, r = 0.05, \rho_1 = 0.055, \rho_2 = 0.005,$
$c_{L1} = 0.005, c_{L2} = 0.005$

In both cases, the grey region represents the set of initial conditions generating unbounded trajectories. Moreover, in the case of the chaotic attractor shown in Figure 8b, it is very close to the basin boundary, hence the system is very vulnerable as a small parameter change may cause a contact between the bounded attractor and its basin boundary, thus giving rise to a global bifurcation at which the chaotic attractor is transformed into a chaotic repellor, after which almost all the initial conditions will generate diverging trajectories. In an economic interpretation, divergence means that the two banks, and so the loan market, cannot find a suitable adjustment around the Nash equilibrium, thus the duopoly system collapses. This situation happens for very large values of the ρ_i parameter, thus in the presence of a severe bank financial distress. We may also assist to a transitory phase, whose length depends on the parameter values, where the two banks try to find a compromise in their loan supply along a transient motion around the Nash equilibrium. The transitory phase characterized by system instability, may stabilize in the long-run. In this respect, Figure 9 shows the loan quantity and profit evolution for both banks in this particular situation. Bank 2, characterized by slightly larger costs but far lower probability of default than bank 1, offers a greater amount of loans in equilibrium (i.e. in the long-run, after about 10 periods). Figure 9 shows that despite damped oscillations of quantities and profits [7], both banks reach the Nash equilibrium.

Figure 9: The banks' profit evolution

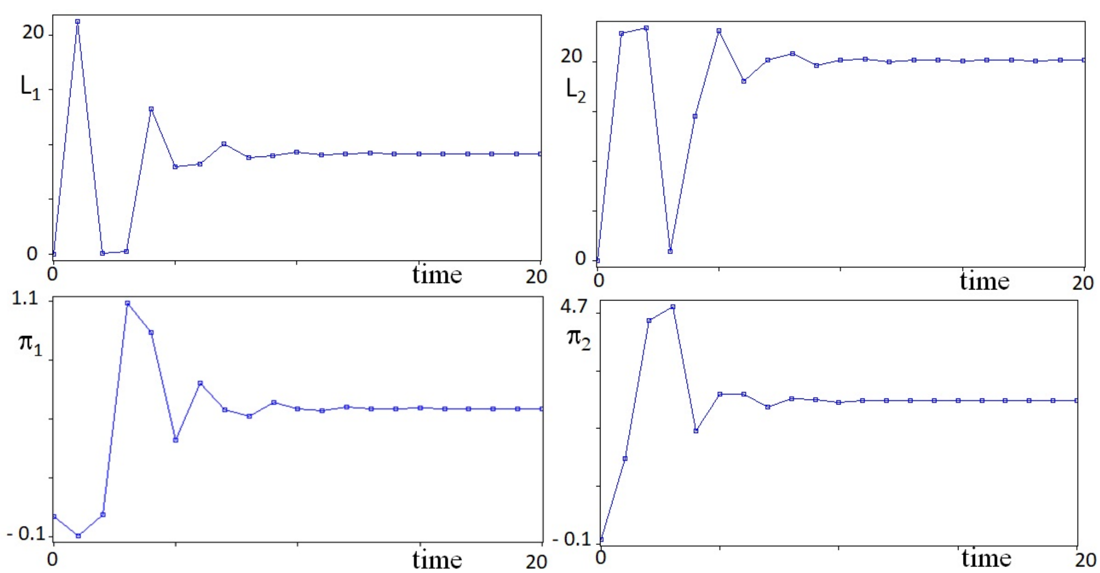


Figure 9: $v_1 = 15$, $v_2 = 15$, $\alpha = 0.05$, $y = 200$, $\beta = 15$, $r_B = 0.01$, $r = 0.05$, $\rho_1 = 0.08$, $\rho_2 = 0.01$, $c_{L1} = 0.003$, $c_{L2} = 0.005$

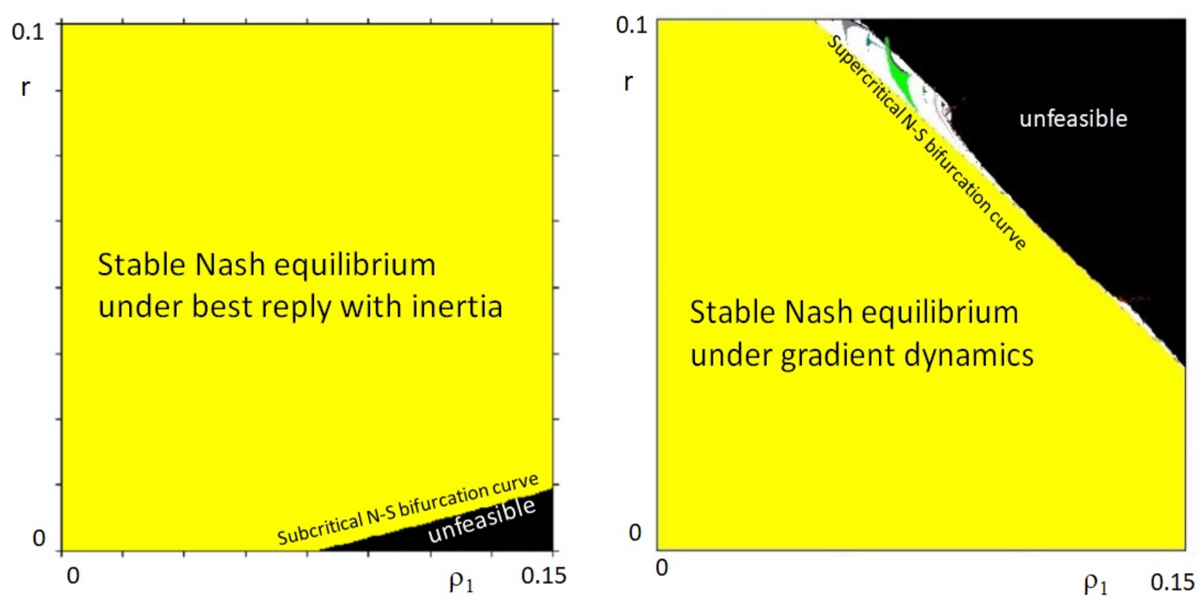
[7] Although bank 1 exhibits negative profit in period 1, it remains in the market and its loan supply stabilizes to a positive value.

6 Economic implications

In this Section, we focus on economic implications of the dynamic adjustments proposed, showing the similarities and differences between the two models. First of all, our analyses suggests that the impact of interest rate on banking stability depends on the level of rationality that characterizes bank competition. In the model, the parameter r is exogenously determined by the Central Bank according to its inflation and output targets (Bacchiocchi and Giombini, 2021), so that different values of the parameter r correspond to expansive or restrictive monetary policies for low and high r values, respectively. We start by investigating the impact of monetary policies, as captured by different r values, on the volume of loan provided, NPLs, and market stability, in the two models.

In Figures 10a and 10b we compare the dynamics of the two models (adaptive best reply and gradient dynamics) to the same range of interest rates. We take as reference bank 1, so the x-axis measures the amount of expected NPLs ρ_1 (i.e. proxy of the expected financial riskiness of its lending activity), while the y-axis shows different monetary policies r.

Figure 10: A stability comparison for different monetary policies



(a) $\lambda_1 = 0.7, \lambda_2 = 0.7, \alpha = 0.03, y = 200, \beta = 15,$ $r_B = 0.01, \rho_2 = 0.005, c_{L1} = 0.005, c_{L2} = 0.005$

(b) $v_1 = 15, v_2 = 15, \alpha = 0.03, y = 200, \beta = 15,$ $r_B = 0.01, \rho_2 = 0.005, c_{L1} = 0.005, c_{L2} = 0.005$

Figure 10a shows the impact of monetary policies assuming the adaptive best reply dynamics. We obtain that divergence occurs (for high levels of ρ_1) in the presence of expansive monetary measure, i.e. for very low levels of the interbank rate r.[8]

Overall the system is stable for a wide range of r and ρ_1 values, suggesting that the transmission mechanisms of monetary policies works properly in the presence of inertia and banks that compete by an (adaptive) best reply strategy.

Figure 10b shows the impact of monetary policies on the share of NPLs in the presence of banks that compete by means of gradient dynamics. We obtain that the yellow stability region shrinks, and the system moves to instability (in white) or divergence (in black) as long as the interbank rate r increases, for a larger set of ρ_1 values than those of the previous Figure 10a.

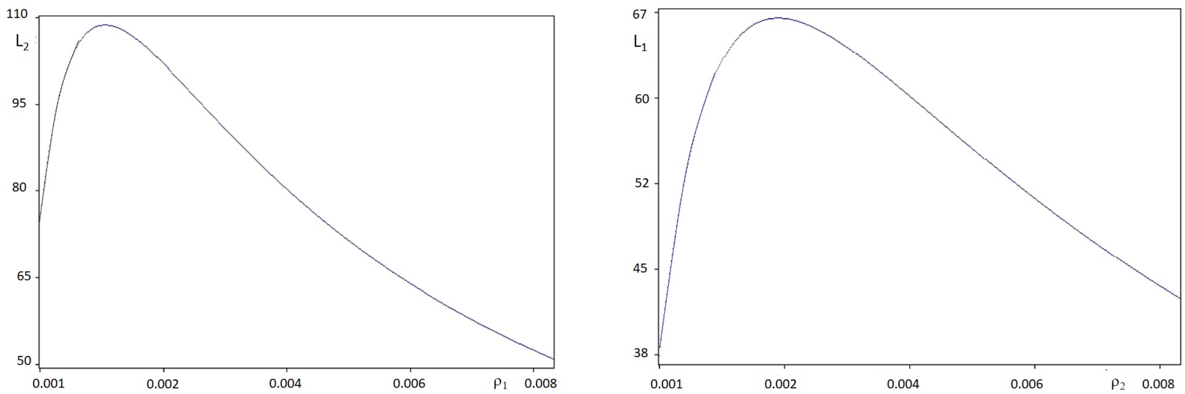
This finding suggests that in the presence of a lower degree of banks rationality (i.e: gradient dynamics), the monetary policy set by the Central Bank performs worsen than in the presence of more rational agents (i.e.: adaptive best reply). Likely, in the former case the transmission mechanisms (that works through the price or quantity channels) encounters obstacles related to bounded rationality of banks. These obstacles refer to the capacity of banks to modify their loan supply, potentially affecting the potency of forward guidance and leading to a powerful mitigation of the effects of monetary policy. As discussed in Farhi and Werning (2019), under forward guidance the intended interest rate path is directly and exhaustively communicated by the central bank. However, expectations for other endogenous macroeconomic variables, such as output or inflation, are not under the direct control of the central bank nor directly announced and, thus, agents can only form beliefs about them indirectly. As a consequence, banks react late to restrictive monetary policies (i.e. high r values) that would aim at reducing loan quantities by increasing loan costs. Therefore, as long as banks do not react to restrictive monetary policies by reducing loan supply, a larger amount of loans fails to be reimbursed because of the increased cost (high r) leading the system to instability and divergence.

Secondly, our analyses shows that the market dynamics depend on bank interdependence for both models considered. That is, the bank i financial stress has a non-linear impact on the credit supply of the competitor j. For low ρ_i values the competitor j increases its market share at the expense of bank i, but when the

[8] In Figure 10a we used λ_1 and λ_2 equal to 0.7, but the same qualitative result occurs in the best reply case with lambdas equal to 1. The only difference is given by the fact that the yellow basin of attraction is smaller, thus lower values of ρ_1 can cause divergence, coherently with the stability argument seen in the previous Section 4.

expected NPLs share ρ_i exceeds a certain threshold, the difficulty of the bank i becomes detrimental also for the competitor j, that reduces its loan supply.

Figure 11: The banks' interdependence



(a) $v_1 = 15, v_2 = 15, \alpha = 0.03, y = 200,$ $\beta = 15, r_B = 0.01, r = 0.001, \rho_2 = 0.01,$ $c_{L1} = 0.002, c_{L2} = 0.003$

(b) $v_1 = 15, v_2 = 15, \alpha = 0.03, y = 200,$ $\beta = 15, r_B = 0.01, r = 0.001, \rho_1 = 0.02,$ $c_{L1} = 0.002, c_{L2} = 0.003$

Focusing on the gradient dynamics, Figure 11a shows the impact of ρ_1 on the credit supply of the competitor, L_2; while Figure 11b highlights the effect of ρ_2 on the credit supply of the bank 1, L_1. The threshold or apex of the curve could be different depending on the value of the other parameters, especially banks' variable cost c_{L1} and c_{L2}. This finding confirms that whatever is the bank taken as reference, the financial stress of a credit institution translates into a potential suffering situation for all the other banks in the market, leading to what is known in the financial literature as credit crunch (i.e. reduction of the overall loan supply).

The effect works through two channels: the interbank channel and the signalling channel via loan interest rate r_L. As modeled in the bank's profit function (3), the banks lend each other in the interbank market. If one bank i perceives an high risk on its credit recovery, it will likely reduce not only the volume of loans provided to households and firms L_i, but also to the banks' counterpart j, M_i.

In this situation, the other bank(s) counterpart j would react by shrinking their loan supply too.

The other channel works through the cost of loans. In fact, an higher expected loan default means a greater insolvency risk for the bank i. It can be interpreted as an additional cost to bear in the lending activity. For this reason, the struggling credit institution i will shrink its loan supply re-balancing its portfolio towards the other type of assets modeled (i.e. financial investment B). [9]

The reduced loan supply of one of the two credit institutions in our duopoly banking market pushes up the loan rate r_L, that in such cases acts as a signal of expected financial difficulties, and bring down the overall loan demand from private, see equation (8). The result is that, in equilibrium, all the banks in the industry, including the financial healthier competitor(s) j, face a smaller market, thus being forced to provide a lower loan quantity.

These considerations imply that the higher the concentration of institutions in the banking sector, the larger the aforementioned credit crunch effect. In this vein, in a duopoly, the financial tensions of one of the two banks have a strong impact on the unique competitor. If the market is more fragmented or competitive (i.e. with a greater number of banks) the impacts on the other competitors, by means of the interbank and signalling channels, are relatively lower.

A final remark comes from the speed required to reach the unique Nash equilibrium.

Figures 12a and 12b show the time series of L (i.e. the overall supply of loans in this market) assuming the adaptive best reply, and the gradient dynamics, respectively. We obtain that the speed of convergence is faster in the presence of the (sub-optimal) optimizing behavior[10] of Figure 12a, than in the presence of the local adjustment that characterizes Figure 12b.

The result is consistent with the following underlying assumption.

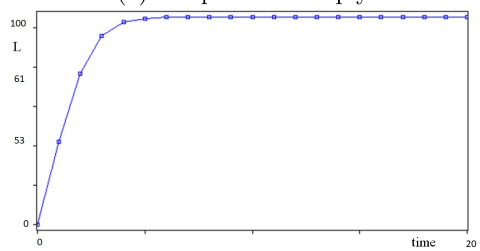
A greater degree of agents' rationality (i.e. adaptive best reply) leads to a faster banks' reaction to changes in competitors' loan offers and banking market conditions, while in the case of more limited knowledge and computational ability (i.e. gradient dynamics), the adjustment is relatively slower.

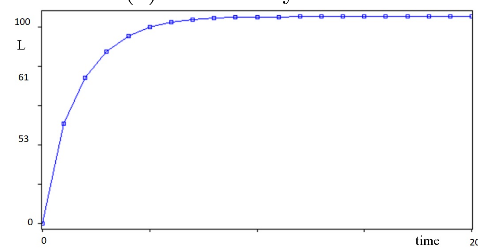
[9] In this simple model we focus just on the loan market without considering the possible effect of re-balancing the banks portfolio or changing its dimension (i.e. quantitative changes). This requires a deeper analysis of the interdependence between loan, deposit and bond markets.

[10] *Ceteris paribus*, in Figure 12a, with λ_1 and λ_2 equal to 1 (i.e. the "pure" Best Reply), the equilibrium is reached after 3 periods of time only.

Figure 12: The speed to the Nash equilibrium

(a) Adaptive best reply   (b) Gradient dynamics





(a) $\lambda_1 = 0.7, \lambda_2 = 0.7, \alpha = 0.03,$
$y = 200, \beta = 15, r_B = 0.01, r = 0.001,$
$\rho_1 = 0.03, \rho_2 = 0.02, c_{L1} = 0.002,$
$c_{L2} = 0.003.$

(b) $v_1 = 15, v_2 = 15, \alpha = 0.03,$
$y = 200, \beta = 15, r_B = 0.01, r = 0.001,$
$\rho_1 = 0.03, \rho_2 = 0.02, c_{L1} = 0.002,$
$c_{L2} = 0.003.$

7 Conclusions and future research

This paper analyzed two duopoly models to describe banks that compete in a loan market described by an isoelastic demand function as in Puu (1991). The two dynamic models are characterized by different kinds of boundedly rational adjustments to increase their profits under different assumptions on limited information and computational ability, as well as in the presence of NPLs. We first discussed on the adaptive best reply mechanism where each bank reacts with inertia to competitor's decision. Then, we focused on a dynamic adjustment mechanism that involves a lower degree of rationality, by considering the gradient dynamics, based on the assumption that the banks adjust their loans supply over time proportionally to their marginal profits. The main mathematical properties of similar discrete-time dynamic models have already been studied in the literature, see e.g. Puu (1991) and Agliari et al. (2005) for the former model, Bischi et al. (1999b) and Bischi et al. (2001) for the latter one. However, the meaning of the dynamic variables as well as the structure of the parameters' space is quite different. In particular, the marginal costs are replaced by the aggregate parameters k_1 and k_2, that include the share of loans expected by the banks ρ_i, and the interest rate on interbank market r.

Moreover, a comparison between the two models has not been analyzed in the literature, and such a comparison concerning local and global stability properties of the two models, is particularly interesting when competition between banks is considered. In both cases we obtain that bank heterogeneity, which derives from either different cost structures, or different shares of NPLs, or both, affects the stability of the equilibrium.

In term of economic implications, the models suggest that in the presence of a larger degree of bounded rationality of banks (i.e.: gradient dynamics), the monetary policy set by the Central Bank performs worsen than in the presence of more rational agents (i.e.: adaptive best reply). Likely, in the former case the transmission mechanisms (that works through the price or quantity channels) encounters obstacles related to too limited bank rationality. The latter leads the system to divergence or instability for relatively high levels of interest rates and share of expected NPLs. Secondly, our analyses showed that bank interdependence affects the market dynamics so that the financial stress of a credit institution could translate into suffering situation for all the other banks in the market, leading to a credit crunch.

In terms of future research agenda, some additional elements are worth exploring.

First of all, the identification of the combinations of banking operations harbingers of financial distress or of corporate insolvencies is of paramount relevance. Secondly, attention could be deserved to the analysis of the impact of a reserve requirement change, or the effect of monetary policy on financial assets yields, on deposits and loans interest rates. Moreover, future agenda could focus on the possible interactions among the different markets in which the bank operates. Indeed, the cost function $C(L, D, B)$ could be modeled so that costs would be not perfectly separable, and interaction effects among markets could be explored. In this latter case, both different kinds of boundedly rational adjustments, and NPLs would affect not only the equilibrium of the loan market but also of the financial system as a whole. Additionally, costs may be nonlinear to capture the effects of economies or dis-economies of scale.

Last, but non least, the two models analyzed in this paper also provide arguments about a question often addressed in the literature on dynamic games, concerning the possibility that a repeated game will eventually lead to a Nash equilibrium despite the fact that players are boundedly rational in the short run. This is an evolutionary interpretation of the Nash equilibrium, and traditionally, answers to this question have been given in terms of the local stability of Nash equilibria. However, even if only a way to behave rationally exists (represented by immediate convergence, in one shot, to a Nash equilibrium), several kinds of boundedly rational adaptive adjustment mechanisms may be observed, characterized by different stability properties. Thus, a comparison between different adjustment mechanisms, related with different information sets or computational abilities, or other features, is always interesting in this context. Moreover, in a nonlinear model, a study of local stability only may not be sufficient to perform such a comparison. For example, a study of the extension of the basin of attraction of a stable equilibrium can give information about its robustness with respect to exogenous perturbations, but this requires a global analysis of the dynamical system, i.e. a study not based on linear approximations. Since for general higher-dimensional systems such results are hard to come by, we limited ourselves to the case of two banks.

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