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A Common External Framework for Kleene and McCarthy Algebras

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ABSTRACT

Algebras of disjoint alternatives (ADAs), introduced by Manes, are extensions of McCarthy algebras by a unary operator acting as an oracle for the halting problem, and provide a semantics for the equational theory of the *if-then-else* construct.

We study the variety of ADAs in the signature without the constant k of Manes' formulation, and prove a representation theorem: every ADA has an essentially unique representation as an algebra of pairs (B, F) , where B is a Boolean algebra and F a distinguished filter of B . This refines Manes' construction, which captures the ADAs possessing a fixed point of the negation, i.e. the case $F = B$.

Since regular double Stone algebras admit a similar representation due to Katriňák, we derive that the two varieties are term-equivalent, and hence so are ADAs, three-valued Łukasiewicz algebras, and three-valued MV-algebras. As consequences, we obtain a categorical equivalence between ADAs and Boolean algebras with a distinguished filter, extending Manes' equivalence, and the amalgamation property for the variety of ADAs.

The representation accommodates, in a single framework, the connectives of Kleene's strong, McCarthy's sequential, and weak Kleene logics, modeling three distinct forms of error propagation in computation. Finally, we outline exploratory connections with rough sets and assume-guarantee contracts.

1 | Introduction

There is a long and influential tradition in the study of algebraic structures capable of modeling computational phenomena. In his pioneering work [1], McCarthy lays the foundations of an abstract analysis of computational processes. Certainly, one of the main motivations was providing a foundational framework for programming languages. Central features of computation, that lie outside classical logic, are the concepts of non-termination or undefinedness. This has naturally led to three-valued logical systems, where, in addition to the classical truth values, a third value is used to model partiality, undefinedness, and computational failure.

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Among the various calculi proposed in the literature, McCarthy’s conditional logic assumes a prominent position. Its connectives are non-commutative and reflect the operational behavior of short-circuit evaluation: expressions are evaluated from left to right, and evaluation stops as soon as a definitive result is obtained. This evaluation strategy has a long-standing and influential tradition whose origins can be traced back to Hilbert and the early works of Church and Turing. Nowadays, it is implemented in several major programming languages, including C, Lisp, and Prolog, and it has proved both computationally efficient and semantically expressive.

The algebraic counterpart of McCarthy’s calculus was subsequently introduced under different names, and pursued in independent forms by Konikowska and Guzmán. In Ref. [2], it appears as a *McCarthy algebra*, while in Ref. [3] it is called a *C-algebra*. Despite the terminological variation, these notions refer to one and the same algebraic concept: a regular, non-commutative extension of Boolean algebras designed to capture conditional evaluation in a three-valued setting.¹

Building on this stream of reasoning, Manes [4] introduced the notion of an *algebra of disjoint alternatives* (ADA), obtained by enriching McCarthy algebras by an additional unary operation acting as an oracle for the halting problem. ADAs provide a refined algebraic framework in which both defined and undefined behaviors of processes can be handled in a structurally controlled manner. Moreover, ADAs were introduced as an abstract tool to model the equational theory of the *if-then-else* construct in high-level programming languages.

An interesting feature of both McCarthy algebras and ADAs is their representability in terms of algebras of pairs. McCarthy algebras embed into algebras of pairs of disjoint subsets of a set (Theorem 3), and Manes extends this construction to ADAs by considering pairs of disjoint elements of a Boolean algebra B , i.e. structures of the form (B, B) in the notation of the present paper. However, this construction does not account for the whole variety: as we shall see (Lemma 1 and Theorem 7), the algebras of the form (B, B) are precisely the ADAs possessing a fixed point of the negation. A representation theorem covering every ADA is, to the best of our knowledge, missing from the literature. Filling this gap is the first aim of this paper.

Throughout the paper, we work with ADAs in the k -free signature $(\otimes, \oplus, ', \downarrow, 0, 1)$, dropping the constant k of Manes’ original definition [4, Definition 17]. As we explain in Remark 2, this choice is not merely cosmetic, but it is a matter of choice: the resulting variety properly contains the class of k -free reducts of Manes’ ADA, which coincides with the class of ADAs possessing a fixed point of the negation. It is precisely this enlargement that calls for, and renders possible, the refined representation in terms of pairs (B, F) with an arbitrary filter F . Throughout the paper, filters are not necessarily proper; in particular, the improper filter $F = B$ is allowed.

Another fundamental class of algebraic structures arising in the study of three-valued logics is that of regular double Stone algebras (RDSAs). The study of RDSAs provides methods for modeling uncertain and partial information. Other algebraic approaches to uncertainty are based on intuitionistic and interval-valued fuzzy structures, including the ideal-theoretic constructions for semirings and hemirings studied in Refs. [5, 6]. From a logical viewpoint, regular double Stone algebras generalize classical logic to a substructural logic in which the principles of non-contradiction and excluded middle may fail [7].

Regular double Stone algebras appear naturally in rough set theory, capturing the interplay between lower and upper approximations [8, 9]. Rough sets play a central role in artificial intelligence and related areas, including machine learning, data mining, and approximate reasoning [10, 11].

Moreover, these structures can be regarded as an algebraic counterpart of three-valued Łukasiewicz logic [12], a paradigmatic system for reasoning under indeterminacy [13].

Recent works have further highlighted that regular double Stone algebras are also relevant to the perspective of quantum logics (see Refs. [14, 15]), insofar as they represent “sharp contexts” in generalizations of orthomodular lattices, in much the same way as Boolean algebras represent classical contexts within the theory of orthomodular lattices (see also Ref. [16]).

From their connection with three-valued Łukasiewicz logic, regular double Stone algebras can be equivalently viewed as Kleene algebras enriched with an additional unary external operator. This viewpoint provides a useful bridge with other algebraic frameworks and it makes it possible to compare regular double Stone algebras with varieties that may appear at first glance rather distant (see e.g. Remark 1).

Like ADAs, RDSAs are tightly connected with Boolean algebras equipped with filters, and they admit representation results of a similar flavor. The aim of this paper is to make this parallel precise and to take advantage of its potential in order to obtain a refined representation of ADAs. This structural parallel suggests a deeper matching between the two classes.

For readability, let us briefly emphasize the main contributions of the present paper. First, we prove a representation theorem (Theorem 7): every ADA is isomorphic, in an essentially unique way, to an algebra of pairs (B, F) , where B is a

Boolean algebra, and F is a distinguished filter of B . As a byproduct, Manes’ construction is recovered as the special case $F = B$. The proof relies on suitable notions of sharp, dense, and dually dense elements of an ADA, which we introduce in analogy with the structure theory of regular double Stone algebras. Second, since RDSAs admit a representation of the very same nature, due to Katriňák [17], we establish that the varieties of ADAs and RDSAs are term-equivalent (Corollary 1). Consequently, ADAs are also term-equivalent to three-valued Łukasiewicz algebras and to three-valued MV-algebras. Third, as applications of the term-equivalence, we show that the category of ADAs is equivalent to the category of Boolean algebras with a distinguished filter, extending Manes’ equivalence with Boolean algebras. We also derive the amalgamation property for the variety of ADAs and the superamalgamation property for its expansion by the constant k . These algebraic properties are closely related to interpolation phenomena: in several standard settings of algebraic logic, amalgamation corresponds to Robinson-type properties and, under additional assumptions, to Craig interpolation, while superamalgamation corresponds to stronger forms of Craig interpolation.

An additional motivation for studying ADAs and RDSAs comes from the fact that they naturally accommodate several classical three-valued computational semantics within a unifying framework. In particular, by virtue of Corollary 1, these structures can be regarded as essentially one and the same mathematical object. Such a perspective makes it possible to reason under a single framework that supports equivalently operations whose behaviors may correspond to the connectives of Kleene’s strong three-valued logics and McCarthy’s sequential logic. Moreover, it is well-known that one can define, within this setting, an additional pair of operations capturing the behavior of the connectives of weak Kleene logics [3, 18].

This unifying framework is particularly significant as it reconciles three distinct conceptualizations of “undefinedness” in computation.

Kleene, for instance, introduced his three-valued semantics in the context of the theory of recursive functions [19], where the third truth value represents an indeterminate outcome of a partial computation. The corresponding connectives are commutative and well suited to model parallel or independent computations, in which an undefined result in one branch does not prevent the overall evaluation and may be resolved by additional information.

In contrast, McCarthy developed his three-valued logic to model the behavior of *partial predicates* in sequential computation, with particular attention to runtime errors [1]. His semantics follows a strict left-to-right, short-circuit evaluation strategy, characteristic of many programming languages, where the second argument of a connective is evaluated only when necessary. A third equally relevant paradigm is represented by weak Kleene logics, where the undefined value is treated as *infectious*: any error or non-terminating component propagates and “contaminates” the entire computation [20].

Taken together, these three systems provide distinct models of error propagation: indeterminacy, critical sequential error, and infectious failure. By showing that ADAs and RDSAs are term-equivalent, this paper offers a unified algebraic environment where these diverse perspectives can be analyzed and compared within a common framework. Related approaches, developed from different perspectives and with different aims, can be found in the work of Avron and Konikowska [21], Bergstra and Ponse [20], as well as [22–24].

This paper is structured as follows.

In Section 2, we provide a modicum of preliminaries to keep the paper as far as possible self-contained.

In Section 3, we propose a refinement of the representations of McCarthy algebras and ADAs as presented in Refs. [3, 4, 25].

In Section 4, we deepen the study of ADAs in terms of particular designated subsets of their universes in order to provide a full-fledged description of this variety in terms of pairs (B, F) , where B is a Boolean algebra and F is a designated filter of B , culminating in Theorem 7.

In fact, Theorem 7 will be of great relevance to the term-equivalence between ADAs and RDSAs discussed in Section 5.

In Section 6, we derive some direct consequences of the term-equivalence: a categorical equivalence between ADAs and Boolean algebras with a distinguished filter, and the amalgamation property for ADAs, together with the superamalgamation property for ADA_k , the class of ADAs in the language extended by the constant k . Finally, in Section 7, we summarize the results and the method of the paper, and outline two exploratory directions concerning assume–guarantee contracts and rough set theory.

2 | Preliminaries

In the present section we provide some basic notions and results needed throughout the paper. For further background, we refer the reader to Refs. [26, 27].

2.1 | Regular Double Stone Algebras

For standard facts about double Stone algebras we refer the reader to Grätzer [28] or Balbes and Dwinger [29].

Definition 1 [28, 29]. An algebra $L = (L, \wedge, \vee, \sim, 0, 1)$ is a *p-algebra* if $(L, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and $\sim$ satisfies:

$$x \wedge y = 0 \text{ if and only if } x \leq y^\sim.$$

Dually, an algebra $L = (L, \wedge, \vee, +, 0, 1)$ is a *dual p-algebra* if $(L, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and $+$ satisfies:

$$x \vee y = 1 \text{ if and only if } x \geq y^+.$$

An algebra $L = (L, \wedge, \vee, \sim, +, 0, 1)$ is a *double p-algebra* if the following conditions are satisfied:

- (1) $L = (L, \wedge, \vee, \sim, 0, 1)$ is a *p-algebra*.
- (2) $L = (L, \wedge, \vee, +, 0, 1)$ is a *dual p-algebra*.

It may be worth mentioning that the class of double *p-algebras* is a variety [29]. Let us now introduce the notions of Stone and double Stone algebras.

Definition 2 [28, 29].

- (1) A *p-algebra* $L = (L, \wedge, \vee, \sim, 0, 1)$ is a *Stone algebra* if it satisfies:

$$x^\sim \vee x^{\sim\sim} = 1. \quad (\text{Stone Condition})$$

- (2) A *dual p-algebra* $L = (L, \wedge, \vee, +, 0, 1)$ is a *dual Stone algebra* if it satisfies:

$$x^+ \wedge x^{++} = 0. \quad (\text{Dual Stone Condition})$$

- (3) An algebra $L = (L, \wedge, \vee, \sim, +, 0, 1)$ is a *double Stone algebra* if $(L, \wedge, \vee, \sim, 0, 1)$ is a Stone algebra and $(L, \wedge, \vee, +, 0, 1)$ is a dual Stone algebra.

Varlet [30] investigated the following important condition on double Stone algebras:

$$\text{if } x^\sim = y^\sim \text{ and } x^+ = y^+ \text{ then } x = y. \quad (\text{Regularity})$$

Following Varlet [30], we call a double Stone algebra *regular* if it satisfies (Regularity). In fact, “regularity” is an appropriate name. Actually, in Ref. [30], it is proven that (Regularity) is equivalent to congruence regularity: if two congruences coincide on a congruence class, then they are in fact the same congruence. In 1973, Katriňák [31] proved that (Regularity) is equivalent to the following condition:

$$x \wedge x^+ \leq y \vee y^\sim, \quad (\text{M1})$$

which is obviously equivalent to an identity in the presence of the lattice operations.

Thus, the class of regular double Stone algebras is a variety.

Given a regular double Stone algebra L , three distinguished subsets, namely $S_K(L)$, $D^\sim(L)$ and $D^+(L)$, play a central role. We define the set $S_K(L)$ of *sharp elements* of L by

$$S_K(L) = \{x \in L : x = x^{\sim\sim}\} = \{x^\sim : x \in L\} = \{x \in L : x^+ = x^\sim\}.$$

If L is a regular double Stone algebra, then $S_K(L)$ forms the largest Boolean subalgebra of L .²

Next, we define the set

$$D^\sim(L) = \{x \in L : x^\sim = 0\},$$

whose elements are called *dense elements of L*.

Finally, we define the set $D^+(L)$ of *dually dense elements of L* by:

$$D^+(L) = \{x \in L : x^+ = 1\}.$$

2.2 | A Representation Theorem For RDS

We now recall a representation theorem from [17] (see also Ref. [32]).

Given a Boolean algebra $B = (B, \wedge, \vee, ', 0, 1)$ and a filter F on B , let us consider the following set:

$$(B, F) = \{(x, y) \in B^2 : (x \leq y) \& (x \vee y' \in F)\}. \quad (A)$$

The set (B, F) can be turned into an algebra in the language $(\wedge, \vee, \sim, +, 0, 1)$.

Let us consider

$$(B, F) = ((B, F), \wedge, \vee, \sim, +, 0, 1),$$

where $\wedge, \vee$ are defined componentwise, and the unary operations $\sim$ and $+$ are set as follows:

$$(x, y)^\sim = (y', y'); \quad (x, y)^+ = (x', x'), \text{ where } (x, y) \in (B, F). \quad (1)$$

It turns out that the algebra (B, F) is a regular double Stone algebra. Moreover, Theorem 1 provides a rather useful representation [17, 33] (see also Ref. [15]).

Theorem 1. *Every regular double Stone algebra A is isomorphic to $(S_K(A), g(D^\sim(A)))$, where the isomorphism f is given, for each $a \in A$, by*

$$f(a) = (a^{++}, a^{\sim\sim}), \quad (2)$$

and $g : D^\sim(A) \rightarrow S_K(A)$ is such that for any $a \in D^\sim(A)$:

$$g(a) = a^{++}.$$

Additionally, in any regular double Stone algebra A , a unary operation $'$ can be defined as given in Equation (3):

$$x' = x^\sim \vee (x \wedge x^+). \quad (3)$$

The operation $'$ is in fact a Kleene negation (Definition 3).

Definition 3 [34]. A *Kleene algebra* $A = (A, \wedge, \vee, ', 0, 1)$ is a bounded distributive lattice and $'$ is an antitone and involutive operation that satisfies:

$$x \wedge x' \leq y \vee y'. \quad (\text{Kleene})$$

An intuitive reading of (Kleene) is that any “contradiction” is dominated by a “tautology”. On the algebra (B, F) , $'$ is defined by:

$$(x, y)' = (y', x').$$

Remark 1. In a regular double Stone algebra A there are three negation-like operations, namely $\sim$, $+$, and $'$. These operations are not independent from each other. In fact, if two operations $\dagger, \# \in \{', \sim, +\}$ are fixed, then the remaining one is term definable from them.

Consequently, it is possible to pass from the standard language of RDSAs, based on the operations $\sim$ and $+$, to an equivalent language involving $\sim$ and $'$. In this setting, the operation $\sim$ retains its usual role, while $'$ behaves as a Kleene negation, so that $(A, \wedge, \vee, ', 0, 1)$ forms a Kleene algebra.³

Moreover, (Regularity) can be equivalently reformulated as

$$x'^{\sim} = y'^{\sim} \text{ and } x^{\sim\sim} = y^{\sim\sim} \text{ imply } x = y.$$

In other words, by exploiting the term equivalence with 3-valued Łukasiewicz algebras (Definition 4 and Theorem 2), regular double Stone algebras can be viewed as Kleene algebras enriched with an additional unary operator

$$\Diamond x = x^{\sim\sim}.$$

Definition 4 [35, 36]. A 3-valued Łukasiewicz algebra is an algebra $A = (A, \wedge, \vee, ', \Diamond, 0, 1)$, where $(A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and $'$ is an antitone and involutive operation, and A satisfies:

- (1) $\Diamond x \vee x' = 1$;
- (2) $x' \wedge x = x' \wedge \Diamond x$;
- (3) $(x \wedge y)' = x' \vee y'$;
- (4) $\Diamond(x \wedge y) = \Diamond x \wedge \Diamond y$.

Let us observe that 3-valued Łukasiewicz algebras are also referred to as *Łukasiewicz–Moisil algebras*.

In Ref. [37], Varlet and Katriňák in Ref. [38] independently proved Theorem 2.

Theorem 2. *Regular double Stone algebras and 3-valued Łukasiewicz algebras are term equivalent.*

We close the present section discussing an important subset of regular double Stone algebras.

Definition 5 [39]. Let A be a regular double Stone algebra. The *core* of A is defined as the intersection of the set of dense elements and the set of dually dense elements, that is,

$$D^{\sim}(A) \cap D^{+}(A).$$

Let us notice that if a regular double Stone algebra A has a non-empty core, then it possesses rather remarkable properties. In fact, it can be seen that algebras of this sort are each and all of the form (B, B) , for a certain Boolean algebra B . Actually, the element $(0,1)$ will be the unique member of the core of (B, B) . Moreover, $(0, 1)' = (0, 1)$, see Equation (3). Lemma 1 summarizes these facts.

Lemma 1 [15]. *Let A be a regular double Stone algebra and $x \in A$. Then,*

- (1) *if $x \in D^{\sim}(A) \cap D^{+}(A)$, then x is of the form $(0, 1)$.*
- (2) *the cardinality of the core of A is at most 1 [39].*
- (3) *x belongs to the core of A if and only if A is of the form (B, B) , for a Boolean algebra B , and $x = x'$, i.e. x is a fixed point.*

Lemma 1 expresses the fact that a regular double Stone algebra A admits at most one fixed point $k = k'$ which would be both dense and dually dense, and moreover

$$D^{\sim}(A) \cap D^{+}(A) = \{k\}.$$

Actually, for a regular double Stone algebra, the conditions of having a non-empty core, satisfying Condition (K),

$$k^\sim = 0 \text{ and } k^+ = 1 \quad (\text{K})$$

and possessing a fixed point of $'$ are all equivalent. A prominent example of a regular double Stone algebra having a non-empty core is the three-element algebra whose Hasse diagram is in Display (3)

$$\begin{array}{c} 0^\sim = 1 = k^+ = 0^+ \\ | \\ k \\ | \\ 1^+ = 0 = k^\sim = 1^\sim \end{array} \quad (3)$$

and this algebra generates regular double Stone algebras as a variety.

2.3 | McCarthy Algebras and ADAs

For a wider account on McCarthy algebras and algebras of disjoint alternatives (ADAs), we refer the reader to Refs. [2, 3, 4, 25, 40]. It is worth recalling that what is called a *McCarthy algebra* in Ref. [2] and a *C-algebra* in Ref. [3] refers to the very same object. Both terminologies describe the algebraic counterpart of the three-valued logical calculus introduced by McCarthy in Ref. [1]. The change in terminology reflects differences in presentation rather than in the underlying mathematical object. We remark that in Ref. [3] Guzmán and Squier present for the first time a remarkably compact and complete equational basis for McCarthy algebras.

Definition 6 [3, Lemmas 1.1 and 1.4]. An algebra $L = (L, \otimes, \oplus, ', 0, 1)$ is called a *McCarthy algebra* if L satisfies:

- (1) $x'' = x$;
- (2) $(x \otimes y)' = x' \oplus y'$;
- (3) $(x \otimes y) \otimes z = x \otimes (y \otimes z)$;
- (4) $x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$;
- (5) $(x \oplus y) \otimes z = (x \otimes z) \oplus (x' \otimes y \otimes z)$;
- (6) $x \oplus (x \otimes y) = x$;
- (7) $(x \otimes y) \oplus (y \otimes x) = (y \otimes x) \oplus (x \otimes y)$;
- (8) $0' = 1$;
- (9) $1 \otimes x = x$.

It is well known that the variety of McCarthy algebras is generated by the three-element McCarthy algebra [3]. We denote it by

$$\mathbf{M}_3 = \langle \{0, k, 1\}; \otimes, \oplus, ', 0, 1 \rangle,$$

and its operations are given by the following tables:

$$\begin{array}{c|ccc} \otimes & 0 & k & 1 \\ \hline 0 & 0 & 0 & 0 \\ k & k & k & k \\ 1 & 0 & k & 1 \end{array} \quad \begin{array}{c|ccc} \oplus & 0 & k & 1 \\ \hline 0 & 0 & k & 1 \\ k & k & k & k \\ 1 & 1 & 1 & 1 \end{array} \quad \begin{array}{c|cc} x & 0 & k & 1 \\ \hline x' & 1 & k & 0 \end{array} \quad (4)$$

In particular, k is the unique fixed point of the negation.

Lemma 2 collects some identities that are satisfied in every McCarthy algebra. For their proofs, we refer the reader to Ref. [3].

Lemma 2. *Every McCarthy algebra satisfies the following laws:*

- $x \otimes x = x$;
- $x \otimes y = x \otimes (x' \oplus y)$;
- $x \oplus (x' \otimes x) = x$;
- $(x \oplus x') \otimes y = (x \otimes y) \oplus (x' \otimes y)$;
- $(x \oplus x') \otimes x = x$.

In Ref. [4], Manes introduces an extension of McCarthy algebras by means of an external unary operation, leading to the notion of algebras of disjoint alternatives (ADAs). These structures arise when McCarthy algebras are equipped with an oracle for the halting problem. In fact, programming languages admit the ternary operator

$$\text{if } p, \text{ then } f, \text{ else } g,$$

where p is a test which may not halt in general.

Let us stress that Definition 7 differs slightly from Manes' original one [4, Definition 17]. Manes works in a signature expanded by an additional constant k , which is a fixed point of negation. We work instead in the k -free signature. Accordingly, at the level of the additional ADA axioms, we omit the corresponding clause

$$k^\downarrow = 0.$$

The precise relationship between the two formulations is clarified in Remark 2.

Definition 7. An *algebra of disjoint alternatives (ADA)* is a McCarthy algebra equipped with a unary operation $^\downarrow$ satisfying:

- (1.) $1^\downarrow = 1$ and $0^\downarrow = 0$;
- (2.) $x \otimes y^\downarrow = x \otimes (x \otimes y)^\downarrow$;
- (3.) $x^\downarrow \oplus (x^\downarrow)' = 1$;
- (4.) $x^\downarrow \oplus x = x$.

Lemma 3 shows that the additional operation $^\downarrow$ is uniquely determined on the three-element McCarthy algebra. Consequently, throughout the paper we shall refer unambiguously to *the* three-element ADA.

Lemma 3. *The three-element McCarthy algebra admits a unique expansion into an ADA.*

Proof. Existence is provided by Manes' three-element ADA [4]. We prove uniqueness. By axiom (1) in Definition 7,

$$0^\downarrow = 0, \quad 1^\downarrow = 1.$$

Hence, only the value of $k^\downarrow$ remains to be determined.

Suppose first that $k^\downarrow = 1$. Axiom (4) would then give

$$1 \oplus k = k,$$

whereas Display (4) gives $1 \oplus k = 1$, a contradiction.

Suppose instead that $k^\downarrow = k$. By axiom (3),

$$k \oplus k' = 1.$$

Since $k' = k$, Display (4) yields

$$k \oplus k' = k \oplus k = k,$$

again a contradiction.

Therefore, the only possible value is

$$k^\downarrow = 0.$$

Hence the expansion is unique. □

From now on, the resulting algebra will simply be called the *three-element ADA*.

Lemma 4 is a version of [4, Theorem 2.(3)], and it will be of repeated usage throughout the paper.

Lemma 4. *Up to isomorphism, the subdirectly irreducible ADAs are precisely*

- (1) *the two-element Boolean algebra, equipped with the identity operation $^\downarrow$;*
- (2) *the three-element ADA.*

Consequently, the variety of ADAs is generated by the three-element ADA. Equivalently, an identity holds in every ADA if and only if it holds in the three-element ADA.

Proof. Manes' proof of [4, Theorem 2.(3)] carries over verbatim, since it never uses the distinguished constant. The only difference is that the two-element Boolean algebra, excluded in Ref. [4] because of the presence of the constant k , is here a subdirectly irreducible ADA. Since it is a subalgebra of the three-element ADA, the generation claim follows. □

Let us recall two results from Ref. [4] that remain valid in the reduced signature. The identities in Lemma 6 can be verified in the three-element ADA and therefore hold throughout the variety, since the latter is generated by the three-element ADA (Lemma 4). The same subdirect representation shows that the fixed-point set of $^\downarrow$ forms a Boolean algebra, as stated in Lemma 5.

Lemma 5. *Let A be an ADA. Then, the subset*

$$A^\downarrow = \{x \in A : x^\downarrow = x\}$$

forms a Boolean algebra.

We also recall some identities that will be repeatedly used throughout the paper.

Lemma 6 [4]. *Every ADA satisfies the following identities:*

- $x = (x \oplus x') \otimes x^\downarrow$;
- $x \oplus x' = x \oplus 1$;
- $(x \otimes y)^\downarrow = x^\downarrow \otimes y^\downarrow$;
- $x \oplus x = x$;
- $(x \oplus y) \otimes (y \oplus x \oplus 1) = (y \oplus x) \otimes (x \oplus y \oplus 1)$;
- $(x \oplus y)^\downarrow = x^\downarrow \oplus (x'^\downarrow \otimes y^\downarrow)$;
- $x^\downarrow \downarrow = x^\downarrow$.

Remark 2. In [4], the signature contains a constant, denoted 2 there and k here, satisfying $k' = k$. Axioms (17b)–(17d) of [4, Definition 17] coincide with axioms (2)–(4) of Definition 7, while axiom (17a) reads

$$0^\downarrow = 0, \quad k^\downarrow = 0, \quad 1^\downarrow = 1.$$

Thus, Definition 7 is obtained from Manes' definition by removing the constant k and the corresponding clause $k^\downarrow = 0$. This omission creates no ambiguity in the three-element case: by Lemma 3, the remaining axioms uniquely determine the operation $\downarrow$ on the three-element McCarthy algebra and, in particular, force $k^\downarrow = 0$. Hence we may refer unambiguously to the three-element ADA. Moreover, Lemma 4 shows that this algebra generates the whole variety. Nevertheless, removing the constant substantially enlarges the class of algebras under consideration.

We denote by ADA_k the variety in the expanded signature.

An ADA is the k -free reduct of an ADA_k algebra if and only if it possesses a fixed point c of negation. Such a fixed point is necessarily unique, as is already known for McCarthy algebras. It remains to observe that every such c satisfies $c^\downarrow = 0$. Although Lemma 3 establishes this fact only for the fixed point k of the three-element ADA, its conclusion extends to arbitrary ADAs by Lemma 4. Indeed, under every subdirect projection, the image of c is again a fixed point of negation. The two-element factor is therefore excluded, whereas in every three-element factor the image of c is k , whose $\downarrow$ -value is 0. Hence every subdirect coordinate of $c^\downarrow$ is 0, and consequently $c^\downarrow = 0$. Thus, interpreting k as c yields the unique expansion of the given ADA to an ADA_k algebra.

By Theorem 7, the representation (B, F) has a fixed point precisely when $(0, 1) \in (B, F)$, i.e. when $F = B$: the reducts of ADA_k algebras are thus exactly the ADAs of the form (B, B) .

The two varieties are genuinely different: there is no two-element ADA_k [4, p. 384], while every Boolean algebra, with $\downarrow$ the identity map, is an ADA, namely $(B, \{1\})$. In particular, the reduct class is not closed under subalgebras, so it is not a variety in the k -free signature; by Lemma 4, it does however, generate the variety of ADAs.

Summing up: the variety of ADAs is neither Manes' variety nor the class of its k -free reducts, but the strictly larger variety generated by the latter, with Manes' ADAs corresponding exactly to the special case $F = B$. Dropping the constant is thus essential: only in the k -free setting do pairs (B, F) with an arbitrary filter F arise.

It may be worth observing that in Ref. [3] the authors prove a representation result for McCarthy algebras in terms of pairs of sets, although it is not explicitly isolated as a representation theorem. For the reader's convenience, we restate it in Theorem 3.

Theorem 3. *For every McCarthy algebra A there exists a set S such that A is isomorphic to a subalgebra of the McCarthy algebra*

$$\{(P_1, P_2) : P_1, P_2 \subseteq S, P_1 \cap P_2 = \emptyset\}$$

with operations:

- $(P_1, P_2)' = (P_2, P_1)$;
- $(P_1, P_2) \otimes (Q_1, Q_2) = (P_1 \cap Q_1, P_2 \cup (P_1 \cap Q_2))$;
- $(P_1, P_2) \oplus (Q_1, Q_2) = (P_1 \cup (P_2 \cap Q_1), P_2 \cap Q_2)$;
- $1 = (S, \emptyset)$;
- $0 = (\emptyset, S)$.

In Ref. [4] the construction in Theorem 3 is extended to the case of ADAs by considering, in place of an arbitrary set S , a Boolean algebra $B = (B, \wedge, \vee, ', 0, 1)$, and by defining the additional operation $\downarrow$ as follows:

$$(x_1, x_2)^\downarrow = (x_1, x_1').$$

This extension is used in Ref. [4] to establish Theorem 4. Let us stress that the result concerns the variety ADA_k (see Remark 2): the presence of k forces the representing pairs to be of the form (B, B) , and $B \mapsto (B, B)$ is the object part of the equivalence. In the k -free setting, the equivalence involves, instead, Boolean algebras with a distinguished filter (Corollary 4).

Theorem 4. *The categories of ADA_k and Boolean algebras are equivalent.*

3 | On the Construction of Algebras of Disjoint Alternatives

In Refs. [3, 4, 25], the authors provide a representation of McCarthy algebras and ADAs in terms of algebras of pairs of sets. In particular, concerning ADAs, Manes in Ref. [4] starts from a Boolean algebra B and considers the set

$$\{(x_1, x_2) \in B^2 : x_1 \wedge x_2 = 0\}.$$

The present paper proposes an adaptation of a long-standing construction that will be a refinement of Manes', whose target will be an algebra in the signature $(\otimes, \oplus, ', 0, 1)$. In fact, the construction in Ref. [4] characterizes only the algebras with a fixed point for $'$, see Lemma 1.

Indeed, we introduce an additional condition that a pair (x_1, x_2) must satisfy in order to be included in the construction. A further, albeit minor, difference lies in the fact that we will consider pairs (x_1, x_2) such that $x_1 \leq x_2$, rather than pairs satisfying $x_1 \wedge x_2 = 0$. Over a Boolean algebra, the two presentations correspond under complementation of the second coordinate, since

$$x_1 \wedge x_2 = 0 \quad \text{if and only if} \quad x_1 \leq x_2'.$$

To approach the construction, let $B = (B, \wedge, \vee, ', 0, 1)$ be a Boolean algebra, and F a filter of B . Then, let us consider the following set:

$$(B, F) = \{(x, y) \in B^2 : (x \leq y) \& (x \vee y' \in F)\}. \quad (\text{A})$$

On (B, F) we define the binary operations as follows:

$$(x_1, x_2) \otimes (y_1, y_2) = (x_1 \wedge y_1, x_2 \wedge (x_1' \vee y_2)), \quad (5)$$

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 \vee (x_2' \wedge y_1), x_2 \vee y_2),$$

$$(x_1, x_2)' = (x_2', x_1'); \quad (x_1, x_2)^\downarrow = (x_1, x_1). \quad (6)$$

Lemma 7 is a rather straightforward technical result showing that when particular elements are involved, the operations simplify considerably.

Lemma 7. *Let B be a Boolean algebra, and let $F \subseteq B$ be a filter. Consider the construction given in Display (A) and the operations defined by equations in Display (5). Then, for all $(x_1, x_2), (y_1, y_2) \in (B, F)$ such that $x_1 = x_2$ and $y_1 = y_2$, the following equalities hold:*

$$(x_1, x_2) \otimes (y_1, y_2) = (x_1 \wedge y_1, x_2 \wedge y_2),$$

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 \vee y_1, x_2 \vee y_2).$$

Proof. Let $(x_1, x_2), (y_1, y_2) \in (B, F)$ be such that $x_1 = x_2$ and $y_1 = y_2$. We compute the operation $\otimes$ as defined in Equation (5):

$$\begin{aligned} (x_1, x_1) \otimes (y_1, y_1) &= (x_1 \wedge y_1, x_1 \wedge (x_1' \vee y_1)) \\ &= (x_1 \wedge y_1, (x_1 \wedge x_1') \vee (x_1 \wedge y_1)) \\ &= (x_1 \wedge y_1, x_1 \wedge y_1). \end{aligned}$$

The computation for the operation $\oplus$ is analogous. □

Theorem 5 and Theorem 6 provide a foundational step toward a representation theorem, establishing that, for every Boolean algebra B and every filter F of B , the structure (B, F) is an algebra of disjoint alternatives.

Theorem 5. Let B be a Boolean algebra, and let $F \subseteq B$ be a filter of B . Then the structure

$$((B, F), \otimes, \oplus, ', 0, 1),$$

is a McCarthy algebra.

Proof. By Theorem 3, every McCarthy algebra is isomorphic to a subalgebra of a McCarthy algebra of pairs constructed over a Boolean algebra. In particular, the operations $\otimes$, $\oplus$, and $'$ defined in Display (5) are known to satisfy McCarthy's axioms when defined over (B, B) .

Therefore, it suffices to show that the present construction is closed under the operations. Indeed, if F is a filter of B , one verifies that for all $(x_1, x_2), (y_1, y_2) \in (B, F)$, the elements obtained by applying $\otimes$, $\oplus$, and $'$ still belong to (B, F) .

Once closure is established, the validity of the axioms follows directly from the fact that these identities already hold in the full algebra of pairs over B .

Let $(x_1, x_2), (y_1, y_2) \in (B, F)$. Concerning $\oplus$, we need to show that:

$$x_1 \vee (x_2' \wedge y_1) \leq x_2 \vee y_2; \quad (7)$$

$$x_1 \vee (x_2' \wedge y_1) \vee (x_2 \vee y_2)' \in F. \quad (8)$$

The inequality in Equation (7) is straightforward. To prove the inclusion in Display (8), note that since $(x_1, x_2), (y_1, y_2) \in (B, F)$, we have:

$$x_1 \vee x_2' \in F, \quad y_1 \vee y_2' \in F.$$

Since F is closed under finite meets and upward closure, it follows that:

$$(x_1 \vee x_2') \wedge (y_1 \vee y_2') \in F,$$

and

$$\begin{aligned} ((x_1 \vee x_2') \wedge (y_1 \vee y_2')) &\leq (x_1 \vee x_2') \wedge (x_1 \vee y_1 \vee y_2') \\ &= x_1 \vee (x_2' \wedge (y_1 \vee y_2')) \\ &= x_1 \vee (x_2' \wedge y_1) \vee (x_2' \wedge y_2') \\ &= x_1 \vee (x_2' \wedge y_1) \vee (x_2 \vee y_2)' \end{aligned}$$

i.e. $x_1 \vee (x_2' \wedge y_1) \vee (x_2 \vee y_2)' \in F$. Analogously for $\otimes$. Also, the closure with respect to $'$ is immediate, and therefore our claim obtains. $\square$

Similarly, in Theorem 6 we will see how to turn the McCarthy algebra described in Theorem 5 into an ADA.

Theorem 6. Let B be a Boolean algebra, and let $F \subseteq B$ be a filter of B . Then the structure

$$((B, F), \otimes, \oplus, ', \downarrow, 0, 1),$$

where the unary operation $\downarrow$ is defined by:

$$(x_1, x_2)^\downarrow = (x_1, x_1)$$

forms an algebra of disjoint alternatives.

Proof. By virtue of Theorem 5, the structure

$$((B, F), \otimes, \oplus, ', 0, 1)$$

is a McCarthy algebra. Moreover, by the results in Ref. [4], the expansion of the full algebra of pairs over a Boolean algebra B with the operation

$$(x_1, x_2)^\downarrow = (x_1, x_1)$$

forms an algebra of disjoint alternatives.

Therefore, it suffices to verify that the present construction is closed under the operation $\downarrow$. By definition of the construction, for every $x \in B$ the pair (x, x) belongs to (B, F) , independently of the choice of the filter F . Hence $(x_1, x_1) \in (B, F)$. $\square$

4 | A Representation Theorem For Algebras of Disjoint Alternatives

In this section we establish the second part of the representation result, showing that every algebra of disjoint alternatives is isomorphic to a structure of the form (B, F) , where B is a Boolean algebra, and F is a Boolean filter.

To achieve this, we introduce certain distinguished subsets naturally associated with any ADA, namely the sets of sharp, dense, and dually dense elements. These notions will serve as the main tools in the representation argument. Moreover, they provide a bridge to other algebraic frameworks in which such concepts are well-known and assume a fundamental role.

Recall from Section 2 that the sets of sharp, dense, and dually dense elements play a crucial role in the structure theory of regular double Stone algebras. The aim of this section is to extend these notions to algebras of disjoint alternatives. This will lead to a full representation of the variety of ADAs through the construction described in Section 3.

Definition 8 [4]. Let A be an ADA. We call

$$A^\downarrow = \{x^\downarrow : x \in A\}$$

the set of *sharp* elements of A (in analogy with $S_K(A)$ for RDSAs).

It is known from [4, Lemma 4] that $A^\downarrow$ forms a Boolean algebra.

For notational convenience, we introduce a new operator dual to $\downarrow$. We define the unary operation $\uparrow$ in any ADA by

$$x^\uparrow = x'^{\downarrow'}. \quad (9)$$

Lemma 8 describes the relationships between each element of an ADA and its projections via the operators $\downarrow$ and $\uparrow$.

Lemma 8. *Let A be an algebra of disjoint alternatives. Then, for all $x, y \in A$, the following properties hold:*

- (1) $x^\downarrow \otimes x = x^\downarrow$ and $x \otimes x^\downarrow = x$;
- (2) $x \oplus x^\downarrow = x$;
- (3) $x^\uparrow \otimes x = x$ and $x \otimes x^\uparrow = x$;
- (4) $x \oplus x^\uparrow = x$ and $x^\uparrow \oplus x = x^\uparrow$.

Proof. (1) Concerning $x^\downarrow \otimes x = x^\downarrow$, we compute

$$x^\downarrow \otimes x = x^\downarrow \otimes (x^\downarrow \oplus x) = x^\downarrow.$$

In order to show that $x \otimes x^\downarrow = x$, we have

$$x = x \otimes 1 = x \otimes 1^\downarrow = x \otimes (x \otimes 1)^\downarrow = x \otimes x^\downarrow.$$

(2) As regards $x \oplus x^\downarrow = x$ it can be seen that

$$\begin{aligned}
 x \oplus x^\downarrow &= x \oplus (x^\downarrow \otimes x) \\
 &= (x \otimes x^\downarrow) \oplus (x^\downarrow \otimes x) \\
 &= (x^\downarrow \otimes x) \oplus (x \otimes x^\downarrow) \\
 &= x^\downarrow \oplus x \\
 &= x.
 \end{aligned}$$

Conditions (3) and (4) are dual, taking advantage of De Morgan property. □

Lemma 9 establishes a rather relevant property that holds in every ADA. This quasi-identity is analogous to well-known regularity conditions arising in other algebraic settings, such as (Regularity) for double Stone algebras. It highlights the central role played by the set of sharp elements in the structure of every ADA.

Lemma 9 (Regularity quasi-identity). *Let A be an algebra of disjoint alternatives. For all $x, y \in A$, if*

$$x^\downarrow = y^\downarrow \quad \text{and} \quad x^\uparrow = y^\uparrow,$$

then

$$x = y.$$

Proof. Assume that $x^\downarrow = y^\downarrow$ and $x^\uparrow = y^\uparrow$. We first observe that

$$x = x \otimes x^\downarrow = x \otimes (x^\downarrow \otimes y) = (x \otimes x^\downarrow) \otimes y = x \otimes y.$$

Similarly, we have $y = y \otimes x$.

Moreover,

$$x = x \oplus x^\uparrow = x \oplus (x^\uparrow \oplus y) = (x \oplus x^\uparrow) \oplus y = x \oplus y,$$

and analogously, $y = y \oplus x$.

Combining these identities together, we obtain

$$\begin{aligned}
 x &= x \oplus y \\
 &= (x \otimes y) \oplus (y \otimes x) \\
 &= (y \otimes x) \oplus (x \otimes y) \\
 &= y \oplus x \\
 &= y.
 \end{aligned}$$
□

Definition 9 introduces the set of dense elements. Together with the set of sharp elements, this notion will allow us to obtain a complete representation theorem for ADAs.

Definition 9. Let A be an ADA. We define the set of *dense* elements $D(A)$ as

$$D(A) = \{x \in A : x'^\downarrow = 0\}.$$

Dually, one may introduce the corresponding notion of dually dense elements.

Definition 10. Let A be an ADA. We define the set of *dually dense* elements of A as

$$D^*(A) = \{x \in A : x'^{\uparrow} = 1\}.$$

Lemma 10 will be useful to our arguments.

Lemma 10. *Let A be an ADA. Then, for every $x \in A$, the following hold*

$$x^{\downarrow} \otimes x^{\uparrow} = x^{\downarrow} \text{ and } x^{\downarrow'} \otimes x'^{\uparrow} = x'^{\uparrow}.$$

Proof. It can be seen that from the equation $x = x^{\downarrow} \oplus x$ it follows that

$$\begin{aligned} x^{\uparrow} &= (x^{\downarrow} \oplus x)^{\uparrow} \\ &= x^{\downarrow\uparrow} \oplus x^{\uparrow} \\ &= x^{\downarrow} \oplus x^{\uparrow} \end{aligned}$$

by a dual version of Lemma 6. The remaining equation follows analogously, see Display (9). □

Lemma 11 shows that, for every element $x \in A$, certain canonical combinations of x and x' belong to the sets of dense and dually dense elements, respectively.

Lemma 11. *Let A be an ADA. Then, for every $x \in A$, the following holds*

$$(x \oplus x')^{\downarrow} = 0.$$

That is, the element $x \oplus x'$ belongs to the set of dense elements of A . Moreover,

$$(x \otimes x')^{\uparrow} = 1.$$

That is, the element $x \otimes x'$ belongs to the set of dually dense elements of A .

Proof. Let $x \in A$. Then,

$$\begin{aligned} (x \oplus x')^{\downarrow} &= (x' \otimes x)^{\downarrow} \\ &= x'^{\downarrow} \otimes x^{\downarrow} \\ &\leq x^{\downarrow'} \otimes x^{\downarrow} \\ &= 0 \end{aligned}$$

where the inequality is due to Lemma 10. The second statement follows dually. □

Lemma 12 shows that every ADA A has all its structural information encoded in the Boolean algebra $A^{\downarrow}$ of sharp elements. Indeed, not only the sharp elements do form a Boolean algebra, but the set of dense elements can be projected injectively into $A^{\downarrow}$, and their image forms a filter of $A^{\downarrow}$.

Lemma 12. *Let A be an ADA. The set of dense elements $D(A)$ can be embedded injectively into the set of sharp elements $A^{\downarrow}$ via the function*

$$f : D(A) \rightarrow A^{\downarrow}, \quad f(x) = x^{\downarrow}.$$

Moreover, the image of $D(A)$ under f is a filter in $A^{\downarrow}$.

Proof. Let $x, y \in D(A)$ such that $f(x) = f(y)$. Then $x^\downarrow = y^\downarrow$. Since $x, y \in D(A)$, we have $x'^\downarrow = 0 = y'^\downarrow$, hence $x'^{\downarrow\downarrow} = y'^{\downarrow\downarrow}$, and by regularity (Lemma 9), it follows that $x = y$. Therefore, f is injective.

Now, let $x, y \in D(A)$. We show that $x \otimes y \in D(A)$:

$$(x \otimes y)'^\downarrow = (x' \oplus y')^\downarrow = x'^\downarrow \oplus (x''^\downarrow \otimes y'^\downarrow) = 0 \oplus (x^\downarrow \otimes 0) = 0,$$

so $x \otimes y \in D(A)$. Moreover,

$$f(x \otimes y) = (x \otimes y)^\downarrow = x^\downarrow \otimes y^\downarrow.$$

We now prove that $f(D(A))$ is a filter in $A^\downarrow$. We have already shown that $f(D(A))$ is closed under $\otimes$ and contains the top element $1 = f(1)$, since $1'^\downarrow = 0$.

Finally, let $f(x) \in f(D(A))$ and suppose $y \in A^\downarrow$ satisfies $f(x) \leq y$. Consider the element $y \oplus x$. We show that $y \oplus x \in D(A)$:

$$(y \oplus x)'^\downarrow = (y' \otimes x')^\downarrow = y'^\downarrow \otimes x'^\downarrow = y' \otimes 0 = 0,$$

so $y \oplus x \in D(A)$, and we compute:

$$\begin{aligned} f(y \oplus x) &= (y \oplus x)^\downarrow \\ &= y^\downarrow \oplus (y'^\downarrow \otimes x^\downarrow) \\ &= y \oplus (y' \otimes f(x)) \\ &= y \oplus 0 \\ &= y. \end{aligned}$$

Therefore, $y \in f(D(A))$, and we conclude that $f(D(A))$ is a filter in $A^\downarrow$. □

A dual version of Lemma 12 says that $D^*(A)$ can be projected into an ideal of the algebra $A^\downarrow$ via the mapping $g(x) = x^\uparrow$. In order to state the uniqueness part of Theorem 7 precisely, we first fix the appropriate notion of isomorphism for the pairs involved in the construction of Display (A).

Definition 11. Let B_1, B_2 be Boolean algebras, and let $F_1 \subseteq B_1, F_2 \subseteq B_2$ be filters. An *isomorphism of pairs*

$$h : (B_1, F_1) \rightarrow (B_2, F_2)$$

is an isomorphism of Boolean algebras $h : B_1 \rightarrow B_2$ such that $h(F_1) = F_2$.

Theorem 7 provides a full-fledged representation of every ADA in terms of a Boolean algebra equipped with a distinguished filter. In this sense, the entire structure of an ADA is determined by the Boolean algebra $A^\downarrow$ together with the filter arising from its dense elements.

Theorem 7. Every ADA is isomorphic to a structure of the form (B, F) , where B is a Boolean algebra and $F \subseteq B$ is a filter of B . Moreover, the pair (B, F) is uniquely determined up to isomorphism of pairs.

Proof. Let A be an ADA, and consider $A^\downarrow$ and $D(A)$ as defined in Lemma 12. It is known that $A^\downarrow$ is a Boolean algebra. Let us fix:

$$B = A^\downarrow.$$

Due to Lemma 12, the map f embeds injectively $D(A)$ into $A^\downarrow$, and $f(D(A))$ is a filter of $A^\downarrow = B$. Let us set:

$$F = f(D(A)).$$

We can now form the algebra (B, F) and define a mapping

$$\phi : A \rightarrow (B, F) \text{ by } \phi(x) = (x^\downarrow, x^\uparrow).$$

We first verify that ϕ is well defined. Let $x \in A$. By Lemma 10,

$$x^\downarrow \leq x^\uparrow.$$

Moreover,

$$\begin{aligned} x^\downarrow \oplus (x^\uparrow)' &= x^\downarrow \oplus (x')^\downarrow \\ &= (x \oplus x')^\downarrow, \end{aligned}$$

where the first equality follows from the definition of $^\uparrow$, and the second from Lemma 6. By Lemma 11,

$$x \oplus x' \in D(A).$$

Since $f(d) = d^\downarrow$ for every $d \in D(A)$, it follows that

$$(x \oplus x')^\downarrow \in f(D(A)) = F.$$

Therefore,

$$(x^\downarrow, x^\uparrow) \in (B, F),$$

and hence ϕ is well defined.

By regularity (Lemma 9), if

$$\phi(x) = (x^\downarrow, x^\uparrow) = (y^\downarrow, y^\uparrow) = \phi(y),$$

then $x = y$. Therefore, ϕ is injective. Regarding the operation $\otimes$,

$$\begin{aligned} \phi(x \otimes y) &= ((x \otimes y)^\downarrow, (x \otimes y)^\uparrow) \\ &= ((x \otimes y)^\downarrow, (x \otimes y)'^{\downarrow'}) \\ &= (x^\downarrow \otimes y^\downarrow, (x' \oplus y')^{\downarrow'}) \\ &= (x^\downarrow \otimes y^\downarrow, (x'^{\downarrow} \oplus (x''^{\downarrow} \otimes y'^{\downarrow}))') \\ &= (x^\downarrow \otimes y^\downarrow, x'^{\downarrow'} \otimes (x^{\downarrow'} \oplus y'^{\downarrow'})) \\ &= (x^\downarrow, x'^{\downarrow'}) \otimes (y^\downarrow, y'^{\downarrow'}) \\ &= (x^\downarrow, x^\uparrow) \otimes (y^\downarrow, y^\uparrow). \end{aligned}$$

For $^\downarrow$:

$$\begin{aligned} \phi(x^\downarrow) &= (x^{\downarrow\downarrow}, x^{\downarrow\uparrow}) \\ &= (x^{\downarrow\downarrow}, x^{\downarrow'\downarrow'}) \\ &= (x^\downarrow, x^{\downarrow''}) \\ &= (x^\downarrow, x^\downarrow) \\ &= \phi(x)^\downarrow. \end{aligned}$$

For the negation,

$$\begin{aligned}
\phi(x') &= (x'^{\downarrow}, x''^{\downarrow'}) \\
&= (x'^{\downarrow}, x^{\downarrow'}) \\
&= (x'^{\downarrow''}, x^{\downarrow'}) \\
&= (x^{\downarrow}, x'^{\downarrow'})' \\
&= (x^{\downarrow}, x^{\uparrow})' \\
&= \phi(x)'.
\end{aligned}$$

Preservation of $\oplus$ follows from preservation of $\otimes$ and $'$, since $x \oplus y = (x' \otimes y')'$. Preservation of 0 and 1 is immediate.

We now show that ϕ is surjective. Let $(x, y) \in (B, F)$. By definition, $x, y \in B$, with $x \leq y$, and $x \vee y' \in F$.

Since x and y are Boolean elements of $A^{\downarrow} = B$, the McCarthy operation $\oplus$ coincides with the Boolean join on these elements. Hence,

$$x \vee y' = x \oplus y' \in F = f(D(A)) \subseteq A^{\downarrow}.$$

By Lemma 12, f is injective, so $x \oplus y'$ has a unique preimage in $D(A)$, which we denote by $f^{-1}(x \oplus y')$.

Now consider

$$z = y \otimes f^{-1}(x \oplus y').$$

We claim that z is the required element, that is,

$$\phi(z) = (z^{\downarrow}, z^{\uparrow}) = (x, y).$$

This would show that ϕ is surjective.

On the one hand,

$$\begin{aligned}
z^{\downarrow} &= (y \otimes f^{-1}(x \oplus y'))^{\downarrow} \\
&= y^{\downarrow} \otimes (f^{-1}(x \oplus y'))^{\downarrow} \\
&= y \otimes f \circ f^{-1}(x \oplus y') \\
&= y \otimes (x \oplus y') \\
&= (y \otimes x) \oplus (y \otimes y') \\
&= (y \otimes x) \oplus 0 \\
&= y \otimes x \\
&= x.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
z^{\uparrow} &= z'^{\downarrow'} = (y \otimes f^{-1}(x \oplus y'))'^{\downarrow'} \\
&= (y' \oplus (f^{-1}(x \oplus y'))')^{\downarrow'} \\
&= (y'^{\downarrow} \oplus (y''^{\downarrow} \otimes (f^{-1}(x \oplus y'))^{\downarrow}))' \\
&= (y' \oplus (y \otimes (f^{-1}(x \oplus y'))^{\downarrow}))' \\
&= y'' \otimes (y' \oplus (f^{-1}(x \oplus y'))^{\downarrow'}) \\
&= y \otimes (y' \oplus 1)
\end{aligned}$$

$$= y \otimes 1$$

$$= y.$$

Perhaps a few words will clarify the argument. We have

$$(f^{-1}(x \oplus y'))'^{\downarrow'} = 1$$

since the element $f^{-1}(x \oplus y')$ is dense. Therefore,

$$(f^{-1}(x \oplus y'))'^{\downarrow'} = 0' = 1.$$

Moreover, $y'^{\downarrow} = y'$, since $y \in A^{\downarrow} = B$.

Thus, $\phi(z) = (z^{\downarrow}, z^{\uparrow}) = (x, y)$, which establishes that ϕ is surjective.

Finally, uniqueness. Let

$$\psi : (B_1, F_1) \longrightarrow (B_2, F_2)$$

be an isomorphism arising from two representations of the same ADA. For $i = 1, 2$, let S_i and D_i denote, respectively, the sets of sharp and dense elements of (B_i, F_i) . Since these sets are defined by the equations

$$a^{\downarrow} = a \quad \text{and} \quad (a')^{\downarrow} = 0,$$

we have $\psi(S_1) = S_2$ and $\psi(D_1) = D_2$.

By Lemma 7,

$$S_i = \{(x, x) : x \in B_i\},$$

and $\varepsilon_i(x) = (x, x)$ is a Boolean isomorphism from B_i onto S_i . Hence

$$h = \varepsilon_2^{-1} \circ (\psi \upharpoonright_{S_1}) \circ \varepsilon_1 : B_1 \longrightarrow B_2$$

is a Boolean isomorphism.

Moreover,

$$D_i = \{(x, 1) : x \in F_i\}, \quad \varepsilon_i(F_i) = \{d^{\downarrow} : d \in D_i\}.$$

Since ψ preserves $^{\downarrow}$ and maps D_1 onto D_2 , it follows that

$$\psi(\varepsilon_1(F_1)) = \varepsilon_2(F_2),$$

and therefore $h(F_1) = F_2$. Thus (B_1, F_1) and (B_2, F_2) are isomorphic as pairs. □

Therefore, by virtue of Theorem 7, any ADA A can be uniquely described in terms of the Boolean algebra $A^{\downarrow}$ together with the filter

$$f(D(A)) = \{x^{\downarrow} : x \in D(A)\}.$$

We conclude the section with a series of technical results that further elucidate the internal structure of algebras of disjoint alternatives by exhibiting identities valid in every such algebra.

Lemma 13. *Let A be an ADA, and let $x, y \in A$. Then, the following identities hold:*

$$x \oplus x' = x \oplus x'^{\downarrow} = x \oplus 1,$$

$$x \otimes x' = x \otimes x'^{\downarrow} = x \otimes 0.$$

Proof. By Theorem 7,

$$\begin{aligned}
 x \oplus x' &= (x_1, x_2) \oplus (x_1, x_2)' \\
 &= (x_1, x_2) \oplus (x_2', x_1') \\
 &= (x_1 \vee (x_2' \wedge x_2'), x_2 \vee x_1') \\
 &= (x_1 \vee x_2', 1).
 \end{aligned}$$

Also,

$$\begin{aligned}
 x \oplus x'^{\downarrow} &= (x_1, x_2) \oplus (x_2', x_2') \\
 &= (x_1 \vee (x_2' \wedge x_2'), x_2 \vee x_2') \\
 &= (x_1 \vee x_2', 1).
 \end{aligned}$$

Finally,

$$\begin{aligned}
 x \oplus 1 &= (x_1, x_2) \oplus (1, 1) \\
 &= (x_1 \vee (x_2' \wedge 1), x_2 \vee 1) \\
 &= (x_1 \vee x_2', 1).
 \end{aligned}$$

The other equations are dual. □

Let us observe that Lemma 13 and Lemma 14 describe particular cases in which McCarthy operations assume a particularly smooth behavior. In particular, Lemma 14 shows that in specific cases the operations are in fact commutative, and therefore in these cases they behave as lattice operation in the proper sense.

Lemma 14. *Let A be an ADA, and let $x \in A$. Then the following identities hold:*

$$\begin{aligned}
 x \oplus x' &= x' \oplus x, \\
 x \otimes x' &= x' \otimes x, \\
 x \oplus x^{\downarrow} &= x^{\downarrow} \oplus x, \\
 x \otimes x^{\uparrow} &= x^{\uparrow} \otimes x.
 \end{aligned}$$

Proof. We prove the first equation.

$$\begin{aligned}
 x \oplus x' &= (x_1 \vee x_2', 1) \\
 &= (x_2' \vee (x_1 \wedge x_1), x_1' \vee x_2) \\
 &= (x_2', x_1') \oplus (x_1, x_2) \\
 &= x' \oplus x.
 \end{aligned}$$
□

Lemma 15, whose proof is left to the interested reader, is a reformulation of (Kleene) and (M1).

Lemma 15. *Let A be an ADA, and let $x, y \in A$. Then the following identity holds:*

$$y \oplus y'^{\downarrow} \oplus (x \otimes x'^{\downarrow}) = y \oplus y'^{\downarrow}.$$

5 | Term-Equivalence of ADAs and RDSAs

As we have seen, the representation theorem (Theorem 7) provides a transparent description of the structure of algebras of disjoint alternatives. Furthermore, the same theorem also suggests a deeper connection with other well-known algebraic frameworks. Indeed, both ADAs and regular double Stone algebras admit representations in terms of structures of the form (B, F) , where B is a Boolean algebra, and F is a distinguished filter, see Theorem 1.

Actually, it is known from Ref. [3] that every Kleene algebra can be transformed into a McCarthy algebra, a correspondence that, in particular, allows one to move from regular double Stone algebras to ADAs. This naturally raises the question whether an inverse translation exists, and whether these translations establish a term-equivalence between the two classes.

Theorem 11 and Corollary 1 provide a positive answer to these questions.

Before presenting the main translation results, we briefly recall some known facts about regular double Stone algebras that will be relevant to our discourse. In view of the term-equivalence established later, Theorem 2 and Theorem 8 will automatically transfer to algebras of disjoint alternatives.⁴

Theorem 8. *3-valued Łukasiewicz algebras and 3-valued MV-algebras are term equivalent.*

In fact, given a regular double Stone algebra A , one obtains a term equivalence with 3-valued MV-algebras by defining, for all $x, y \in A$,

$$x \oplus y = (x^{\sim\sim} \vee y) \wedge (x \vee y^{\sim\sim}),$$

and taking the negation $\neg$ to be the operation $'$ defined in Equation (3).

Theorem 9 follows by adapting the result of Guzmán and Squier [3] to the setting of regular double Stone algebras.

Theorem 9. *Let A be a regular double Stone algebra, then it can be turned into a McCarthy algebra if we define:*

$$\begin{aligned} x \oplus y &= x \vee (x^{\sim} \wedge y), \\ x \otimes y &= x \wedge (x^+ \vee y), \\ x' &= x^{\sim} \vee (x \wedge x^+). \end{aligned}$$

Proof. Straightforward from Ref. [3] and Equation (3). □

We can now easily prove Theorem 10, which states the final missing piece: every regular double Stone algebra can be immediately turned into an algebra of disjoint alternatives.

Theorem 10. *Let A be a regular double Stone algebra, and consider the associated McCarthy algebra defined in Theorem 9. Let us consider, for every $x \in A$,*

$$x^\downarrow = x^{++}.$$

Then, the resulting algebra is an algebra of disjoint alternatives.

Proof. By Theorem 9, A can be endowed with the structure of a McCarthy algebra.

By the representation results in Theorem 1 and Theorem 7, every element x can be identified with a pair (x_1, x_2) . A direct computation shows that

$$x^{++} = (x_1, x_1) = x^\downarrow. \quad \square$$

Let us now present Theorem 11, which will be the main result of the current section.

Theorem 11. *Let A be an algebra of disjoint alternatives. Then A can be turned into a regular double Stone algebra by defining:*

$$\begin{aligned}x \wedge y &= (x^{\uparrow'} \oplus y) \otimes x, \\x \vee y &= (x^{\downarrow'} \otimes y) \oplus x, \\x^{\sim} &= x'^{\downarrow}, \\x^+ &= x^{\downarrow'}.\end{aligned}$$

Proof. Instead of verifying directly that these operations satisfy the axioms of regular double Stone algebras, we argue via the representation in Theorem 7.

By Theorem 7, every ADA A is isomorphic to a structure of the form (B, F) . Moreover, regular double Stone algebras admit a representation in terms of pairs (B, F) as well (see Theorem 1). Therefore, it suffices to show that the above operations coincide with the standard operations of regular double Stone algebras on elements of (B, F) .

Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be elements of (B, F) . We compute:

$$\begin{aligned}x \wedge y &= ((x^{\uparrow'} \oplus y) \otimes x) \\&= (((x'_2, x'_2) \oplus (y_1, y_2)) \otimes (x_1, x_2)) \\&= (x'_2 \vee (x_2 \wedge y_1), x'_2 \vee y_2) \otimes (x_1, x_2) \\&= ((x'_2 \vee y_1) \wedge x_1, (x'_2 \vee y_2) \wedge ((x'_2 \vee y_1)' \vee x_2)) \\&= ((x'_2 \vee y_1) \wedge x_1, (x'_2 \vee y_2) \wedge ((x_2 \wedge y_1') \vee x_2)) \\&= (x_1 \wedge y_1, x_2 \wedge y_2).\end{aligned}$$

Hence, $x \wedge y$ coincides with the meet operation of regular double Stone algebras.

The proof for $\vee$ is dual.

For the unary operations, we have:

$$x^{\sim} = x'^{\downarrow} = (x'_2, x'_1)^{\downarrow} = (x'_2, x'_2),$$

which coincides with the pseudocomplement in the corresponding representation. Moreover,

$$x^+ = (x^{\downarrow})' = (x_1, x_1)' = (x'_1, x'_1),$$

which is in fact the dual pseudocomplement.

Thus, all operations agree with those of regular double Stone algebras under the representation, and the claim follows. $\square$

Theorem 10 and Theorem 11 provide term translations in the two opposite directions. Moreover, by Theorem 1 and Theorem 7, both classes are represented by algebras based on the same pairs (B, F) . On these representations, the term operations introduced in Theorem 10 and Theorem 11 coincide with the corresponding operations defined above. Consequently, applying the two translations successively recovers the original basic operations in either signature. We therefore obtain the following corollary.

Corollary 1. *The varieties of ADAs and regular double Stone algebras are term-equivalent.*

Finally, from Theorem 2, and Theorem 8 we straightforwardly claim Corollary 2, and Corollary 3.

Corollary 2. *The varieties of ADAs and 3-valued Łukasiewicz algebras are term-equivalent.*

Corollary 3. *The varieties of ADAs and 3-valued MV algebras are term-equivalent.*

6 | Consequences of the Term-Equivalence

In this section we collect some direct consequences of Corollary 1, namely results that transfer automatically from the variety of regular double Stone algebras to ADAs.

It is well-known that the algebraic category of regular double Stone algebras **RDS** is equivalent to the category $\mathbf{B}_F$ whose objects are pairs (B, F) , where B is a Boolean algebra and F is a fixed filter of B , and whose morphisms $f : (B_1, F_1) \rightarrow (B_2, F_2)$ are Boolean homomorphisms $f : B_1 \rightarrow B_2$ such that $f(F_1) \subseteq F_2$ (see Refs. [9, 15]). As a direct consequence of the term-equivalence in Corollary 1, we obtain the following result, which extends Theorem 4 of Manes [4] (cf. Remark 2).

Corollary 4. *The category of ADAs is categorically equivalent to the category of Boolean algebras with a distinguished filter.*

This result can be useful for comparative studies with other algebraic classes, especially in contexts where purely algebraic tools are not sufficient. Among three-valued algebraic structures, a particularly relevant example is the quasivariety of Bochvar algebras [41]. These algebras provide an algebraic semantics for Bochvar's external logic [42] and are obtained by expanding involutive bisemilattices, the algebraic counterparts of weak Kleene logic, with the external unary operator J_2 .

The relationship between Bochvar algebras and regular double Stone algebras has been recently investigated in Ref. [18], and the term-equivalence established in the present paper allows those results to be read, equivalently, at the level of ADAs.

A further group of consequences concerns the amalgamation property. In abstract algebraic logic, amalgamation is closely related to Robinson-style properties and, under suitable additional hypotheses, to Craig-type interpolation [43]. From this point of view, the following results, again direct consequences of Corollary 1, are of some interest [44].⁵

Theorem 12. *The variety of ADAs has the amalgamation property.*

Moreover, let us consider the expansion ADA_k introduced in Remark 2. Then, we obtain:

Theorem 13. *The variety of ADA_k has the superamalgamation property.*

In suitable algebraizable settings, the superamalgamation property provides the algebraic counterpart of strengthened forms of Craig interpolation. Determining the precise logical counterpart of Theorem 13, however, requires the specification of a deductive system having ADA_k as its equivalent algebraic semantics and the verification of the hypotheses of the relevant transfer theorem.

7 | Conclusions and Future Work

In this paper we have provided a complete representation theorem for algebras of disjoint alternatives: every ADA is isomorphic, in an essentially unique way, to an algebra of pairs (B, F) , where B is a Boolean algebra and F is a distinguished filter of B (Theorem 7). Manes' original construction over pairs of disjoint elements is recovered as the special case $F = B$, which characterizes, by Lemma 1, precisely the ADAs possessing a fixed point of the negation. Since regular double Stone algebras admit a representation of the very same form (Theorem 1), we have derived that the varieties of ADAs and RDSAs are term-equivalent (Corollary 1), and hence that ADAs, 3-valued Łukasiewicz algebras, and 3-valued MV-algebras are, up to term-equivalence, one and the same mathematical object (Corollaries 2 and 3). As applications, we have obtained a categorical equivalence between ADAs and Boolean algebras with a distinguished filter (Corollary 4), extending Manes' equivalence with Boolean algebras (Theorem 4), together with the amalgamation property for ADAs and the superamalgamation property for ADA_k (Theorems 12 and 13).

On the methodological side, our approach is based on the sets of sharp, dense, and dually dense elements, notions borrowed from the structure theory of regular double Stone algebras. The representation theorem shows that these subsets determine the whole ADA: the sharp elements form a Boolean algebra, while the projections of the dense elements determine a distinguished filter on it. This shared structural description explains the term-equivalence between ADAs and regular double Stone algebras, despite the marked difference between their primitive operations, which are lattice-theoretic and commutative in the latter case, and sequential and non-commutative in the former.

The significance of the equivalence, from a computational standpoint, is that a single algebraic environment simultaneously supports operations capturing the behavior of the connectives of three well-known evaluation paradigms: Kleene's strong logic, modeling indeterminacy in partial or parallel computation [19]; McCarthy's logic, modeling sequential evaluation with short-circuit behavior [1]; and Bochvar's/weak Kleene logic, modeling an infectious failure propagating through a computation [20, 41]. These three systems correspond to different readings of the third truth value, uncertainty, critical runtime error, and meaninglessness, respectively, and, by Corollary 1, they become interdefinable perspectives on one and the same structure.

The representation by pairs (B, F) also suggests that the third value need not be assigned a unique computational interpretation. The Boolean algebra B describes the underlying classical structure, while the filter F identifies a distinguished collection of elements. Different choices of F may therefore provide different ways of organizing partial or admissible information within the same general algebraic framework.

We conclude by indicating two directions for future work in which the constructions developed here may prove useful.

The first direction concerns the theory of assume–guarantee contracts, a formal framework supporting the compositional design of complex systems [45–48]. A contract is a pair $C = (A, G)$, where A represents the assumptions on the environment and G the guarantees provided by the component whenever the assumptions are satisfied. A standard construction associated with a Boolean algebra B is given by

$$C(B) = \{(x, y) \in B^2 : x \vee y = 1\},$$

which corresponds, up to the order-theoretic reformulation $x' \leq y$, to the special case (B, B) considered in this work, and hence always yields an algebra with a fixed point. In the light of Theorem 7, it may be of interest to consider the more general construction

$$\{(x, y) \in B^2 : (x \vee y = 1) \& (x' \vee y' \in F)\},$$

where F is a filter of B : this refinement could allow for a more accurate modeling of situations arising in applications. Moreover, alongside the operations typically considered on algebras of contracts, conjunction and disjunction, composition and merging (which correspond to operations of 3-valued MV-algebras), quotient and separation, implication and coimplication [47], it could be fruitful to import McCarthy-style operations, arising from the present work and from Ref. [3], in order to model situations in which the *order of evaluation* plays a crucial role, capturing computational features analogous to short-circuit evaluation in programming languages. For instance, one may consider operations of the form:⁶

$$\begin{aligned} (x_1, x_2) \otimes (y_1, y_2) &= (x_1 \wedge y_1, x_2 \wedge (x'_1 \vee y_2)), \\ (x_1, x_2) \oplus (y_1, y_2) &= (x_1 \vee (x'_2 \wedge y_1), x_2 \vee y_2), \end{aligned} \tag{10}$$

as well as operations inspired by weak Kleene logics, whose distinctive feature is that the third truth value is *infectious*:

$$\begin{aligned} (x_1, x_2) \sqcap (y_1, y_2) &= (x_1 \wedge y_1, (x_2 \vee y_2) \wedge (x_2 \vee y'_1) \wedge (x'_1 \vee y_2)), \\ (x_1, x_2) \sqcup (y_1, y_2) &= ((x_1 \wedge y_1) \vee (x_1 \wedge y'_2) \vee (x'_2 \wedge y_1), x_2 \vee y_2). \end{aligned} \tag{11}$$

A systematic study of these augmented algebras of contracts, specifically regarding how the new operations interact with standard contract refinement and composition, is left as a subject for future work.

The second direction concerns rough set theory [10, 11], whose well-known connections with regular double Stone algebras [8, 9, 49] transfer, by Corollary 1, to ADAs. Here, each set is approximated by a lower and an upper approximation, and the pairs (B, F) of Theorem 7 provide a natural environment for such approximation pairs. Just as in the case of contracts, McCarthy-style operations could be introduced in this setting to model forms of approximate reasoning in which the order of evaluation of the available information is relevant. We regard the development of this perspective, too, as a possible subject for future investigations.

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Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Endnotes

¹Throughout this paper, we will adopt the name *McCarthy algebras*.

²Sometimes in the literature, $S_K(L)$ goes under the name of *skeleton of L* .

³For a further reading on varieties and (pseudocomplemented) expansions of Kleene algebras, we may refer the reader to Ref. [50].

⁴Actually, Theorem 8 is, independently, due to Iorgulescu [51] and Mundici [52].

⁵For brevity's sake, we refer the reader to Refs. [53] and [54] for an account of amalgamation and superamalgamation properties.

⁶We remark that the operations in Display (10) and Display (11) are defined with respect to the construction described in Display (A).

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