Asset price-GDP cross feedback. The role of dividend policies in a dynamic setting

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Abstract

This paper investigates the interconnections between real economy and the financial market in a discrete dynamical framework. We focus on the effect of the dividend policy decided by managers and the expectations of investors about future payments. We found that an excessively high dividend payout ratio erodes the economy, while in the opposite case instability and fluctuations arise. By means of analytical results and numerical simulations we show that the positive return of government bonds leads to instability, while investors' expectations modify the path of the asset price.

Keywords: Economic growth, Dividend payout ratio, Behavioural finance, Economic development, Nonlinear dynamics, OLG model

1. Introduction

The irrelevance of dividends has been affirmed by [1] based on the hypothesis of a frictionless capital market. Recent literature has demonstrated that the market is imperfect and that dividend policies affect the value of the firms and their asset prices. Decisions and announcements over dividend payout ratio modify information asymmetry (see, for example, [2] and [3]) and may affect investor preferences ([4]). Dividend policy and dividend volatility are ambiguously connected to stock price movements: empirical evidence negatively correlates

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dividend payout ratio and stock price volatility (see, among all, [5], [6], [7], [8] and [9]). Conversely, a positive correlation between dividends and price changes is suggested in [10] and [11]. The unclear relationship between dividends and stock returns is a central issue in finance since it could support the decision making of managers and traders. Dividends move capital generated in real economy to financial markets, affecting the resources (retained earnings) allocated for investments in fixed capital that, in its turn, influences economic growth (see, among all, [12], [13] and [14]). In this view, the decision over dividend payments is crucial and the twofold effect of dividend payout ratio on real economy and financial markets needs to be investigated (on the lack of knowledge about the transmission of shocks between the two sectors see [15]).

This paper contributes to literature by investigating how the Constant Dividend Policy might influence the long run evolution of real economy and financial markets, in a discrete dynamical framework. We assume that physical capital increases, depending on (1) the retained earnings allocated to physical investments, (2) the expectations of individuals about future asset price. Note that (1) depends on the dividend policy since resources allocated in physical capital decreases as the dividend payment increases, while (2) depends on dividends since the asset price is related to dividend payout ratio and the expectations regarding future payments.

Our simplified model assumes that physical capital of a representative firm depends - in each period - on the choice of managers regarding the portion of earnings paid as dividends and the amount allocated to new investments. Dividends contribute to price formation in the stock market, in which individuals choose between the firm asset and a risk free bond issued by a fiscal authority. The allocation depends on individual risk preferences and beliefs over future dividend payments.

We found that an excessively high dividend payout ratio erodes the economy, while in the opposite case instability and fluctuations arise. Additionally, we show that positive returns for government bonds lead to instability. This counterintuitive idea reflects empirical evidence: real economy and asset prices evolve over time and rarely remain stable, while bond returns are usually positive. Our findings relate the instability to the sign of the bond yield, suggesting that, if all the other parameters would allow stability, a negative return would be needed in order to reach the equilibrium. Moreover, we showed that unfulfilled expectations affect the path of the economy: an expectation of lower dividends tends to rise the asset price after dividends are realised, conversely an expectation of higher dividends lowers the asset price in the long run.

The rest of the paper is organized as follows: Section 2 introduces the model; Section 3 presents analytical results on existence and stability of the equilibria, while Section 4 provides empirical analysis and numerical simulations. Section 5 concludes the paper.

2. The model

We start our analysis by building the model. We define a stylized framework that describes the interaction between real economy and financial markets and, subsequently, we introduce different dividend policies. Although a simplified system does not capture the complex relation between financial and real sectors, this option allows us to analytically understand how managers' decisions affect the outcome in terms of asset price volatility and capital evolution.

2.1. Evolution of capital

As far as the capital accumulation is concerned, we assume that the economy evolves following a one-sector growth model with overlapping generations of individuals. We base our framework on the model discussed in [16] that is presented in Section 2.1.1, while in Section 2.1.2 we extend the model in order to take into consideration the dividend payout ratio and its interconnections with the financial market.

2.1.1. The baseline model

In this section, we introduce the overlapping generations model presented in [16] that describes the capital accumulation over time. Its structure will be extended in Section 2.1.2.

At each period $t \in \mathbb{N}$, three goods exist: capital $K_t \geq 0$, labour $L_t > 0$, and a physical good produced from capital and labour, that is either consumed or invested to build future capital. The physical good is the numeraire.
At time t, $L_t > 0$ individuals are born and they live for two periods, therefore in each period two generations (young and old) coexist.
Young individuals supply inelastically one unit of labour to firms and receive a real wage $w_t > 0$ that is allocated between consumption $c_t \geq 0$ and investment $s_t \geq 0$ in the firm:

$$w_t = c_t + s_t \,.$$

When old, i.e. $t+1$, individuals retire and their income is the return from the investment made at time t and it is assumed that they consume their income entirely. The income of the old individual is

$$D_{t+1} := R_{t+1} s_t \tag{1}$$

where $R_{t+1} := 1 + r_{t+1} \geq 0$ is the return factor on savings from t to $t+1$. Individual preferences are represented by an additively separable utility function that depends on expected returns R^e_{t+1} (and, hence, expected future consumptions $D^e_{t+1} := R^e_{t+1} s_t$) and present consumption c_t

$$U(c_t, D_{t+1}) := u(c_t) + \eta u(D^e_{t+1}) \quad \eta > 0\,. \tag{2}$$

Considering that $c_t = w_t - s_t$ and $D^e_{t+1} = R^e_{t+1} s_t$, each young individual selects

$$s_t = s(w_t, R^e_{t+1}) \tag{3}$$

that maximises his\her utility (2).

Generations grow at rate $n > -1$, therefore

$$L_t = (1+n) L_{t-1}.$$

A representative firm produces at every period t. In $t = 0$ the capital K_0 is already installed while for $t \geq 1$ the capital K_t is built from previous investment in the firms $I_{t-1} := L_{t-1}s_{t-1}$ so that the productive capital in t is

$$K_t = I_{t-1} = L_{t-1}s_{t-1} \tag{4}$$

while the old generation receives profit.
The wage equals the marginal product of labour: given the production function $F(K, L)$, the wage is equal to the change in output per unit change in labour measured by the partial derivative of F with respect to L. Therefore

$$w_t := F_L(K_t, L_t).$$

Shareholders receive the marginal product of capital $F_K(K_t, L_t)$ therefore the total capital income is

$$\Pi_t := F_K(K_t, L_t)K_t \tag{5}$$

(on marginal productivity of capital and profits see also [16] and [17]).
Old individuals receive profit:

$$R_t s_{t-1} L_{t-1} = R_t K_t = \Pi_t$$

and an easy computation leads to

$$R_t = F_K(K_t, L_t)\,.$$

The accumulation of capital is

$$K_{t+1} = L_t s(w_t, R^e_{t+1}). \tag{6}$$

where $s(w_t, R^e_{t+1})$ is given in (3) and is the solution of the maximisation problem for the utility function (2).
[16] assumes the production technology is represented by the Constant Elasticity of Substitution (CES) production function

$$\bar{F}(K, L) := A\left[\alpha K^{-\rho} + (1-\alpha)L^{-\rho}\right]^{-\frac{1}{\rho}}, \quad A > 0, \alpha \in (0, 1), \rho > -1, \rho \neq 0 \tag{7}$$

and has elasticity of substitution between capital and labour

$$ES = \frac{1}{1+\rho}.$$

Capital depreciates at the rate $\delta \in [0,1]$ and, after production, the non-depreciated capital is equal to the good produced, so that the total production function is

$$F(K_t, L_t) := \bar{F}(K_t, L_t) + (1-\delta)K_t.$$

Without loss of generality, the author assumes $\delta = 1$ considering a time period of about 20 years. Moreover, equation (7) is homogeneous of degree one, the production function can be written in intensive form as

$$f(k_t) := F(k_t, 1) = A\left[\alpha k_t^{-\rho} + 1 - \alpha\right]^{-\frac{1}{\rho}}$$

where $k := \frac{K}{L}$ is capital per worker. Due to the homogeneity properties of the production technology, the wage may be written as:

$$w(k_t) = A(1-\alpha)\left(\alpha k_t^{-\rho} + 1 - \alpha\right)^{-\frac{1+\rho}{\rho}} .$$

As far as expectations are concerned, the authors study the complete model using myopic expectations. Although the assumption of myopic expectations is unrealistic, this assumption allows one to study the framework in explicit form. Moreover, macro-economic literature considers myopic foresight a good tool for analysing stability scenario. Under myopic expectations it has

$$R_{t+1}^e := R_t = f'(k_t).$$

Then the evolution of capital over time presented in (6) can be written in its intensive form as

$$k_{t+1} = \frac{1}{1+n} s\left(w(k_t), f'(k_t)\right) \tag{8}$$

where the saving function $s(\cdot)$ given in (3) depends on the characteristics of the utility function (2). This will be discussed in the following sections.

2.1.2. *The extended model*

The model presented in the previous section assumes that all the profits are paid as dividends to old individuals, that are the owners of the representative firm. Despite this, dividend policy is part of the corporate strategy: managers have to decide the portion of profit allocated as dividends, while the residual part is allocated for investment in corporate physical capital (usually used to acquire or upgrade physical assets). Consequently we assume that part of the profits Π_t defined in (5) is invested in physical capital and contributes to capital evolution, and modifies the rule (8) as

$$k_{t+1} = \frac{1}{1+n}\left(s_t + j_t\right) \tag{9}$$

where j_t is the portion of profit invested in the firm (expressed in terms of capital per worker) while s_t is the saving function. Their properties will be discussed below and in Section 2.2 respectively.

One of the most common dividend policies is the *Constant Dividend Policy* (CDP) under which the company pays a constant portion of the profit at every period (for dividend policies see [18]). Recalling that

$$\Pi_t = F_K(K_t, L_t)K_t$$

and given the properties of the production function, the profit can be written as:

$$\pi_t^{(k)}(k_t) := K_t f'(k_t) = A\alpha\left[\alpha k_t^{-\rho} + 1 - \alpha\right]^{-\frac{1+\rho}{\rho}} k_t^{-\rho-1} K_t\,. \tag{10}$$

Under CDP, a portion $\beta \in [0,1]$ of the profit is allocated to dividend payment. Notice that the assumption in [16] where, at each period t, old individuals own all the firm capital implies that at each period the firm adjusts its capital structure toward seasoned equity offering, i.e. issuing new shares at every period. Conversely, in our framework the number of issued shares x is fixed over time and existing shares are traded in the financial market, as will be discussed in Section 2.2.

Then the investment in physical capital at every period may be written as

$$J(K_t, L_t) := \Pi_t(K_t, L_t) - \beta\Pi_t(K_t, L_t). \tag{11}$$

Previous equation states that physical investment, in terms of capital per worker, is given by profit minus the dividend payment. Notice that for $\beta = 1$ the model is equal to the baseline framework discussed in previous section, therefore it might be considered a generalisation of [16]. Substituting equation (10) in (11) and considering $k_t = \frac{K_t}{L_t}$, the investment in the firm per worker is

$$j(k_t) = (1-\beta)A\alpha \left[\alpha k_t^{-\rho} + 1 - \alpha\right]^{-\frac{1+\rho}{\rho}} k_t^{-\rho}. \tag{12}$$

2.2. Evolution of asset price

As discussed in Section 2.1.2, the representative firm has $x > 0$ outstanding shares and the unitary price of the share at time t is denoted by p_t. Moreover, a risk-free bond issued by the fiscal authority exists and it pays a constant return $R_f := 1 + r_f > 0$ per unit.

Firm shares pay a dividend

$$d_{t+1} := \frac{\beta}{x} A\alpha \left[\alpha k_t^{-\rho} + 1 - \alpha\right]^{-\frac{1+\rho}{\rho}} k_t^{-\rho} \tag{13}$$

where previous equations descend from (12).

Since the aim of this work is to investigate the effect of different dividend policies for the volatility of the economy, we simplify the framework discussed in Section 2.1.1 by assuming no consumption at a young age. The assumption of null consumption is unrealistic but frequently applied in asset price intertemporal equilibrium models (see for example [19] on which our model is based, or the Merton's Bellman equation in [20]); notice that a less naive choice, such as a fixed young consumption would not enrich the results of the model, therefore we set $c_t = 0$. The budget constraint of a young individual i is then

$$w(k_t) = y^{(i)} + p_t x^{(i)}$$

where $y^{(i)}$ and $x^{(i)}$ are the amount of consumption goods that are invested in the risk free and risky asset respectively and the price of the risk free asset is assumed equal to 1. The income of the old individual is

$$D_{t+1} = R_f y^{(i)} + q_{t+1} x^{(i)}$$

where $q_{t+1} := p_{t+1} + d_{t+1}$ is the gross return (cum-dividend price). Notice that the investment made by individuals - differently from the case discussed in [16] - is allocated in financial assets, therefore $s_t = 0$, $\forall t$.

Young individuals have linear mean-variance preferences and maximise the expected utility over future consumption with respect to subjective beliefs over future asset price. Even though in our framework the employed dividend policy decided by managers will be carried out in every future period, individuals do not have complete information and believe that the dividend policy might change over time. Individuals know that managers employ CDP, but depending on available information (such as public announcements) they believe that a new amount γd_{t+1}, $\gamma \geq 0$ will be applied with probability $\theta \in (0, 1)$.

An extensive literature on agents' heterogeneity shows how different beliefs can generate rich dynamics; see for example the seminal work of [21] in which the heard effect is studied, the findings in [22] in which agents rationally adapt their beliefs over time depending on past performances, or the results in [23] where chartists and fundamentalists interact in the cryptocurrency market.

We assume that individuals have rational expectations regarding future prices. Notice that the assumption of rational expectation is unrealistic, although it will be considered below since the focus of the work is on the effect of dividends over asset price.

The expected income depends on the expected value of the dividend payment: individuals believe that an amount d_{t+1} will be paid with probability $1 - \theta$, while the amount γd_{t+1} will be paid with probability θ, therefore the expected value of the dividend payment per share is $[1 + (\gamma - 1)\theta]d_{t+1}$ from which it follows that the expected income at time $t + 1$ is

$$\mathbb{E}[D_{t+1}] = R_f y^{(i)} + p_{t+1} x^{(i)} + [1 + (\gamma - 1)\theta] d_{t+1} x^{(i)}$$

with subjective variance

$$\mathbb{V}[D_{t+1}] = \theta(1 - \theta)(\gamma - 1)^2 \left(d_{t+1} x^{(i)} \right)^2$$

and the expected utility given the linear mean-variance preferences is

$$U\left(\mathbb{E}[D_{t+1}], \mathbb{V}[D_{t+1}]\right) = R_f y^{(i)} + \{p_{t+1} + [1+(\gamma-1)\theta]d_{t+1}\}x^{(i)} - \frac{a}{2}\theta(1-\theta)(\gamma-1)^2\left(d_{t+1}x^{(i)}\right)^2$$

where $a > 0$ is related to risk aversion. Notice that $y^{(i)} = w(k_t) - p_t x^{(i)}$, therefore the amount $x_*^{(i)}$ that maximises the utility is

$$x_*^{(i)} = \frac{p_{t+1} + [1+(\gamma-1)\theta]d_{t+1} - R_f p_t}{a\theta(1-\theta)(\gamma-1)^2 d_{t+1}^2}$$

and the total demand of asset shares is

$$x^* = \frac{p_{t+1} + [1+(\gamma-1)\theta]d_{t+1} - R_f p_t}{a\theta(1-\theta)(\gamma-1)^2 d_{t+1}^2} L_t.$$

As [24] discusses, a natural market-clearing condition is the one that chooses the highest price at which demand and supply are equal:

$$p_{t+1} = \max\left\{p \in \mathbb{R}_+ \,\middle|\, \frac{p_{t+1} + [1+(\gamma-1)\theta]d_{t+1} - R_f p_t}{a\theta(1-\theta)(\gamma-1)^2 d_{t+1}^2} L_t + \varepsilon_t = x\right\} \tag{14}$$

where ε_t is a stochastic noise denoting the quantity of assets held by noise traders after trading, and it captures the effect of all other sources that may influence the price p_t. The market clearing condition (14) allows one to obtain the explicit form for the asset price:

$$p_{t+1} = R_f p_t + \left[\frac{a\theta(1-\theta)(\gamma-1)^2 d_{t+1}(x-\varepsilon_t)}{L_t} - 1 - (\gamma-1)\theta\right] d_{t+1}. \tag{15}$$

The presented model for the evolution of asset price depending on subjective beliefs over future gross return may be considered a simplification of [25] where multiple assets and heterogeneous beliefs are investigated.

2.3. A model for real economy and finance

Given the assumption in Sections 2.1.2 and 2.2 we are now able to build an inclusive model for real economy and the financial market.

At time $t = 0$ a representative firm with k_0 capital per worker issues x shares and there exist L_0 young individuals that receive a wage $w(k_0)$ and choose their portfolio $(y^{(i)}, x^{(i)})$ of risk free and risky asset, respectively. As discussed in Section 2.1.2, capital increases since part of the profits are retained by the firm.

Retained profits given in (12) contributes to increase firm capital and the rule that governs the physical capital evolution reads

$$k_{t+1} = \frac{(1-\beta)A\alpha\left[\alpha k_t^{-\rho} + 1 - \alpha\right]^{-\frac{1+\rho}{\rho}} k_t^{-\rho}}{1+n}$$

where equation (12) is substituted in (9). Note that below it will be assumed that the population remain constant over time in order to disregard the effects due to changes in labour force, therefore $n = 0$. As far as the financial market is concerned, the evolution of asset price has been discussed in previous section. Let us recall that individuals believe that the dividend payment could change over time with probability θ, and given the market clearing condition, the evolution of asset price is given by equation (15) where dividends d_{t+1} are defined as in (13).

This research is focused on the deterministic component of the price evolution while the stochastic noise might not allow the detection of a characteristic structure in the dynamic system, since the evolution could behave like a stochastic process for sufficiently high noise, therefore in the following we set $\varepsilon_t = 0$, $\forall t$. Moreover, we assume constant labour force $L_t = L$ $\forall t$ to isolate the effect of the labour force growth rate on asset price and capital. Notice that, as in [25], the previous equation allows negative prices. The unrealistic assumption of negative asset prices has been discussed in literature since the work of [26]. In this work we allow prices to be negative, but we highlight that a future development is needed to investigate the case with only non-negative prices.

We are now able to specify the evolution of the economy over time as

$$T = \begin{cases} k_{t+1} = \mathcal{K}' := (1-\beta)A\alpha\left[\alpha + (1-\alpha)k_t^{\rho}\right]^{-\frac{1+\rho}{\rho}} k_t \\ p_{t+1} = \mathcal{P}' := R_f p_t + \left[\frac{a\theta(1-\theta)(\gamma-1)^2 d_{t+1} x}{L} - 1 - (\gamma-1)\theta\right] d_{t+1} \end{cases} \tag{16}$$

where dividends d_{t+1} are given by (13) and under the assumption $n = s = 0$ previously discussed. The dynamics of price and capital will be analysed in the following sections.

3. General equilibrium and dynamic evolution

In this section we provide an analytical discussion for system T in order to verify the existence and stability of equilibria for the general equilibrium model. Note that the system described in (16) incorporates many simplifications over the evolution of capital and asset price in real economy. Despite so, thanks to its relatively low complexity it allows one to understand the effect of dividend payment in real economy and the financial market.

As the existence of equilibria is concerned, the following Proposition holds true (proof in Appendix).

Proposition 1. *Map T in (16) always has the fixed point $E_0 = (0,0)$. Moreover, let $\beta_0 = \frac{\alpha^{\frac{1}{\rho}}}{A}$. Then, if one of the following is satisfied:*

- $-1 < \rho < 0 \wedge \beta > 1 - \beta_0$;
- $\rho > 0 \wedge \beta < 1 - \beta_0$;

a non-null fixed point $E^* = (k^*, p^*)$ exists, with

$$k^* = \left\{ \frac{[(1-\beta)A\alpha]^{\frac{\rho}{1+\rho}} - \alpha}{1-\alpha} \right\}^{\frac{1}{\rho}}, \tag{17}$$

$$p^* = \frac{\beta k^*}{(1-R_f)(1-\beta)x} \left[\frac{a\theta(1-\theta)(\gamma-1)^2 \beta k^*}{L(1-\beta)} - 1 - (\gamma-1)\theta \right]. \tag{18}$$

Meaningful equilibria are the stable ones, i.e. those equilibria that might be reached by the economy. Given the variables discussed, the following can be proved (proof in Appendix)

Proposition 2.

Consider β_0 as defined in Proposition 1. Then

- *Fixed point E_0 is locally asymptotically stable if $R_f < 1$ and $\rho > 0 \wedge \beta > 1 - \beta_0$.*
- *Fixed point E^*, when exists, is locally asymptotically stable if $R_f < 1$ and one of the following holds true:*

– $\rho \leq 1$;

– $\rho > 1 \wedge \beta > 1 - \left(\frac{1+\rho}{\rho-1}\right)^{\frac{1+\rho}{\rho}} \beta_0$.

We are now able to describe how the economy evolves depending on parameter values.

Recall that the elasticity of substitution, that is a measure of the ease with which capital and labour may be substituted in production, is given by $ES = \frac{1}{1+\rho}$. For $\rho \in (-1, 0)$, we consider economies with $ES > 1$. Notice that, since the work of [27], it has been highlighted that the relationship between income and wage per worker leads, empirically, to an elasticity of substitution significantly lower than one. Therefore, below we will focus on the case $\rho > 0$.

In case $\rho > 0$, if $\beta > 1 - \beta_0$ and $R_f < 1$ the economy disappears (i.e. the fixed point E_0 is locally attractive). The intuition behind this result is as follows: a high substitutability between input factors ($\rho < 0$) guarantees the absence of poverty traps since the intended level of output may be guaranteed without further investments in capital due to the relatively easiness to substitute physical capital and labour. In the most realistic case, $\rho > 0$, constraints on the substitution between capital and labour are more severe. In this scenario, when the risk-free bond has a negative return, a high portion of investors selects the risky asset increasing its price. If, in addition, the dividend payout ratio is high, the economy bears an excessively high cost for financing its activity. In the long run, a poverty trap is created and the system converges to the null equilibrium, characterized by zero capital and zero asset price. As β decreases crossing the value $1 - \beta_0$ while $\rho > 0$, the origin loses stability and a new and stable fixed point E^* arises. Instability is generated when the dividend payout ratio is lower than $1 - \left(\frac{1+\rho}{\rho-1}\right)^{\frac{1+\rho}{\rho}} \beta_0$, jointly with $\rho > 1$, or when the return of the risk free asset is positive, i.e. $R_f > 1$. Notice that for $\beta < 1 - \left(\frac{1+\rho}{\rho-1}\right)^{\frac{1+\rho}{\rho}} \beta_0$ the first eigenvalue of the Jacobian matrix crosses the value -1, therefore a period doubling bifurcation occurs and oscillations for capital emerge.

The interaction between elasticity of substitution and the cost to finance the activity (driven by the dividend payout ratio and the return of the risk free

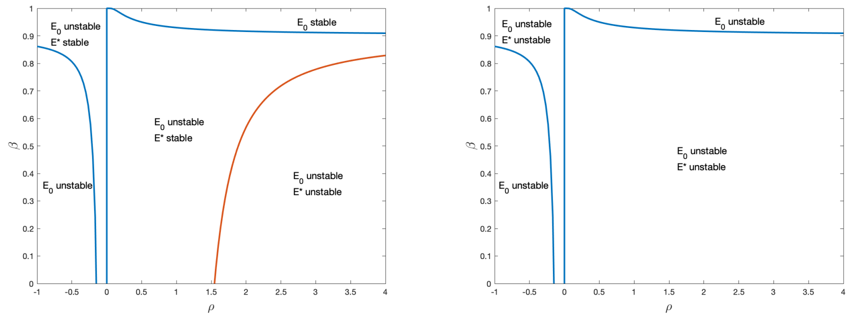

Figure 1: Existence and stability of the fixed points in the parameter plane (ρ, β). The blue line represents the curve such that $\beta = 1 - \beta_0$, the orange line represents the curve such that $\beta = 1 - \left(\frac{1+\rho}{\rho-1}\right)^{\frac{1+\rho}{\rho}} \beta_0$. Left panel: $R_f < 1$; Right panel: $R_f > 1$. Baseline parameter values as in Table 4.

asset) influences the long run evolution of the system: when the substitutability between input factors is high ($\rho < 0$), i.e. the production is more flexible, if the dividend payout ratio β is low, the growth is unbounded (the unique equilibrium is E_0 and it is unstable). Increasing the dividend payout ratio, a positive equilibrium emerges; unbounded growth is then limited (E^* becomes stable) if the return of the bond is negative: once again a higher fraction of investors selects the risky asset and consequently the price of the asset increases. The cost to finance the production becomes too high (given high β), containing the growth. In case of less flexibility in the composition of the capital/labour mix for production ($\rho > 1$) if the dividend payout ratio is high the representative firm (and, hence, the economy) survives if $R_f > 1$ (neither E_0 nor E^* are stable therefore the economy does not disappear). In this case, the demand for the risky asset is restrained and, although high dividends are paid, growth is possible. When R_f falls below 1, due to the increased number of investors, the portion of income paid as dividends becomes too high and unsustainable, leading the whole economy into the poverty trap. In all the cases, R_f acts as a bound for growth.

Figure 1 shows the existence and stability of the fixed points in the parameter plane (ρ, β). Notice that the fixed point E^* does not exist in the two regions in which there is no reference to it, while the orange curve represents one of the conditions for the stability of E^* and therefore it only appears in the left panel (E^* may only be stable for $R_f < 1$). So far we proved that an excessively high dividend payout ratio erodes the economy when $\rho > 1$, while in the case of excessively low β or $R_f > 1$ instability arises. Analytical results need to be discussed in light of empirical evidence.

4. Empirical analysis and numerical simulations

Here we provide empirical analysis and numerical simulations to complete the analytical results discussed in the previous section.
In the previous section it has been proved that a negative return of the risk free asset is a necessary condition for the stability of the two equilibria. As far as the non trivial equilibrium is concerned, according to the mathematical results, when the other conditions are fulfilled, $R_f < 1$ stabilises the asset price. The idea that a positive bond return leads to instability might be counterintuitive due to our experience, and a brief discussion is needed. A stable economy, in our framework, is an economy in which the capital per capita does not change over time and, in the same way, the asset price has zero volatility leading to zero returns over time. Empirical analysis suggests that real economy might fluctuate, decrease or increase but it rarely remains stable over time, while asset prices are intrinsically characterized by their volatility. These behaviours are combined with the common assumption (and experience) of a positive return of bonds. In this view, our findings relate to the instability of the economy and the return of risk free asset, proving that - if all the other parameters would allow stability - a negative sign for bond return would be needed in order to converge to the equilibrium (that, depending on other parameter values, might be the non-trivial equilibrium E^* or the so-called poverty trap E_0). Decades ago a negative return of bond was not conceivable, but the dramatic crisis of 2008 changed this perspective: as is well known, in order to stimulate consumptions and

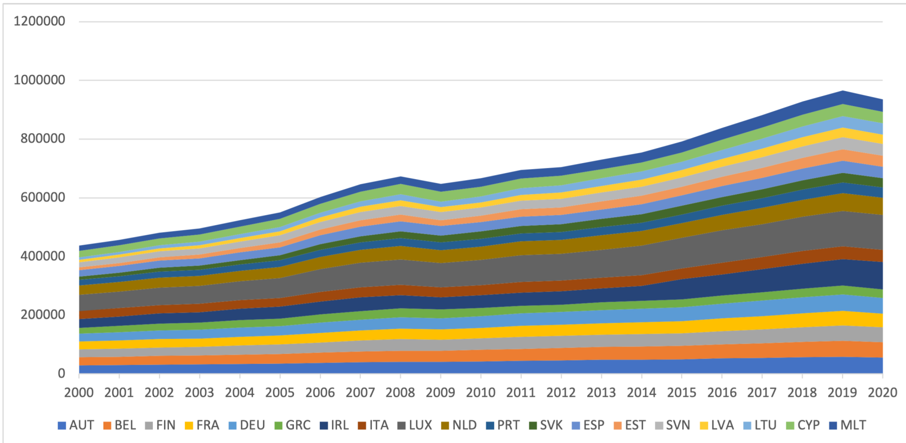

Figure 2: Euro Area, Gross domestic product (GDP) (indicator). Source: OECD (2021), doi: 10.1787/dc2f7aec-en (Accessed on 30 June 2021).

investments, the FED and ECB applied an expansive monetary policy injecting liquidity into the market. The massive purchase of assets implemented by the central banks led to negative returns. Returns below zero for bonds, our $R_f < 1$, is no longer a rarity: in 2021, about 27% of the bond market traded with negative yields. Our simplified analysis suggests that when the bond return falls below zero, the asset price tends to converge to the equilibrium price that, depending on other parameter values, might be given by the non trivial equilibrium E^* or the poverty trap E_0. Empirical evidence for this behaviour is not easy to capture: the first assumption here is that all the other parameters satisfy the stability condition and therefore, as capital per capita is concerned, the economy has reached, or is reaching, the equilibria. In order to select, for our analysis, a Country for which a stable and attractive fixed point exists, we consider the GDP per capita from 2000 to 2020 of the Countries belonging to the Euro Area (Figure 2).

As a first analysis we assume the economy has already reached the equilibria. Considering our simplified analytical model, the capital per capita would be

	AUT	BEL	FIN	FRA	DEU	GRC	IRL
g_i	0.7812	0.7449	0.7907	0.7626	0.8152	0.9766	1.4468
	ITA	LUX	NLD	PRT	SVK	ESP	
g_i	0.6346	0.9338	0.7492	0.8274	1.1946	0.8854	
	EST	SVN	LVA	LTU	CYP	MLT	
g_i	1.7103	1.0367	1.7294	1.8710	1.0583	1.0400	

Table 1: g_i value for the 19 Euro Area Countries.

constant over time. Using the GDP per capita as an estimate of k_t, two aspects have to be considered: (1) GDP per capita is computed over the total population while k_t is computed over labour force (assumed constant), (2) GDP is a nominal indicator and the comparison over time is misleading since increasing values of GDP might be caused by changes in prices instead of real growth. Being $C = \{c_1, c_2, \ldots, c_{19}\}$ the set of 19 Countries belonging to the Euro Area according to the OECD (and listed in Figure 2), with $|C| = 19$ we select as stable the Country $c_i \in C$ for which the sum of the percentage yearly growth, in absolute value, is the lowest:

$$g_i = \sum_{t=2001}^{2020} \left| \frac{GDP(t)_i}{GDP(t-1)_i} - 1 \right|$$

$$\text{Stable Country } = c_i \in C | g_i < g_j \forall j \in \{1, 2, \ldots, |C|\}, j \neq i.$$

where $GDP(t)_i$ is the GDP of Country i at year t. The results are in Table 1.

The Country with the lowest g_i is Italy. Notice that the time step considered in this work encloses a decade, therefore as far as the bond is concerned, at least 10Y Government bonds need to be considered. Italian Government bonds with maturity at 10 or more years have never paid negative returns. Despite so, we decided to analyse the relation with return close to 0. Notice that, from the end of 2016 to the beginning of 2020 the inflation rate has been positive in Italy with values in the range $(0, 2)$, while the yield of the government bond is given in real terms. Therefore a yield close to 0 could be understood as a negative yield in nominal terms. We found empirical evidence considering 10Y Government

Time period	Interval	Bond yield	FTSE MIB price volatility
05/02/16-04/03/16	5 weeks	≥ 1	352.547
11/03/16-08/04/16	5 weeks	< 1	290.621
15/04/16-13/05/16	5 weeks	≥ 1	149.441
20/05/16-24/06/16	5 weeks	≥ 1	208.995
11/06/16-14/10/16	16 weeks	< 1	106.591
21/10/16-03/02/17	16 weeks	≥ 1	1.614.666
14/06/19-23/08/19	11 weeks	≥ 1	422.937
30/08/19-08/11/19	11 weeks	< 1	353.795
24/01/20-06/03/20	7 weeks	< 1	2.043.188
13/03/20-24/04/20	7 weeks	≥ 1	368.195
13/03/20-17/07/20	19 weeks	≥ 1	2.068.917
24/07/20-27/11/20	19 weeks	< 1	947.798

Table 2: Italy 10Y bond yield - FTSE MIB price volatility, relation.

bonds and the benchmark stock market index (representing the asset price of the representative firm in our paper).

We considered the 10Y Government bond (plain vanilla fixed coupon) bid yield and FTSE-MIB price history, both with weekly frequency from July 2001 (data Thomson & Reuters). Until March 2016 the bond paid return higher than 1% therefore comparisons are not possible. As visible in Table 2, for 5 weeks in the range March-April 2016 the return has been lower than 1, while price volatility has been $V = 290621$. Comparing the volatility for the previous 5 weeks it is visible that the volatility was higher when the bond return was above 1. In the following 5 weeks (15/04/16-13/05/16) the volatility seems to continue decreasing; despite this, in the subsequent 5 weeks (20/05/16-24/06/16) the price increases its fluctuations again. We hypothesize that the effect of bond return close to zero on volatility persists for a short period after R_f crosses the positive threshold. This would explain why in the period April-June 2016 V increased slowly, while in the following analysis, considering longer intervals, the volatility

Time period	Interval	Bond yield	BEL 20 price volatility
15/03/19-20/06/19	16 weeks	positive	19.596
19/07/19-01/11/19	16 weeks	negative	9.341
27/11/20-12/02/21	12 weeks	negative	4.667
09/04/21-25/07/21	12 weeks	positive	5.331

Table 3: Belgium 10Y bond yield - BEL 20 price volatility, relation.

rises immediately above the value experienced during low bond returns. Bond yield falls again in July 2016, for 16 weeks, and asset price volatility is consistently lower during the period in which returns are lower than 1, compared to the following time period (a comparison with the previous 16 weeks is not possible since the bond had high returns only for the previous 11 weeks). When bond return crosses the value once again, in June 2019, the volatility drops passing from $V = 422937$ to $V = 353795$. As in the previous case, a comparison with the time interval that comes after the period with a return lower than one is not possible due to a new fall of R_f after a few weeks. The phenomenon does not repeat itself in early 2020 but the period coincides with the beginning of the Covid-19 spread in Italy, when financial markets also experienced unexpected behaviours. Returns below one emerges again in July 2020, and once again price volatility is lower in this period when compared with the previous one. This preliminary analysis might not be significant since, despite the return is close to 0, the yield does not assume negative values. For this reason we repeat the analysis with the Country with lowest g_i, Italy excluded: Belgium. We considered the bid yield of the BEL 10Y Government Bond (plain vanilla fixed coupon) and the price volatility of the BEL 20 index (the benchmark stock market index of Euronext Brussels), both with weekly frequency, from January 2018 (data Thomson & Reuters). Belgium experienced negative bond returns starting from the second half of 2019. Positive and negative returns alternate frequently, therefore a comparison considering the same interval of time has been possible for two time periods (see Table 3). These preliminary findings show that a connection is possible between the return of investment grade bonds and the asset price of a representative firm. Despite this, further analysis is needed: price volatility is a good indicator for a price that has already reached its equilibrium (in this case the volatility should be zero), but it might not be the correct indicator in the case of a price that is in the path of convergence. Due to the aim of this work, further and more detailed analysis will be carried out in a dedicated research on the relation with empirical data. Numerical simulations, instead, show clearly the effect of the bond yield on price volatility: in Figure 3 the volatility (variance) of the asset price p_t is computed considering 500 time steps and it is reported in the case of positive and negative returns for the risk free asset. For $R_f > 1$ the non trivial fixed point, when it exists, is always unstable and the volatility level is high. For $R_f < 1$ the scenario drastically changes with low volatility for all the combinations in the parameter plane. In both cases the volatility increases as ρ increases, but for $R_f > 1$ the range of values that the volatility may reach (depending on the combination between ρ and β) is wider.

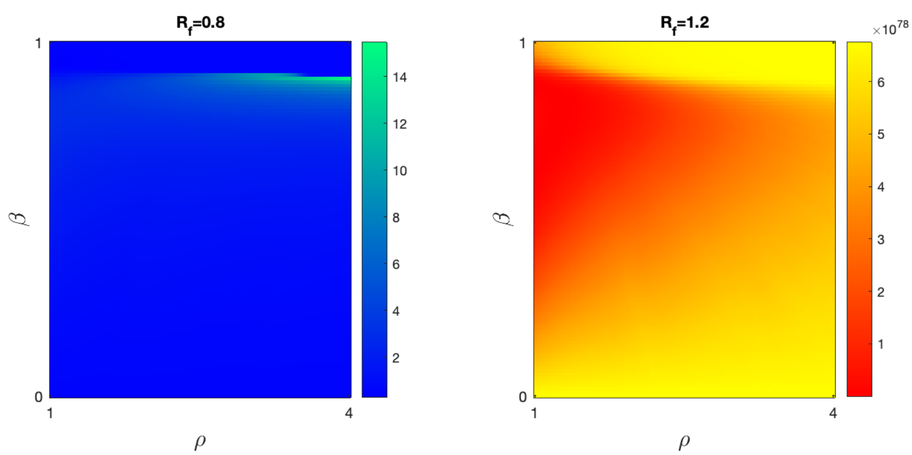

Figure 3: Volatility (variance) of p_t computed on 500 time steps, moving ρ and β, for negative (left panel) and positive (right panel) returns of the risk free bond. Baseline parameter values as in Table 4.

Parameter	Range	Baseline value	Source
ρ	$(1, 4)$	2.5	[28]
α	(0.425, 0.999)	0.712	[28]
A	$(9.9, 10.4)$	10.15	[29]
L	(0.9729.35)	7.4	[30]
x	-	3.5	[31]
R_f	-	1.09	Thomson & Reuters
β	$(0.4, 0.46)$	0.43	[32]

Table 4: Baseline parameter values.

As far as the dividend payout ratio is concerned, analytical results in Section 3 showed that β affects the existence and stability of the equilibria: when the dividend payout ratio is excessively high while $\rho > 1$, the economy is trapped in a poverty trap, while in the opposite case fluctuations emerge. Nothing can be said a priori regarding the effect of beliefs: the role of parameters θ and γ, respectively related to the perceived probability of a different payout ratio and expected amount of such a payout, needs to be investigated by means of numerical simulations.

Table 4 shows the parameter values that will be used as a baseline below. [28] highlight that many researchers discussed how, considering a CES production function, meaningful results arise in the case of elasticity of substitution lower than one. Therefore below we will focus on this case. Moreover the authors esteems the elasticity of substitution $\sigma \in (0.2, 1.3)$. Being $\sigma = \frac{1}{1+\rho}$ and considering $\sigma < 1$ we set $\rho \in (1, 4)$. In the same work, the author esteem parameter α as dependent on σ and with $\alpha = 0.999$ for $\sigma = 0.2$ and $\alpha = 0.210$ for $\sigma = 1.3$. Using linear interpolation we compute the values for the σ in the range $(0.2, 1)$. As far as parameter A is concerned, we considered the range of Multifactor Productivity values computed for the Euro area (specifically Belgium, Luxembourg, Austria, Netherlands, Italy, Spain, France, Portugal, Germany, Greece and Finland) by [29] with base USA 2015 = 10. For labour the average labour force

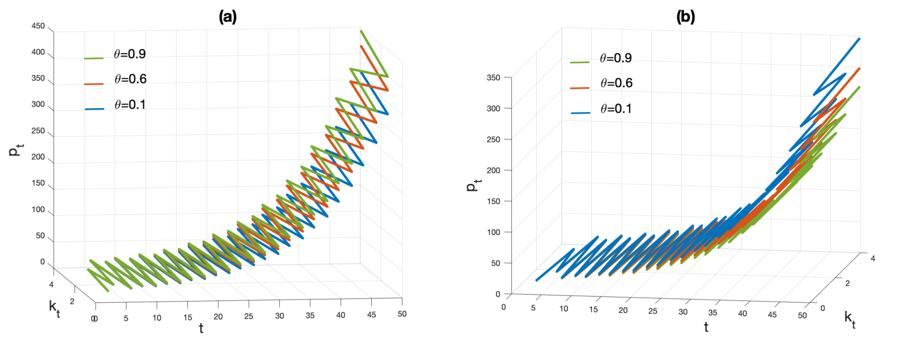

Figure 4: Capital and asset price evolution moving θ. Panel (a): $\gamma = 0.7$; Panel (b): $\gamma = 1.3$. Baseline parameter values, $a = 0.5$.

computed for the Euro area in 2020 expressed in millions has been used. For the total amount of outstanding shares, it has been computed the average (per Country) number of outstanding listed shares issued by EU residents computed at the end of 2020 based on data in [31], expressed in hundreds of billions. As far as the return of the risk free bond is concerned, we considered the average return of the 20Y German Government Bond, computed quarterly, for the period March 2017-March 2019 (the time interval has been selected in order to avoid the effect of the Covid-19 pandemic on financial markets), with data being extracted from Thomson & Reuters. Lastly, for the dividend payout ratio we considered the Average Dividend Payout Ratio of Western European Companies for the period 2014-2018 as reported in [32] based on Suisse HoltLens data. In the previous section we identified which parameters might generate fluctuations in real economy and financial markets, here we aim to analyse the effect of those parameters that do not have a direct effect on stability, as the erroneously expected payout ratio (parametrized by γ) and its expected probability (parametrized by θ). As visible in Figure 4, the effect of the two parameters needs to be analysed jointly: when individuals believe that the dividend payout ratio might be lower than the real one (i.e. $\gamma < 1$), the stronger the perception (i.e. the higher θ), the higher the asset price after dividends are paid; conversely

- when individuals expect higher dividend payments - the stronger the expectation (i.e. the higher θ), the lower the price after dividends are paid. Notice that a wide range of literature analyses the connection between dividend expectations and asset price, mainly highlighting how markets react to announcements (see, among others, [33], [34], [35], [36]). Here we show the long run effect of expectations, i.e. the effect after dividends are realised: unfulfilled expectations rise the asset price when the dividend is higher than expected and they lower it otherwise.

5. Conclusion

We study the discrete time evolution of a simplified economy in which physical capital and asset price evolves depending on (1) the dividend payments decided by managers, (2) the expectations of individuals regarding future cum-dividend prices and (3) the return of a risk free bond. We found that the dividend payout ratio produces a twofold effect: an excessively high dividend payout ratio erodes the economy, while in the opposite case instability and fluctuations arise. Additionally, positive returns for government bonds lead to instability. This counterintuitive idea reflects empirical evidence: real economy rarely remains stable over time, asset prices are characterized by their volatility and bond returns are usually positive. In this view, our findings relate the instability of capital and asset price to the return of risk free bond suggesting that, if all the other parameters would allow stability, a negative sign for bond return would be needed in order to reach the equilibrium (either a nontrivial equilibrium or a poverty trap), in which capital and asset price do not change over time. Moreover, we showed that expectation over dividends does not affect the existence and stability of an economy but it affects its path. Particularly we highlighted that unfulfilled expectations might have a twofold effect: an expectation of lower dividends tends to rise the asset price after dividends are realised, conversely an expectation of higher dividends lowers the asset price in the long run.

Declarations

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Code availability: Not applicable

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Appendix

Appendix A. Proof of Proposition 1

Fixed points for map T given by (16) are the solution of

$$\begin{cases} k_{t+1} = k_t \\ p_{t+1} = p_t \end{cases} .$$

Notice that for $k_{t+1} = k_t = 0$ and $p_{t+1} = p_t = 0$ the system is satisfied therefore the trivial fixed point $E_0 = (0,0)$ always exists. By setting $k_{t+1} = k_t = k$ and $p_{t+1} = p_t = p$, the equilibria for map T are couples (k,p) that solve

$$\begin{cases} \left[\alpha + (1-\alpha)k^{\rho}\right]^{-\frac{1+\rho}{\rho}} = \frac{1}{(1-\beta)A\alpha} \\ \\ p = \frac{d(k^*)}{1-R_f}\left[\frac{a\theta(1-\theta)(\gamma-1)^2 d(k^*)x}{L} - 1 - (\gamma-1)\theta\right] \end{cases} \tag{A.1}$$

where $d(k^*) = \frac{\beta}{x}A\alpha\left[\alpha + (1-\alpha)(k^*)^{\rho}\right]^{-\frac{1+\rho}{\rho}} k^*$ and k^* is the value of k that solves the first equality of the system.

Consider $\mathcal{M}(k) := \left[\alpha + (1-\alpha)k^{\rho}\right]^{-\frac{1+\rho}{\rho}}$ and $m := \frac{1}{(1-\beta)A\alpha}$. Then, the first equation in system (A.1) is solved iff

$$\mathcal{M}(k) = m\,. \tag{A.2}$$

Function $\mathcal{M}(k)$ is such that $\mathcal{M}'(k) = -(1+\rho)(1-\alpha)k^{\rho-1}\left[\alpha + (1-\alpha)k^{\rho}\right]^{-\frac{1}{\rho}-2} < 0$. Moreover, for $\rho \in (-1,0)$ it has $\mathcal{M}(0) = +\infty$, $\lim_{k\to+\infty}\mathcal{M}(k) = \alpha^{-\frac{1+\rho}{\rho}}$ while for $\rho > 0$ it has $\mathcal{M}(0) = \alpha^{-\frac{1+\rho}{\rho}}$, $\lim_{k\to+\infty}\mathcal{M}(k) = 0$.

Being $\alpha^{-\frac{1+\rho}{\rho}} > m$ if $\alpha^{-\frac{1}{\rho}} > \frac{1}{(1-\beta)A}$ if follows that

- in case $\rho \in (-1,0)$, $k_{t+1} = k_t$ if $\beta > 1 - \frac{\alpha^{\frac{1}{\rho}}}{A}$;
- in case $\rho > 0$, $k_{t+1} = k_t$ if $\beta < 1 - \frac{\alpha^{\frac{1}{\rho}}}{A}$

When the fixed point exists, it is given by

$$k^* = \left\{\frac{\left[(1-\beta)A\alpha\right]^{\frac{\rho}{1+\rho}} - \alpha}{1-\alpha}\right\}^{\frac{1}{\rho}} \tag{A.3}$$

.

For the second equation in (A.1) it has $d(k^*) = \frac{\beta k^*}{x(1-\beta)}$, where k^* is defined in (A.3), and the unique solution is given by

$$p^* = \frac{\beta k^*}{(1-R_f)(1-\beta)x}\left[\frac{a\theta(1-\theta)(\gamma-1)^2\beta k^*}{L(1-\beta)} - 1 - (\gamma-1)\theta\right] \tag{A.4}$$

.

Consequently, up to two fixed points exist:

- $E_0 = (0,0)$ for all parameter values;
- $E^* = (k^*, p^*)$ if one of the following is satisfied:

  $-1 < \rho < 0 \wedge \beta > 1 - \frac{\alpha^{\frac{1}{\rho}}}{A}$;

  $\rho > 0 \wedge \beta < 1 - \frac{\alpha^{\frac{1}{\rho}}}{A}$.

Appendix B. Proof of Proposition 2

A fixed point of map T is stable if all the eigenvalues of the Jacobian matrix, computed at the fixed point, are in modulus less than one. It has

$$J = \begin{bmatrix} \frac{\partial \mathcal{K}'}{\partial k} & 0 \\ \\ \frac{\partial \mathcal{P}'}{\partial k} & \frac{\partial \mathcal{P}'}{\partial p} \end{bmatrix} \tag{B.1}$$

is the Jacobian matrix is triangular and its eigenvalues are the entries of the main diagonal. Therefore fixed points are stable if it is verified

$$\begin{cases} \left|\frac{\partial\partial \mathcal{K}'}{\partial k}\right| < 1 \\ \\ \left|\frac{\partial \mathcal{P}'}{\partial p}\right| < 1 \end{cases} . \tag{B.2}$$

As far as the first inequality in (B.2) is concerned, note that

$$G(k) := \frac{\partial \mathcal{K}'}{\partial k} = (1-\beta)A\alpha\left[\alpha + (1-\alpha)k^\rho\right]^{-\frac{1}{\rho}-2}\left[\alpha - \rho(1-\alpha)k^\rho\right].$$

For $\rho > 0$ it has $|G(0)| < 1$ for $\beta > 1 - \frac{\alpha^{\frac{1}{\rho}}}{A}$ while for $\rho \in (-1, 0)$ it has $|G(0)| > 1$ for all parameter values. As far as k^* is concerned it has

$$G(k^*) = (1+\rho)\alpha[(1-\beta)A\alpha]^{\frac{-\rho}{1+\rho}} - \rho$$

Consider the case $\rho \in (-1, 0)$. The fixed point exists if $\beta > 1 - \frac{\alpha^{\frac{1}{\rho}}}{A}$.
It has $\lim_{\beta \to 1 - \frac{\alpha^{\frac{1}{\rho}}}{A}} G(k^*) = 1$, $\frac{\partial G(k^*)}{\partial \beta} < 0$, $\lim_{\beta \to 1} G(k^*) = -\rho \in (0, 1)$ therefore $|G(k^*)| < 1$ for all parameter values.
Consider the case $\rho > 0$. The fixed point exists if $\beta < 1 - \frac{\alpha^{\frac{1}{\rho}}}{A}$.
$\lim_{\beta \to 0} G(k^*) = (1+\rho)\left(\frac{\alpha}{A^\rho}\right)^{\frac{1}{1+\rho}} - \rho$, $\frac{\partial G(k^*)}{\partial \beta} > 0$, $\lim_{\beta \to 1 - \frac{\alpha^{\frac{1}{\rho}}}{A}} G(k^*) = 1$, therefore $|G(k^*)| < 1$ for $\rho \in (0, 1]$ or $\rho > 1 \wedge \beta > 1 - \left(\frac{1+\rho}{\rho-1}\right)^{\frac{1+\rho}{\rho}} \frac{\alpha^{\frac{1}{\rho}}}{A}$.
As far as the second inequality in (B.2) is concerned note that $\left|\frac{\partial \mathcal{P}'}{\partial p}\right| < 1$ for $R_f < 1$. We can conclude that

- Fixed point E_0 is stable if $R_f < 1$ and $\rho > 0 \wedge \beta > 1 - \frac{\alpha^{\frac{1}{\rho}}}{A}$;
- Fixed point E^*, when exists, is stable if $R_f < 1$ and one of the following holds true:
  - $\rho \leq 1$;
  - $\rho > 1 \wedge \beta > 1 - \left(\frac{1+\rho}{\rho-1}\right)^{\frac{1+\rho}{\rho}} \frac{\alpha^{\frac{1}{\rho}}}{A}$.