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RECENT AND NEW PERSPECTIVES IN INTERVAL ANALYSIS

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Abstract

Introduced in the 1950s as a method to deal with the uncertainty of errors, interval analysis, a kind of generalization of real analysis in which real intervals replace real numbers, despite the progress made by research, still has some crucial open problems, including the need to standardize theory through a robust and consistent framework for both analysis and algebra. Therefore, this work aims to pursue a dual objective. First, it intends to offer an updated state of the art on the concepts, problems and techniques of interval analysis, with a specific focus on some theoretical aspects and on the calculus of interval-valued functions of a single real variable. Through an intensive use of the so-called midpoint-radius representation, more advantageous than conventional notations, the possible types of partial orders in the space of compact real intervals are studied; the use of the gH-difference and gH-differentiability is also extremely useful, above all for the concepts related to the study of functions: limits, derivatives, monotonicity, as well as the analysis of extreme points, concavity and convexity. The various topics, revisited and enriched with innovative notations (e.g., a new representation of complex numbers), acquire a more complete meaning and new application possibilities open up (e.g., at the q-calculus). The other goal to aspire to is to deepen the investigation from an algebraic point of view, also through unconventional approaches. In particular, thanks to the introduction of a new partial order with polarity characteristics with respect to already acquainted orders, it is possible to determine hitherto unexplored algebraic structures: some quite known, such as semirings and pre-semirings, others more unusual, like the so-called combined structures. Moreover, from a study on the complementation properties, interval Boolean structures are also configured and, finally, the construction of an interval quotient set leads us towards even more solid structures, such as a pseudoring. The graphic representations, which constitute a fundamental part of the work, accompany the entire discussion, providing interesting and explanatory examples that ensure greater clarity and expository completeness.

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Introduction

In mathematics, there are real numbers, a real arithmetic for combining them, and a real analysis for studying their properties. Interval mathematics is a generalization in which real intervals replace real numbers, interval arithmetic replaces real arithmetic, and interval analysis (IA) replaces real analysis.

This method was introduced as an attempt to handle interval (non statistical, non probabilistic) uncertainty that appears in many mathematical or computer models of some deterministic real-world phenomena, even if there are already examples of its use during ancient times, just think of the method used by Archimedes for the determination of π.

Considering more recent times, the first monograph dealing with the interval analysis and which in a certain sense marks its birth, is the famous 1966 book by R. Moore ([64]), even if, among the early contributions in the interval-valued calculus, it is interesting to mention a paper by S. Markov ([56]) which remained essentially un-cited for more than 30 years and was "rediscovered" only after the publication of [78], [79] and [81]. In addition, taking into consideration the interval and fuzzy cases jointly, some relevant works in this area are given by [20], [51], [68] and [95], while other important contributions are found in papers on gH-differentiability (see, e.g., [7], [8], [15], [80], [82]) as well as those concerning interval and fuzzy optimization and decision making (e.g., [18], [29], [38], [69], [87] and [93]), while a recent generalization to the multidimensional convex case is proposed in [83].

In general we can say that, during the last fifty years, all the main mathematical areas have been more or less touched by this new approach, often assembling elements belonging to different contexts, which brought out both similarities and differences between the classical concepts and those based on the interval approach.

A relevant example is given by the creation of well-developed algebraic interval structures that, compared with classical theory, reveal new characteristics and original properties (e.g., non distributivity of operations, existence of several partial orders with different interpretations or emergence of multiple solutions to well-posed problems) which necessarily require revisiting the conventional background. Therefore, what is really missing and extremely indispensable is a unitary, complete and coherent scheme, where to place the various components. Likewise, crucial open problems are still present in the theory. For example, the concept of derivative of a real-valued function, one of the most important in classical real analysis, was expected to have a similar notion when considering interval-valued functions. However, such a concept, both theoretically well founded and also applicable to concrete situations, has only recently been introduced, despite almost half a century of (otherwise very important) research development.

Consequently, the motivations of this research are two-fold:

- first of all we intend to offer an updated state of art about the concepts, problems and the techniques in interval analysis, with a specific focus on its mathematical aspects recently addressed by research, especially those concerning the theoretical aspects of calculus in the setting of interval-valued functions of a single real variable;

- on the other hand, the aim is to contribute directly to the study of the aforementioned theoretical aspects, in particular by deepening the investigation from an algebraic point of view, also through approaches that go beyond the classical representations (entirely innovative and original approaches will be proposed, thanks to which it will be possible to endow the theory with a renewed and powerful algebraic framework hitherto absent in literature).

We will not explicitly address the issues in how to solve problems such as algebraic, differential, integral equations or others when intervals are involved. An account on these topics can be found in the very extended literature on interval analysis and related fields; see, e.g., [1], [56], [58], [64], [65] and the references therein.

Finally, it should be noted how recently the interest for this topic had increased significantly, in particular after the IEEE 1788–2015 Standard for Interval Arithmetic and the implementation of specific tools and classes in the C++, Julia (among others) programming languages, or in computational systems such as MATLAB, Mathematica, or in specific packages such as CORA 2016 (see [2]). In particular, the research activity in the calculus for interval-valued functions (of one or more variables) is now very extended, especially with regard to the more general calculus for fuzzy-valued functions (started in [71]), with applications to almost all fields of applied mathematics.

That said, in order to achieve the purposes stated above, it was decided to divide this work into two parts which are presented in a preparatory way to each other and which are in turn divided into three chapters each, all closely interconnected.

In the first part some recent results of the theory will be taken into consideration: these will be revisited, expanded and enriched with new meanings and details, as well as innovative application proposals.

In particular, in Chapter 1 we will give an overview on theory of interval analysis following the so-called classical approach, retracing the main steps that characterized its development, from the sporadic beginnings to the most recent theoretical and applicative results, to then dwell on the description of the salient aspects of the theory. In doing this, the two main types of interpretation will be taken into consideration: numerical and set. After introducing the basic aspects of the calculus and the main properties connected to them, we will move on to a quick overview of more elaborate structures, such as vectors, matrices and elements of complex calculus, after which we can finally introduce the concept of interval-valued functions with the main problems connected to them. All this will be of fundamental importance in order to address the particular themes that will be treated in this work as it will offer the right interpretative tools necessary for its correct understanding.

However, it is the following two chapters that will allow us to get to the heart of the theory. Here, in fact, mainly based on the results reported in two recent papers ([84] and [85]), new ideas and approaches regarding the interval analysis and calculus for interval-valued functions of a single real variable will be presented. By making intensive use of the so-called midpoint-radius representation, in Chapter 2 an innovative approach will be developed concerning the numerous possible types of partial orders in the space $\mathcal{K}_\mathcal{C}$ of compact real intervals. In this regard, the use of a comparative index recently proposed in the literature, as well as the concept of gH-difference, will be of fundamental importance.

Then, in Chapter 3, these concepts will be applied to the calculus of interval-valued functions for which the midpoint-radius notation will be adopted again, due to the numerous advantages it offers compared to more conventional notations. Concepts such as limits and continuity will also be introduced as well as those of gH-derivative and monotonicity, up to an in-depth study on the extremal points, concavity and convexity of interval-valued functions. Great importance will be given to graphical representations (mainly obtained through the use of software such as MATLAB) which will facilitate understanding of the theory and make the examples more explanatory. This also concerns the second part of the chapter in which, a new notation for representing complex intervals will be proposed, the peculiarities and advantages of which will be fully exploited also through an unprecedented graphic representation. Finally, an entire section will be dedicated to the presentation of a possible example of application of interval analysis to a topic, the q-calculus, which today is of great interest in the scientific community.

Nevertheless, the most innovative stage of the whole work is reached within the second part, where new algebraic structures are introduced, breaking the classical schemes so far proposed in literature and providing good ideas for applicative outlets, especially in the logic-computer field.

In fact, although over the years many authors have ventured into the search for algebraic systems within which to configure the interval theory, however, even today these structures have not been completely axiomatized. Therefore, in attempting to help fill this gap, in Chapter 4 some innovative approaches will be introduced towards the determination of interval algebraic systems. This will help define that solid and complete framework that the method required, implementing those more abstract and basic aspects that any mathematical theory needs. In particular, thanks to the introduction of a new partial order with polarity characteristics with respect to partial orders already known in literature, it will be possible to arrive at the notion of interval completion lattice and, above all, making use of some not obvious extensions of the space $\mathcal{K}_{\mathcal{C}}$, we will be able to outline algebraic structures hitherto unexplored in interval theory, some quite well-known, such as semirings, pre semirings or hemirings, others more unusual, ranging from lattice-ordered structures to the so-called clodum (Chapter 5).

Moreover, following a careful and original study on the complementation properties, through the use of interesting and rather extravagant models, in Chapter 6 we will be capable of configuring even Boolean interval-type structures (such as Boolean lattices, Boolean algebras and Boolean rings). Finally, the construction of the quotient set of $\mathcal{K}_{\mathcal{C}}$, obtained thanks to an ingenious definition of the equivalence relationship between intervals, will lead us to the determination of further structures, the highest point of which is represented by an example of interval quotient pseudoring.

The graphic representations, which constitute a fundamental part of the work, will accompany the entire discussion, providing interesting and explanatory examples; moreover, they will offer the possibility of approaching the discussion also through a visual register which, alongside the analytical one, will ensure a clearer and more complete exposition of the topics and their more precise location as mathematical objects within a complex general framework.

Part I

Recent perspectives in interval analysis

Chapter 1

Basic results

In this chapter we give an overview on theory of interval arithmetic and analysis. Definitions, notations and basic facts are introduced and briefly explained following the so-called classical approach, as defined in [64], [65], [17], [49] and [1].

After a concise introduction on the reasons that originated the interval analysis (Section 1.1), in Section 1.2 we will retrace the main steps that have characterized its historical evolution, from its origins to the present day, helping to determine its current structure as well how to outline future prospects. But it will be in Section 1.3 that we will finally enter the theory itself by describing the elements that characterize the interval method following the two principal approaches: axiomatic and set. Starting from the basic concepts, the main algebraic operations with their properties will be presented and then more complex structures, such as vectors and matrices, will be analyzed as well as the principal elements of complex interval calculus will be presented. Whereupon, after introducing the concept of interval-valued functions, some important well-known applications will be shown, to then conclude with a mention of the numerous alternative theories related to the interval method.

1.1 What is interval analysis

Interval analysis (also known as *interval arithmetic*, *interval mathematics* or *interval computation*), IA for short, was introduced as an attempt to handle interval (non statistical, non probabilistic) uncertainty that appears in many mathematical or computer models of some deterministic real-world phenomena. Therefore, it may be viewed as a kind of bridge between deterministic problem solving and problems with generalized uncertainty.

In particular this latter, being the quantitative estimation of errors in measured data, which is common to all scientific practice, can always be considered a serious danger to the search for a reliable scientific knowledge; for that reason IA, providing mathematical tools for controlling the whole range of errors in computation (rounding, truncation and input) improved its status and scientific reputation over the years, as one of the most suitable potential weapon against uncertainty in science and technology.

More specifically, IA arose exactly from the attempt to compute the error bounds of the numerical solutions of a finite state machine (or, more simply, a computer) for which the roundoff error was automatically accounted for by the computer itself. This led to the investigation of computations with intervals as the entity that enabled automatic error analysis.

Consequently, as active field of research and application, IA is relatively a new mathematical discipline, defined as a separate branch of study only at the end of the 50s, thanks to Ramon E. Moore's technical reports and his doctoral dissertation ([62] and [63]). Moreover, to better explain what IA really is and to introduce how it really emerged in mathematics, we can take inspiration by George W. Walster's article *Introduction to Interval Arithmetic* [91], as brilliantly suggested in [35].

Indeed, it was 1957 when Ramon Edgar Moore, an employee of Lockheed Missiles and Space Co. Inc., conceived interval arithmetic as an approach to bound rounding errors in mathematical computation, even thought was only forty years later, at April 19, 1997, during the meeting of Sun Microsystems' Interval Arithmetic University R & D program, that he explained his thinking. In particular he clarified that he was considering how scientists and engineers represent measurements and computed results through the notation

$$\widehat{x} \pm \widetilde{x},$$

where $\widehat{x}$ is the result (or measurement) and $\widetilde{x}$ is the error tolerance.

The problem was that, while representing fallible values by the $\widehat{x} \pm \widetilde{x}$ form is convenient, computing with it is not, even in a very simple case such as calculating the area of a square. Indeed, when the errors due to finite precision arithmetic are taken into account at the same time, then complexity increases further. Moreover, error analyses of large scientific, engineering and commercial algorithms are sufficiently complex that they are often not conducted and the result is that machine computing with floating-point arithmetic is not closely linked to mathematics, engineering, science or commerce.

So, Moore's idea was the following: since $\widehat{x} \pm \widetilde{x}$ consists of two numbers, $\widehat{x}$ and $\widetilde{x}$, why not use two different numbers to represent exactly the same information? This means that, instead of $\widehat{x} \pm \widetilde{x}$, it was possible to use $\widehat{x} + \widetilde{x}$ and $\widehat{x} - \widetilde{x}$, which define the extremes of an interval containing the exact quantity x. It was such a simple, yet brilliant, idea that allowed to start IA as a very important branch of applied mathematics.

1.2 Historical background

In this section we will give an overview of the main steps that have characterized the historical evolution of interval analysis, from its origins to the present day, thanks to which it was possible to elaborate its current structure and to outline future prospects.

1.2.1 Childhood of interval analysis

Even though interval method developed from the 1950s and 1960s, it was not a completely new phenomenon in mathematics as it has appeared several times under different names in the course of history. Indeed a famous and very old example of an interval enclosure is given by the method due to Archimedes where he considered inscribed and circumscribed polygons, each of n sides, of a circle with radius 1 to obtain an interval containing the number π. By choosing n large enough, an interval of arbitrary small width can be found in this way containing π.

Considering more recent times it can be argued that it was John Charles Burkill, in 1924, the first to deal with interval analysis, as in his article [12] he examines functions of intervals (which necessarily include arithmetic since the algebraic operations are functions); nevertheless, intervals as entities were not the focus of this work and we must wait a 1931 article by Rosalind Cicely Young, a doctoral candidate at the University of Cambridge, to finally see a first description of the rules for calculating with intervals and other subset of real numbers. In particular Young developed the arithmetic for sets of numbers and was interested in properties of limits. Moreover, she worked out the commutative, associative, and distributive law of scalars (real numbers) over intervals.

Then the method developed, particularly thanks to some apparently independently different works, as an approach to putting bounds on rounding and measurement errors in mathematical computation and thus developing numerical methods that yield reliable results.

Indeed one of the first references to interval arithmetic as a tool in numerical computing can already be found in a German work [28] (originally published in 1951) where the rules for the Interval Arithmetic (when only positive numbers are considered) are specified and applied to what today is called interval arithmetic evaluation of rational expressions.

During the same period Paul S. Dwyer (University of Michigan) particularized Young's work to compact sets of numbers (intervals). He discussed matrix computations using interval arithmetic in his book [21], a 1951 text on linear algebra where the interval method was introduced as an integral part of roundoff error analysis; likewise, five years later, in [92], Mieczyslaw Warmus suggested some formulas in order to provide a valid theoretical support to numerical computation with intervals.

But nevertheless, the most important paper for the development of IA has probably been published in 1958 by the Japanese scientist Teruo Sunaga [88]. Here not only the algebraic rules for the basic operations with intervals can be found but also a systematic investigation of the procedures which they carry out: the general principle of bounding the range of a rational function over an interval by using only the endpoints through interval arithmetic evaluation is already discussed; furthermore, interval vectors and corresponding operations are introduced (as multidimensional intervals). Sunaga also presented the idea of computing an improved enclosure for the zero of a real function by what is today called interval Newton method. Finally, he discussed how enclosing the value of a definite integral by bounding the remainder term through the tools of interval arithmetic and computing a pointwise enclosure for the solution of an initial value problem by remainder term enclosing.

We can adfirm that both, Warmus and Sunaga, had the full development of axiomatic interval arithmetic. Sunaga also recognized the importance of interval arithmetic in computational mathematics but did not proceed further whereas, in the same year, Patrick C. Fischer (see [23]) reported a computer program that uses two computer words that automate propagated and roundoff errors: one holds the approximate value of the variable while the other word holds the value representing the bound of previous computations and roundoff errors of the approximation. It was the first effort at implementation of IA to computers.

In any case, although written in English, these results did not find much attention until the first monograph dealing with interval analysis appeared: the celebrated book of the 1966 written by Ramon E. Moore [64], which marked the birth of modern interval arithmetic. Moore's book was derived from the research of his Ph.D. thesis and therefore was mainly concentrated on bounding solutions of initial value problems for ordinary differential equations although it contained a lot of general ideas.

Its merit was that starting with a simple principle, it provided a general method for automated error analysis, not just errors resulting from rounding: the same way classical arithmetic operates on real numbers, interval arithmetic defines a set of operations on intervals.

1.2.2 Popularization of IA

After the appearance of Moore's book several research groups from different countries started to investigate the theory and applications of IA and during the last decades the role of compact intervals as independent objects has continuously increased in numerical analysis, when verifying or enclosing solutions of various mathematical problems or when proving that such problems cannot have a solution in a particular given domain. This was possible by viewing intervals extensions of real or complex numbers, by introducing interval-valued functions and interval arithmetics and by applying appropriate fixed point theorems.

In the following twenty years German researchers carried out pioneering work around Götz Alefeld and Ulrich Kulisch at the University of Karlsruhe and later also at the Bergische University of Wuppertal. Among these, Karl Nickel explored more effective implementations, while Arnold Neumaier improved containment procedures for the solution set of systems of equations.

Important results were also achieved by Eldon R. Hansen, who dealt with interval extensions for linear equations and then provided crucial impacts to global optimisation, including what is now known as Hansen's method, perhaps the most widely used interval algorithm; on the other hand Helmut Ratschek and Jon George Rokne, using intervals to provide applications for continuous values, developed branch and bound methods which, until then, had only been applied to integer values. Finally, it will be Rudolf Lohner, in 1988, who will develop Fortran-based software for reliable solutions for initial value problems using ordinary differential equations.

Nevertheless, it is important to stress the fact that, in order to broaden interval modes to the various fields of mathematics, one of the main problems is that the classical techniques of analysis cannot be transferred one-to-one into interval-valued algorithms, as dependencies between numerical values are usually not taken into account; so an accurate work is required, often with the expense that we have to sacrifice some useful properties of ordinary arithmetic and analysis.

1.2.3 Areas of possible fruitful research

In spite of all the problems and disadvantages, nowadays IA is a well-established area not only providing mathematical and computational tools for modelling systems with uncertainties or controlling rounding errors in computations, but also the field of fuzzy sets method relied more and more on interval formulation (through the so-called α-cut approach). Indeed, we know that, apart from the strict statements $x \in X$ and $x \notin X$, intermediate values are also possible, to which real numbers are assigned. Hence, a distribution function could spread uncertainty, which can be understood as a further interval.

In addition, very interesting convergences can be found even in quantum structures, in the framework of the so-called unsharp approach to quantum theory, which proposes to describe some apparent mysteries of the quantum world as special cases of some general interval or fuzzy phenomena, whose behaviour has not yet been fully understood.

On the other hand also experimental and computational physics represent interesting application areas of IA as interval computations are used to handle the gathered uncertain data about observed physical phenomena and interval algorithms are used to solve problems arising from experimental and theoretical physics. Moreover, being hundred of times faster than Monte Carlo method, interval computations are succesfully applied also in electrical engineering and in control theory, and besides, we cannot forget to mention their use to verify computed numerical solutions in chaotic dynamical systems or their application to visualize strange attractors in discrete chaotic systems.

Other important sources of interval research originated from the work on reasoning with time intervals in economics where the notion of arrangement interval relations has been developed and extensively studied in particular for managing uncertainty in optimization problems and in decision management.

And again, throughout the last decades reliable computing, validated numerics and interval problems with differential equations have been discussed in several monographs and research papers and another major approach to a set of similar problems is that of differential inclusions and multivalued analysis, also able to deal with discontinuous dynamical systems which do not fully fit into the interval analysis topic.

We can say that interval arithmetic represents an elegant tool which has been used to solve an impressive array of problems. For instance, it was thanks to IA that, in 2002, Warwick Tucker (see [89]) was able to show that the Lorenz equations do possess a strange attractor, thus solving a long-outstanding problem: the Steve Smale's 14th conjecture (see [77]).

In conclusion, if in the early stages IA had to do with rounding, truncation and input errors of numerical computation, later it was realized that its potential went beyond simple calculation, being able to deal with problems that are inaccessible to conventional approaches, such as its use in a proof of Kepler's conjecture on the densest packing of spheres in space; in fact, using interval arithmetic, the problem was solved by Thomas C. Hales in 2000 (see [31] and [32]), thus allowing to finally find the definitive proof of a conjecture that for almost 400 yeard had been looking for a confirmation.

1.3 Elements of classical theory

In this section we finally enter the theory itself by describing the elements that characterize the interval method following its two main approaches: axiomatic and set.

Starting from the basic concepts, the principal algebraic operations with their properties will be presented and then more complex structures, such as vectors and matrices, will be analyzed as well as the principal elements of complex interval calculus will be described. Whereupon, after introducing the concept of interval-valued functions, some important well-known applications will be shown. We will conclude the section with a mention of the numerous alternative theories related to the interval method.

1.3.1 The two approaches

First of all, let recall what a closed real interval $[a, b]$ is. Indeed, although various other types of real intervals (open, half-open) appear throughout mathematics, our work will center primarily on closed intervals and in particular here the term interval will mean real closed interval.

So, an interval $[a, b]$ is defined as the set of all real numbers included between extremes a and b, denoted by

$$X = \{x \in \mathbb{R} \mid a \leqslant x \leqslant b\}$$

and belonging to the real number line with the usual meaning of the order relation $\leqslant$. But, as anticipated in Section 1.1, there is also another definition of an interval: as a pair of real numbers lying at the endpoints of the interval itself. Thus, we can say that, from the point of view of IA, intervals on the real line have a dual nature: as sets of real numbers and as a new kind of number represented by pairs of real numbers. In this second case it is possible to define a whole interval arithmetic, developed axiomatically and consistent with the set interpretation.

Consequently, IA can be mainly derived in two different ways which correspond to two different approaches.

1. The "number (or axiomatic) approach", where intervals are considerated as numbers with two components and where operations are defined axiomatically. In general the arithmetic associated with this approach is simple as it is expressed by operations which are at most four times more complex than the corresponding real numbers operations (for details see [49]). Nevertheless, the arithmetic obtained is not without some critical issues as it is subdistributive and it does not have inverse elements with respect to addition and multiplication; furthermore, as we will show later, the risk of obtaining a kind of overestimation is substantial and in order to reduce it the exponential complexity could extremely increase.

2. The “set (or extension principle) approach”, which derives from the so-called Moore’s united extension and considers intervals as sets. This method originates an arithmetic, known as constraint interval arithmetic, which, instead of operations, requires a procedure and is expressed through global optimization of the united extension function of the arithmetic operations. However, its algebraic structure is distributive and possesses both additive and multiplicative inverse, but is not so simple to be executed because of its global optimization nature.

Indeed, as described above, on the real line, with the usual meaning of the order relation $\leqslant$, an interval $[a, b]$ is the set of all real numbers $\{x \in \mathbb{R} : a \leqslant x \leqslant b\}$; on the other hand, as a number, an interval X consists of a pair of numbers $\{a, b\}$, which denote the left and right endpoints of the interval. Naturally, analysis on intervals, since intervals are sets, requires set-valued functions, limits, integration, and differentiation theory: all this is done via the so-called united extension (see [64]).

Proceding step by step we can say that the original formulation of the number approach was defined axiomatically in 1956 by Young, Dwayer, Warmus and then, independently, by Sunaga. It was only in a second moment that Moore rediscovers this approach and extends it to rounded interval arithmetic, allowing its utilization in computational mathematics.

Indeed, in his first works, Moore adopted the set outlook using a kind of extension principle for intervals, called united extension. Representing one way to define arithmetic on intervals, the extension principle is especially crucial because it defines how real-valued expressions are represented in the context of intervals and can be viewed as one of the main unifying concepts between interval analysis and fuzzy set theory. Moreover, it represents one way to define arithmetic on intervals.

Generally, we can say that an extension principle defines how to obtain functions whose domains are sets. Achieving this for real numbers is clear while it is more complex for sets because in this case well-defined entities must be defined.

W. Strother was the first, in his paper [86], to define the united extension for set-valued functions for domains possessing specific topological structures; thus, Moore did nothing but applied Strother’s united extension to intervals and in doing so, he retained the name united extension as the extension principle particularized to intervals. That is, Moore’s united extension (or the interval extension principle) consists on a set-valued function whose domain is the set of intervals and the range is an interval for those underlying functions that are continuous.

In particular, considering a real-valued function f of a single real variable x, Moore would like to know the precise range of values taken by $f(x)$ as x varies through a given interval X; this means that the image of X under the mapping f is

$$f(X) = \{f(x) : x \in X\}. \tag{1.1}$$

The next definition represents the first step.

Definition 1.3.1. *([64]) Let $f : M_1 \to M_2$ be a mapping between sets, and denote by $S(M_1)$ and $S(M_2)$ the families of subsets of M_1 and M_2, respectively. The united extension of f is the set-valued mapping $\bar{f} : \ S(M_1) \to S(M_2)$ such that*

$$\bar{f}(X) = \{f(x) : x \in X, \ X \in S(M_1)\}.$$

Note that

$$\bar{f}(X) = \bigcup_{x \in X} \{f(x)\},$$

that is, $\bar{f}(X)$ contains the same elements as the set image $f(X)$. For this reason it is usually apply the term united extension to set images such as the one described in (1.1).

Definition 1.3.2. *An interval arithmetic based on united extension principle is called extension interval arithmetic.*

However later, while implementating Sunaga's work, Moore applied the united extension for different pairs of arbitrary intervals X and Y, asserting that for all arithmetic functions $\circ \in \{+, -, \times, \div\}$, we have the following fact

$$X \circ Y = \{x \circ y \mid x \in X, \ y \in Y\}.$$

Hence, he abandons the united extension definition and develops axioms because of the simplification they lead of the operations since they need not account for multiple occurrences. Nevertheless, on the other hand, using the axioms create a kind of overestimation which represents a severe problem too. Therefore, despite the simplicity, Moore was aware from the beginning of the problems of overestimation associated with multiple occurrences of the same variable in an expression. Furthermore, it was also obvious that, from the axiomatic approach, $X - X$ is never 0 and that $X \div X$ is never 1 unless X is a real number.

Actually, in this kind of approach, interval arithmetic and associated semantics deal with intervals $[a, b]$, for which $a < b$, has some interesting applications to fuzzy arithmetic since the axioms of IA can be applied to each α-cut of a fuzzy set membership function.

In general the axioms of IA, as they were articulate by Warmus ([92]), Sunaga ([88]) and, in a second moment, by Moore ([62]), can be summarized as follows:

1. addition $[a, b] + [c, d] = [a + c, b + d]$;
2. subtraction $[a, b] - [c, d] = [a - d, b - c]$;

3. multiplication $[a, b] \times [c, d] = [min\{ac, ad, bc, bd\}, max\{ac, ad, bc, bd\}]$;
4. division $[a, b] \div [c, d] = [a, b] \times \left[\frac{1}{d}, \frac{1}{c}\right]$, where $0 \notin [c, d]$.

These axioms can be traced back to a single formulation.

Definition 1.3.3. *For all arithmetic functions $\circ \in \{+, -, \times, \div\}$ and for any two intervals X and Y, the binary algebraic operations are defined by*

$$X \circ Y = \{x \circ y \mid x \in X, y \in Y\}$$

where, in the case of division, $0 \notin Y$.

Similarly to Definition 1.3.2, we have the following one.

Definition 1.3.4. *An interval arithmetic based on axioms expounded by Definition 1.3.3 is called axiomatic interval arithmetic.*

As reported above, the basic axioms associated with interval arithmetic and their properties were more fully developed in 1966 by Moore (see [64]) and the difference from the axioms of real numbers inevitabily emerged such as subdistributivity.

It is interesting to say that the four axioms associated to the four operations essentially compute the maximum and minimum values of the set $\{x \circ y \mid \exists x \in X, \exists y \in Y\}$ and the two intervals X and Y are considered independent. As a result, axiomatic interval arithmetic is a type of min/max calculus since the values only involve the endpoints (min/max) of the intervals.

As well described in [49], we can say that the power of the axiomatic approach to IA lies especially in its simplicity of application since its complexity is at most four times that of real value arithmetic.

Conversely, this approach leads to overestimations in general because it takes every instantiation of the same variable independently and simple notions such as $X - X = 0$ and $X \div X = 1$, with $0 \notin X$, are desirable properties which can be maintained only if the united extension is used to define IA.

In addition it is interesting to mention that there is also a third method, which derived directly from the united extension rather than axiomatically and redefine intervals into a form equivalent to the graph of a function of one variable and two coefficients. The ensuing arithmetic is called "constraint interval arithmetic" and within it an interval $[a, b]$ is defined as the graph of the following real single-valued function

$$X(\lambda_x) = \lambda_x a + (1 - \lambda_x)b, \;\; 0 \leqslant \lambda_x \leqslant 1. \tag{1.2}$$

Basically, since in (1.2) the input numbers a and b are known, they are considered as coefficients, whereas λ_x is varying, although constrained between 0 and 1, hence the name constraint interval arithmetic. Note that $X(\lambda_x)$ defines a set representation explicitly, and the associated arithmetic is developed on sets of numbers. The algebraic operations are defined as follows:

$$X \circ Y = \{(\lambda_x a_x + (1-\lambda_x)b_x) \circ (\lambda_y a_y + (1-\lambda_y)b_y),\ 0 \leqslant \lambda_x \leqslant 1,\ 0 \leqslant \lambda_y \leqslant 1\},$$

where $a_x = minX$, $b_x = maxX$, and $\circ \in \{+, -, \times, \div\}$.

The result is that constraint interval arithmetic represents the complete implementation of the united extension, and it provides an algebra that possesses an additive inverse, a multiplicative inverse, and a distributive law. Moreover, one of its big benefits consists on its semantically correspondence with classical real arithmetic as any sentence of real arithmetic can be converted in another of constraint interval arithmetic which is semantically equivalent.

In general, as it will be well exposed in Subsection 1.3.7, we can say that over the years, various interval arithmetic approaches have been developed in addition to the axiomatic and united extension methods and also different representations of intervals have been invented with the common goal of simplifying operations and obtain more accurate results in arithmetic.

1.3.2 Definitions and basic concepts

According to the approaches described in Subsection 1.3.1, the basic algebraic operations for real numbers can be extended to intervals. In this section, we shall formulate the basic relations and algebraic procedures on them. The definitions follow the line drawn by most IA texts, such as [17] and [65].

In Subsection 1.3.1 we have already defined what a real interval is, adopting the convention of denoting intervals by capitol letters and their endpoints by lowercase letters. In particular, from here on, we establish that the left and right endpoints of a real interval A will be denoted by a^- and a^+ respectively.

Thus, the interval A can be defined as a number:

$$A = [a^-, a^+]$$

or as a set:

$$A = \{x \in \mathbb{R} \mid a^- \leqslant x \leqslant a^+\}.$$

Definition 1.3.5. *Let $a^-, a^+ \in \mathbb{R}$ be real numbers such that $a^- \leqslant a^+$. An interval A is the set of all real numbers included between extremes a^- and a^+, denoted by*

$$A = [a^-, a^+] = \{x \in \mathbb{R} \mid a^- \leqslant x \leqslant a^+\}, \tag{1.3}$$

where a^- and a^+ are called, respectively, the lower and upper bound (endpoints) of $[a^-, a^+]$.

In general, as well reported in [79], considering the more general case in which we have a metric vector space $\mathbb{V}$ with the induced topology and in particular the space $\mathbb{V} = \mathbb{R}^n$ $(n \geqslant 1)$ of real vectors equipped with standard addition and scalar multiplication operations, we denote by $\mathcal{K}(\mathbb{V})$ and $\mathcal{K}_{\mathcal{C}}(\mathbb{V})$ the spaces of nonempty compact and compact convex sets of $\mathbb{V}$ (see also [57] and [73] for more details).

In this thesis we will almost exclusively examine the particular case in which $n = 1$ and, from here on, the notations $\mathcal{K}(\mathbb{R})$ and $\mathcal{K}_{\mathcal{C}}(\mathbb{R})$ will be simply denoted by $\mathcal{K}$ and $\mathcal{K}_{\mathcal{C}}$.

Accordingly, we define by $\mathcal{K}_C$ the family of all bounded closed intervals in $\mathbb{R}$, i.e.,

$$\mathcal{K}_C = \left\{ \left[a^-, a^+\right] \mid a^-, a^+ \in \mathbb{R} \text{ and } a^- \leqslant a^+ \right\}. \tag{1.4}$$

Two intervals $A = [a^-, a^+]$ and $B = [b^-, b^+]$ are said to be equal if they are the same sets. Operationally, this happens if their corresponding endpoints are equal:

$$A = B \Leftrightarrow a^- = b^- \text{ and } a^+ = b^+. \tag{1.5}$$

Remark 1.3.1. *Note that the equality in $\mathcal{K}_C$ is defined in terms of equality in $\mathbb{R}$. This definition is a special case of the axiom of extensionality of axiomatic set theory (i.e., two sets are equal if and only if they have precisely the same elements) from the fact that every interval is an ordered set.*

An interval A is said to be symmetric if $a^- = -a^+$.

We say that an interval A is degenerate if $a^- = a^+$. Such an interval contains a single real number a and, by convention, we identify the degenerate interval with the real number itself:

$$[a, a] = a$$

In this sense, we may write $[0, 0] = 0$.

Furthermore, referring to the set-type operations, we can say that the intersection of two intervals A and B is empty if either $a^+ < b^-$ or $b^+ < a^-$. In this case A and B have no points in common and we write

$$A \cap B = \varnothing,$$

where $\varnothing$ denotes the empty set.

If $A \cap B \neq \varnothing$ we define the intersection between A and B as the interval

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\} = [max\{a^-, b^-\}, min\{a^+, b^+\}]. \tag{1.6}$$

Similarly, we define:

- the union between A and B, which is not necessary an interval, as

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}; \tag{1.7}$$

- the interval hull between A and B as

$$A \uplus B = [min\{a^-, b^-\}, max\{a^+, b^+\}]. \tag{1.8}$$

It is trivial to verify that, for any two intervals A and B, we have:

$$A \cup B \subseteq A \uplus B \tag{1.9}$$

where $A \cup B = A \uplus B \;\Leftrightarrow\; A \cap B \neq \varnothing$.

Other important terms which will be useful in the sequel consist of the following:

- the width of the interval A, defined by

$$w(A) = a^+ - a^-; \tag{1.10}$$

- the absolute value of the interval A, given by

$$|A| = max\{|a^-|, |a^+|\}; \tag{1.11}$$

- the midpoint of the interval A, denoted by

$$\widehat{a} = \frac{1}{2}(a^- + a^+). \tag{1.12}$$

Hence any symmetric interval has midpoint 0.

We also know that the real numbers are ordered by the relation $<$ which is transitive; so a kind of corresponding relation can be defined for intervals in $\mathcal{K}_C$:

$$A < B \Leftrightarrow a^+ < b^- \tag{1.13}$$

and transitive property still holds: if $A < B$ and $B < C$ then $A < C$.

It follows that we can define an interval $A = [a^-, a^+]$ as:

- positive if $a^- > 0$;

- negative if $a^+ < 0$.

This means that:

$$A > 0 \Leftrightarrow a > 0, \forall a \in A. \tag{1.14}$$

We observe that not every two elements of $\mathcal{K}_C$ can be compared by the relation $<$; indeed if $A \cap B \neq \varnothing$, according to (1.13), it is not possible to write neither $A < B$ nor $B < A$. That is, only a partial order (i.e., can only compare certain elements) is possible to be defined with respect to $<$, with the consequence that the order $<$ is strictly partial on $\mathcal{K}_C$.

Another transitive order relation for intervals is represented by set inclusion:

$$A \subseteq B \Leftrightarrow b^- \leqslant a^- \text{ and } a^+ \leqslant b^+. \tag{1.15}$$

Of course, also in this case we have a partial order as not every pair of intervals is comparable under set inclusion.

For instance, if A and B are overlapping intervals, then A is not contained in B, nor is B contained in A. However, $A \cap B$ is contained in both A and B.

1.3.3 Algebraic operations and properties of IA

The notion of the degenerate interval permits us to regard the system of closed intervals as an extension of the real number system. Indeed, there is an obvious one-to-one pairing

$$[a, a] \leftrightarrow a$$

between the elements of the two systems.

According to the Definition 1.3.3 and from the fact that intervals are ordered sets of real numbers, it is possible to find an operational way to implement binary and unary algebraic operations that, for practical applications, can be simplified further and be formulated in terms of the interval endpoints.

Therefore, for every two real intervals $A = [a^-, a^+]$ and $B = [b^-, b^+]$ and for every real number k, it is immediate to define the classic Minkowski operations as shown below:

- addition $A \oplus_M B = [a^- + b^-, a^+ + b^+]$;
- scalar multiplication $k \cdot A = \begin{cases} [k \cdot a^-, k \cdot a^+] \text{ if } k \geqslant 0 \\ [k \cdot a^+, k \cdot a^-] \text{ if } k < 0 \end{cases}$;
- negation $-A = -1 \cdot A = [-a^+, -a^-]$;
- subtraction $A \ominus_M B = A \oplus_M (-1) \cdot B = [a^- - b^+, a^+ - b^-]$;
- multiplication $A \otimes_M B = [min\{a^- \cdot b^-, a^- \cdot b^+, a^+ \cdot b^-, a^+ \cdot b^+\}, max\{a^- \cdot b^-, a^- \cdot b^+, a^+ \cdot b^-, a^+ \cdot b^+\}]$;
- reciprocal $A^{-1} = \dfrac{1}{A} = \left[min\left\{\dfrac{1}{a^-}, \dfrac{1}{a^+}\right\}, max\left\{\dfrac{1}{a^-}, \dfrac{1}{a^+}\right\}\right]$, providing $0 \notin A$;
- division $A \oslash_M B = \dfrac{A}{B} = A \otimes_M \dfrac{1}{B}$
  $= \left[min\left\{\dfrac{a^-}{b^-}, \dfrac{a^-}{b^+}, \dfrac{a^+}{b^-}, \dfrac{a^+}{b^+}\right\}, max\left\{\dfrac{a^-}{b^-}, \dfrac{a^-}{b^+}, \dfrac{a^+}{b^-}, \dfrac{a^+}{b^+}\right\}\right]$;

- square $A^2 = \begin{cases} [min\{(a^-)^2, (a^+)^2\}, max\{(a^-)^2, (a^+)^2\}] & \text{if } 0 \notin A \\ [0, max\{(a^-)^2, (a^+)^2\}] & \text{otherwise} \end{cases}$;

- square rooth $\sqrt{A} = \left[\sqrt{a^-}, \sqrt{a^+}\right]$, providing $a^- \geqslant 0$.

Generally, the subscript $(\cdot)_M$ used in the notation of Minkowski-type operations between intervals will be removed, and classical addition, subtraction, multiplication and division will be denoted by $\oplus$, $\ominus$, $\otimes$ and $\oslash$ respectively, but we will insert the subscript in cases where these operations are used in combination with others.

The definitions of the basic interval arithmetic operations lead to a number of familiar looking algebraic properties. By virtue of our definition of an interval, the properties of real numbers are obviously assumed. It is easy to verify (see [65]) that, with the axiomatic approach, some important properties still hold, as already anticipated in Subsection 1.3.1.

- Interval addition and multiplication are commutative and associative, as for any $A, B, C \in \mathcal{K_C}$:

$$A \oplus B = B \oplus A \text{ and } A \oplus (B \oplus C) = (A \oplus B) \oplus C;$$

$$A \otimes B = B \otimes A \text{ and } A \otimes (B \otimes C) = (A \otimes B) \otimes C.$$

- The degenerate intervals $0 = [0, 0]$ and $1 = [1, 1]$ are addittive and multiplicative identity elements in the system of intervals as, for any $A \in \mathcal{K_C}$:

$$0 \oplus A = A \oplus 0 = A \text{ and } 1 \otimes A = A \otimes 1 = A.$$

- The degenerate interval $0 = [0, 0]$ is an absorbing element for interval multiplication as, for any $A \in \mathcal{K_C}$:

$$0 \otimes A = A \otimes 0 = 0.$$

- Cancellation law holds for interval addition, as for any $A, B, C \in \mathcal{K_C}$:

$$A \oplus C = B \oplus C \Rightarrow A = B.$$

- Inclusion monotonicity of interval arithmetic holds:
  let A_1, A_2, B_1, and B_2 be elements of $\mathcal{K_C}$ such that $A_1 \subseteq B_1$ and $A_2 \subseteq B_2$, then we have

$$A_1 \oplus A_2 \subseteq B_1 \oplus B_2 \text{ and } A_1 \ominus A_2 \subseteq B_1 \ominus B_2.$$

  An immediate consequence is the following important special case:
  let A and B be intervals of $\mathcal{K_C}$ with $a \in A$ and $b \in B$, then we have

$$a + b \in A \oplus B \text{ and } a - b \in A \ominus B.$$

On the other hand it is fundamental to remark the fact that there is also a large number of properties, typical of the real numbers, which are lost or modified (see [65]), such as the following.

- Nonexistence of additive and multiplicative inverse elements, except for degenerate intervals:
  - (i) subtraction is not the inverse of addition as for any $A \in \mathcal{K}_{\mathcal{C}}$, it is $A \ominus A \neq 0$; indeed we have
  $$A \ominus A = [a^- - a^+, a^+ - a^-]$$
  which equals $0 = [0, 0]$ only if $a^- = a^+$;
  - (ii) similarly, division is not the inverse of multiplication as for any $A \in \mathcal{K}_{\mathcal{C}}$, it is $\frac{A}{A} \neq 1$; indeed we have that $\frac{A}{A} = 1$ only if $a^- = a^+$.
- Squaring is tighter than multiplication by itself since, from the definitions given above, it follows that
$$A^2 \subseteq A \otimes A.$$
- Multiplication is not distributive with respect to addition, since for any real intervals A, B, C, it is
$$A \otimes (B \oplus C) \neq (A \otimes B) \oplus (A \otimes C)$$
as you can easily verify with the following counterexample.

  Example 1.3.1. $A = [-1, 1], B = [-3, -2]$ *and* $C = [2, 3]$. *So we have* $A \otimes (B \oplus C) = [-1, 1] \otimes ([-3, -2] \oplus [2, 3]) = [-1, 1] \otimes [-1, 1] = [-1, 1]$; *while* $(A \otimes B) \oplus (A \otimes C) = ([-1, 1] \otimes [-3, -2]) \oplus ([-1, 1] \otimes [2, 3]) = [-3, 3] \oplus [-3, 3] = [-6, 6]$.

  Nevertheless, a sub-distributive law hods (see [65]):
$$A \otimes (B \oplus C) \subseteq (A \otimes B) \oplus (A \otimes C)$$
for any$A, B, C \in \mathcal{K}_{\mathcal{C}}$.
- Contrary to what happened with the interval addition, cancellation law fails for interval multiplication, that is, for any$A, B, C \in \mathcal{K}_{\mathcal{C}}$,
$$A \otimes B = A \otimes C \nRightarrow B = C.$$
An easy counterexample is obtained using the same intervals as the previous one.

Example 1.3.2. *$A = [-1, 1], B = [-3, -2]$ and $C = [2, 3]$.
Despite having that $A \otimes B = [-1, 1] \otimes [-3, -2] = [-3, 3]$ as well as $A \otimes C = ([-1, 1] \otimes [2, 3]) = [-3, 3]$; however B and C are different from each other.*

Finally, about the algebraic system of interval arithmetic, the following facts are valid (see [17]).

Proposition 1.3.1. *Let $(\mathcal{K}_\mathcal{C}, \circ)$ be the algebraic system of intervals. Then the two structures $(\mathcal{K}_\mathcal{C}, \oplus)$ and $(\mathcal{K}_\mathcal{C}, \otimes)$ are abelian monoids.*

Indeed, as we have seen above, interval addition and multiplication are associative and commutative in $\mathcal{K}_\mathcal{C}$ and the degenerate intervals 0 and 1 represent additive and multiplicative identity elements respectively.

So, the set $\mathcal{K}_\mathcal{C}$ forms an abelian monoid under both addition and multiplication (for a more complete list of relations associated with $\mathcal{K}_\mathcal{C}$ see [56]). However, there are two very important pieces of evidence that have risen in this section:

- additive and multiplicative inverses do not always exist for interval numbers;

- there is no distributivity between addition and multiplication except for certain special cases.

This clearly means that it is necessary sacrificing some useful properties of ordinary arithmetic but at the same time it allows to investigate other peculiarities typical of mathematical entities such as intervals.

Historically, the study and evolution of the algebraic structures and the topological properties were developed by Moore himself in [64] and [65] as well as foundations for validated methods for solutions of equations in [66]. Later, as the field expanded, one of the main applications of interval analysis is in validation methods which are computational process for solutions of equations in a mathematically valid way. This means that solutions are generated with guaranteed bounds. Then the implementation of interval arithmetic on the computer leads to rounded interval arithmetic and its resultant algebraic structure was derived in [46].

1.3.4 Vectors, matrices and systems of linear equations

We define an n-dimensional interval vector as

$$\mathbf{A} = (A_1, A_2, ..., A_n) \text{ where } A_i = [a_i^-, a_i^+] \text{ for any } i = 1, ..., n.$$

In particular we have that:

- a 2-dimensional interval vector $\mathbf{A} = ([a_1^-, a_1^+], [a_2^-, a_2^+])$ can be represented by a rectangle in the plane x, y:

$$\mathbf{A} = \{(a_1, a_2) | a_1^- \leqslant a_1 \leqslant a_1^+ \text{ and } a_2^- \leqslant a_2 \leqslant a_2^+\};$$

- a 3-dimensional interval vector $\mathbf{A} = (A_1, A_2, A_3)$ can be represented by a box in the space x, y, z;

- an n-dimensional interval vector $\mathbf{A} = (A_1, A_2, ..., A_n)$ can be thought of as an n-dimensional "box" in the space $x_1, x_2, ...x_n$.

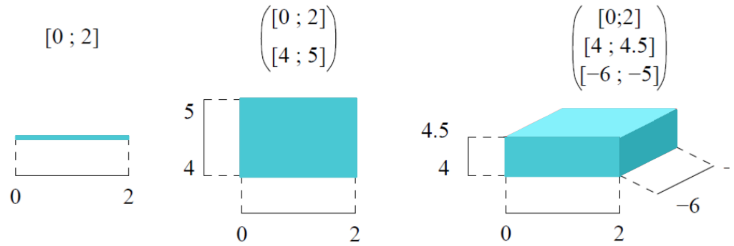


Figure 1.1: Examples of interval vectors in dimension 1, 2 and 3.

It follows that, with suitable modifications, many of the notions for ordinary intervals can be extended to interval vectors:

1. If $\mathbf{a} = (a_1, ..., a_n)$ is a real vector and $\mathbf{A} = (A_1, ..., A_n)$ is an interval vector, then we write:

   $\mathbf{a} \in \mathbf{A} \quad \Leftrightarrow \quad a_i \in A_i$ for all $i = 1, ..., n$.

2. The intersection of two interval vectors is empty if and only if the intersection of any of their corresponding components is empty; that is: $\mathbf{A} \cap \mathbf{B} = \varnothing \quad \Leftrightarrow \quad A_i \cap B_i = \varnothing$ for any i.
   Otherwise, for $\mathbf{A} = (A_1, ..., A_n)$ and $\mathbf{B} = (B_1, ..., B_n)$ we have that: $\mathbf{A} \cap \mathbf{B} = (A_1 \cap B_1, ..., A_n \cap B_n)$ which is an interval vector too.

3. If $\mathbf{A} = (A_1, ..., A_n)$ and $\mathbf{B} = (B_1, ..., B_n)$ are interval vectors, then we have that: $\mathbf{A} \subseteq \mathbf{B} \quad \Leftrightarrow \quad A_i \subseteq B_i$ for all $i = 1, ..., n$.

We also recall some important definitions below. If $\mathbf{A} = (A_1, ..., A_n)$ is an interval vector, then:

- the width of $\mathbf{A}$ is the largest of the widths of any of its component intervals:

$$w(\mathbf{A}) = \max_i w(A_i), \quad i = 1, ..., n;$$

- the midpoint of $\mathbf{A}$ is denoted by

$$\widehat{\mathbf{a}} = (\widehat{a}_1, ..., \widehat{a}_n),$$

where $\widehat{a}_i = \frac{1}{2}(a_i^- + a_i^+)$ is the midpoint of the interval vector element A_i, for $i = 1, ..., n$;

- the norm of $\mathbf{A}$ is given by

$$\|\mathbf{A}\| = \max_i |A_i|, \quad i = 1, ..., n,$$

which is a kind of generalization of absolute value.

In a completely analogous way to what has been done for vectors it is possible to define an $(n \times m)$-interval matrix as

$$[\mathbf{A}] = (A_{ij})_{i:1,...,n;j:1,...,m} \text{ where } A_{ij} = [a_{ij}^-, a_{ij}^+].$$

Also in this case is possible to extend to interval matrices some of the notions for ordinary intervals, so if $[\mathbf{A}] = (A_{ij})_{i:1,...,n;j:1,...,m}$ in an interval matrix, then:

- the width of $[\mathbf{A}]$ is the largest of the widths of any of its component intervals: $w([\mathbf{A}]) = \max_{ij} w(A_{ij})$;
- the midpoint of $[\mathbf{A}]$ is denoted by the real matrix

$$[\widehat{\mathbf{A}}] = (\widehat{a}_{ij})_{i:1,...,n;j:1,...,m},$$

where $\widehat{a}_{ij} = \frac{1}{2}(a_{ij}^- + a_{ij}^+)$ is the midpoint of the interval matrix element A_{ij}, for $i = 1, ..., n; j = 1, ..., m$;

- the norm of $[\mathbf{A}]$ is given by

$$\|[\mathbf{A}]\| = \max_i \sum_j |A_{ij}|, \quad i = 1, ..., n; j = 1, ..., m,$$

which is a kind of interval extension of the maximum row sum norm for real matrices.
Note that if $[\mathbf{B}]$ is any real matrix contained in an interval matrix $[\mathbf{A}]$ (so that $[\mathbf{B}] \subseteq [\mathbf{A}]$), then $\|\mathbf{B}\| \leqslant \|[\mathbf{A}]\|$.

The importance of interval vectors and matrices arises in particular when we have to deal with linear interval systems which consist of a matrix interval extension $[\mathbf{A}] = (A_{ij})_{i:1,...,n;j:1,...,m}$ and an interval vector $\mathbf{B} = (B_1, B_2, ..., B_n)$.

As described in details in [50], the problem we address is the extension to intervals of the usual real-valued (also called crisp) system of linear equations problem: $\mathbf{A}\cdot\mathbf{x} = \mathbf{b}$.

What we want is to find the smallest cuboid $\mathbf{X} = (X_1, X_2, ..., X_m)$, for all vectors $\mathbf{x} = (x_1, x_2, ..., x_m)$ for which there is a pair $(\mathbf{A}, \mathbf{b})$ with $\mathbf{A} \in [\mathbf{A}]$ and $\mathbf{b} \in \mathbf{B}$ satisfying $\mathbf{A}\cdot\mathbf{x} = \mathbf{b}$.

In case of quadratic systems, i.e., when $n = m$, it could be possible to find an interval vector X, which covers all possible solutions, using the interval-valued Gauss method which represents the interval version of the linear algebra method known as Gaussian elimination (see [65]). In any case, using the interval entities $[\mathbf{A}]$ and $\mathbf{B}$ repeatedly in the calculation, this procedure could produce poor results which only provides first rough estimates of some problems. Indeed, even if the result contains the entire solution set, it also has a large area outside it. So, a rough solution $\mathbf{X}$ can often be improved by an interval version of the Gauss–Seidel method, an alternate strategy commonly used in practice which is extensively described in [65].

1.3.5 Elements of complex IA

The classical interval arithmetic theory can be extended, via complex interval numbers, to determine regions of uncertainty in computing with complex numbers. Indeed, as an interval can also be defined as a set of points at a given distance from the centre, it is easy to extend this definition from real numbers to complex numbers. So, let define what a complex interval is, according to [17].

Definition 1.3.6. *Let A_Z and B_Z be real intervals. A complex interval $\mathbf{Z}$ is the set of all ordinary complex number $a_z + ib_z$, for all $a_z \in A_Z$ and $b_z \in B_Z$, that is:*

$$\mathbf{Z} = A_Z + iB_Z = \{\mathbf{z} = a_z + ib_z \mid a_z \in A_Z,\ b_z \in B_Z\}$$

where A_Z and B_Z are, respectively, the interval real part and the interval imaginary part of $\mathbf{Z}$ while $i = [i, i]$ is the interval imaginary unity.

Geometrically, a complex interval may be concieved as a 2-dimensional interval vector

$$\mathbf{Z} = (A_Z, B_Z)$$

where A is the real interval element and B is the imaginary interval element. Then the complex interval is represented by a rectangle in the complex plane, that is a kind of rectangle of certainty.

In the complex case we denote by $\mathcal{K}_{\mathcal{C}}(\mathbb{C})$ the set of complex intervals.

Note that, as an interval is a real closed interval and a complex number is an ordered pair of real numbers, there is no reason to limit the application of interval arithmetic to the measure of uncertainties in computations with real numbers. Therefore, the basic algebraic operations for real intervals can be extended to complex numbers and it is not surprising that complex interval arithmetic is similar to the ordinary complex one.

Considering a complex interval $\mathbf{Z} = A_Z + iB_Z = (A_Z, B_Z)$, is interesting to remark the following notational conventions:

- if $B_Z = [0, 0]$ then $\mathbf{Z}$ is a pure real interval;

- if $A_Z = [0, 0]$ then $\mathbf{Z}$ is a pure imaginary interval;

- if $0 \notin A_Z^2 + B_Z^2$ then $\mathbf{Z}$ is a nonzero complex interval;

- if $A_Z = [a_z, a_z]$ and $B_Z = [b_z, b_z]$ (so they are both real point interval), then $\mathbf{Z}$ is nothing more than a point complex interval. It follows that every element $[a_z, a_z] + i[b_z, b_z]$ is an isomorphic copy of an element $a_z + ib_z \in \mathbb{C}$. By convention, we agree to identify them.

We also define the conjugate of a complex interval $\mathbf{Z} = A_Z + iB_Z = (A_Z, B_Z)$ as

$$\bar{\mathbf{Z}} = A_Z - iB_Z = (A_Z, -B_Z).$$

Similarly to ordinary complex numbers, also complex intervals cannot be ordered with respect to inequality relation $<$, while equality relation for two complex intervals $\mathbf{X} = (A_X, B_X) = A_X + iB_X$ and $\mathbf{Y} = (A_Y, B_Y) = A_Y + iB_Y$, is defined in the following way:

$$A_X + iB_X = A_Y + iB_Y \Leftrightarrow A_X = A_Y \text{ and } B_X = B_Y.$$

In general complex interval arithmetic can be defined in terms of real interval arithmetic in a way which is similar to how ordinary complex arithmetic is defined in terms of the real one; therefore, using procedures similar to those seen in the case of real intervals, it is possible to implement binary and unary operations even in the case of complex intervals.

It follows that, for any two complex intervals $\mathbf{X} = (A_X, B_X) = A_X + iB_X$ and $\mathbf{Y} = (A_Y, B_Y) = A_Y + iB_Y$ and for every real number k, the following facts are immediate (see [17]):

- addition $\mathbf{X} \oplus \mathbf{Y} = (A_X \oplus A_Y, B_X \oplus B_Y) = (A_X \oplus A_Y) \oplus i(B_X \oplus B_Y)$;

- scalar multiplication $k \cdot \mathbf{X} = (k \cdot A_X, k \cdot B_X)$;

- negation $-\mathbf{X} = (-A_X, -B_X)$;

- subtraction $\mathbf{X} \ominus \mathbf{Y} = \mathbf{X} \oplus (-1) \cdot \mathbf{Y}$;

- multiplication $\mathbf{X} \otimes \mathbf{Y} = (A_X \otimes A_Y \ominus B_X \otimes B_Y) + i(A_X \otimes B_Y \oplus B_X \otimes A_Y)$;

- reciprocal $\mathbf{X}^{-1} = \dfrac{1}{\mathbf{X}} = \dfrac{A_X \ominus iB_X}{A_X^2 \oplus B_X^2} = \dfrac{\bar{\mathbf{X}}}{A_X^2 \oplus B_X^2}$ if $\mathbf{X}$ is a nonzero complex interval;
- division $\dfrac{\mathbf{X}}{\mathbf{Y}} = \mathbf{X} \otimes \left(\dfrac{1}{\mathbf{Y}}\right)$ if $\mathbf{Y}$ is a nonzero complex interval.

Other important definitions are the following.

1. The complex width of a complex interval $\mathbf{X} = A_X + iB_X$, where $A_X = [a_x^-, a_x^+]$ and $B_X = [b_x^-, b_x^+]$, is defined by

$$w(\mathbf{X}) = w(A_X) + iw(B_X) = (a_x^+ - a_x^-) + i(b_x^+ - b_x^-).$$

2. The complex midpoint of a complex interval $\mathbf{X} = A_X + iB_X$, where $A_X = [a_x^-, a_x^+]$ and $B_X = [b_x^-, b_x^+]$, is denoted by

$$\widehat{\mathbf{X}} = \widehat{a}_x + i\widehat{b}_x = \frac{1}{2}(a_x^- + a_x^+) + \frac{i}{2}(b_x^- + b_x^+).$$

We remark the fact that for a complex interval $\mathbf{X} = A_X + iB_X$ with $B_X = [0, 0]$, the operations for complex interval are reduced to the corresponding operations for real intervals.

In addition, according to [17], it can also be shown that, as happens in the real interval case, there is no distributivity between addition and multiplication of complex intervals except for certain special cases, and inverse elements do not always exist.

But in complex interval arithmetic there are two other useful properties of ordinary complex arithmetic which fail. Indeed the additive and multiplicative properties of ordinary complex conjugates do not hold for complex interval conjugates.

Finally, it should be remarked the fact that interval arithmetic can be also extended, in an similar way, to other multidimensional number systems such as quaternions and octonions, but with the compromise that other useful properties of ordinary arithmetic have to be sacrificed.

1.3.6 Basic notions of interval-valued functions

The application of functions to intervals is strictly connected to the extension principle used by Moore in [65]. Interval arithmetic can be used to define the bounds of the image of a continuous function f, defined on a closed interval $X = [x^-, x^+]$ as its preimage. Indeed, since intervals are the connected subsets of $\mathbb{R}$, their image obtained by a continuous function is also an interval. In particular we can say that the range of values of a real-valued continuous function f of a single real variable $x \in X$ is contained in an interval $Y = [y^-, y^+]$, that is $f(X) \subseteq Y$.

This kind of interval exists since f is a continuous function and X is a compact set, which means that f reaches a minimum and maximum and all the values between them.

Things are particularly easy when we deal with monotonic functions, either increasing or decreasing. As shown in Figure 1.2, an increasing function send an interval $X = [x^-, x^+]$ into the interval $f(X) = [f(x^-), f(x^+)]$.

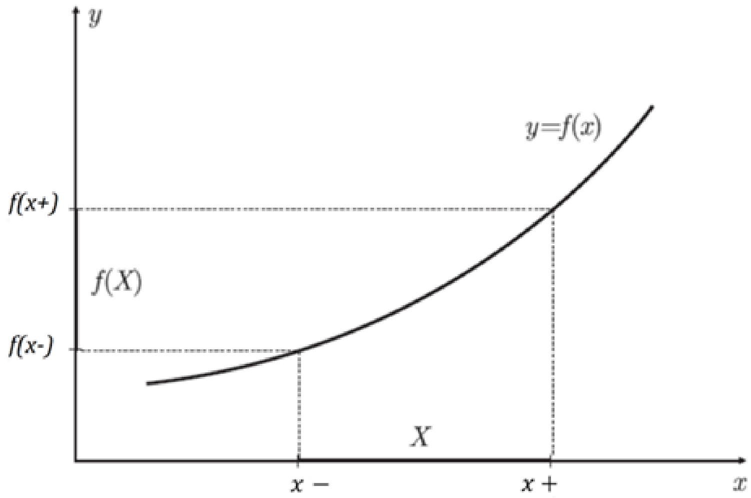


Figure 1.2: Example of interval extension of a monotonic function.

Consequently, when we have to find the interval extension of monotonic functions, we only need to calculate the value of the endpoints of the interval $X = [x^-, x^+]$, as the image of interval itself is:

$$f(X) = f([x^-, x^+]) = [min\{f(x^-), f(x^+)\}, max\{f(x^-), f(x^+)\}].$$

All this means that the following basic features for interval-valued functions $f(X)$ with $X = [x^-, x^+]$ can be easily defined:

- power function $f(x) = x^2,\ x \in \mathbb{R}$

$$\Rightarrow f(X) = \{x^2 | x \in X\} = \begin{cases} [(x^-)^2, (x^+)^2] & \text{if } \ 0 \leqslant x^- \leqslant x^+ \\ [(x^+)^2, (x^-)^2] & \text{if } \ x^- \leqslant x^+ < 0 \\ [0, max\{(x^-)^2, (x^+)^2\}] & \text{if } \ x^- \leqslant 0 \leqslant x^+ \end{cases}$$

or in the more general case $f(x) = x^n,\ x \in \mathbb{R}$,
for $n \in \mathbb{N}$ even

$$\Rightarrow f(X) = \{x^n | x \in X\} = \begin{cases} [(x^-)^n, (x^+)^n] & \text{if } \ 0 \leqslant x^- \leqslant x^+ \\ [(x^+)^n, (x^-)^n] & \text{if } \ x^- \leqslant x^+ < 0 \\ [0, max\{(x^-)^n, (x^+)^n\}] & \text{if } x^- \leqslant 0 \leqslant x^+ \end{cases}$$

while in case of $n \in \mathbb{N}$ odd

$$f(X) = \{x^n | x \in X\} = [(x^-)^n, (x^+)^n];$$

- exponential function $f(x) = e^x,\ x \in \mathbb{R}$

$$\Rightarrow f(X) = \{e^x | x \in X\} = [e^{x^-}, e^{x^+}]$$

or in the more general case $f(x) = a^x,\ x \in \mathbb{R}$

$$\Rightarrow f(X) = \{a^x | x \in X\} = \begin{cases} [a^{x^-}, a^{x^+}] & \text{if} \quad a > 1 \\ [a^{x^+}, a^{x^-}] & \text{if} \quad 0 < a < 1 \end{cases};$$

- logarithmic function $f(x) = ln(x),\ x \in \mathbb{R}^+$

$$\Rightarrow f(X) = \{ln(x) | x \in X\} = [ln(x^-), ln(x^+)] \text{ if } x^- > 0$$

or in the more general case $f(x) = log_a(x),\ x \in \mathbb{R}^+$

$$\Rightarrow f(X) = \{log_a(x) | x \in X\} = \begin{cases} [log_a(x^-), log_a(x^+)] & \text{if} \quad a > 1 \\ [log_a(x^+), log_a(x^-)] & \text{if} \quad 0 < a < 1 \end{cases};$$

- square root function $f(x) = \sqrt{x},\ x \geqslant 0$

$$\Rightarrow f(X) = \{\sqrt{x} | x \in X\} = [\sqrt{x^-}, \sqrt{x^+}] \text{ if } x^- \geqslant 0;$$

- sine function $f(x) = sin(x),\ x \in \mathbb{R}$
(as it is not monotonic, we consider its restriction to the set $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ where the function is increasing)

$$\Rightarrow f(X) = \left\{sin(x) | x \in X \subseteq \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right\} = \left[sin(x^-), sin(x^+)\right],$$

cosine function is similar, considering $X \subseteq [0, \pi]$.

Actually, dealing with monotonic functions, it is easy to verify that the result is exactly the same we have obtained considering the endpoints of the interval.

More generally, it is sufficient to consider the endpoints of the interval X, paying attention to the so-called critical points within the interval being those points where the monotonicity of the function changes direction. In this way we have been able to define a few interval-valued functions by selecting a real-valued function f and computing the range of $f(x)$ with x varied through some interval X: the result is defined to be the set image $f(X)$.

But it is also possible to use another process which consists in extending a real-valued function f by applying its formula directly to interval arguments.

Definition 1.3.7. *Let f be a continuous real-valued function of a single real variable x, that is*

$$f : \mathbb{R} \longrightarrow \mathbb{R},$$

$$x \longmapsto f(x)$$

and let F be a function which sends interval X to interval $F(X)$

$$F : \mathcal{K}_{\mathcal{C}} \longrightarrow \mathcal{K}_{\mathcal{C}},$$

$$X \longmapsto F(X).$$

We say that F is an interval extension of f if, for degenerate interval arguments $[x, x]$, we have that F corresponds to f, that is,

$$F([x, x]) = f(x).$$

Following the approach described in [65], we first consider the monotonic case and take an increasing (or decreasing) function

$$f : \mathbb{R} \longrightarrow \mathbb{R} \text{ such that } \forall x_1, x_2 \in X \mid x_1 \leqslant x_2 \Rightarrow f(x_1) \leqslant (\geqslant) f(x_2).$$

In order to better explain, let us start with the following example.

Example 1.3.3. *Consider the continuous real-valued function f given by*

$$f(x) = 1 - x, \ \textit{with}\ x \in \mathbb{R} \ \textit{and}\ X = [x^-, x^+] \ \textit{with}\ x^- < x^+.$$

Note that a function is defined by two things: a domain and a rule. In that case they are both specified as the elements of the domain are reals numbers and the mapping rule is: $x \longmapsto 1 - x$. Note also that, taken in isolation, the entity $f(x) = 1 - x$ is a formula, not a function. Often this distinction is ignored and we tend to interpret it as a function whose domain should be taken as the largest possible set over which the formula makes sense (in this case, all of $\mathbb{R}$). Nevertheless, to the definition of f, the domain is just as essential as the formula.

Now we take the formula $f(x) = 1 - x$ that describes the function above and apply it to interval arguments. We obtain the following interval-valued function:

$$F(X) = 1 - X \ \textit{with}\ X = [x^-, x^+],$$

which is an extension of the initial real-valued function. What we have done is to enlarged the domain in order to include nondegenerate intervals X as well as the degenerate intervals $[x, x] = x$.

According to the laws of interval arithmetic, we calculate the extension of f by applying its formula to interval X:

$$F(X) = 1 - X = [1, 1] - [x^-, x^+] = [1, 1] + [(-x)^+, (-x)^-] = [1 - x^+, 1 - x^-].$$

On the other hand, as x increases through the interval $[x^-, x^+]$, the values of $f(x)$ decrease through this interval from $1-x^-$ to $1-x^+$ as

$$f(x^-) = 1 - x^- > 1 - x^+ = f(x^+).$$

Then, by definition we have:

$$f(X) = f([x^-, x^+]) = [min\{f(x^-), f(x^+)\}, max\{f(x^-), f(x^+)\}]$$

$$= [1 - x^+, 1 - x^-]$$

in other words, we obtain: $F(X) = f(X)$ *with* $f(X) = \{f(x)|x \in X\}$. *Therefore we have found the united extension of* $f : f(X) = 1 - X$.

Things change when we are dealing with general functions as shown in the Example 1.3.4.

Example 1.3.4. *Consider the following real-valued function*

$$f(x) = x^2 - x + 1 \text{ where } x \in [-2, 1] = [x^-, x^+] = X.$$

We know that in ordinary real arithmetic, it is possible to rewrite the function in different ways, such as: $x^2 - x + 1 = x(x-1) + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$. *Hence we obtain three real-valued functions, mathematically equal, which can be defined as:*

$f(x) = x^2 - x + 1, \quad g(x) = x(x-1) + 1 \quad$ *and* $\; h(x) = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$.
Now, let form corresponding interval extension to $X = [x^-, x^+]$*:*
$F(X) = X^2 - X + 1, \quad G(X) = X(X-1) + 1 \quad$ *and* $\; H(X) = \left(X - \frac{1}{2}\right)^2 + \frac{3}{4}$.
Applying these formulas to the given interval $X = [x^-, x^+] = [-2, 1]$ *and using the algebraic rules defined in Subsection 1.3.3, we obtain the following results:*

$$F([-2,1]) = [-2,1]^2 - [-2,1] + [1,1] = [0,4] + [-1,2] + [1,1] = [0,7];$$

$$G([-2,1]) = [-2,1]\cdot([-2,1] - [1,1]) + [1,1] = [-2,1]\cdot[-3,0] + [1,1] = [-3,6] + [1,1] = [-2,7];$$

$$H([-2,1]) = \left([-2,1] - \left[\frac{1}{2}, \frac{1}{2}\right]\right)^2 + \left[\frac{3}{4}, \frac{3}{4}\right] = \left[-\frac{5}{2}, \frac{1}{2}\right]^2 + \left[\frac{3}{4}, \frac{3}{4}\right] = \left[0, \frac{25}{4}\right] + \left[\frac{3}{4}, \frac{3}{4}\right] = \left[\frac{3}{4}, 7\right].$$

In conclusion we get three different solutions even thought these expressions would be equivalent in ordinary arithmetic. Indeed, considering

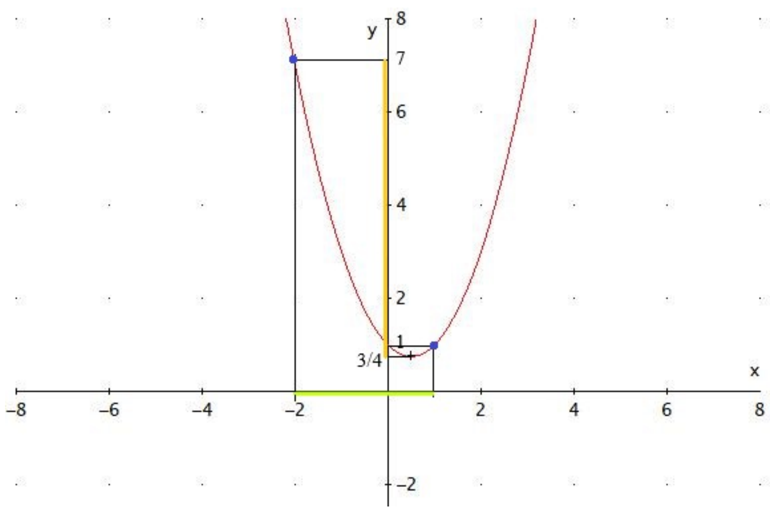


Figure 1.3: The real-valued function: $f(x) = x^2 - x + 1$.

$X = [-2, 1]$, *we obtain that, as* x *increases from* -2 *to* 1, *the tree maps* F, G *and* H *decrease from* 7 *to* $\frac{3}{4}$, *then increase back to* 1 *as shown in Figure 1.3.*

As a result we have that

$$f([-2, 1]) = g([-2, 1]) = h([-2, 1]) = \left[\frac{3}{4}, 7\right].$$

On the other hand, considering interval-valued extensions of f, g *and* h, *we have just obtain that*

$$H([-2, 1]) = \left[\frac{3}{4}, 7\right] \subseteq F([-2, 1]) = [0, 7]) \subseteq G([-2, 1] = [-2, 7]$$

It is evident that neither of the two functions F *and* G *map the interval* $[-2, 1]$ *into* $\left[\frac{3}{4}, 7\right]$ *and this is due to the lack of distributivity and additive and multiplicative symmetric in interval arithmetic. It is evident that the united extension of the original function* f *results from the third equivalent formula* h *as* $H([-2, 1]) = \left[\frac{3}{4}, 7\right]$

This means that the different formulas generate different extensions but not all can be considered the united extension.

So it is clear that the quality of the interval arithmetic evaluation as an enclosure of the range of f over an interval X is strongly dependent on how the expression for $f(x)$ is written. This represents one of the main obstacle to the application of interval arithmetic; indeed if an interval occurs several times in a calculation and each occurrence is taken independently then this fact can lead to an expansion of the resulting intervals.

Actually the exact range of values could be achieved if each variable appears only once and if function f is continuous inside the box but, unfortunately, not every function can be written in this way. Therefore, there is the effective risk of an over-estimation (also called *interval dependency problem*) of the value range which could prevent more significant conclusions.

Nevertheless, as described in [35], interval analysis does not suffer from any restriction to a particular class of functions that it can be applied to and this is thanked to the fundamental theorem of interval arithmetic.

Theorem 1.3.1 (Moore's fundamental theorem [65])**.** *For any bounded real-valued function f defined by an arithmetical expression*

$$f : \mathbb{R} \longrightarrow \mathbb{R},\ x \longmapsto f(x),$$

the corresponding interval-valued extension F

$$F : \mathcal{K}_\mathcal{C} \longrightarrow \mathcal{K}_\mathcal{C},\ X \longmapsto F(X)$$

is an inclusion function of f, that is, for any compact interval $X \in \mathcal{K}_\mathcal{C}$, the following inclusion applies:

$$F(X) \supseteq f(X) = \{f(x)|x \in X\}. \tag{1.16}$$

Note that, unlike $F(X)$, in general $f(X)$ does not necessarily have to be an interval.
Furthermore, until now we have limited the processes to functions of a single interval variable X but there is no reason to avoid more general functions. Therefore we can consider a function depending on n interval variables.

Definition 1.3.8. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function, then the function $F : \mathcal{K}_\mathcal{C}{}^n \longrightarrow \mathcal{K}_\mathcal{C}$ is an interval extension of f if*

$$F(X_1, ..., X_n) \supseteq f(X_1, ..., X_n) = \{f(x_1, ..., x_n)|x_i \in X_i, i := 1, ..., n\}.$$

By an interval extension of f, we mean an interval-valued function F of n interval variables $X_1, ..., X_n$ such that for real arguments $x_1, ..., x_n$ we have

$$F(x_1, ..., x_n) = f(x_1, ..., x_n).$$

Remark 1.3.2. *As shown in Example 1.3.4, we obtained extensions F of real rational functions f by replacing:*

- *the real variable x with an interval variable X;*

- *the real arithmetic operations with corresponding interval operations.*

The result F is called the natural interval extension of f.

The same procedure can be performed with functions of n variables and the most important observation we made can be repeated even with general functions: two rational expressions which are equivalent in real arithmetic may not be equivalent in interval arithmetic.

However, the inclusion property provides a robust rejection test, i.e.,

$$0 \notin F(X) \Rightarrow 0 \notin f(X).$$

This also means that, given a function f and a bounding box B defined as a product of n intervals, we have a very simple experiment to prove that the box B does not intersect the image (or surface) of f: $0 \notin F(B) \Rightarrow 0 \notin f(B)$.

As described in [35] and [36], "point sampling" fails as a rejection test on non-monotonic intervals.

It is important to say that while many methods exist for isolating monotonic regions, inclusion methods using interval can be considered as the most general and strong as they evaluate an inclusion extension of the implicit function (see Figure 1.4) and use that for spatial rejection or evaluating monotonicity. In particular when the function f is non-monotonic on an interval I, in order to assure a convex hull $CH(I)$ over the range, calculating the lower and upper components of a domain interval could be not sufficient. Of course, things are different with an inclusion extension F of f, which include all minima and maxima of the function within that interval. Note that these can be used for any computable function, but require implementation of an inclusion arithmetic library.

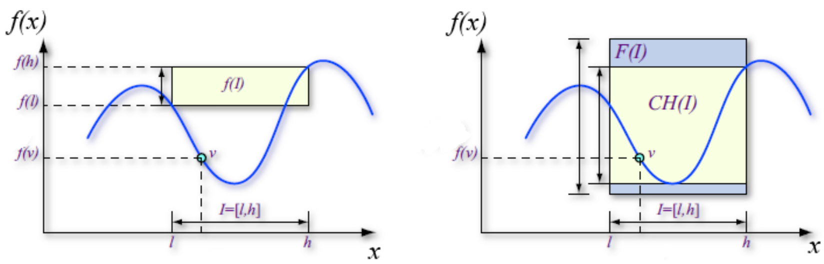


Figure 1.4: Example of the inclusion property when the function is non-monotonic. Left: calculation through lower and upper component of the domain interval. Right: inclusion extension which includes all minima and maxima of the function. (Figure reproduced from [36]).

Thus, just as IA can be used to find bounds on the image of a continuous function defined on a closed interval as its preimage, so interval methods are applied to a wide range of mathematical concepts as well as important calculation estimates, such as bounding the error term in Taylor's series or in evaluating definitive integrals (as well explained in [17]); however, it is important to remember that the methods of classical numerical analysis cannot be transferred one-to-one into interval-valued algorithms, as dependencies between numerical values are usually not considered.

This is a general rule that must be taken into account especially when dealing with central concepts of analysis such as convergence, continuity or differentiability (which will be dealt with in details in Chapters 2 and 3). In particular, since the definitions of such concepts depend on having an adequate way to express distance, it is advisable first of all to provide an appropriate notion of interval metric as a measure of the "distance" between the objects.

Therefore, in order to introduce continuity and convergence in the context of interval analysis, we define a measure of distance between two real intervals $X = [x^-, x^+]$ and $Y = [y^-, y^+]$ as

$$d(X, Y) = max\{|x^- - y^-|, |x^+ - y^+|\}, \tag{1.17}$$

from which we have the following definition.

Definition 1.3.9. *Let $\{X_k\}$ be a sequence of intervals. We say that $\{X_k\}$ is convergent if there exists an interval X^* such that, for every $\varepsilon > 0$, there is a natural number N_ε, such that $d(X_k, X^*) < \varepsilon, \forall k > N$.*
As in the case of real sequences, we write

$$X^* = \lim_{k\to\infty} (X_k) \ \ or \ \ X_k \to X^*$$

and refer to X^* as the limit of $\{X_k\}$.

To conclude this section we must remember that the real interval system represents an extension of the real number system. In fact, as reported in [65], the correspondence $[x, x] \longleftrightarrow x$ can be considered as a mapping which preserves distances:

$$d([x, x], [y, y]) = max\{|x - y|, |x - y|\} = |x - y|, \forall x, y.$$

For this reason, it is called an isometry, and it follow that the real line is isometrically embedded in the metric space of intervals.

1.3.7 Alternative theories

As anticipated in Subsection 1.3.1 and according to [17] and [49], there are some alternative theories of IA which were mainly introduced to extend the algebraic structure associated with intervals or to decrease the dependency effect.

In general, different approaches have been developed in addition to the axiomatic and united extension (the *constrain interval arithmetic* is an example of these) and various representations of intervals have been created too, with the intention of make operations simpler and define more precise results. Among these last, we mentioned in particular the midpoint-error form, as we will extensively use it in the following. Originally called *range arithmetic*, this notation was first developed by Oliver Aberth in 1988 and represents an interval X (also called range number) as

$$X = \widehat{x} \pm \widetilde{x} = [x^-, x^+]$$

where $\widehat{x} = \dfrac{x^- + x^+}{2}$ is the midpoint and $\widetilde{x} = \dfrac{x^+ - x^-}{2}$ represents the error (or radius) of the interval.

Instead, regarding alternative theories, it is important to remark that one of the most important problem about interval arithmetic was to extend it to unbounded intervals that may be entered or result from a division by 0: this is what the *extended interval arithmetic*, proposed by Kahan in 1968, aimed to do, trying to incorporate plus and minus infinity as endpoints of intervals; successively, also the space of improper intervals, with the so-called *directed interval arithmetic*, was developed, including a particular kind of intervals with negative width, known as nonregular intervals, which are used to complete the set of real intervals to its closure.

Later, in order to solve the problem of overestimation that characterizes the axiomatic approach, the so-called *generalized interval arithmetic* was introduced. In this particular approach an interval is represented by

$$X = [x^-, x^+] = y \pm [-k, +k], \ k \geqslant 0$$

so that $X = [y-k, y+k]$ and the peculiarity of the arithmetic associated with it is the reduction of the effects of dependencies. More recent generalizations of this approach are represented by the *affine arithmetic* and the *Taylor model arithmetic*: the first minimized the effect of overestimation defining an interval $X = [x^-, x^+]$ by its affine representation $x = x_0 + x_1\epsilon_1 + ... + x_n\epsilon_n$, so that

$$X = [x_0 - \xi, x_0 + \xi],$$

where $\xi = \sum_{i=1}^{n} |x_i|$, while the Taylor model, which represents one of the most succesful approach to deal with dependancy problem, is a method to do arithmetic on functions as it is able to provide enclosures of any function by a Taylor polynomial.

Furthermore, during the same years, *triplex arithmetic* and its probabilistic generalization, *quantile arithmetic*, were introduced to carry more information about the uncertanty of the bounds which are denoted by endpoints. In particular a triplex interval is symbolized as $X = [x^-, \widehat{x}, x^+], \ x^- \leqslant \ \widehat{x} \leqslant \ x^+$, where $\widehat{x}$ is called the main value and the arithmetic associated with, expecially the quantile one, approximates distributions whose support is an interval.

For the three-point quantile arithmetic, the distribution af a variable X, is:

$$f_X(x) = \begin{cases} \alpha Y & \text{if } x = x_1, \text{where } P(X \leqslant x_1) = \alpha \\ 1 - 2\alpha & \text{if } x = x_2, \text{where } P(X \leqslant x_2) = \dfrac{1}{2} \\ \alpha Y & \text{if } x = x_3, \text{where } P(X \leqslant x_3) = 1 - \alpha \\ 0 & \text{otherwise} \end{cases}$$

where the parameter $0 \leqslant \alpha \leqslant \frac{1}{2}$ is generally fixed a priori. We obtain that $x^- \leqslant x_1 \leqslant x_2 \leqslant x_3 \leqslant x^+$, where the support of the distribution is $[x^-, x^+]$ and x_1, x_2, x_3 represent the α-quantile, the median and the $(1-\alpha)$-quantile of the continuous variable X. It can be demostrated that 0 and 1 represent the additive and multiplicative identities and that quantile arithmetic is commutative but not associative and not even subdistributive; however, if $F(X)$ is the quantile expression of the rational expression $f(x)$, the following property holds: $f(x) \subseteq F(X)$.

Finally, we cite two more types of interval approach: the *ellipsoid arithmetic* developed by Arnold Neumaier in 1993, which is based on approximating enclosing affine trasformations of ellipsoids that are in turn enclosed in an ellipsoid, and the *variable precision arithmetic*, developed in order to bound solutions to problems which require more precision than guaranteed by the common floating-point arithmetic.

We terminate this collection merely pointing out a very important method, constructed in 1985 by Ernest Gardenyes, which can be considered a structural, algebraic and logic completion of the classical intervals as it deals with a basic problem of the classical theory: the lost of semantic of quantification over real variables. Indeed, the so-called *modal interval analysis* (MIA for short) provides a set of semantically equivalent interval sentences to a bigger subset of the corresponding real sentences.

As well explained in [17], [34] and [75], what MIA intends to do is to define a modal interval by associating a quantifier to a classical interval, and to introduce the fundamental relationship of inclusion between modal intervals by including it among the sets of predicates they accept. So, a modal interval consists in a classical interval, which defines its domain, and a quantifier, which defines its modality. We can say that, just as a real number x can be represented by the pair consisting of the absolute value and the sign $(|x|, \pm)$, in the same way a modal interval can be identified as a pair of a set teoretical interval and a logic quantifier $([x^-, x^+], *)$, where $* \in \{\forall, \exists\}$. So a modal interval represents the set of all true sentences with respect to the quantification $*x \in [x^-, x^+]$.
To better explain this we can consider the following two sentences:

$$\forall x \in [x^-, x^+],\ \forall y \in [y^-, y^+],\ \ \exists z \in \mathbb{R} \mid z = x \circ y,$$

$$\forall x \in [x^-, x^+],\ \exists y \in [y^-, y^+],\ \ \exists z \in \mathbb{R} \mid z = x \circ y,$$

where $\circ \in \{+, \times\}$.

We know that, according to the classical interval theory, these two sentences can be translated only in one single way, that is,

$$\exists Z \in \mathcal{K}_{\mathcal{C}} \mid Z = [x^-, x^+] \circ [x^-, x^+]$$

which, from a semantic point of view, is equivalent to the first sentence. Therefore, in classical theory, the meaning of the second sentence is lost as the semantics of the existential quantifier is not included in the set definition of the operations; on the other hand, being MIA based on predicate logic and set theory, it can easily overcome the problem.

The result is that we obtain a theory where, despite its complicated construction, additive and multiplicative inverse exist; however, even this approach is not without its problems as, differently from the theory of constraint intervals, it is not possible to translate every sentence of real arithmetic into a semantically equivalent modal sentence without loosing dependency information.

Chapter 2

Orders and representations for intervals

The contents presented in this chapter is inspired in particular by the results of the first part of a recent work consisting of two distinct and interrelated papers ([84] and [85]), concerning interval analysis and the calculus for interval-valued functions of a single real variable.

Starting with a recently proposed comparison index, we develop an innovative general setting for partial orders in the space $\mathcal{K}_{\mathcal{C}}$ of compact real intervals. We adopt extensively the midpoint-radius representation of intervals in the real half-plane and show its usefulness in calculus. However, the contents of this chapter have been expanded and enriched with various new elements, which offer innovative and interesting interpretative ideas.

More specifically, the basic properties of the space of real intervals are described in Section 2.1, while Section 2.2 introduces several partial orders for intervals, discusses their properties in terms of the midpoint representation and in relation to lattice theory. The fundamental role of gH-difference in characterizing the partial orders is also shown and there are numerous references to graphical representations relating to this notation.

2.1 The metric space $\mathcal{K}_C$ of real intervals

In this section we introduce the basic properties of the space of real intervals and, as already shown in Subsection 1.3.3, we focus on the fact that in interval arithmetic the standard Minkowski addition and multiplication are not invertible operations. However, the need to determine the inverse elements of these operations has proved to be extremely important as they are fundamental within the whole interval analysis, with particular reference to important applications such as the solution of equations, the concepts of differentiability, of interval integral, differential equations and so on.

Therefore, the attempt to find such inverse elements has always been one of the main objectives of the interval analysis and a good example of this is represented by the operations introduced by Hukuhara in [37], which we briefly present in Subsection 2.1.1.

2.1.1 The gH-operations in $\mathcal{K}_C$

As defined in (1.4), we denote by $\mathcal{K}_C$ the family of all bounded closed intervals in $\mathbb{R}$.

To describe and represent basic concepts and operations for real intervals, the well-known *midpoint-radius* (or simply *midpoint*) representation is very useful: for a given interval $A = [a^-, a^+]$, let us define the midpoint $\widehat{a}$ and radius $\widetilde{a}$, respectively, by

$$\widehat{a} = \frac{a^+ + a^-}{2} \text{ and } \widetilde{a} = \frac{a^+ - a^-}{2},$$

so that $a^- = \widehat{a} - \widetilde{a}$ and $a^+ = \widehat{a} + \widetilde{a}$. It follows that, using midpoint notation, interval $A = [a^-, a^+]$ can also be denoted by $A = (\widehat{a}; \widetilde{a})$; therefore, we can redefine:

$$\mathcal{K}_C = \{(\widehat{a}; \widetilde{a}) \mid \widehat{a}, \widetilde{a} \in \mathbb{R} \text{ and } \widetilde{a} \geqslant 0\}.$$

Given $A = [a^-, a^+],\ B = [b^-, b^+] \in \mathcal{K}_C$ and $\tau \in \mathbb{R}$, we have the following classical (Minkowski-type) addition, scalar multiplication and difference:

- $A \oplus_M B = [a^- + b^-, a^+ + b^+]$,
- $\tau A = \{\tau a : a \in A\} = \begin{cases} [\tau a^-, \tau a^+], & \text{if } \tau \geqslant 0 \\ [\tau a^+, \tau a^-], & \text{if } \tau < 0, \end{cases}$
- $-A = (-1)A = [-a^+, -a^-]$,
- $A \ominus_M B = A \oplus_M (-1)B = [a^- - b^+, a^+ - b^-]$.

Switching to midpoint notation, we get that:

$$A \oplus_M B = \left(\frac{(a^+ + b^+) + (a^- + b^-)}{2}; \frac{(a^+ + b^+) - (a^- + b^-)}{2} \right)$$

$$= \left(\frac{a^+ + a^-}{2} + \frac{b^+ + b^-}{2}; \frac{a^+ - a^-}{2} + \frac{b^+ - b^-}{2}\right) = \left(\widehat{a} + \widehat{b}; \widetilde{a} + \widetilde{b}\right).$$

Proceeding in a similar way also in the other cases, we obtain that, using midpoint notation, the previous operations, for $A = (\widehat{a}; \widetilde{a})$, $B = (\widehat{b}; \widetilde{b})$ and $\tau \in \mathbb{R}$ are given by:

- $A \oplus_M B = (\widehat{a} + \widehat{b}; \widetilde{a} + \widetilde{b})$,
- $\tau A = (\tau\widehat{a}; |\tau|\widetilde{a})$,
- $-A = (-\widehat{a}; \widetilde{a})$,
- $A \ominus_M B = (\widehat{a} - \widehat{b}; \widetilde{a} + \widetilde{b})$.

Generally, as already mentioned in Subsection 1.3.3, the subscript $(\cdot)_M$ in the notation of Minkowski-type operations will be removed, and classical addition and subtraction will be denoted by $\oplus$ and $\ominus$, respectively, but we will insert the subscript in cases where these operations are used in combination with other ones.

Therefore, as well explained in [79] and partially anticipated in Subsection 1.3.3, given two subsets A and B and a real number τ, it is well known that addition $\oplus$ is associative, commutative and with neutral element 0; hereafter 0 will also denote the singleton $\{0\}$. However, the opposite $-A$ of A, obtained thanks to scalar multiplication when $\tau = -1$, is such that

$$A \oplus (-A) = (\widehat{a}; \widetilde{a}) \oplus (-\widehat{a}; \widetilde{a}) = (\widehat{a} - \widehat{a}; \widetilde{a} + \widetilde{a}) = (0; 2\widetilde{a}) \neq (0; 0) = 0,$$

i.e., the opposite of A is not its inverse in Minkowski addition (unless $A = (\widehat{a}; 0) = \{\widehat{a}\}$ is a singleton).

A first implication of this fact is that, in general, even if it is true that $(A \oplus C = B \oplus C) \Leftrightarrow A = B$, addition/subtraction simplification is not valid, i.e., $(A \oplus B) \ominus B \neq A$.

In order to overcome this situation, the following H-difference was introduced by Hukuhara in [37]:

$$A \ominus_H B = C \Leftrightarrow A = B \oplus C (= B \oplus_M C).$$

An important property of $\ominus_H$ is that

$$A \ominus_H A = 0, \forall A \in \mathcal{K_C} \text{ and } (A \oplus B) \ominus_H B = A, \forall A, B \in \mathcal{K_C};$$

we have that H-difference is unique, i.e., if there exists an interval C such that $C \oplus B = A$, then if C exists it is unique (see [71]) and we call it the Hukuhara difference of A and B (H-difference for short): $A \ominus_H B$.

Nevertheless, for $A \ominus_H B$ to exist a further condition is necessary: A must contain a translate $\{c\} + B$ of B. In general we have that $A \ominus B \neq A \ominus_H B$ and from an algebraic point of view, the difference of two sets may be interpreted in terms of addition (as we have seen above) or in terms of negative addition:

$$A \ominus_H B = C \Leftrightarrow B = A \oplus (-1C)(= A \ominus_M C)$$

where $(-1)C$ is the opposite set of C. These two conditions are compatible to each other and suggest a further generalization of Hukuhara difference. Therefore, we denote the generalized Hukuhara difference (gH-difference for short) of two intervals A and B as:

$$A \ominus_{gH} B = C \Longleftrightarrow \begin{cases} (i) \quad A = B \oplus_M C, \\ or \\ (ii) \quad B = A \ominus_M C. \end{cases} \tag{2.1}$$

It is easy to show that (i) and (ii) of (2.1) are both valid if and only if $C = (\widehat{c}; 0) = \{\widehat{c}\}$ is a singleton.

The gH-difference of two intervals always exists and is equal to

$$\begin{aligned} A \ominus_{gH} B &= [\min\{a^- - b^-, a^+ - b^+\}, \max\{a^- - b^-, a^+ - b^+\}] \\ &= (\widehat{a} - \widehat{b}; |\widetilde{a} - \widetilde{b}|) \subseteq A \ominus_M B. \end{aligned}$$

In a similar way we can define the gH-addition for intervals as

$$A \oplus_{gH} B = A \ominus_{gH} (-B) \tag{2.2}$$

so that we get

$$\begin{aligned} A \oplus_{gH} B &= A \ominus_{gH} (-B) \\ &= [\min\{a^- + b^+, a^+ + b^-\}, \max\{a^- + b^+, a^+ + b^-\}] \\ &= (\widehat{a} + \widehat{b}; |\widetilde{a} - \widetilde{b}|) \subseteq A \oplus_M B. \end{aligned}$$

In conclusion we have that the Minkowski addition $\oplus$ is associative and commutative and with neutral element $0 = \{0\}$ but, as already mentioned, in general in Minkowski addition the opposite of A is not the inverse of A (unless $A = \{a\}$ is a singleton) and an important implication of this fact is that additive simplification is not valid, i.e., $(A \oplus B) \ominus B \neq A$.

Conversely, considering the gH-difference, we always have

$$A \ominus_{gH} A = 0 \text{ and } (A \oplus_M B) \ominus_{gH} B = A, \forall A, B \in \mathcal{K}_C$$

(and other properties that will be given in the following, when needed).

Note also that:

- $\alpha A \ominus_M \beta A = (\alpha + \beta)A$ only if $\alpha\beta \geqslant 0$ (except for trivial cases),
- $A \ominus_{gH} B = A \ominus_M B$ or $A \oplus_{gH} B = A \oplus_M B$ only if A or B are singletons.

Remark 2.1.1. *The introduction of two additions $\oplus_M$, $\oplus_{gH}$ and two differences $\ominus_M$, $\ominus_{gH}$ for intervals is not motivated here as an attempt to define a "true" arithmetic in $\mathcal{K}_C$; for example, $\oplus_M$ and $\oplus_{gH}$ are both commutative with neutral element 0, but only $\oplus_M$ is associative. The four operations are each-other strongly related and their properties motivate the (appropriate) use of them in the context of interval analysis and calculus.*

It is possible to repeat a similar procedure as regards the multiplication and division operations: indeed the gH-difference can be used to introduce a division of real intervals.

In Subsection 1.3.3 we have seen how Minkowski multiplication, reciprocal and division are defined. In particular we noticed that the multiplicative inverse (it is not the inverse in the algebraic sense) of an interval $B = [b^-, b^+]$ with $b^- > 0$ or $b^+ < 0$ (i.e. $0 \notin B$), is defined by

$$B^{-1} = \left[\frac{1}{b^+}, \frac{1}{b^-}\right].$$

So, we can denote the generalized Hukuhara division (gH-division for short) of two intervals A and B as follows:

$$A \oslash_{gH} B = C \Longleftrightarrow \begin{cases} (i) \quad A = B \otimes_M C, \\ or \\ (ii) \quad B = A \oslash_M C = A \otimes_M C^{-1}. \end{cases} \tag{2.3}$$

If both cases (i) and (ii) of (2.3) are valid, we have:
$C \otimes_M C^{-1} = C^{-1} \otimes_M C = \{1\}$, i.e., $C = \{\widehat{c}\}$ and $C^{-1} = \left\{\frac{1}{\widehat{c}}\right\}$ with $\widehat{c} \neq 0$, that is, C is a singleton.
It is immediate to see that $A \oslash_{gH} B$ always exists and is unique for given $A = [a^-, a^+],\ B = [b^-, b^+]$ with $0 \notin B$.

Remark 2.1.2. *If $0 \in]b^-, b^+[$, the gH-division is undefined, while for intervals $B = [0, b^+]$ or $B = [b^-, 0]$ the division is possible but we obtain unbounded results C which have the infinitive form $C =]-\infty, c^+]$ or $C = [c^-, +\infty[$: we can work with $B = [\varepsilon, b^+]$ or with $B = [b^-, -\varepsilon]$ and we obtain the result by the limit for $\varepsilon \to 0^+$.*

Finally, as was done for addition, even in the case of multiplication it is possible to give a generalized Hukuhara version, using the gH-division operation that we have just introduced.

So we define the generalized Hukuhara multiplication (gH-multiplication for short) as follows:

$$A \otimes_{gH} B = \begin{cases} \{0\} & \text{if } 0 \in A, 0 \in B, \\ A \oslash_{gH} B^{-1} & \text{if } 0 \notin B, \\ B \oslash_{gH} A^{-1} & \text{if } 0 \notin A. \end{cases} \tag{2.4}$$

Note that, in case $0 \notin A$ and $0 \notin B$, we have:

$$A \oslash_{gH} B^{-1} = B \oslash_{gH} A^{-1}.$$

2.1.2 The interval metric space

One of the things we are interested in is discussing the notion of continuity and convergence in the context of interval analysis but, as we know, for this it is first necessary to define a suitable metric.

In this regard, let us briefly recall that, assuming S is any set and that a real-valued function d is defined such that for any two elements $x, y \in S$ the following statements hold:

1 $d(x, y) = 0$ if and only if $x = y$,

2 $d(x, y) = d(y, x)$,

3 $d(x, y) \leqslant d(x, z) + d(z, y)$ for any $z \in S$,

the function d is a *metric* on S which, in turn, is called *metric space.*

It is well known that the three conditions listed above can be considered as the essential characteristics of the distance between objects of S; therefore, what we really need now is to define a distance between intervals in $\mathcal{K}_{\mathcal{C}}$. For this purpose we will use the well-known Pompeiu–Hausdorff distance. In this regard let us remember that, in general, if A and B are two closed and bounded sets and $a \in A$, then the distance between the point a and the set B is given by

$$d(a, B) = \min\{d(a, b) : b \in B\} = \min_{b \in B} |a - b|$$

where $d(a, b)$ is the (Euclidean) distance between the points a and b.

So we can give the following definition.

Definition 2.1.1. *Considering two intervals $A, B \in \mathcal{K}_C$, the Pompeiu–Hausdorff distance $d_H : \mathcal{K}_C \times \mathcal{K}_C \to \mathbb{R}^+ \cup \{0\}$ is defined by*

$$d_H(A, B) = \max\left\{\max_{a \in A} d(a, B), \max_{b \in B} d(b, A)\right\}$$

with $d(a, B) = \min_{b \in B} |a - b|$.

The following properties are well known:

$$\begin{aligned} d_H(\tau A, \tau B) &= |\tau| d_H(A, B), \forall \tau \in \mathbb{R}, \\ d_H(A \oplus C, B \oplus C) &= d_H(A, B), \\ d_H(A \oplus B, C \oplus D) &\leqslant d_H(A, C) + d_H(B, D). \end{aligned}$$

It is also known (see [79, 82]) that

$$d_H(A, B) = \|A \ominus_{gH} B\| \tag{2.5}$$

where, for $C \in \mathcal{K}_C$, the quantity

$$\|C\| = \max\{|c|\,; c \in C\} = d_H(C, \{0\})$$

is called the *magnitude* of C.

Furthermore, an immediate property of the gH-difference for $A, B \in \mathcal{K}_C$ is

$$d_H(A, B) = 0 \iff A \ominus_{gH} B = 0 \iff A = B. \tag{2.6}$$

Then, we have that all the conditions necessary to define a metric are satisfied; therefore, we have that $(\mathcal{K}_C, d_H)$ is a metric space.

In addition, as stated in [7] (Theorem 8.5), [19] and [48] (Proposition 1.3.1), it is well known that $(\mathcal{K}_C, d_H)$ is a complete metric space. Indeed, the concepts of a convergent sequence of intervals $(A_n)_{n\in\mathbb{N}}$, $A_n \in \mathcal{K}_C$ can be considered in the classical sense in the metric space $\mathcal{K}_C$, endowed with the d_H distance.

Definition 2.1.2. *We say that* $\lim_{n\to\infty} A_n = A$ *if and only if, for any real* $\varepsilon > 0$*, there exists an* $n_\varepsilon \in \mathbb{N}$ *such that* $d_H(A_n, A) < \varepsilon$ *for all* $n > n_\varepsilon$*.*

Consequently, the following equivalence is always true, as it is a trivial application of (2.6):

$$\lim_{n\to\infty} A_n = A \text{ if and only if } \lim_{n\to\infty} (A_n \ominus_{gH} A) = 0. \tag{2.7}$$

2.1.3 Innovative graphical representations for intervals

Before concluding Section 2.1, we briefly propose further innovative graphical representations for the set of real intervals, thanks to which it is possible to interpret an interval as a point with all the advantages that this entails.

In Subsection 2.1.1 two different notations have been introduced to describe the intervals $A \in \mathcal{K}_\mathcal{C}$:

a) endpoint notation: $A = [a^-, a^+]$

b) midpoint-radius notation: $A = (\widehat{a}; \widetilde{a})$

where $\widehat{a} = \frac{1}{2}(a^- + a^+)$ and $\widetilde{a} = \frac{1}{2}(a^+ - a^-)$, so that $a^- = \widehat{a} - \widetilde{a}$, $a^+ = \widehat{a} + \widetilde{a}$.

By making use of the aforementioned notations, it is interesting to observe how intervals can be represented as points both in the *midpoint half-plane* $(\widehat{x}; \widetilde{x})$ or in the *extremes plane* $[x^-, x^+]$ respectively, as shown in Figure 2.1.

In this regard we can even consider the two points $A = [a^-, a^+]$ and $A_M = (\widehat{a}; \widetilde{a})$ as different representations of the same element, respectively in the midpoint half-plane and in the extremes plane (see Figure 2.2). In particular, in this second type of representation, it can be seen that the lower extreme a^- of the interval $A = [a^-, a^+]$ is positioned on the horizontal axis

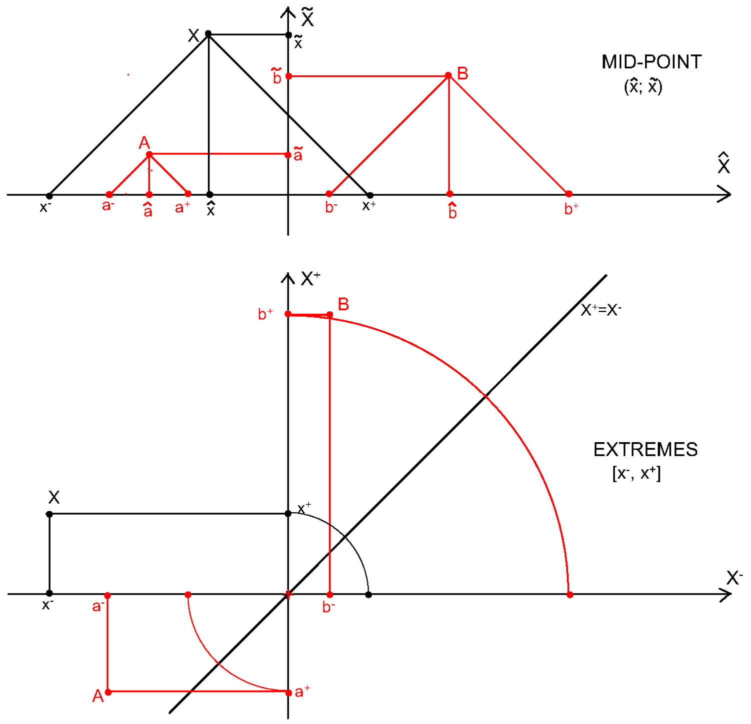


Figure 2.1: Example of graphical representation of intervals by points in the midpoint plane $(\widehat{x};\widetilde{x})$ (top) and in the extremes plane $[x^-, x^+]$ (bottom).

of the plane while the upper extreme a^+ on the vertical axis. We also note how, unlike the midpoint half-plane, in this case the entire plane is used.

Moreover, it is also possible to redefine the above concepts using matrices. Indeed, as

$$\begin{cases} \widehat{a} = \frac{1}{2}(a^- + a^+) \\ \widetilde{a} = \frac{1}{2}(-a^- + a^+), \end{cases}$$

we can consider the associated matrix $M = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$ so that, for all $A = [a^-, a^+] \in \mathcal{K}_{\mathcal{C}}$ we have

$$\begin{bmatrix} \widehat{a} \\ \widetilde{a} \end{bmatrix} = M \cdot \begin{bmatrix} a^- \\ a^+ \end{bmatrix}.$$

Since $det(M) = \frac{1}{2} \neq 0$, it follows that the inverse matrix M^{-1} exists and corresponds to

$$M^{-1} = \frac{1}{det(M)} \cdot \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = 2 \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = 2M^T,$$

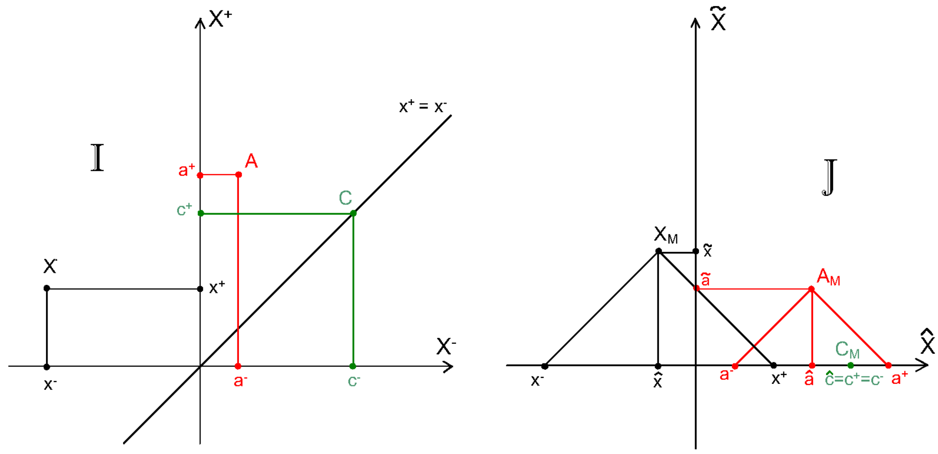


Figure 2.2: Example of representation of the same intervals in the midpoint plane (right) and in the extremes plane (left).

where M^T is the transposed matrix of M.

So we get: $M \cdot M^{-1} = M \cdot 2M^T = I$, that is: $M \cdot M^T = M^T \cdot M = \frac{1}{2} \cdot I$. In fact, it can be trivially verified that

$$M \cdot M^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \frac{1}{2} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{2} \cdot I_2.$$

In summary we have

$$\begin{bmatrix} \widehat{a} \\ \widetilde{a} \end{bmatrix} = M \cdot \begin{bmatrix} a^- \\ a^+ \end{bmatrix} \quad \text{for all} \quad A = [a^-, a^+]$$

and, on the other hand,

$$\begin{bmatrix} a^- \\ a^+ \end{bmatrix} = 2M^T \cdot \begin{bmatrix} \widehat{a} \\ \widetilde{a} \end{bmatrix} \quad \text{for all} \quad A = (\widehat{a}; \widetilde{a})\,.$$

Hence, according to Figure 2.2, we can define the sets:

$$\mathbb{I} = \{X = [x_1, x_2] : x_1 \leqslant x_2\} \quad \text{and} \quad \mathbb{J} = \{Y = (y_1; y_2) : y_2 \geqslant 0\}$$

where the matrix M gives us the bijective map: $M : \mathbb{I} \xrightarrow[su]{1-1} \mathbb{J}$, such that

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto M \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

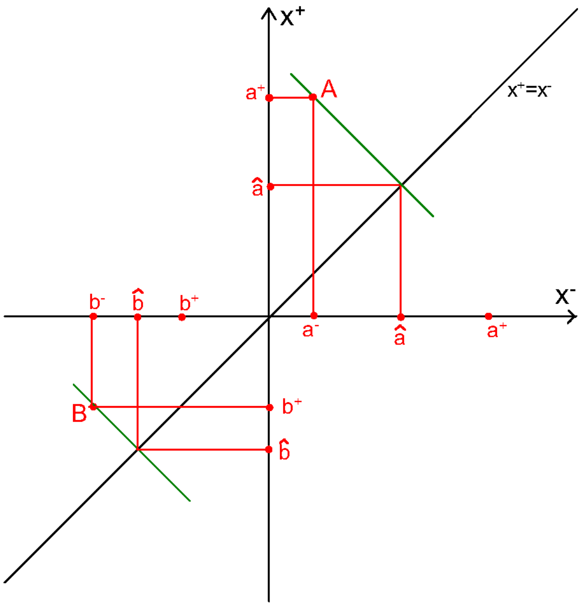


Figure 2.3: Visualization in the extremes plane $[x^-, x^+]$ of the distance between the points representing two intervals (A and B) and the line $x^+ = x^-$.

represents the composition of a rotation and a contraction.

Indeed, as illustrated in Figure 2.3, the distance from point A to the line $x^+ = x^-$ is $d = \dfrac{a^+ - a^-}{\sqrt{2}} = \sqrt{2} \cdot \widetilde{a}$, from which it follows $\widetilde{a} = \frac{d}{\sqrt{2}}$. This means that M represents the composition between a rotation and a contraction of $\mathbb{I}$ into $\mathbb{J}$.

Indeed, if we define

$$N = \sqrt{2} \cdot M = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} cos\frac{\pi}{4} & sin\frac{\pi}{4} \\ -sin\frac{\pi}{4} & cos\frac{\pi}{4} \end{bmatrix},$$

it follows that

$$M = N \cdot \frac{1}{\sqrt{2}} I_2,$$

where matrix N stands for a 45 degree clockwise rotation, while $\dfrac{1}{\sqrt{2}} I_2$ represents a contraction by a factor $\dfrac{1}{\sqrt{2}}$.

In this work we will make extensive use of the graphic representation in the half-plane $(\widehat{x}; \widetilde{x})$ which will be privileged with respect to that in the plane $[x^-, x^+]$.

2.2 Order relations for intervals

The problem of ordering intervals has been a topic of intense research in several areas and in this section we will introduce several partial orders for intervals and we will also discuss their properties in terms of the midpoint representation and in relation to lattice theory. Then the fundamental role of gH-difference in characterizing the partial orders will be shown and finally we will provide numerous references to graphical representations related to this notation.

2.2.1 Orders and lattices in classical theory: basic recalls

Before proceeding, let us recall some elements of classical theory (the material in this section follows [10], [24] and [54]).

It is well known that a *partially ordered set* $(P, \leq)$, *poset* for short, is a set P with a binary relation $\leq$ that is a partial order, i.e., it satisfies the following properties: $\forall x, y, z \in P$

1) $x \leq x$, $\forall x \in P$ (reflexive);

2) if $x \leq y$ and $y \leq x$, then $x = y$, $\forall x, y \in P$ (antisymmetric);

3) if $x \leq y$ and $y \leq z$, then $x \leq z$, $\forall x, y, z \in P$ (transitive).

Furthermore we say that P is a *totally ordered set* when $\leq$ has the following additional property:

4) $\forall x, y \in P$, we have $x \leq y$ or $y \leq x$ (linearity).

We also recall the fact that to each partial order $\leq$ on set P, an inverse partial order $\leq'$ exists, defined by

$$x \leq' y \Leftrightarrow x \geq y.$$

The so-called duality principle follows: if $(P, \leq)$ is a poset, then $(P, \leq')$ is a poset too, called *dual* poset.

Now let $S \subseteq P$ be a subset of a poset P.
We say that an upper bound (resp. lower bound) of S is an element $b \in P$ such that $x \leq b$ (resp. $x \geq b$), $\forall x \in S$; moreover if $b \in S$, then b is the greatest element or *maximum* (resp. least element or *minimum*) of S.
An element $m \in S$ is said to be maximal (resp. minimal) if in S there is no element that is greater (resp. smaller) than m.
Finally, the *least upper bound* (*lub*) of S is called its *supremum* and is denoted by $supS$. Using the principle of duality, it is also possible to define *the greatest lower bound* (*glb*) of S, called its *infimum* and indicated with $infS$. If the supremum and the infimum exist, they are unique.

Definition 2.2.1. *A poset* $(L, \leq)$ *is a lattice when any of its elements* x *and* y *have a supremum, denoted by* $x \vee y$*, and an infimum, denoted by* $x \wedge y$*. For this reason we often denote the lattice structure by* $(L, \vee, \wedge)$*.*

A Lattice $(L, \leq)$ is *bounded* if it has a maximum, denoted by 1, and a minimum, denoted by 0, which satisfy

$$0 \leq x \leq 1, \quad \text{for every } x \in L.$$

Furthermore, a lattice L is *complete* if each of its subsets has a supremum and an infimum in L, as best described below.

Definition 2.2.2. *([24]) A poset* $(L, \leq)$ *is a complete lattice if an only if, for all subset* $Y \subseteq L$*,* $\sup(Y)$ *and* $\inf(Y)$ *exist.*

Note that any nonempty complete lattice is universally bounded because it contains its greatest element (the *unit*) and its least element (the *zero*).

In any lattice $(L, \leq)$, by replacing the partial order with its dual $\leq'$ and by exchanging the roles of the supremum and infinum (considering the dual operations), it is possible to form a new lattice $(L, \leq')$ or $(L, \vee', \wedge')$, called the *dual lattice*.

The duality principle assures that for every definition and property that applies to the lattice $(L, \leq)$ there is a dual one that applies to the dual lattice by interchanging $\leq$ with $\geq$ and $\vee$ with $\wedge$.

As will be covered in more detail in Subsection 4.1.2, the lattice operations $\vee$ and $\wedge$ satisfy many properties. The four fundamentals are:

- $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$, $\forall x, y \in L$ (commutativity);
- $x \vee (y \vee z) = (x \vee y) \vee z$ and $x \wedge (y \wedge z) = x \wedge (y \wedge z)$, $\forall x, y, z \in L$ (associativity);
- $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$, $\forall x, y \in L$ (absorption);
- $x \vee x = x$ and $x \wedge x = x$, $\forall x \in L$ (idempotence).

Conversely, a set L equipped with two binary operations $\vee$ and $\wedge$ that satisfy these four properties is a lattice whose supremum is $\vee$, infimum is $\wedge$, and partial order $\leq$ is given by:

- $x \leq y \Leftrightarrow y = x \vee y$ and $x \leq y \Leftrightarrow x = x \wedge y$, $\forall x, y \in L$ (consistency).

A lattice L is called *distributive* if, for any finite index set J the following property holds:

- $y \wedge \left(\bigvee_{i \in J} x_i\right) = \bigvee_{i \in J} (y \wedge x_i)$ and $y \vee \left(\bigwedge_{i \in J} x_i\right) = \bigwedge_{i \in J} (y \vee x_i)$, $\forall x_i, y \in L$ (distributivity).

If it also holds for an infinite index set, then the lattice is called *infinitely distributive*.

In a lattice L with universal bounds 0 and 1, an element $x \in L$ is said to have a complement $x^c \in L$ if

$$x \vee x^c = 1 \text{ and } x \wedge x^c = 0.$$

If all the elements of L have complements, then L is called *complemented*. A lattice is called *Boolean* if it is complemented and distributive. In any Boolean lattice the complement of each element is unique and involutive: $(x^c)^c = x$. However, all this will be resumed and deepened later in a more accurate way in the second part of this work.

2.2.2 Partial order relations for intervals

In order to compare intervals, in addition to the classical relation defined in (1.13), several types of partial orders can be introduced; in classifying them, particular attention is paid to those cases in which there is a (partial or total) overlap between the intervals involved since in such cases the comparison is not as immediate as in the basic classical case of disjoint intervals.

As well described in [30], if we consider $A = [a^-, a^+] = (\widehat{a}; \widetilde{a})$ and $B = [b^-, b^+] = (\widehat{b}; \widetilde{b}) \in \mathcal{K_C}$ with $a^-, a^+, b^-, b^+, \widehat{a}, \widetilde{a}, \widehat{b}, \widetilde{b} \in \mathbb{R}$ $(\widetilde{a}, \widetilde{b} \geqslant 0)$, it is possible to define eight different types of orders, briefly listed below and well represented in the Figure 2.4.

- *Upper versus Lower* order (UL-order for short), denoted by $\precapprox_{UL}$:

$$A \precapprox_{UL} B \ \Leftrightarrow \ a^+ \leqslant b^-.$$

  This order which, as already said, corresponds to the classical case (1.13), requires that the two intervals are separated (i.e., $a \leqslant b$, $\forall a \in A, b \in B$); it is clear that this order does not present particular interpretation difficulties since, in the case of a problem of minimum every possible value of A is to be preferred over those of B, while in a problem of maximum every value of B is better than all those of A .

- *Lower and Upper* order (LU-order for short), denoted by $\precapprox_{LU}$:

$$A \precapprox_{LU} B \ \Leftrightarrow \ a^- \leqslant b^- \text{ and } a^+ \leqslant b^+. \tag{2.8}$$

- *Center and Max-Width* (CW_M-order for short), denoted by $\precapprox_{CW_M}$:

$$A \precapprox_{CW_M} B \ \Leftrightarrow \ \widehat{a} \leqslant \widehat{b} \text{ and } \widetilde{a} \geqslant \widetilde{b}. \tag{2.9}$$

- *Center and min-Width* (CW_m-order for short), denoted by $\precapprox_{CW_m}$:

$$A \precapprox_{CW_m} B \ \Leftrightarrow \ \widehat{a} \leqslant \widehat{b} \text{ and } \widetilde{a} \leqslant \widetilde{b}. \tag{2.10}$$

- *Lower and Center* ($LC-$order for short), denoted by $\precapprox_{LC}$:

$$A \precapprox_{LC} B \ \Leftrightarrow \ \widehat{a} \leqslant \widehat{b} \text{ and } a^- \leqslant b^-. \tag{2.11}$$

- *Upper and Center* ($UC-$order for short), denoted by $\precapprox_{UC}$:

$$A \precapprox_{UC} B \ \Leftrightarrow \ \widehat{a} \leqslant \widehat{b} \text{ and } a^+ \leqslant b^+. \tag{2.12}$$

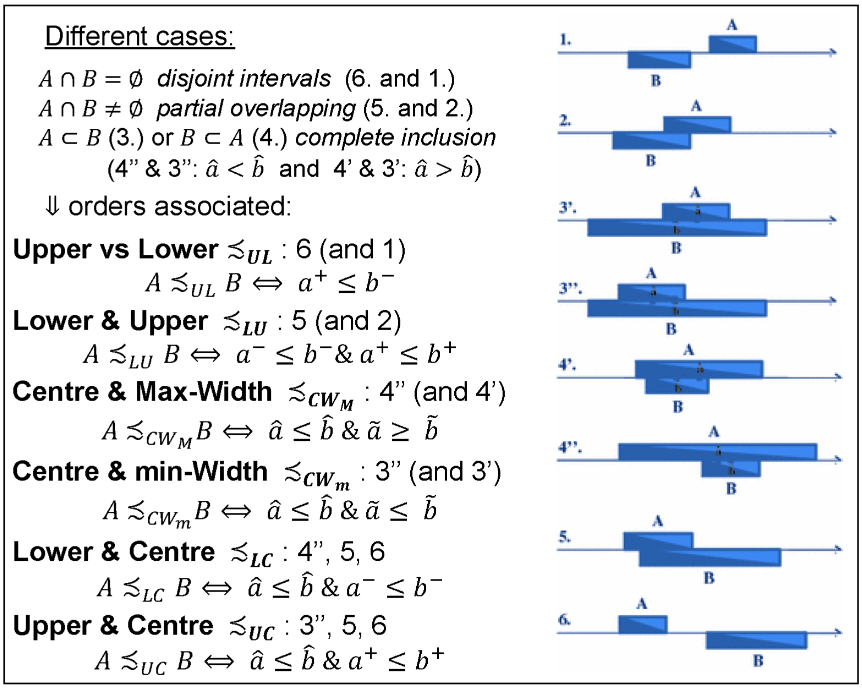


Figure 2.4: The eight possible positions between two intervals $A = [a^-, a^+] = (\widehat{a}; \widetilde{a})$ and $B = [b^-, b^+] = (\widehat{b}; \widetilde{b}) \ \in \mathcal{K_C}$ with the partial orders associated with them.

Furthermore, the following properties are valid for the partial orders just described (see [30]):

1) $(A \precapprox_{LU} B$ and $B \precapprox_{CW_m} A)$ iff $A = B$;

2) $(A \precapprox_{LU} B$ and $B \precapprox_{CW_M} A)$ iff $A = B$;

3) $(A \precapprox_{LU} B$ and $A \precapprox_{CW_M} B)$ iff $A \precapprox_{LC} B$;

4) $(A \precapprox_{LU} B$ and $A \precapprox_{CW_m} B)$ iff $A \precapprox_{UC} B$;

5) If $A \precapprox_{CW_M} B$, then $A \precapprox_{LC} B$;

6) If $A \precapprox_{CW_m} B$, then $A \precapprox_{UC} B$;

7) $A \precapprox_{LU} B$ iff $(A \precapprox_{LC} B$ and $A \precapprox_{UC} B)$.

2.2.3 A comparison index based on the gH-difference

It is known that several authors have introduced interval-based comparison indices to help in decision making with interval imprecision or uncertainty; a comparison index is nothing more than a useful tool in choosing between two or more intervals as it represents the uncertain or imprecisely defined outcome of a decision problem. In order to define it, we must first consider the order induced by the gH-difference and the natural order on the real numbers.

Given an interval $C = [c^-, c^+] = (\widehat{c}; \widetilde{c}) \in \mathcal{K_C}$, $p \in \mathbb{R}, p \geqslant 1$, we define the (modified) p-norm as

$$\|C\|_p = (|\widehat{c}|^p + |\widetilde{c}|^p)^{\frac{1}{p}}$$

while in the case where $p = \infty$, we can define $\|C\|_\infty = max\{|\widehat{c}|, \widetilde{c}\}$ which represents the infinity norm or maximum norm.

Furthermore, we also have that for each p the following properties hold:

- $\|C\|_p \geqslant 0$ and $\|C\|_p = 0 \iff C = 0$,
- $\|C + D\|_p \leqslant \|C\|_p + \|D\|_p$.

In particular we are now interested in the 2-norm of C, given by

$$\|C\|_2 = \sqrt{\widehat{c}^2 + \widetilde{c}^2} = \frac{\sqrt{2}}{2}\sqrt{(c^-)^2 + (c^+)^2}$$

such that, of course, we have:
$\|C\|_2 \geqslant 0$ and $\|C\|_2 = 0 \iff C = 0$, $\|C + D\|_2 \leqslant \|C\|_2 + \|D\|_2$.

In order to include commonly used order relations, a new comparison index has been defined, based on the generalized Hukuhara difference.

Considering as usual $A = [a^-, a^+] = (\widehat{a}; \widetilde{a})$ and $B = [b^-, b^+] = (\widehat{b}; \widetilde{b})$, their gH-difference can be expressed in endpoint notation as

$$A \ominus_{gH} B = \left[(A \ominus_{gH} B)^-, (A \ominus_{gH} B)^+\right];$$

while in midpoint notation it is

$$A \ominus_{gH} B = \left((\widehat{A \ominus_{gH} B}); (\widetilde{A \ominus_{gH} B})\right)$$

where $(\widehat{A \ominus_{gH} B}) = \widehat{a} - \widehat{b}$ stands for the midpoint and $(\widetilde{A \ominus_{gH} B}) = |\widetilde{a} - \widetilde{b}|$ for the radius.

A good property for the gH-difference is that it always exists for any pairs of intervals and is useful to analyse the basic order relations in terms of arithmetic interval operations.

According to [30], some properties relating the orders and the gH-difference are immediate to prove, such as the following:

1) $A \precsim_{LU} B$ iff $(A \ominus_{gH} B)^+ \leqslant 0$;

2) if $A \precsim_{CW_M} B$ or $A \precsim_{CW_m} B$ or $A \precsim_{LC} B$ or $A \precsim_{UC} B$, then $(A \ominus_{gH} B)^- \leqslant 0$ and $(\widehat{A \ominus_{gH} B}) \leqslant 0$

and a generic comparison index, based on gH-difference, has been suggested too.

Definition 2.2.3. *([30]) Given two distinct intervals $A \neq B$, the gH-comparison index of order $p > 0$ is defined as*

$$CI_p(A,B) = \frac{\widehat{A \ominus_{gH} B}}{\|A \ominus_{gH} B\|_p}$$

where $A \ominus_{gH} B$ is the gH-difference, $\forall A, B$.

In addition, the main properties of the index have also been given.

Proposition 2.2.1. *([30]) Given two distinct intervals $A \neq B$, we have $\forall p > 0$*

1. $CI_p(A,B) \in [-1,1]$,
2. $CI_p(A,B) = -CI_p(B,A)$,
3. $CI_p(A,B) = 0 \Longleftrightarrow \widehat{a} = \widehat{b}$,
4. $|CI_p(A,B)| = 1 \Longleftrightarrow (\widetilde{a} = \widetilde{b}$ *and* $\widehat{a} \neq \widehat{b})$,
5. $CI_p(A,B) \geqslant 0 \Longleftrightarrow \widehat{a} \geqslant \widehat{b}$,
6. *An invariance of scale holds:* $CI_p(kA,kB) = \begin{cases} CI_p(A,B) & \text{if} \quad k > 0 \\ CI_p(B,A) & \text{if} \quad k < 0, \end{cases}$
7. $CI_p(A \oplus C, B \oplus C) = CI_p(A,B)$.

However, here only the particular case of comparison index based on gH-difference and the Euclidean 2-norm is considered. The definition and basic properties are given below.

Definition 2.2.4. *([84]) Given two distinct intervals $A \neq B$, the gH-comparison index is defined as*

$$CI_{gH}(A,B) = \frac{\widehat{A \ominus_{gH} B}}{\|A \ominus_{gH} B\|_2}; \tag{2.13}$$

it has the following properties:

1. $CI_{gH}(A,B) \in [-1,1]$,
2. $CI_{gH}(A,B) = -CI_{gH}(B,A)$,

3. $CI_{gH}(A,B) = 0 \Longleftrightarrow \widehat{a} = \widehat{b}$,

4. $|CI_{gH}(A,B)| = 1 \Longleftrightarrow (\widetilde{a} = \widetilde{b}$ *and* $\widehat{a} \neq \widehat{b})$,

5. $CI_{gH}(A,B) \geqslant 0 \Longleftrightarrow \widehat{a} \geqslant \widehat{b}$,

6. $CI_{gH}(kA,kB) = \begin{cases} CI_{gH}(A,B) & \textit{if} \quad k>0 \\ CI_{gH}(B,A) & \textit{if} \quad k<0, \end{cases}$

7. $CI_{gH}(A \oplus C, B \oplus C) = CI_{gH}(A,B)$.

We can write

$$CI_{gH}(A,B) = \frac{\widehat{A \ominus_{gH} B}}{\|A \ominus_{gH} B\|_2} = \frac{\widehat{a} - \widehat{b}}{\sqrt{(\widehat{a} - \widehat{b})^2 + (\widetilde{a} - \widetilde{b})^2}}$$

and, assuming the condition $\widehat{a} \neq \widehat{b}$, we define the following *gH-comparison ratio*

$$\gamma_{A,B} = \frac{\widetilde{a} - \widetilde{b}}{\widehat{a} - \widehat{b}} = \gamma_{B,A}. \tag{2.14}$$

Note that for the comparison ratio $\gamma_{A,B}$ the following properties are immediate:

(a) an invariance of scale holds: $\gamma_{kA,kB} = \begin{cases} \gamma_{AB} & \text{if } \ k>0 \\ -\gamma_{AB} & \text{if } \ k<0, \end{cases}$

(b) $\gamma_{A+C,B+C} = \gamma_{A,B}$.

The comparison ratio $\gamma_{A,B}$ can be determined for all the possible positions of two intervals $A = [a^-, a^+] = (\widehat{a}; \widetilde{a})$ and $B = [b^-, b^+] = (\widehat{b}; \widetilde{b})$ and in particular it characterizes how the two intervals overlap each other (see Figure 2.4). We have the following cases.

1. Case 1 is an unambiguous one as the two intervals do not overlap and the strict dominance is verified: we have $B \precapprox_{UL} A$, which occurs when $B \precapprox_{LU} A$ and $\widehat{a} - \widehat{b} \geqslant \widetilde{a} + \widetilde{b}$, that is $a^- \geqslant b^+$.

2. In case 2 we have $B \precapprox_{LU} A$ or, equivalently $b^- \leqslant a^-$ and $b^+ \leqslant a^+$, from which we have $a^- - b^- \geqslant 0$, $a^+ - b^+ \geqslant 0$ and, obviously, $\widehat{a} - \widehat{b} > 0$. It is immediate to see that
$$0 \leqslant \frac{a^- - b^-}{\widehat{a} - \widehat{b}} = \frac{(\widehat{a} - \widetilde{a}) - (\widehat{b} - \widetilde{b})}{\widehat{a} - \widehat{b}} = \frac{\widehat{a} - \widehat{b}}{\widehat{a} - \widehat{b}} - \frac{\widetilde{a} - \widetilde{b}}{\widehat{a} - \widehat{b}} = 1 - \gamma_{A,B} \tag{2.15}$$
as well as
$$0 \leqslant \frac{a^+ - b^+}{\widehat{a} - \widehat{b}} = \frac{(\widehat{a} + \widetilde{a}) - (\widehat{b} + \widetilde{b})}{\widehat{a} - \widehat{b}} = \frac{\widehat{a} - \widehat{b}}{\widehat{a} - \widehat{b}} + \frac{\widetilde{a} - \widetilde{b}}{\widehat{a} - \widehat{b}} = 1 + \gamma_{A,B} \tag{2.16}$$
which means that $|\gamma_{A,B}| \leqslant 1$.

3. Case 3 occurs when $\widetilde{a} \leqslant \widetilde{b}$ (so $\widetilde{a} - \widetilde{b} \leqslant 0$) and has to be split up into the two sub-cases $3'$ and $3''$ depending on the relative positions of the midpoints:

   $3'$. in the first case, which corresponds to $B \precsim_{CWm} A$, we have $\widehat{a} > \widehat{b}$, that is $\widehat{a} - \widehat{b} > 0$; therefore, we have $\gamma_{A,B} \leqslant 0$;

   $3''$. on the other hand, in the case $A \precsim_{CWm} B$ we have that $\widehat{a} < \widehat{b}$, which means $\widehat{a} - \widehat{b} < 0$; so we have $\gamma_{A,B} \geqslant 0$.

4. Also case 4, which occurs when $\widetilde{a} \geqslant \widetilde{b}$, from which $\widetilde{a} - \widehat{b} \geqslant 0$, should be divided into two sub-cases 4' and 4" still dependent on the relative positions of the midpoints:

   $4'$. the first case stands for $B \precsim_{CW_M} A$, we have $\widehat{a} > \widehat{b}$, that is $\widehat{a} - \widehat{b} > 0$; therefore, we have $\gamma_{A,B} \geqslant 0$;

   $4''$. similarly, in the second case, which corresponds to $A \precsim_{CW_M} B$, we have that $\widehat{a} < \widehat{b}$, so that $\widehat{a} - \widehat{b} < 0$; therefore, we have $\gamma_{A,B} \leqslant 0$.

5. In case 5, in a completely analogous way to case 2, we have $A \precsim_{LU} B$ or, equivalently $a^- \leqslant b^-$ and $a^+ \leqslant b^+$, from which we have $a^- - b^- \leqslant 0$, $a^+ - b^+ \leqslant 0$ and obviously $\widehat{a} - \widehat{b} < 0$. Also this time it is immediate to verify the validity of (2.15) and (2.16) so that once again we have $|\gamma_{A,B}| \leqslant 1$.

6. Case 6, similarly to the first, is unambiguous and the strict dominance $A \precsim_{UL} B$ is verified as we have $A \precsim_{LU} B$ and $\widehat{b} - \widehat{a} \geqslant \widetilde{a} + \widetilde{b}$, that is $b^- \geqslant a^+$.

Accordingly, we can conclude that, if $\widehat{a} \neq \widehat{b}$ and $|\gamma_{A,B}| \leqslant 1$, it is possible to base our choice on the value of $\widehat{a}$ and $\widehat{b}$ as there is no risk; on the other hand, when $|\gamma_{A,B}| > 1$ there is a risk that requires more careful analysis.

To sum up, assuming that $\widehat{a} < \widehat{b}$, it is possible to prove that, in terms of $\gamma_{A,B}$, the five order relations we have defined above can be characterized as follows (see also Figure 2.5)

(a) $A \precsim_{LU} B \;\Leftrightarrow\; \gamma_{A,B} \in [-1, 1]$;

(b) $A \precsim_{CW_M} B \;\Leftrightarrow\; \gamma_{A,B} \leqslant 0$;

(c) $A \precsim_{CW_m} B \;\Leftrightarrow\; \gamma_{A,B} \geqslant 0$;

(d) $A \precsim_{LC} B \;\Leftrightarrow\; -1 \leqslant \gamma_{A,B} \leqslant 0$;

(e) $A \precsim_{UC} B \;\Leftrightarrow\; 0 \leqslant \gamma_{A,B} \leqslant 1$.

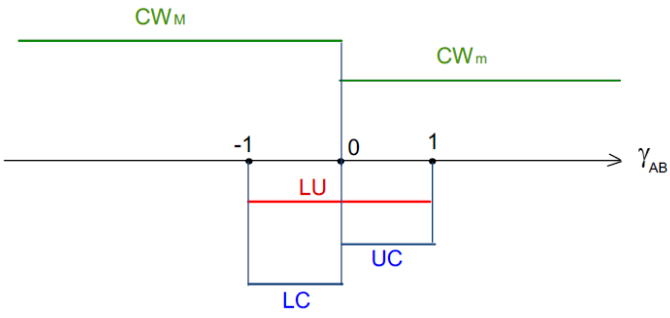


Figure 2.5: Characterization of the order relations in terms of $\gamma_{A,B}$ (assuming that $\widehat{a} < \widehat{b}$).

Therefore, it is evident that the comparison ratio $\gamma_{A,B}$ is very useful in the characterization of different order relations for intervals and its use is extremely convenient when we are dealing with maximum or minimum problems. Indeed if we consider two distinct intervals $A \neq B$ we can use the (partial) order relations to decide if A is "less" than B, or if A is "greater" than B, or if A and B are incomparable with respect to the order considered, as will be fully discussed in the Subsection 2.2.4.

2.2.4 Optimization problems

The notions of "smaller than" and "greater than" are strictly related to the order relation that we want to use to rank intervals; in particular it could happen that, with respect to the selected (partial) order, the two intervals cannot be compared. Therefore, in this case, it is not easy to choose the best range. This is especially the case when the insides of the intervals overlap. In fact, if we are minimizing and $a^+ \leqslant b^-$, then the whole interval A is smaller than interval B because $a \leqslant b$ for all possible values $a \in A$ and $b \in B$. This means that A can be chosen for the minimum or B for the maximum. When, on the other hand, there is an overlap of intervals, the choice will depend on their relative position and in this case having precise selection criteria available is of great help in identifying a final decision.

Specifically, we have that the following cases may occur.

If $\widetilde{a} = \widetilde{b}$ the comparison is easy as indeed, being $A \neq B$, either $\widehat{a} < \widehat{b}$ or $\widehat{a} > \widehat{b}$ and the decision can be based simply on the comparison of the midpoint values.

If $\widetilde{a} \neq \widetilde{b}$ and $\widehat{a} = \widehat{b}$, then A and B are incomparable with respect to any order relation; indeed, in that case, the intervals are equally centered and one of them is strictly included in the other (we can eventually have a preference for the bigger or the smaller one, but there is no simple way to quantify how much one is better or worse than the other).

The interesting and more complex case to analyze is obviously when $\widetilde{a} \neq \widetilde{b}$ and $\widehat{a} \neq \widehat{b}$. Consider first the comparison "A is less than B", formally "decide if $A \prec B$ or not". If $\widehat{a} < \widehat{b}$ and A and B do not overlap with internal points, i.e., when $a^+ \leqslant b^-$, it is reasonable to accept $A \prec B$, as no element in A is greater than any elements in B (see first case in Figure 2.6); instead, some indecision is justified if the two intervals overlap internally (as shown in the other three cases of Figure 2.6).

$\boldsymbol{A} \precsim_{\boldsymbol{UL}} \boldsymbol{B}$: $a^+ \leq b^-$ (disjoint intervals)
$\Rightarrow a \leq b \;\; \forall a \in A, \forall b \in B$ no risk

$\boldsymbol{A} \precsim_{\boldsymbol{LU}} \boldsymbol{B}$: $a^- \leq b^-$ & $a^+ \leq b^+$ (partial overlapping)
$\Rightarrow$ Risk: $\exists a \in A, \exists\, b \in B \mid a > b$

$\boldsymbol{A} \precsim_{CW_m} \boldsymbol{B}$: $\tilde{a} \leq \tilde{b}$ (complete inclusion $A \subset B$)
$\Rightarrow$ **Risk (left)** : $\exists b \in B \mid a > b \;\; \forall a \in A$

$\boldsymbol{A} \precsim_{CW_M} \boldsymbol{B}$: $\tilde{a} \geq \tilde{b}$ (complete inclusion $B \subset A$)
$\Rightarrow$ **Risk (right)** : $\exists a \in A \mid a > b \;\; \forall b \in B$

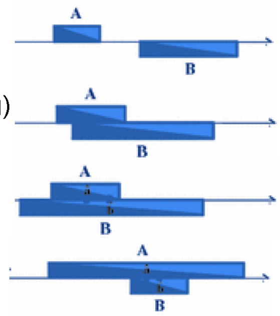


Figure 2.6: Risk in a minimization problem (assuming that $\widehat{a} < \widehat{b}$).

We can analyze this situation using the comparison ratio $\gamma_{A,B}$; we distinguish two cases, (I) $\widehat{a} < \widehat{b}$, $\widetilde{a} < \widetilde{b}$ and (II) $\widehat{a} < \widehat{b}$, $\widetilde{a} > \widetilde{b}$.

Case (I): ($\widehat{a} < \widehat{b}$ and $\widetilde{a} < \widetilde{b}$ so that $\gamma_{A,B} > 0$).

We can easily check the validity of (2.15). Therefore, if $a^- \leqslant b^-$, that is $1 - \gamma_{A,B} \geqslant 0$, so $\gamma_{A,B} \leqslant 1$ (or, even better $0 < \gamma_{A,B} \leqslant 1$), then there is no element in B which is smaller than all elements in A (second case shown in Figure 2.6).

But if $a^- > b^-$, that is $1 - \gamma_{A,B} < 0$, so that $\gamma_{A,B} > 1$, then elements of B exist on the left side of B which are smaller than all $a \in A$ (third case shown in Figure 2.6) and the ratio $\dfrac{a^- - b^-}{\widehat{a} - \widehat{b}} = 1 - \gamma_{A,B}$ measures how much elements of B are better than all elements of A, with respect to how much the central value of A is better that the central value of B.

In some sense, $1 - \gamma_{A,B}$ gives a relative measure of a possible "loss" $a^- - b^- > 0$ if we chose A against B based on central values (expecting a mid-value "gain" $\widehat{b} - \widehat{a}$).

Case (II): ($\widehat{a} < \widehat{b}$ and $\widetilde{a} > \widetilde{b}$ so that $\gamma_{A,B} < 0$). Similarly to the previous case, we can immediately verify the validity of (2.16). This means that if $a^+ \leqslant b^+$, that is $\gamma_{A,B} + 1 \geqslant 0$, so $\gamma_{A,B} \geqslant -1$ (or, even better $-1 \leqslant \gamma_{A,B} < 0$), then there is no element in A which is greater than all elements in B.

But if $a^+ > b^+$, that is $\gamma_{A,B} + 1 < 0$, so $\gamma_{A,B} < -1$, then elements of A exist on the right side of A which are greater than all $b \in B$ (fourth case shown in Figure 2.6) and the ratio $\dfrac{a^+ - b^+}{\widehat{a} - \widehat{b}} = 1 + \gamma_{A,B}$ measures how many elements of A are worse than all elements of B, with respect to how much the central value of A is better than the central value of B.
In some sense, $\gamma_{A,B} + 1$ gives a relative measure of a possible "loss" $a^+ - b^+ > 0$ if we chose A against B based on the central values (expecting a mid-value "gain" $\widehat{b} - \widehat{a}$).

Summarizing, we can say that in accepting $A \prec B$ on the basis of the comparison $\widehat{a} < \widehat{b}$ of the midpoint values, a possibly positive (worst-case) loss appears when $\gamma_{A,B} > 1$ or when $\gamma_{A,B} < -1$; we then have the following interpretation of the comparison ratio $\gamma_{A,B}$.

- If $\widehat{a} < \widehat{b}$ and $-1 \leqslant \gamma_{A,B} \leqslant 1$, no possible worst-case loss appears in accepting $A \prec B$.
- If $\widehat{a} < \widehat{b}$ and $\gamma_{A,B} > 1$, a possible worst-case loss in accepting $A \prec B$ appears because some values of B (on the left side) are less than all values of A (i.e., some values of B are better than all values of A); the quantity $1 - \gamma_{A,B} < 0$ gives a relative measure of the possible loss with respect to the midpoint gain.
- If $\widehat{a} < \widehat{b}$ and $\gamma_{A,B} < -1$, a possible worst-case loss in accepting $A \prec B$ appears because some values of A (on the right side) are greater than all values of B (i.e., some values of A are worse than all values of B); the quantity $1 + \gamma_{A,B} < 0$ gives a relative measure of the possible loss with respect to the midpoint gain.

The gH-comparison index $\gamma_{A,B}$ will be used extensively in the rest of this work. In a similar way we can define a comparison index based on M-difference and 2-norm.

Definition 2.2.5. *([84]) Given two intervals A, B, the M-comparison index is defined as*

$$CI_M(A,B) = \frac{\widehat{A \ominus_M B}}{\|A \ominus_M B\|_2} = \frac{\widehat{a} - \widehat{b}}{\sqrt{(\widehat{a} - \widehat{b})^2 + (\widetilde{a} + \widetilde{b})^2}} \tag{2.17}$$

where $A \ominus_M B$ is the Minkowski difference. Given two distinct intervals $A \neq B$, it has the following properties:

1. $CI_M(A,B) \in [-1,1]$,

2. $CI_M(A,B) = -C_M(B,A)$,

3. $CI_M(A,B) = 0 \Longleftrightarrow \widehat{a} = \widehat{b}$,

4. $|CI_M(A,B)| = 1 \Longleftrightarrow (\widetilde{a} = \widetilde{b} = 0$ *and* $\widehat{a} \neq \widehat{b})$,

5. $CI_M(A,B) \geqslant 0 \Longleftrightarrow \widehat{a} \geqslant \widehat{b}$,

6. $CI_M(kA,kB) = \begin{cases} CI_M(A,B) & \text{if} \quad k > 0 \\ CI_M(B,A) & \text{if} \quad k < 0, \end{cases}$

7. $CI_M(A \oplus C, B \oplus C) = CI_M(A,B)$.

Assuming $\widehat{a} \neq \widehat{b}$, we can define the M-comparison ratio

$$\eta_{A,B} = \frac{\widetilde{a} + \widetilde{b}}{\widehat{a} - \widehat{b}} \ . \tag{2.18}$$

The reciprocal of the ratio $\eta_{A,B}$, called *acceptability index*,

$$Acc(A \leqslant B) = \frac{\widehat{b} - \widehat{a}}{\widetilde{b} + \widetilde{a}}$$

was introduced in [76] and it always exists when $\widetilde{a} + \widetilde{b} > 0$ (i.e., when at least one of A and B is a proper interval).
Given two distinct intervals $A = [a^-, a^+] = (\widehat{a}; \widetilde{a})$ and $B = [b^-, b^+] = (\widehat{b}; \widetilde{b})$ it has the following basic properties:

(1) if $Acc(A \leqslant B) \geqslant 1$, that is $\frac{\widehat{b} - \widehat{a}}{\widetilde{b} + \widetilde{a}} - \frac{\widetilde{b} + \widetilde{a}}{\widetilde{b} + \widetilde{a}} \geqslant 0$, so $\frac{\widehat{b} - \widetilde{b} - (\widehat{a} + \widetilde{a})}{\widetilde{b} + \widetilde{a}} \geqslant 0$ which corresponds to $\frac{b^- - a^+}{\widetilde{b} + \widetilde{a}} \geqslant 0$, we obtain $a^+ \leqslant b^-$ (i.e., all values of A are less than or equal to all values of B);

(2) if $Acc(A \leqslant B) \leqslant -1$, that is $\frac{\widehat{b} - \widehat{a}}{\widetilde{b} + \widetilde{a}} + \frac{\widetilde{b} + \widetilde{a}}{\widetilde{b} + \widetilde{a}} \leqslant 0$, so $\frac{\widehat{b} + \widetilde{b} - (\widehat{a} - \widetilde{a})}{\widetilde{b} + \widetilde{a}} \leqslant 0$ which means $\frac{b^+ - a^-}{\widetilde{b} + \widetilde{a}} \leqslant 0$, we have $a^- \geqslant b^+$ (i.e., all values of A are greater than or equal to all values of B).

When is positive, index $Acc(A \leqslant B)$ gives a measure of acceptability of the inequality $A \leqslant B$: if $Acc(A \leqslant B) = \alpha \in]0,1[$ then $A \leqslant B$ is accepted with degree α.

The two ratios $\gamma_{A,B}$ and $\eta_{A,B}$ are not related each-other in a simple way; the following two numerical examples compare the acceptability index $Acc(A \leqslant B)$ with the gH-comparison ratio $\gamma_{A,B}$ for intersecting intervals A and B. In particular, when an interval is included in the other, we have two possibilities:

(a) $A \subset B$

(a.1) if $A = [3, 9] = (6; 3)$, $B = [4, 12] = (8; 4)$,
we get $Acc(A \leqslant B) = +\frac{2}{7}$, $\gamma_{A,B} = \frac{1}{2}$;

(a.2) if $A = [5, 7] = (6; 1)$, $B = [2, 14] = (8; 6)$,
we get $Acc(A \leqslant B) = +\frac{2}{7}$, $\gamma_{A,B} = \frac{5}{2}$;

(b) $B \subset A$

(b.1) if $A = [1, 13] = (7; 6)$, $B = [8, 10] = (9; 1)$,
then $Acc(A \leqslant B) = +\frac{2}{7}$, $\gamma_{A,B} = -\frac{5}{2}$;

(b.2) if $A = [2, 10] = (6; 4)$, $B = [5, 11] = (8; 3)$,
then $Acc(A \leqslant B) = +\frac{2}{7}$, $\gamma_{A,B} = -\frac{1}{2}$.

In the four cases, the acceptability index has the same value while the gH-comparison ratio has significantly different values; it is then clear that the two indices will not produce comparable results.

2.2.5 The LU-order for intervals

The LU-order for intervals, extensively used in [4], [84] and [85], is well known in the literature. However, we can further refine the definition by introducing also the cases of strict order and strong order, and related annexed propositions, exactly as reported in [84], so as to better highlight the connection with the concept of gH-derivative.

The following definition extends what was stated in (2.8).

Definition 2.2.6. *([84]) Given $A = [a^-, a^+] \in \mathcal{K}_C$, $B = [b^-, b^+] \in \mathcal{K}_C$, we say that*

(i) $A \precapprox_{LU} B$ if and only if $a^- \leqslant b^-$ and $a^+ \leqslant b^+$,

(ii) $A \precsim_{LU} B$ if and only if $A \precapprox_{LU} B$ and ($a^- < b^-$ or $a^+ < b^+$),

(iii) $A \prec_{LU} B$ if and only if $a^- < b^-$ and $a^+ < b^+$.

The corresponding reverse orders are, respectively:

$$A \succapprox_{LU} B \iff B \precapprox_{LU} A, \quad A \succsim_{LU} B \iff B \precsim_{LU} A$$

$$\text{and} \quad A \succ_{LU} B \iff B \prec_{LU} A.$$

Using midpoint notation $A = (\widehat{a}; \widetilde{a})$, $B = (\widehat{b}; \widetilde{b})$, the partial orders (i) and (iii) above can be expressed as

$$(i) \begin{cases} \widehat{a} \leqslant \widehat{b} \\ \widetilde{b} \leqslant \widetilde{a} + (\widehat{b} - \widehat{a}) \\ \widetilde{b} \geqslant \widetilde{a} - (\widehat{b} - \widehat{a}) \end{cases} \quad \text{and } (iii) \begin{cases} \widehat{a} < \widehat{b} \\ \widetilde{b} < \widetilde{a} + (\widehat{b} - \widehat{a}) \\ \widetilde{b} > \widetilde{a} - (\widehat{b} - \widehat{a}) \end{cases}$$

while the partial order (ii) can be expressed in terms of (i) with the additional requirement that at least one of the inequalities is strict.

Proposition 2.2.2. *([84]) Let $A, B \in \mathcal{K}_C$ with $A = (\widehat{a}; \widetilde{a}), B = (\widehat{b}; \widetilde{b})$. We have*

(i.a) $A \precsim_{LU} B$ *if and only if* $\widehat{b} - \widehat{a} \geqslant \left|\widetilde{b} - \widetilde{a}\right|$;

(ii.a) $A \precnsim_{LU} B$ *if and only if* $\widehat{a} < \widehat{b}$ *and* $\widehat{b} - \widehat{a} \geqslant \left|\widetilde{b} - \widetilde{a}\right|$;

(iii.a) $A \prec_{LU} B$ *if and only if* $\widehat{b} - \widehat{a} > \left|\widetilde{b} - \widetilde{a}\right|$;

(i.b) $A \succsim_{LU} B$ *if and only if* $\widehat{a} - \widehat{b} \geqslant \left|\widetilde{b} - \widetilde{a}\right|$;

(ii.b) $A \succnsim_{LU} B$ *if and only if* $\widehat{a} > \widehat{b}$ *and* $\widehat{a} - \widehat{b} \geqslant \left|\widetilde{b} - \widetilde{a}\right|$;

(iii.b) $A \succ_{LU} B$ *if and only if* $\widehat{a} - \widehat{b} > \left|\widetilde{b} - \widetilde{a}\right|$.

Proof. For case (i.a), if $C = A \ominus_{gH} B = [\min\{a^- - b^-, a^+ - b^+\}, \max\{a^- - b^-, a^+ - b^+\}] = \left(\widehat{a} - \widehat{b}; \left|\widetilde{a} - \widetilde{b}\right|\right)$, then $C = \left[\widehat{a} - \widehat{b} - \left|\widetilde{b} - \widetilde{a}\right|, \widehat{a} - \widehat{b} + \left|\widetilde{b} - \widetilde{a}\right|\right] = [c^-, c^+]$. According to Definition 2.2.6, we know that $A \precsim_{LU} B$ if and only if $a^- \leqslant b^-$ and $a^+ \leqslant b^+$, that is, $a^- - b^- \leqslant 0$ and $a^+ - b^+ \leqslant 0$, which means $c^+ = max\{a^- - b^-, a^+ - b^+\} \leqslant 0$. Therefore, $c^+ = \widehat{a} - \widehat{b} + \left|\widetilde{b} - \widetilde{a}\right| \leqslant 0$ which is equivalent to stating that $\widehat{b} - \widehat{a} \geqslant \left|\widetilde{b} - \widetilde{a}\right|$. The other cases are analogous. □

Proposition 2.2.3. *([84]) Let $A, B \in \mathcal{K}_C$ with $A = (\widehat{a}; \widetilde{a})$, $B = (\widehat{b}; \widetilde{b})$. We have*

(i.a) $A \precsim_{LU} B$ *if and only if* $A \ominus_{gH} B \precsim_{LU} 0$;

(ii.a) $A \precnsim_{LU} B$ *if and only if* $A \ominus_{gH} B \precnsim_{LU} 0$;

(iii.a) $A \prec_{LU} B$ *if and only if* $A \ominus_{gH} B \prec_{LU} 0$;

(i.b) $A \succsim_{LU} B$ *if and only if* $A \ominus_{gH} B \succsim_{LU} 0$;

(ii.b) $A \succnsim_{LU} B$ *if and only if* $A \ominus_{gH} B \succnsim_{LU} 0$;

(iii.b) $A \succ_{LU} B$ *if and only if* $A \ominus_{gH} B \succ_{LU} 0$.

Proof. For case (i.a) we remember that if $C = A \ominus_{gH} B = \left(\widehat{a} - \widehat{b}; \left|\widetilde{a} - \widetilde{b}\right|\right)$, then $C = \left[\widehat{a} - \widehat{b} - \left|\widetilde{b} - \widetilde{a}\right|, \widehat{a} - \widehat{b} + \left|\widetilde{b} - \widetilde{a}\right|\right] = [c^-, c^+]$. Since $A \precsim_{LU} B$ if and only if $a^- \leqslant b^-$ and $a^+ \leqslant b^+$, that is, $a^- - b^- \leqslant 0$ and $a^+ - b^+ \leqslant 0$, which means $c^+ = \max\{a^- - b^-, a^+ - b^+\} \leqslant 0$, then we can also write $c^+ = (A \ominus_{gH} B)^+ \leqslant 0$. This implies $c^- = (A \ominus_{gH} B)^- \leqslant 0$ and so $A \ominus_{gH} B \precapprox_{LU} 0$; the other cases are analogous. □

Proposition 2.2.4. *Given $A, B \in \mathcal{K}_C$, we clearly have that*

$$A \prec_{LU} B \implies A \precsim_{LU} B \implies A \precapprox_{LU} B,$$
$$A \succ_{LU} B \implies A \succsim_{LU} B \implies A \succapprox_{LU} B.$$

Definition 2.2.7. *We say that two intervals $A, B \in \mathcal{K}_C$ are LU-incomparable if neither $A \precapprox_{LU} B$ nor $A \succapprox_{LU} B$.*

Proposition 2.2.5. *Let $A, B \in \mathcal{K}_C$ with $A = (\widehat{a}; \widetilde{a})$, $B = (\widehat{b}; \widetilde{b})$. The following statements are equivalent:*

(i) *A and B are LU-incomparable,*

(ii) *$A \ominus_{gH} B$ is not a singleton and $0 \in \operatorname{int}(A \ominus_{gH} B)$,*

(iii) $\left|\widehat{a} - \widehat{b}\right| < \left|\widetilde{b} - \widetilde{a}\right|$,

(iv) *$A \subset \operatorname{int}(B)$ or $B \subset \operatorname{int}(A)$,*

where $\operatorname{int}(E)$ is the set of all interior points in an interval $E \in \mathcal{K}_{\mathcal{C}}$.

Proof. $(i) \iff (ii)$: LU-incomparability means that neither $A \precapprox_{LU} B$ nor $A \succapprox_{LU} B$, i.e., neither $A \ominus_{gH} B \precapprox_{LU} 0$ nor $A \ominus_{gH} B \succapprox_{LU} 0$ and this is equivalent with both $(A \ominus_{gH} B)^+ > 0$ and $(A \ominus_{gH} B)^- < 0$, i.e., $0 \in \operatorname{int}(A \ominus_{gH} B)$.

$(ii) \iff (iii)$: validity of (ii) means $(A \ominus_{gH} B)^- < 0 < (A \ominus_{gH} B)^+$ and this is equivalent to $\widehat{a} - \widehat{b} - \left|\widetilde{a} - \widetilde{b}\right| < 0 < \widehat{a} - \widehat{b} + \left|\widetilde{a} - \widetilde{b}\right|$ or, more simply, to $\left|\widehat{a} - \widehat{b}\right| < \left|\widetilde{a} - \widetilde{b}\right|$; so, (ii) and (iii) are equivalent.

$(ii) \iff (iv)$: observe that $A \ominus_{gH} B = [\min\{a^- - b^-, a^+ - b^+\}, \max\{a^- - b^-, a^+ - b^+\}]$; then $(A \ominus_{gH} B)^- < 0 < (A \ominus_{gH} B)^+$ is equivalent to $a^- - b^- < 0 < a^+ - b^+$ or $a^+ - b^+ < 0 < a^- - b^-$.
But $a^- - b^- < 0 < a^+ - b^+$ is equivalent to $a^- < b^-$ and $a^+ > b^+$, i.e., $B \subset \operatorname{int}(A)$; similarly, $a^+ - b^+ < 0 < a^- - b^-$ is equivalent to $a^+ < b^+$ and $a^- > b^-$, i.e., $A \subset \operatorname{int}(B)$. □

Proposition 2.2.6. *([84]) If A, B, $C \in \mathcal{K}_C$, then*

(i) *$A \precapprox_{LU} B$ if and only if $A \oplus_M C \precapprox_{LU} B \oplus_M C$;*

(ii) *If $A \oplus_M B \precapprox_{LU} C$ then $A \precapprox_{LU} C \ominus_{gH} B$;*

(iii) *If $A \oplus_M B \succapprox_{LU} C$ then $A \succapprox_{LU} C \ominus_{gH} B$.*

Proof. It is easy to check that $A \ominus_{gH} B = (A \oplus_M C) \ominus_{gH} (B \oplus_M C)$, so, according to Proposition 2.2.3, we have $A \precsim_{LU} B$ if and only if $A \ominus_{gH} B \precsim_{LU} 0$, which corresponds to $(A \oplus_M C) \ominus_{gH} (B \oplus_M C) \precsim_{LU} 0$, that is, $(A \oplus_M C) \precsim_{LU} (B \oplus_M C)$ and (i) follows.

For (ii), if $A \oplus_M B \precsim_{LU} C$, that is equivalent to $(A \oplus_M B) \ominus_{gH} C \precsim_{LU} 0$, then $((A \oplus_M B) \ominus_{gH} C)^+ = \max\{a^- + b^- - c^-, a^+ + b^+ - c^+\} \leqslant 0$ and we get $a^- + b^- \leqslant c^-$ and $a^+ + b^+ \leqslant c^+$. Then, $a^- \leqslant c^- - b^-$, $a^+ \leqslant c^+ - b^+$ and from $a^- \leqslant a^+$ we have $a^- \leqslant \min\{c^- - b^-, c^+ - b^+\}$.
On the other hand, $a^+ \leqslant c^+ - b^+ \leqslant \max\{c^- - b^-, c^+ - b^+\}$ and, since $C \ominus_{gH} B = [min\{c^- - b^-, c^+ - b^+\}, max\{c^- - b^-, c^+ - b^+\}]$, we have $a^- \leqslant \min\{c^- - b^-, c^+ - b^+\} \leqslant (C \ominus_{gH} B)^-$ and $a^+ \leqslant \max\{c^- - b^-, c^+ - b^+\} \leqslant (C \ominus_{gH} B)^+$, so we conclude that $A \precsim_{LU} C \ominus_{gH} B$.

For (iii), if $A \oplus_M B \succsim_{LU} C$ then $((A \oplus_M B) \ominus_{gH} C) \succsim_{LU} 0$, from which $((A \oplus_M B) \ominus_{gH} C)^- = \min\{(a^- + b^-) - c^-, (a^+ + b^+) - c^+\} \geqslant 0$ and we get $a^- + b^- \geqslant c^-$ and $a^+ + b^+ \geqslant c^+$. Then, $a^- \geqslant c^- - b^-$, $a^+ \geqslant c^+ - b^+$ and from $a^+ \geqslant a^-$ we have $a^+ \geqslant \max\{c^- - b^-, c^+ - b^+\} = (C \ominus_{gH} B)^+$; on the other hand, $a^- \geqslant c^- - b^- \geqslant \min\{c^- - b^-, c^+ - b^+\} = (C \ominus_{gH} B)^-$ and we conclude, by Definition 2.2.6, that $A \succsim_{LU} C \ominus_{gH} B$. □

2.2.6 The $\precsim_{(\gamma^-,\gamma^+)}$-order for intervals

The three order relations $\precsim_{LU}$, $\precnsim_{LU}$ and $\prec_{LU}$ introduced in Definition 2.2.6 can be generalized in terms of the gH-comparison index as follows, highlighting for each the peculiarities that characterize them.

Definition 2.2.8. *([84]) Given two intervals $A = [a^-, a^+] = (\widehat{a}; \widetilde{a})$ and $B = [b^-, b^+] = (\widehat{b}; \widetilde{b})$ and $\gamma^- \leqslant 0$, $\gamma^+ \geqslant 0$ (eventually $\gamma^- = -\infty$ and/or $\gamma^+ = +\infty$) we define the following order relation, denoted $\precsim_{\gamma^-,\gamma^+}$,*

$$A \precsim_{\gamma^-,\gamma^+} B \Longleftrightarrow \begin{cases} \widehat{a} \leqslant \widehat{b} \\ \widetilde{a} \geqslant \widetilde{b} + \gamma^+ \left(\widehat{a} - \widehat{b}\right) \\ \widetilde{a} \leqslant \widetilde{b} + \gamma^- \left(\widehat{a} - \widehat{b}\right). \end{cases} \tag{2.19}$$

It is immediate to see that the relation $\precsim_{\gamma^-,\gamma^+}$ with $\gamma^- \leqslant 0$, $\gamma^+ \geqslant 0$ is reflexive (i.e., $A \precsim_{\gamma^-,\gamma^+} A$), antisymmetric (i.e., if $A \precsim_{\gamma^-,\gamma^+} B$ and $B \precsim_{\gamma^-,\gamma^+} A$ then $A = B$) and transitive (i.e., if $A \precsim_{\gamma^-,\gamma^+} B$ and $B \precsim_{\gamma^-,\gamma^+} C$ then $A \precsim_{\gamma^-,\gamma^+} C$).
It follows that $\precsim_{\gamma^-,\gamma^+}$ is a partial order and $(\mathcal{K}_C, \precsim_{\gamma^-,\gamma^+})$ is a partial ordered set (*poset* for short).

Definition 2.2.9. *([84]) Given two intervals* $A = [a^-, a^+] = (\widehat{a}; \overline{a})$ *and* $B = [b^-, b^+] = (\widehat{b}; \widetilde{b})$ *and* $\gamma^- \leqslant 0$, $\gamma^+ \geqslant 0$ *(eventually* $\gamma^- = -\infty$ *and/or* $\gamma^+ = +\infty$*) we define the following (strict) order relation, denoted* $\precsim_{\gamma^-,\gamma^+}$*,*

$$A \precsim_{\gamma^-,\gamma^+} B \Longleftrightarrow \begin{cases} \widehat{a} < \widehat{b} \\ \widetilde{a} \geqslant \widetilde{b} + \gamma^+ \left(\widehat{a} - \widehat{b}\right) \\ \widetilde{a} \leqslant \widetilde{b} + \gamma^- \left(\widehat{a} - \widehat{b}\right). \end{cases} \tag{2.20}$$

The relation $\precsim_{\gamma^-,\gamma^+}$, with $\gamma^- \leqslant 0$, $\gamma^+ \geqslant 0$, is not reflexive nor antisymmetric, so we say that it is asymmetric (i.e., only one of $A \precsim_{\gamma^-,\gamma^+} B$ or $B \precsim_{\gamma^-,\gamma^+} A$ can be valid), as well as being transitive.

Definition 2.2.10. *([84]) Given two intervals* $A = [a^-, a^+] = (\widehat{a}; \widetilde{a})$ *and* $B = [b^-, b^+] = (\widehat{b}; \widetilde{b})$ *and* $\gamma^- \leqslant 0$, $\gamma^+ \geqslant 0$ *(eventually* $\gamma^- = -\infty$ *and/or* $\gamma^+ = +\infty$*) we define the following (strong) order relation, denoted* $\prec_{\gamma^-,\gamma^+}$*,*

$$A \prec_{\gamma^-,\gamma^+} B \Longleftrightarrow \begin{cases} \widehat{a} < \widehat{b} \\ \widetilde{a} > \widetilde{b} + \gamma^+ \left(\widehat{a} - \widehat{b}\right) \\ \widetilde{a} < \widetilde{b} + \gamma^- \left(\widehat{a} - \widehat{b}\right). \end{cases} \tag{2.21}$$

The relation $\prec_{\gamma^-,\gamma^+}$ with $\gamma^- \leqslant 0$, $\gamma^+ \geqslant 0$ is asymmetric and transitive.

Definition 2.2.11. *Given* $A, B \in \mathcal{K_C}$*, we clearly have that*

$$A \prec_{\gamma^-,\gamma^+} B \implies A \precsim_{\gamma^-,\gamma^+} B \implies A \precapprox_{\gamma^-,\gamma^+} B,$$
$$A \succ_{\gamma^-,\gamma^+} B \implies A \succsim_{\gamma^-,\gamma^+} B \implies A \succapprox_{\gamma^-,\gamma^+} B.$$

We say that A and B are γ*-incomparable if neither* $A \precapprox_{\gamma^-,\gamma^+} B$ *nor* $A \succapprox_{\gamma^-,\gamma^+} B$ *.*

There are specific values of γ^- and γ^+ which make the order relation $\precsim_{\gamma^-,\gamma^+}$ equivalent to LU-order and other orders suggested in the literature (see [30] for details).

Remark 2.2.1. *According to Definition 2.2.8, in order to have* $A \precapprox_{\gamma^-\gamma^+} B$ *we need* $\widehat{a} \leqslant \widehat{b}$ *and* $\widetilde{b} + \gamma^+ \left(\widehat{a} - \widehat{b}\right) \leqslant \widetilde{a} \leqslant \widetilde{b} + \gamma^- \left(\widehat{a} - \widehat{b}\right)$*. It follows that for the order relation* $\precapprox_{\gamma^-,\gamma^+}$ *with* $\gamma^- \leqslant 0, \gamma^+ \geqslant 0$ *in* $\mathcal{K}_C$*, we have the equivalence*

$$A \precapprox_{\gamma^-,\gamma^+} B \iff (A \precsim_{\gamma^-,\gamma^+} B \text{ or } A = B). \tag{2.22}$$

Proposition 2.2.7. *([84]) Let A and B be two intervals; then it holds that*

(1) $A \precsim_{LU} B \Longleftrightarrow \begin{cases} \widehat{a} < \widehat{b} \\ \widetilde{b} \leqslant \widetilde{a} + (\widehat{b} - \widehat{a}) \\ \widetilde{b} \geqslant \widetilde{a} - (\widehat{b} - \widehat{a}) \end{cases} \iff A \precsim_{-1,1} B,$

i.e., (2.20) with $\gamma^- = -1$ *and* $\gamma^+ = 1$*;*

(2) $A \precsim_{CW_M} B \Longleftrightarrow \widehat{a} < \widehat{b},\, \widetilde{a} \geqslant \widetilde{b} \Longleftrightarrow A \precsim_{-\infty,0} B$, *i.e., (2.20) with* $\gamma^- = -\infty$ *and* $\gamma^+ = 0$*;*

(3) $A \precsim_{CW_m} B \Longleftrightarrow \widehat{a} < \widehat{b},\, \widetilde{a} \leqslant \widetilde{b} \Longleftrightarrow A \precsim_{0,+\infty} B$, *i.e., (2.20) with* $\gamma^- = 0$ *and* $\gamma^+ = +\infty$*;*

(4) $A \precsim_{LC} B \Longleftrightarrow \widehat{a} < \widehat{b},\, a^- \leqslant b^- \Longleftrightarrow \widehat{a} < \widehat{b},\, \widetilde{b} \leqslant \widetilde{a} + (\widehat{b} - \widehat{a}) \Longleftrightarrow A \precsim_{-\infty,1} B$, *i.e., (2.20) with* $\gamma^- = -\infty$ *and* $\gamma^+ = 1$*;*

(5) $A \precsim_{UC} B \Longleftrightarrow \widehat{a} < \widehat{b},\, a^+ \leqslant b^+ \Longleftrightarrow \widehat{a} < \widehat{b},\, \widetilde{b} \geqslant \widetilde{a} - (\widehat{b} - \widehat{a}) \Longleftrightarrow A \precsim_{-1,+\infty} B$, *i.e., (2.20) with* $\gamma^- = -1$ *and* $\gamma^+ = +\infty$.

By varying the two parameters $-\infty \leqslant \gamma^- \leqslant 0$, $0 \leqslant \gamma^+ \leqslant +\infty$, we obtain a continuum of partial order relations for intervals and we have the following equivalences:

Proposition 2.2.8. *([84]) If A and B are two intervals then it holds that*

(1) $A \precapprox_{LU} B \Longleftrightarrow A \precapprox_{-1,1} B$*;*

(2) $A \precapprox_{CW_M} B \Longleftrightarrow A \precapprox_{-\infty,0} B$*;*

(3) $A \precapprox_{CW_m} B \Longleftrightarrow A \precapprox_{0,+\infty} B$*;*

(4) $A \precapprox_{LC} B \Longleftrightarrow A \precapprox_{-\infty,1} B$*;*

(5) $A \precapprox_{UC} B \Longleftrightarrow A \precapprox_{-1,+\infty} B$.

All this can be represented graphically by making use of what has been said in Subsection 2.1.3 and by considering the interval $A = [a^-, a^+] = (\widehat{a}; \widetilde{a})$ as a point in the midpoint half-plane $(\widehat{z}, \widetilde{z})$.

Definition 2.2.12. *For a given interval $A = (\widehat{a}; \widetilde{a})$ and any interval $X = [x^-, x^+] = (\widehat{x}; \widetilde{x})$, we can say that A dominates X (or X is dominated by A) with respect to γ-order $\precapprox_{\gamma^-,\gamma^+}$ (i.e., $\gamma^- \leqslant \gamma_{A,X} \leqslant \gamma^+$) if and only if the following conditions are satisfied:*

1. $\widehat{a} \leqslant \widehat{x}$;
2. $\gamma_{A,X} \leqslant \gamma^+ \Leftrightarrow \widetilde{x} \leqslant \gamma^+(\widehat{x} - \widehat{a}) + \widetilde{a}$;
3. $\gamma_{A,X} \geqslant \gamma^- \Leftrightarrow \widetilde{x} \geqslant \gamma^-(\widehat{x} - \widehat{a}) + \widetilde{a}$.

By varying $\gamma^- \leqslant 0$ and $\gamma^+ \geqslant 0$, we can obtain an infinitive of partial order; for instance, the *LU*-order corresponds as usual to the values $\gamma^- = -1$ and $\gamma^+ = +1$.

Definition 2.2.13. *([84]) For a given interval $A = (\widehat{a}; \widetilde{a})$, we define the following sets of intervals X which are* *(a)* $(\precsim_{\gamma^-,\gamma^+})$*-dominated by A:*

$$\mathbb{D}_<(A;\gamma^-,\gamma^+) = \{X \in \mathcal{K}_C \mid A \precsim_{\gamma^-,\gamma^+} X\}; \tag{2.23}$$

(b) $(\precsim_{\gamma^-,\gamma^+})$*-dominating A:*

$$\mathbb{D}_>(A;\gamma^-,\gamma^+) = \{X \in \mathcal{K}_C \mid X \precsim_{\gamma^-,\gamma^+} A\}; \tag{2.24}$$

(c) $(\precsim_{\gamma^-,\gamma^+})$*-incomparable with A:*

$$\mathbb{I}(A;\gamma^-,\gamma^+) = \{X \in \mathcal{K}_C \mid X \notin \mathbb{D}_<(A;\gamma^-,\gamma^+), X \notin \mathbb{D}_>(A;\gamma^-,\gamma^+)\}. \tag{2.25}$$

From the graphic point of view, the sets $\mathbb{D}_<(A;\gamma^-,\gamma^+)$, $\mathbb{D}_>(A;\gamma^-,\gamma^+)$ and $\mathbb{I}(A;\gamma^-,\gamma^+)$ can be easily represented in the midpoint half-plane $(\widehat{z};\widetilde{z})$ as shown in Figure 2.7.

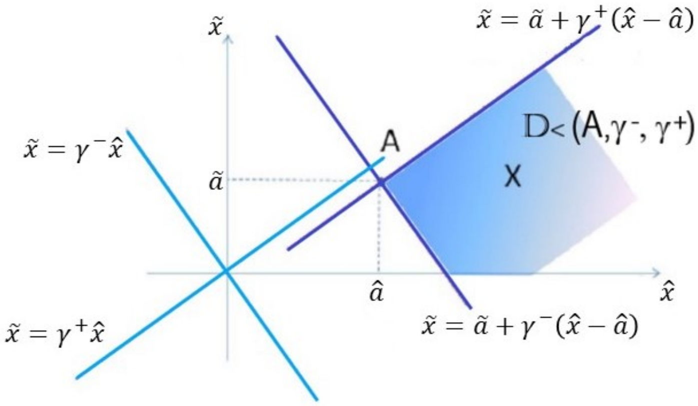


Figure 2.7: The blue area represents the set X of intervals dominated by interval A, i.e., $\mathbb{D}_<(A;\gamma^-,\gamma^+)$.

Proposition 2.2.9. *([84]) For any* $-\infty < \gamma^- < 0$, $0 < \gamma^+ < +\infty$ *and any intervals* $A, B \in \mathcal{K}_C$*, we have*

a. $A \precsim_{\gamma^-,\gamma^+} B$ *if and only if* $\mathbb{D}_<(B;\gamma^-,\gamma^+) \subseteq \mathbb{D}_<(A;\gamma^-,\gamma^+)$*;*

b. $A = B$ *if and only if* $\mathbb{D}_<(A;\gamma^-,\gamma^+) = \mathbb{D}_<(B;\gamma^-,\gamma^+)$*;*

c. $\emptyset = \mathbb{D}_<(A;\gamma^-,\gamma^+) \bigcap \mathbb{I}(A;\gamma^-,\gamma^+) = \mathbb{D}_>(A;\gamma^-,\gamma^+) \bigcap \mathbb{I}(A;\gamma^-,\gamma^+)$*;*

d. $\{A\} = \mathbb{D}_<(A;\gamma^-,\gamma^+) \bigcap \mathbb{D}_>(A;\gamma^-,\gamma^+)$*;*

e. $\mathcal{K}_C = \mathbb{I}(A;\gamma^-,\gamma^+) \bigcup \mathbb{D}_<(A;\gamma^-,\gamma^+) \bigcup \mathbb{D}_>(A;\gamma^-,\gamma^+)$*.*

(For the proof of Propositions 2.2.7, 2.2.8 and 2.2.9, see [84]).

Example 2.2.1. *Considering the midpoint-radius plane* $(\widehat{x};\widetilde{x})$, *the four Figures (2.8, 2.9, 2.10 and 2.11) show, for the given interval* $A = (0;5) = [-5,5]$, *the corresponding set of dominated, dominating and incomparable intervals, respectively the sets:*

$\mathbb{D}_{<}(A;\gamma^-,\gamma^+)$ *(red colored pictures),*

$\mathbb{D}_{>}(A;\gamma^-,\gamma^+)$ *(blue-colored regions),*

$\mathbb{I}(A;\gamma^-,\gamma^+)$ *(in green color),*

for partial orders ($\precapprox_{\gamma^-,\gamma^+}$) *with four different pairs* (γ^-,γ^+).

In particular it is shown:

$(-1,1)$-*dominance (i.e., LU-dominance) in Figure 2.8,*

$(-0.5,0.5)$-*dominance in Figure 2.9,*

$(-1,2)$-*dominance in Figure 2.10,*

$(-1,0.5)$-*dominance in Figure 2.11.*

All the figures consider intervals $X = (\widehat{x};\widetilde{x})$ *in the range* $\widehat{x} \in [-15,15]$ *and* $\widetilde{x} \in [0,20]$.

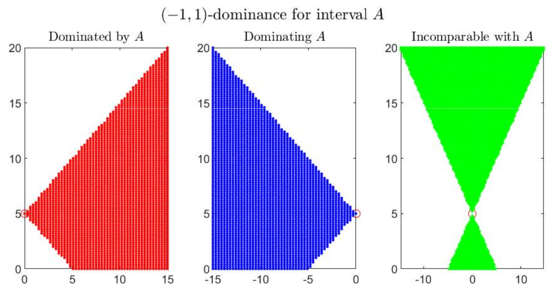


Figure 2.8: $(-1,1)$-dominance (i.e., LU-dominance) for interval A in the midpoint-radius plane $(\widehat{x};\widetilde{x})$: representation of the set of dominated (red), dominating (blue) and incomparable (green) intervals.

By inspecting the four figures, we see that with respect to the LU-order (Figure 2.8, with $\gamma^+ = -\gamma^- = 1$*) or an order with* $\gamma^- + \gamma^+ = 0$ *(Figure 2.9, with* $\gamma^+ = -\gamma^- = 0.5$*), the set of incomparable intervals is symmetric with respect to the vertical line* $\widehat{x} = \widehat{a}$*; when* $\gamma^- + \gamma^+ > 0$ *(Figure 2.10) the right part of the incomparable region, determined by an increase of* $\gamma^+ > 1$*, tends to become more vertical and reduces in favor of the dominated region (red colored) and the dominating region (blue-colored). The opposite effect appears if* $\gamma^+ < 1$ *decreases (Figure 2.11).*

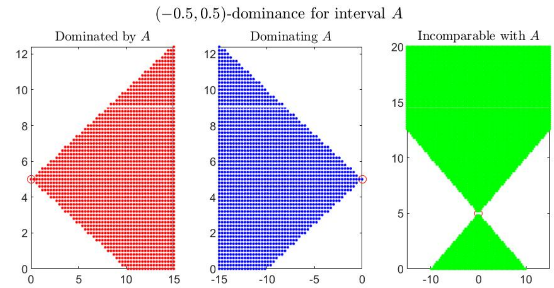


Figure 2.9: $(-0.5, 0.5)$-dominance for interval A in the midpoint-radius plane $(\widehat{x}; \widetilde{x})$: representation of the set of dominated, dominating and incomparable intervals.

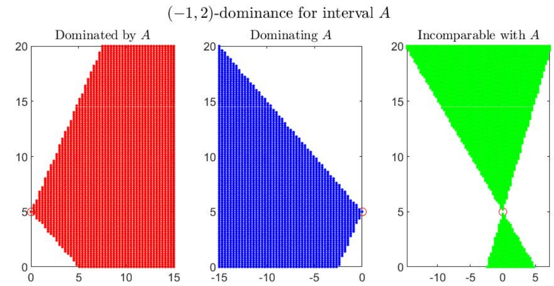


Figure 2.10: $(-1, 2)$-dominance for interval A in the midpoint-radius plane $(\widehat{x}; \widetilde{x})$: representation of the set of dominated, dominating and incomparable intervals.

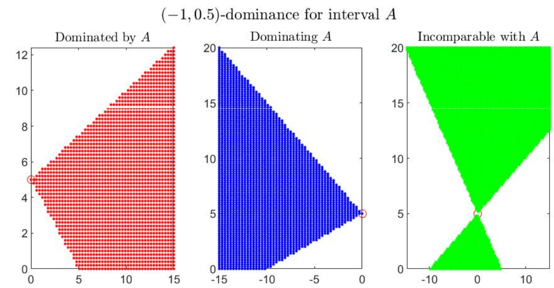


Figure 2.11: $(-1, 0.5)$-dominance for interval A in the midpoint-radius plane $(\widehat{x}; \widetilde{x})$: representation of the set of dominated, dominating and incomparable intervals.

2.2.7 The $\precapprox_{(\gamma^-,\gamma^+)}$-order and lattice theory

The poset structure of $\mathcal{K}_C$, endowed with the partial order $\precapprox_{\gamma^-,\gamma^+}$, can be further analyzed by considering the basic concepts of least upper bound and greatest lower bound introduced in Subsection 2.2.1. For two intervals $A, B \in \mathcal{K}_C$, a (common) *upper bound* is an interval $Z \in \mathcal{K}_C$ such that $A \precapprox_{\gamma^-,\gamma^+} Z$ and $B \precapprox_{\gamma^-,\gamma^+} Z$. A (common) *lower bound* is an interval

$Z \in \mathcal{K}_C$ such that $Z \precsim_{\gamma^-,\gamma^+} A$ and $Z \precsim_{\gamma^-,\gamma^+} B$.

The *least upper bound* for A, B, denoted $lub(A, B)$ or $\sup(A, B)$, is a common upper bound Z such that every other upper bound Z' is such that $Z \precsim_{\gamma^-,\gamma^+} Z'$; analogously, the *greatest lower bound* for A, B, denoted $glb(A, B)$ or $\inf(A, B)$, is a common lower bound Z such that every other lower bound Z' is such that $Z' \precsim_{\gamma^-,\gamma^+} Z$. It is immediate to see that $\inf(A, B)$ and $\sup(A, B)$ always exist (and are unique) for any $A, B \in \mathcal{K}_C$ (see [30] for details). It follows that, according to Definition 2.2.1, the structure $(\mathcal{K}_C, \precsim_{\gamma^-,\gamma^+})$ is a lattice.

If $\mathbb{S} \subset \mathcal{K}_C$ is any subset of intervals, we say that $\mathbb{S}$ is *bounded from below (lower bounded)* with respect to $\precsim_{\gamma^-,\gamma^+}$ if and only if there exists $L \in \mathcal{K}_C$ such that $L \precsim_{\gamma^-,\gamma^+} X$ for all $X \in \mathbb{S}$ and we say that $\mathbb{S}$ is *bounded from above (upper bounded)* with respect to $\precsim_{\gamma^-,\gamma^+}$ if and only if there exists $U \in \mathcal{K}_C$ such that $X \precsim_{\gamma^-,\gamma^+} U$ for all $X \in \mathbb{S}$. If $\mathbb{S} \subset \mathcal{K}_C$ is both lower and upper bounded, we say it is *bounded* (see Figure 2.12).

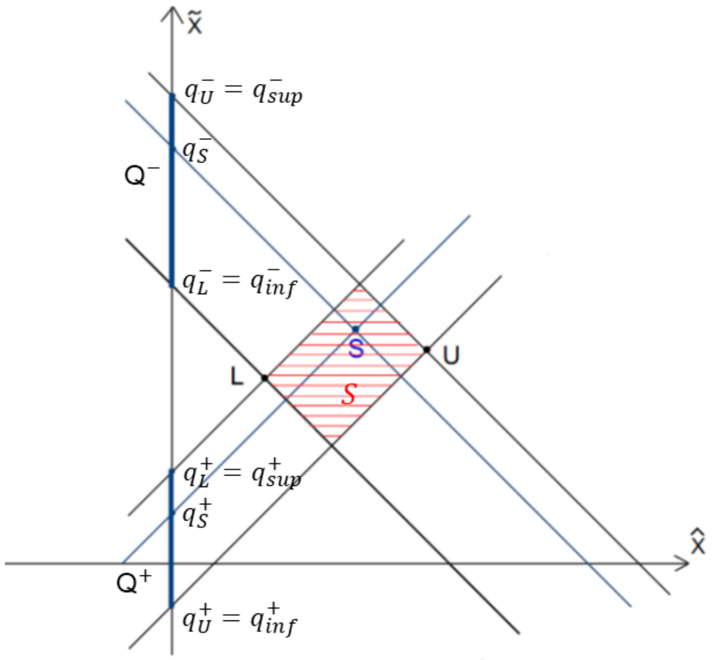


Figure 2.12: Subset of intervals bounded from below and from above.

Now we may define $\overline{\mathcal{K}_\mathcal{C}} = \mathcal{K}_\mathcal{C} \cup \{-\infty, +\infty\}$ where:

$$-\infty \equiv (-\infty; 0) \equiv]-\infty, -\infty[\in \overline{\mathbb{R}}$$

$$+\infty \equiv (+\infty; 0) \equiv]+\infty, +\infty[\in \overline{\mathbb{R}}$$

so that, for all $\gamma^- < 0$, $\gamma^+ > 0$, it is:

$$-\infty \precapprox_{\gamma^-,\gamma^+} A \precapprox_{\gamma^-,\gamma^+} +\infty, \quad \forall A \in \mathcal{K}_{\mathcal{C}}.$$

It follows that, considering $\overline{\mathcal{K}_{\mathcal{C}}}$ endowed with the partial order $\precapprox_{\gamma^-,\gamma^+}$, the structure $(\overline{\mathcal{K}_{\mathcal{C}}}, \precapprox_{\gamma^-,\gamma^+})$ is a bounded lattice (see Figure 2.13).

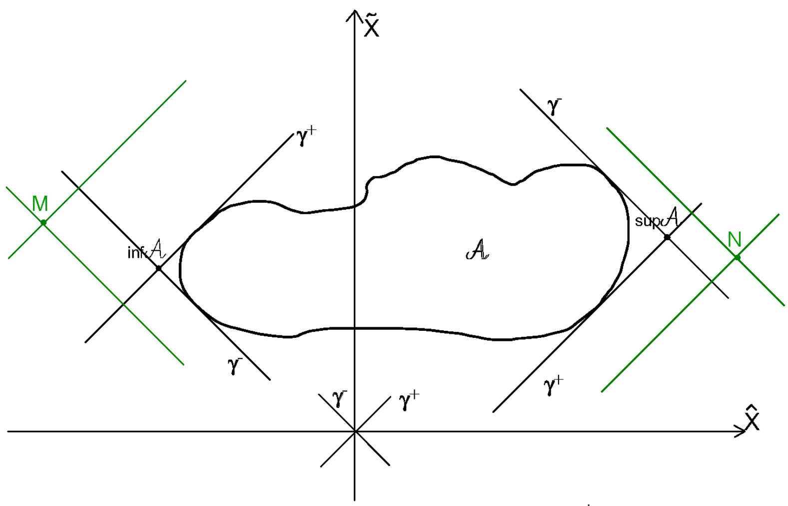


Figure 2.13: Bounded family $\mathcal{A} \subset \mathcal{K}_{\mathcal{C}}$.

This means that for a family $\mathcal{A} \subset \mathcal{K}_{\mathcal{C}}$, such that $\mathcal{A}$ is bounded, (i.e., $\exists\, M, N \in \mathcal{K}_{\mathcal{C}}$ with $M \precapprox_{\gamma^-,\gamma^+} A \precapprox_{\gamma^-,\gamma^+} N$,$\forall A \in \mathcal{A}$), there exist both:

$$\inf(\mathcal{A}) \in \mathcal{K}_{\mathcal{C}} \text{ and } \sup(\mathcal{A}) \in \mathcal{K}_{\mathcal{C}}$$

with

$$M \precapprox_{\gamma^-,\gamma^+} \inf(\mathcal{A}) \text{ and } \sup(\mathcal{A}) \precapprox_{\gamma^-,\gamma^+} N.$$

Definition 2.2.14. *If* $\inf(\mathcal{A}) \in \mathcal{A}$ *we say that it is* $\min(\mathcal{A})$; *if* $\sup(\mathcal{A}) \in \mathcal{A}$ *we say that it is* $\max(\mathcal{A})$.

It is important to remark the fact that every bounded subset of $\mathcal{K}_C$ admits *inf* and *sup*.

Proposition 2.2.10. *([84]) Consider a partial order* $\precapprox_{\gamma^-,\gamma^+}$ *on* $\mathcal{K}_C$ *and let* $\mathbb{S} \subset \mathcal{K}_C$ *be any nonempty bounded subset of intervals. Then, there exist both* $\inf_{\gamma^-,\gamma^+}(\mathbb{S}), \sup_{\gamma^-,\gamma^+}(\mathbb{S}) \in \mathcal{K}_C$ *such that for all* $X \in \mathbb{S}$

$$\inf(\mathbb{S}) \precapprox_{\gamma^-,\gamma^+} X \precapprox_{\gamma^-,\gamma^+} \sup(\mathbb{S}). \tag{2.26}$$

We also have, for all $A \in \mathcal{K}_C$,

$$A = \inf(\mathbb{D}_<(A;\gamma^-,\gamma^+)) \text{ and } A = \sup(\mathbb{D}_>(A;\gamma^-,\gamma^+)). \tag{2.27}$$

Proof. We will prove only (2.26) by a constructive procedure (see Figure 2.12 again for help); the proof of equations in (2.27) is immediate.
Let $L = (\widehat{l}; \widetilde{l}) \in \mathcal{K}_C$ be any lower bound and $U = (\widehat{u}; \widetilde{u}) \in \mathcal{K}_C$ any upper bound for $\mathbb{S}$ and consider the four lines, in the half-plane $(\widehat{x}; \widetilde{x})$, with equations

$$\widetilde{x} = \widetilde{l} + \gamma^{+}\left(\widehat{x} - \widehat{l}\right) \text{ and } \widetilde{x} = \widetilde{l} + \gamma^{-}\left(\widehat{x} - \widehat{l}\right) \text{ (through point } L),$$

$$\widetilde{x} = \widetilde{u} + \gamma^{+}\left(\widehat{x} - \widehat{u}\right) \text{ and } \widetilde{x} = \widetilde{u} + \gamma^{-}\left(\widehat{x} - \widehat{u}\right) \text{ (through point } U).$$

They intersect the vertical axis ($\widehat{x} = 0$) with intercepts, respectively at: $q_L^{+} = \widetilde{l} - \gamma^{+}\widehat{l}$, $q_L^{-} = \widetilde{l} - \gamma^{-}\widehat{l}$, and $q_U^{+} = \widetilde{u} - \gamma^{+}\widehat{u}$, $q_U^{-} = \widetilde{u} - \gamma^{-}\widehat{u}$.

Considering an arbitrary element $S = (\widehat{s}; \widetilde{s}) \in \mathbb{S}$, the two lines trough S with angular coefficients γ^{+} and γ^{-}, with equations $\widetilde{x} = \widetilde{s} + \gamma^{+}\left(\widehat{x} - \widehat{s}\right)$ and $\widetilde{x} = \widetilde{s} + \gamma^{-}\left(\widehat{x} - \widehat{s}\right)$, respectively, have intercepts $q_S^{+} = \widetilde{s} - \gamma^{+}\widehat{s}$ and $q_S^{-} = \widetilde{s} - \gamma^{-}\widehat{s}$ and their sets $Q^{+} = \left\{q_S^{+} | S \in \mathbb{S}\right\}$ and $Q^{-} = \left\{q_S^{-} | S \in \mathbb{S}\right\}$ are both bounded with $q_U^{+} \leqslant q_S^{+} \leqslant q_L^{+}$ and $q_L^{-} \leqslant q_S^{-} \leqslant q_U^{-}$ for all $S \in \mathbb{S}$.
Consequently, there exist the four real numbers $q_{inf}^{+} = \inf Q^{+}$, $q_{sup}^{+} = \sup Q^{+}$, $q_{inf}^{-} = \inf Q^{-}$, $q_{sup}^{-} = \sup Q^{-}$ with $q_{inf}^{+} \leqslant q_{sup}^{+}$ and $q_{inf}^{-} \leqslant q_{sup}^{-}$.
Finally, the intersection point of the two lines $\widetilde{x} = q_{sup}^{+} + \gamma^{+}\widehat{x}$ and $\widetilde{x} = q_{inf}^{-} + \gamma^{-}\widehat{x}$ corresponds to the interval $\inf \mathbb{S} \in \mathcal{K}_C$; analogously, the intersection point of the two lines $\widetilde{x} = q_{inf}^{+} + \gamma^{+}\widehat{x}$ and $\widetilde{x} = q_{sup}^{-} + \gamma^{-}\widehat{x}$ corresponds to the interval $\sup \mathbb{S} \in \mathcal{K}_C$. More precisely, we have

$$\inf \mathbb{S} = \left(\frac{q_{sup}^{+} - q_{inf}^{-}}{\gamma^{-} - \gamma^{+}}; \frac{\gamma^{-} q_{sup}^{+} - \gamma^{+} q_{inf}^{-}}{\gamma^{-} - \gamma^{+}}\right)$$

and

$$\sup \mathbb{S} = \left(\frac{q_{inf}^{+} - q_{sup}^{-}}{\gamma^{-} - \gamma^{+}}; \frac{\gamma^{-} q_{inf}^{+} - \gamma^{+} q_{sup}^{-}}{\gamma^{-} - \gamma^{+}}\right)$$

This completes the proof. □

Therefore, according to Definition 2.2.2, it is possible to conlcude that $(\overline{\mathcal{K}_{\mathcal{C}}}, \precsim_{\gamma^{-},\gamma^{+}})$ is a bounded-complete lattice.

Furthermore, if for a nonempty bounded subset $\mathbb{S} \subset \mathcal{K}_C$ we have that $\inf(\mathbb{S})$ or $\sup(\mathbb{S})$ are elements of $\mathbb{S}$, then there exist the intervals $\min(\mathbb{S})$ or, respectively, $\max(\mathbb{S})$.

As shown in Figure 2.14 and as will be further analyzed in Subsection 4.1.1, interesting bounded subsets in $\mathcal{K}_C$ are the following:

- the "segment" with extremes $A, B \in \mathcal{K}_C$, given by the convex combination of intervals and defined by

$$S(A, B) = \{X_t | X_t = (1 - t)A + tB, t \in [0, 1]\}; \qquad (2.28)$$

- the "interval" (of intervals) with extremes $A, B \in \mathcal{K}_C$, assuming $A \precsim_{\gamma^-,\gamma^+} B$ (here, the dominance is essential), defined by

$$[[A,B]]_{\gamma^-,\gamma^+} = \left\{X \in \mathcal{K}_C | A \precsim_{\gamma^-,\gamma^+} X \precsim_{\gamma^-,\gamma^+} B\right\}. \tag{2.29}$$

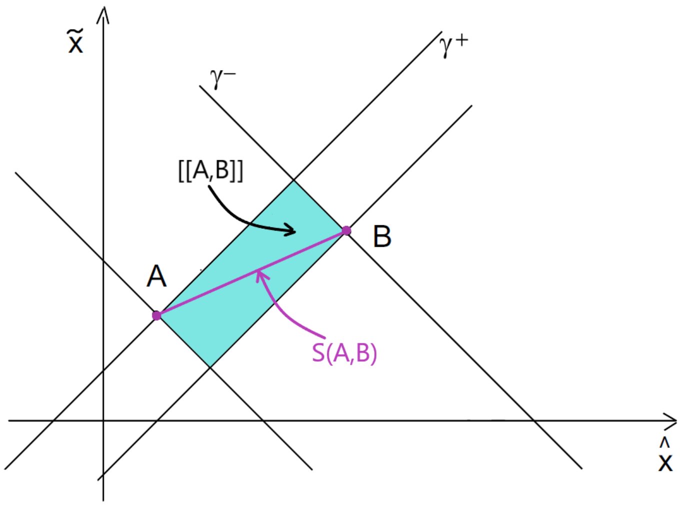


Figure 2.14: Example of "segment" and "interval" of extremes $A, B \in \mathcal{K}_C$, assuming $A \precsim_{\gamma^-,\gamma^+} B$.

If $\mathbb{S}$ is a bounded subset of $\mathcal{K}_C$, we clearly have $\mathbb{S} \subseteq [[\inf(\mathbb{S}), \sup(\mathbb{S})]]_{\gamma^-,\gamma^+}$, with equality if and only if $\mathbb{S} = [[A,B]]_{\gamma^-,\gamma^+}$ with $A = \inf(\mathbb{S}), B = \sup(\mathbb{S})$.

We conclude this section with an interesting property.

Proposition 2.2.11. *([84]) For a given partial order $\precsim_{\gamma^-,\gamma^+}$ with $\gamma^- \leqslant 0$, $\gamma^+ \geqslant 0$, consider the partial order $\precsim_{-\gamma^+,-\gamma^-}$; then, for all $A, B \in \mathcal{K}_{\mathcal{C}}$,*

$$A \precsim_{\gamma^-,\gamma^+} B \Leftrightarrow (-B) \precsim_{-\gamma^+,-\gamma^-} (-A) \tag{2.30}$$

where $-A$ and $-B$ are the opposite intervals of A and B.

Proof. Starting with inequalities (2.19) that define $A \precsim_{\gamma^-,\gamma^+} B$ and recalling that $-A = (-\widehat{a}; \widetilde{a})$ the conclusion follows after a few simple algebraic manipulations:

$$A \precsim_{\gamma^-,\gamma^+} B \iff \begin{cases} \widehat{a} \leqslant \widehat{b} \\ \widetilde{a} \geqslant \widetilde{b} + \gamma^+ \left(\widehat{a} - \widehat{b}\right) \\ \widetilde{a} \leqslant \widetilde{b} + \gamma^- \left(\widehat{a} - \widehat{b}\right) \end{cases} \iff \begin{cases} -\widehat{a} \geqslant -\widehat{b} \\ -\widetilde{a} \leqslant -\widetilde{b} - \gamma^+ \left(\widehat{a} - \widehat{b}\right) \\ -\widetilde{a} \geqslant -\widetilde{b} - \gamma^- \left(\widehat{a} - \widehat{b}\right) \end{cases}$$

$$\iff \begin{cases} -\widehat{b} \leqslant -\widehat{a} \\ -\widetilde{b} \geqslant -\widetilde{a} - \gamma^+ \left(\widehat{b} - \widehat{a}\right) \\ -\widetilde{b} \leqslant -\widetilde{a} - \gamma^- \left(\widehat{b} - \widehat{a}\right) \end{cases} \iff (-B) \precsim_{-\gamma^+,-\gamma^-} (-A). \qquad \square$$

In particular, if $\gamma^- + \gamma^+ = 0$, i.e., $\gamma^+ = -\gamma^- = \gamma \geqslant 0$ so that

$$(\precapprox_{\gamma^-,\gamma^+}) \equiv (\precapprox_{-\gamma^+,-\gamma^-}) \equiv (\precapprox_{\gamma}),$$

we have that for any bounded subset $\mathbb{S} \subset \mathcal{K}_C$,

$$\inf(\mathbb{S}) = -\sup(-\mathbb{S}) \tag{2.31}$$

where the (bounded) subset $-\mathbb{S} \subset \mathcal{K}_C$ is defined by

$$-\mathbb{S} = \{-X | X \in \mathbb{S}\}. \tag{2.32}$$

2.2.8 The $\precapprox_{(\gamma^-,\gamma^+)}$-order and gH-difference

At this point, it is interesting to try to express a partial order $\precapprox_{\gamma^-,\gamma^+}$ in terms of the gH-difference $A \ominus_{gH} B = (\widehat{a} - \widehat{b}; |\widetilde{a} - \widetilde{b}|)$.

Recall that, from (2.19), i.e.,

$$A \precapprox_{\gamma^-,\gamma^+} B \Longleftrightarrow \begin{cases} \widehat{a} \leqslant \widehat{b} \\ \widetilde{a} \geqslant \widetilde{b} + \gamma^+ \left(\widehat{a} - \widehat{b}\right) \\ \widetilde{a} \leqslant \widetilde{b} + \gamma^- \left(\widehat{a} - \widehat{b}\right), \end{cases} \tag{2.33}$$

we can write the reverse order

$$A \succapprox_{\gamma^-,\gamma^+} B \Longleftrightarrow \begin{cases} \widehat{a} \geqslant \widehat{b} \\ \widetilde{a} \leqslant \widetilde{b} + \gamma^+ \left(\widehat{a} - \widehat{b}\right) \\ \widetilde{a} \geqslant \widetilde{b} + \gamma^- \left(\widehat{a} - \widehat{b}\right). \end{cases} \tag{2.34}$$

Remark 2.2.2. *One may think that condition $\widehat{a} \leqslant \widehat{b}$ is redundant in (2.33); indeed, if $\gamma^- < 0$ or $\gamma^+ > 0$, it is implied by the second and third conditions. But if $\widetilde{a} = \widetilde{b} = 0$ and $\gamma^- = \gamma^+ = 0$, the order reduces to the standard order for real numbers while the second and the third conditions reduce to inequalities $0 \geqslant 0$ and $0 \leqslant 0$. For this reason we will always include condition $\widehat{a} \leqslant \widehat{b}$ in (2.33).*
If $\gamma^- < 0$ or $\gamma^+ > 0$ and the second and third conditions are both satisfied with equality, then $A = B$ and vice-versa.

Furthermore, for an interval X, from equation (2.33), we have that

$$X \precapprox_{\gamma^-,\gamma^+} 0 \Longleftrightarrow \begin{cases} \widehat{x} \leqslant 0 \\ \widetilde{x} \geqslant 0 + \gamma^+ (\widehat{x} - 0) \\ \widetilde{x} \leqslant 0 + \gamma^- (\widehat{x} - 0) \end{cases} \Longleftrightarrow \begin{cases} \widehat{x} \leqslant 0 \\ \widetilde{x} \geqslant \gamma^+ \widehat{x} \\ \widetilde{x} \leqslant \gamma^- \widehat{x}. \end{cases}$$

Since the second inequality is always verified as a consequence of the others, we can say that

$$X \precsim_{\gamma^-,\gamma^+} 0 \;\Leftrightarrow\; (\widehat{x} \leqslant 0 \text{ and } \widetilde{x} \leqslant \gamma^- \widehat{x}). \tag{2.35}$$

In a similar way we have that

$$0 \precsim_{\gamma^-,\gamma^+} X \;\Leftrightarrow\; (\widehat{x} \geqslant 0 \text{ and } \widetilde{x} \leqslant \gamma^+ \widehat{x}). \tag{2.36}$$

Therefore, considering the definition of $A \ominus_{gH} B$, we have that:
$A \ominus_{gH} B \precsim_{\gamma^-,\gamma^+} 0$ if and only if $(\widehat{a} - \widehat{b} \leqslant 0$ and $|\widetilde{a} - \widetilde{b}| \leqslant \gamma^-(\widehat{a} - \widehat{b}))$, which are obvious consequences of the first and third conditions ($\widehat{a} \leqslant \widehat{b}$ and $\widetilde{a} \leqslant \widetilde{b} + \gamma^-(\widehat{a} - \widehat{b})$) of (2.33).

The procedure is completely analogous in case $0 \precsim_{\gamma^-,\gamma^+} B \ominus_{gH} A$ and, considering the equality $A \ominus_{gH} B = -(B \ominus_{gH} A)$, we obtain the following results.

Lemma 2.2.1. *([84]) Let $A, B \in \mathcal{K}_C$ and consider the lattice $(\mathcal{K}_C, \precsim_{\gamma^-,\gamma^+})$ with $\gamma^- < 0$ and $\gamma^+ > 0$; then*
(1a) $A \precsim_{\gamma^-,\gamma^+} B \Longrightarrow A \ominus_{gH} B \precsim_{\gamma^-,\gamma^+} 0$
(in the right part of implication, γ^+ is not involved);
(1b) $A \precsim_{\gamma^-,\gamma^+} B \Longrightarrow 0 \precsim_{\gamma^-,\gamma^+} B \ominus_{gH} A$
(in the right part of implication, γ^- is not involved);
(2) $A \precsim_{\gamma^-,\gamma^+} B \Longleftrightarrow \big(A \ominus_{gH} B \precsim_{\gamma^-,\gamma^+} 0$ and $B \ominus_{gH} A \succsim_{-\gamma^+,-\gamma^-} 0\big)$;
(3) Assuming $-\gamma^- = \gamma^+ = \gamma > 0$, then

$$A \precsim_{\gamma} B \Longleftrightarrow (A \ominus_{gH} B \precsim_{\gamma} 0) \Longleftrightarrow (B \ominus_{gH} A \succsim_{\gamma} 0).$$

Remark 2.2.3. *Considering the distinction between type (i) and type (ii) of gH-difference defined in* (2.1)*, several other implications can be established, not used in this work. For example in type (i), it is $\widetilde{a} \geqslant \widetilde{b}$ and we have*

- *$A \ominus_{gH} B \precsim_{\gamma^-,\gamma^+} 0$ if and only if $\Big(\widehat{a} \leqslant \widehat{b}$ and $\widetilde{b} \geqslant \widetilde{a} + \gamma^-(\widehat{b} - \widehat{a})\Big)$*
  (so γ^+ is not involved);
- *$A \ominus_{gH} B \succsim_{\gamma^-,\gamma^+} 0$ if and only if $\Big(\widehat{a} \geqslant \widehat{b}$ and $\widetilde{b} \geqslant \widetilde{a} + \gamma^+(\widehat{b} - \widehat{a})\Big)$*
  (so γ^- is not involved).

Figures 2.15, 2.16, 2.17 and 2.18 show, for the given interval $A = (0; 5) = [-5, 5]$, the gH-differences $C = A \ominus_{gH} X$ comparing it with the corresponding set of dominated, dominating and incomparable intervals in different cases of dominance.
In particular it is shown:

$(-1, 1)$-dominance in Figure 2.15,

$(-0.5, 0.5)$-dominance in Figure 2.16,

$(-1, 2)$-dominance in Figure 2.17,

and $(-1, 0.5)$-dominance in Figure 2.18.

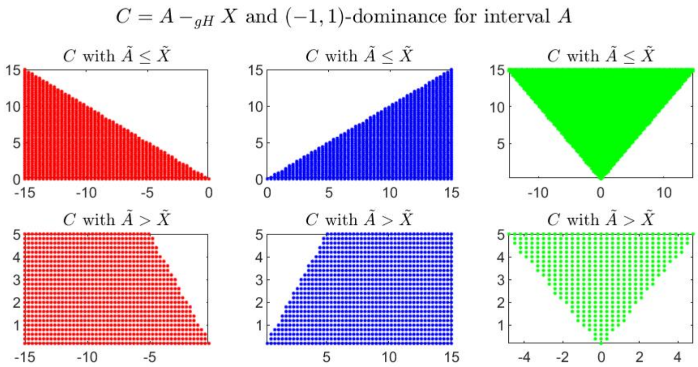


Figure 2.15: $(-1, 1)$-dominance for interval A: representation of the gH-differences $A \ominus_{gH} X$.

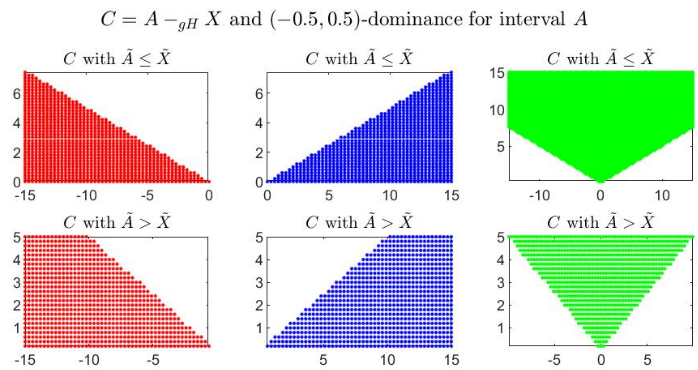


Figure 2.16: $(-0.5, 0.5)$-dominance for interval A: representation of the gH-differences $A \ominus_{gH} X$.

In all figures, the three pictures on top give the gH-differences for intervals X with $\widetilde{x} \leqslant \widetilde{a}$ and the pictures on bottom correspond to the intervals with $\widetilde{x} > \widetilde{a}$.

In Figures 2.15 and 2.16 (where we have $\gamma^- + \gamma^+ = 0$) it can be seen how the two pictures (top and bottom) coincide as they are perfectly superimposable, while in Figures 2.17 and 2.18 we have $\gamma^- + \gamma^+ \neq 0$ and the

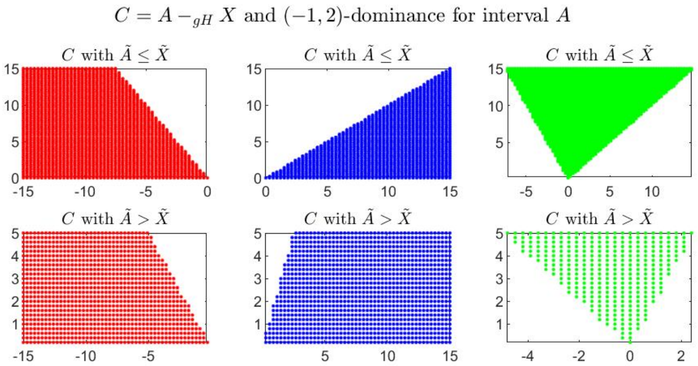


Figure 2.17: $(-1, 2)$-dominance for interval A: representation of the gH-differences $A \ominus_{gH} X$.

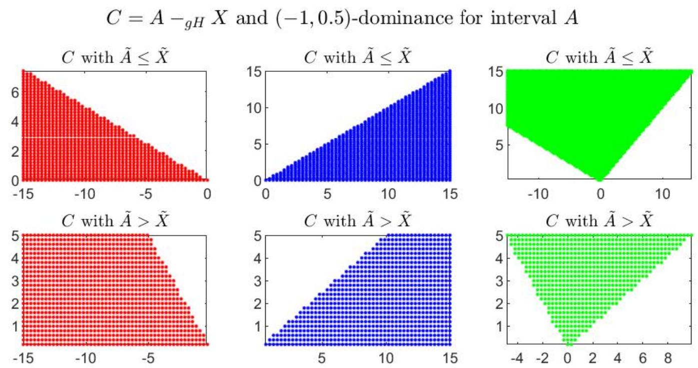


Figure 2.18: $(-1, 0.5)$-dominance for interval A: representation of the gH-differences $A \ominus_{gH} X$.

incomparable sets are not symmetric with respect to the vertical line $\widehat{x} = \widehat{a}$. In this cases, indeed, gH-differences $A \ominus_{gH} X$ are differently asymmetric and the position of $A \ominus_{gH} X$ with respect to 0 does not correspond uniquely to the position of X with respect to A; the distinction is determined by the midpoint values, i.e., when $\widehat{x} \leqslant \widehat{a}$ or $\widehat{x} < \widehat{a}$.

2.2.9 Further developments in $\precsim_{(\gamma^-,\gamma^+)}$-order classification

As described in Subsection 2.2.3, the comparison ratio is very useful in the characterization of different order relations in $\mathcal{K}_{\mathcal{C}}$. Furthermore, in Subsection 2.2.6 we have generalized the LU-order up to the γ case, sorting to the order range, but limiting ourselves to the cases in which $\gamma^- \leqslant 0$, $\gamma^+ \geqslant 0$ (eventually $\gamma^- = -\infty$ and/or $\gamma^+ = +\infty$). Therefore, it is interesting to further extend the cases relating to the γ-order, analysing the whole range of possibilities as regards the values that γ^- and γ^+ can assume.

So let $\gamma^-, \gamma^+ \in \mathbb{R} \cup \{\pm\infty\}$ with $\gamma^- \leqslant \gamma^+$, if $A \in \mathcal{K}_\mathcal{C}$ it is possible to define the following relation between A and a generic interval $X \in \mathcal{K}_\mathcal{C}$:

$$X \precapprox^*_{\gamma^-,\gamma^+} A \Leftrightarrow \quad \begin{array}{r} (1)\ if\ \gamma^- \leqslant \gamma^+ < 0\ : \begin{cases} \widetilde{x} \leqslant \widetilde{a} + \gamma^-(\widehat{x} - \widehat{a}) \\ \widetilde{x} \leqslant \widetilde{a} + \gamma^+(\widehat{x} - \widehat{a}) \end{cases} \\ (2)\ if\ \gamma^- \leqslant 0,\ \gamma^+ \geqslant 0\ : \begin{cases} (\widehat{x} \leqslant \widehat{a} \Leftrightarrow\ \gamma^- = \gamma^+ = 0) \\ \widetilde{x} \leqslant \widetilde{a} + \gamma^-(\widehat{x} - \widehat{a}) \\ \widetilde{x} \geqslant \widetilde{a} + \gamma^+(\widehat{x} - \widehat{a}) \end{cases} \\ (3)\ if\ 0 < \gamma^- \leqslant \gamma^+\ : \begin{cases} \widetilde{x} \geqslant \widetilde{a} + \gamma^-(\widehat{x} - \widehat{a}) \\ \widetilde{x} \geqslant \widetilde{a} + \gamma^+(\widehat{x} - \widehat{a}). \end{cases} \end{array} \tag{2.37}$$

Let us now analyze the various cases, starting with case (2), since it coincides exactly with the case examined so far.

- $\gamma^- \leqslant 0,\ \gamma^+ \geqslant 0,$ (case(2))

  Given $\gamma^- \leqslant 0,\ \gamma^+ \geqslant 0$ (not both zero), we define the order relation, denoted $\precapprox_{\gamma^-,\gamma^+}$, as

  $$X \precapprox_{\gamma^-,\gamma^+} A \Longleftrightarrow \begin{cases} \widetilde{x} \leqslant \widetilde{a} + \gamma^- (\widehat{x} - \widehat{a}) \\ \widetilde{x} \geqslant \widetilde{a} + \gamma^+ (\widehat{x} - \widehat{a}), \end{cases} \tag{2.38}$$

  i.e.,

  $$X \precapprox_{\gamma^-,\gamma^+} A \Longleftrightarrow \begin{cases} \widetilde{x} - \gamma^-\widehat{x} \leqslant \widetilde{a} - \gamma^-\widehat{a} \\ \widetilde{x} - \gamma^+\widehat{x} \geqslant \widetilde{a} - \gamma^+\widehat{a}. \end{cases} \tag{2.39}$$

  We highlight that if $\gamma^- = \gamma^+ = 0$, then the condition $\widehat{x} \leqslant \widehat{a}$ must also be added (which is pleonastic in all other cases).

  Moreover, as stated in Definition 2.2.13, for a given interval $A = (\widehat{a}; \widetilde{a})$, we fix the following sets of intervals X (see Figure 2.19) which are:

  ($\precapprox_{\gamma^-,\gamma^+}$)-dominated by A: $\{X \in \mathcal{K}_C | A \precapprox_{\gamma^-,\gamma^+} X\}$,

  ($\precapprox_{\gamma^-,\gamma^+}$)-dominating A: $\{X \in \mathcal{K}_C | X \precapprox_{\gamma^-,\gamma^+} A\}$,

  ($\precapprox_{\gamma^-,\gamma^+}$)-incomparable with A when $X \in \mathcal{K}_C$ but it does not belongs to any of the two previous sets .

  Note that by changing $\gamma^- \leqslant 0$ and $\gamma^+ \geqslant 0$ we obtain an infinite number of partial orders and by increasing $\gamma^+ \geqslant 0$ and/or decreasing $\gamma^- \leqslant 0$ the incomparability region(s) will be reduced as shown in Figure 2.20. In particular we observe that $\precapprox_{-1,1}$ is the standard LU-order: $A \precapprox_{LU} B$ if and only if $a^- \leqslant b^-$ and $a^+ \leqslant b^+$.

  Then there are some special cases (extremal situations):
  if $\gamma^- = \gamma^+ = 0$, we have:

  $$X \precapprox_{0,0} A \Leftrightarrow (\widehat{x} \leqslant \widehat{a} \text{ and } \widetilde{x} = \widetilde{a}),$$

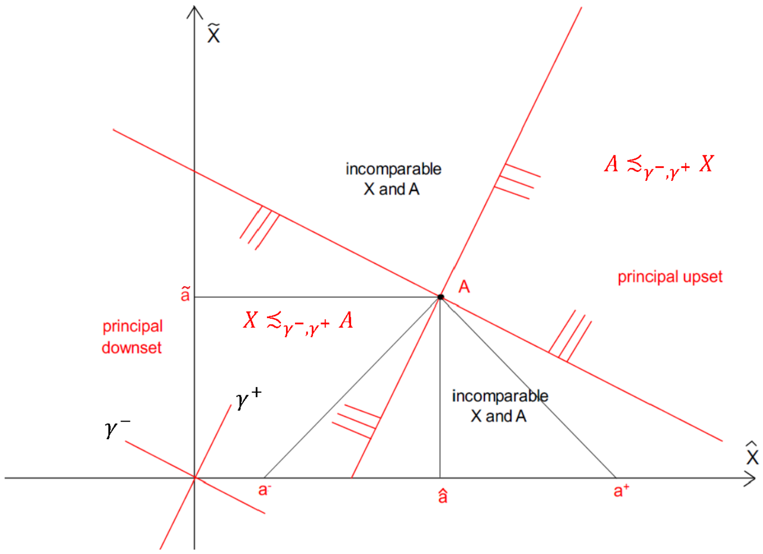


Figure 2.19: Sets of intervals which are dominated by A, dominating A or incomparable with A with respect to order relation $\precapprox_{\gamma^-,\gamma^+}$ if $\gamma^- \leqslant 0$, $\gamma^+ \geqslant 0$.

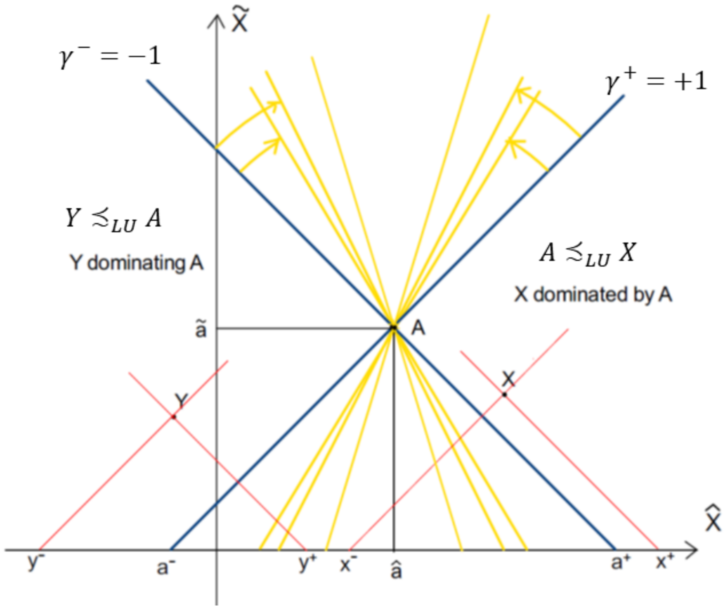


Figure 2.20: The infinite number of partial orders that can be obtained by changing $\gamma^- \leqslant -1$ and $\gamma^+ \geqslant 1$.

if $\gamma^- \longrightarrow -\infty, \gamma^+ \longrightarrow +\infty$ we have:

$$X \precapprox_{-\infty,+\infty} A \Leftrightarrow (\widehat{x} \leqslant \widehat{a}).$$

Other cases are also possible, such as:
$(\gamma^- = 0, \gamma^+ \longrightarrow +\infty); (\gamma^- \longrightarrow -\infty, \gamma^+ = 0)$, etc.

In particular, from (2.14), we have:

- case (2a)
  $\widetilde{x} - \widetilde{a} \leqslant \gamma^-(\widehat{x} - \widehat{a}) \Leftrightarrow$
  $(\widehat{x} < \widehat{a}$ and $\gamma^- \leqslant \gamma_{A,X})$ or $(\widehat{x} > \widehat{a}$ and $\gamma^- \geqslant \gamma_{A,X})$ or $(\widehat{x} = \widehat{a}$ and $\widetilde{x} \leqslant \widetilde{a})$;
- case (2b)
  $\widetilde{x} - \widetilde{a} \geqslant \gamma^+(\widehat{x} - \widehat{a}) \Leftrightarrow$
  $(\widehat{x} < \widehat{a}$ and $\gamma^+ \geqslant \gamma_{A,X})$ or $(\widehat{x} > \widehat{a}$ and $\gamma^+ \leqslant \gamma_{A,X})$ or $(\widehat{x} = \widehat{a}$ and $\widetilde{x} \geqslant \widetilde{a})$.

- $\gamma^- \leqslant \gamma^+ < 0$ (case(1))
  If $\gamma^- \leqslant \gamma^+ < 0$, we define the order relation , denoted $\precapprox_{\gamma^-,\gamma^+}$, (see Figure 2.21) as

$$X \precapprox_{\gamma^-,\gamma^+} A \Longleftrightarrow \begin{cases} \widetilde{x} \leqslant \widetilde{a} + \gamma^-(\widehat{x} - \widehat{a}) \\ \widetilde{x} \leqslant \widetilde{a} + \gamma^+(\widehat{x} - \widehat{a}), \end{cases} \tag{2.40}$$

i.e.,

$$X \precapprox_{\gamma^-,\gamma^+} A \Longleftrightarrow \begin{cases} \widetilde{x} - \gamma^-\widehat{x} \leqslant \widetilde{a} - \gamma^-\widehat{a} \\ \widetilde{x} - \gamma^+\widehat{x} \leqslant \widetilde{a} - \gamma^+\widehat{a}. \end{cases} \tag{2.41}$$

In particular we have:

- case (1a)
  $\widetilde{x} - \widetilde{a} \leqslant \gamma^-(\widehat{x} - \widehat{a}) \Leftrightarrow$
  $(\widehat{x} < \widehat{a}$ and $\gamma^- \leqslant \gamma_{A,X})$ or $(\widehat{x} > \widehat{a}$ and $\gamma^- \geqslant \gamma_{A,X})$ or $(\widehat{x} = \widehat{a}$ and $\widetilde{x} \leqslant \widetilde{a})$;
- case (1b)
  $\widetilde{x} - \widetilde{a} \leqslant \gamma^+(\widehat{x} - \widehat{a}) \Leftrightarrow$
  $(\widehat{x} < \widehat{a}$ and $\gamma^+ \leqslant \gamma_{A,X})$ or $(\widehat{x} > \widehat{a}$ and $\gamma^+ \geqslant \gamma_{A,X})$ or $(\widehat{x} = \widehat{a}$ and $\widetilde{x} \leqslant \widetilde{a})$.

- $0 < \gamma^- \leqslant \gamma^+$ (case(3))
  If $0 < \gamma^- \leqslant \gamma^+$, we define the order relation , denoted $\precapprox_{\gamma^-,\gamma^+}$, (see Figure 2.22) as

$$X \precapprox_{\gamma^-,\gamma^+} A \Longleftrightarrow \begin{cases} \widetilde{x} \geqslant \widetilde{a} + \gamma^-(\widehat{x} - \widehat{a}) \\ \widetilde{x} \geqslant \widetilde{a} + \gamma^+(\widehat{x} - \widehat{a}), \end{cases} \tag{2.42}$$

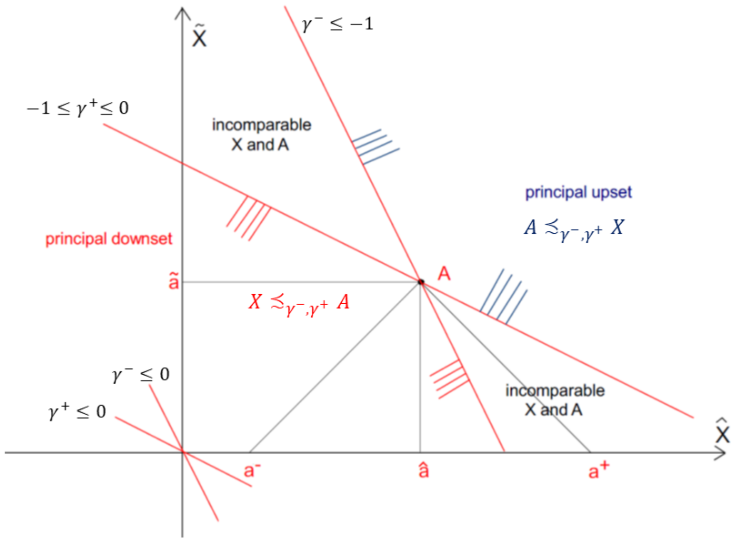


Figure 2.21: The order relation $\precsim_{\gamma^-,\gamma^+}$ if $\gamma^- \leqslant \gamma^+ < 0$. In this case we also have: $\gamma^- \leqslant -1 \leqslant \gamma^+ \leqslant 0$.

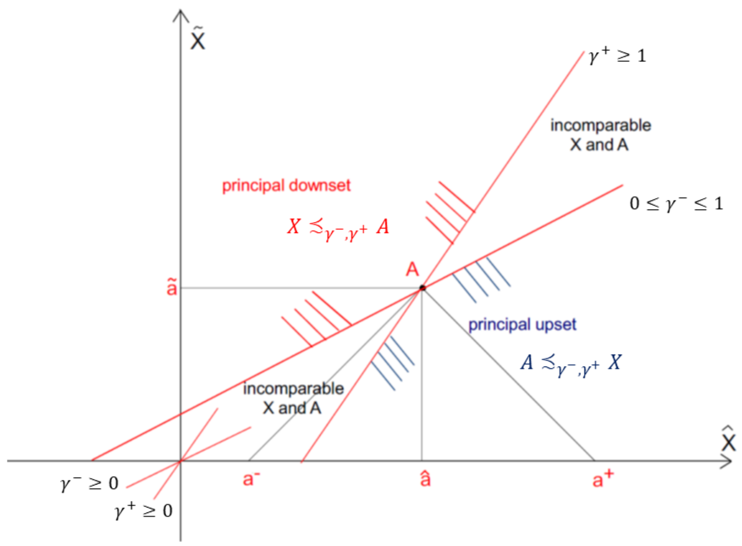


Figure 2.22: The order relation $\precsim_{\gamma^-,\gamma^+}$ if $0 < \gamma^- \leqslant \gamma^+$. In this case we also have: $0 \leqslant \gamma^- \leqslant 1 \leqslant \gamma^+$.

i.e.,

$$X \precsim_{\gamma^-,\gamma^+} A \Longleftrightarrow \begin{cases} \widetilde{x} - \gamma^- \widehat{x} \geqslant \widetilde{a} - \gamma^- \widehat{a} \\ \widetilde{x} - \gamma^+ \widehat{x} \geqslant \widetilde{a} - \gamma^+ \widehat{a}. \end{cases} \tag{2.43}$$

In particular we have:

- case (3a)
  $\widetilde{x} - \widetilde{a} \geqslant \gamma^-(\widehat{x} - \widehat{a}) \Leftrightarrow$
  $(\widehat{x} < \widehat{a}$ and $\gamma^- \geqslant \gamma_{A,X})$ or $(\widehat{x} > \widehat{a}$ and $\gamma^- \leqslant \gamma_{A,X})$ or $(\widehat{x} = \widehat{a}$ and $\widetilde{x} \geqslant \widetilde{a})$;
- case (3b)
  $\widetilde{x} - \widetilde{a} \geqslant \gamma^+(\widehat{x} - \widehat{a}) \Leftrightarrow$
  $(\widehat{x} < \widehat{a}$ and $\gamma^+ \geqslant \gamma_{A,X})$ or $(\widehat{x} > \widehat{a}$ and $\gamma^+ \leqslant \gamma_{A,X})$ or $(\widehat{x} = \widehat{a}$ and $\widetilde{x} \geqslant \widetilde{a})$.

In general we can say that when $X \precapprox_{\gamma^-,\gamma^+} A$ we have:

- if $\gamma < 0$, then $\widetilde{x} - \gamma\widehat{x} \leqslant \widetilde{a} - \gamma\widehat{a}$ (see Figure 2.21);
- if $\gamma > 0$, then $\widetilde{x} - \gamma\widehat{x} \geqslant \widetilde{a} - \gamma\widehat{a}$ (see Figure 2.22).

From here on, we will consider only case (2) $(\gamma^- \leqslant 0,\ \gamma^+ \geqslant 0)$ as it is the one that is best suited for estimating the risk of possible worst case loss (see [30]); indeed it is possible to face two types of risk, due to the possibility of a worstcase loss if we make a choice exclusively on the basis of the midpoint values $\widehat{x}$ and $\widehat{a}$.

Moreover, according to [45], it appears evident that for all $A, B \in \mathcal{K_C}$, where $A = [a^-, a^+] = (\widehat{a}; \widetilde{a})$ and $B = [b^-, b^+] = (\widehat{b}; \widetilde{b})$, the following properties (where $A \leq B$ stands for "A is better than B") are satisfied:

1) reflexivity: $A \leq A$, for all $\in \mathcal{K_C}$;
2) antisymmetry: $A \leq B$ and $B \leq A$ iff $A = B$, for all $A, B \in \mathcal{K_C}$;
3) transitivity: $A \leq B$ and $B \leq C$, then $A \leq C$, for all $A, B, C \in \mathcal{K_C}$;
4) consistency with common sense: if $a^+ \leqslant b^-$ then $A \leq B$, for all $A, B \in \mathcal{K_C}$;
5) scale-invariance: if $A \leq B$ then $cA \leq cB$, for all $A, B \in \mathcal{K_C}$ (that is, if we multiply all the gains by the same positive constant $c > 0$, then whichever gain was larger remains larger, and whichever gain was smaller remains smaller);
6) additivity: $A \leq B$ iff $A \oplus_M C \leq B \oplus_M C$ for all $A, B, C \in \mathcal{K_C}$ (that is, if we add the same amount to the two gains, this will not change which gain is larger);
7) closeness: when the values of a^- and a^+ are close, the corresponding alternatives are practically indistinguishable. Similarly, if we have two

sequences $(A)_n$ and $(B)_n$ so that $(A)_n \leq (B)_n$ and endpoints of both tends to some limits, then, since the limit intervals are indistinguishable from these one for sufficiently large n, we should expect the same relation $\leq$ for the limit intervals. This means that if for all n we have: $(A)_n \leq (B)_n$ and $(a)_n^- \longrightarrow a^-$, $(a)_n^+ \longrightarrow a^+$, $(b)_n^- \longrightarrow b^-$, $(b)_n^+ \longrightarrow b^+$, then $A \leq B$.

On the basis of the properties just stated, as well explained in [45], it follows the result below.

Proposition 2.2.12. *([45]) For a binary relation $\leq$ on the set of all intervals $\mathcal{K_C}$, the following two conditions are equivalent to each other:*

(1) the relation is transitive, reflexive, consistent with common sense, scale-invariant, additive, and closed;

(2) for some values α^-, $\alpha^+ \in \mathbb{R}$, for which $-1 \leqslant \alpha^- \leqslant \alpha^+ \leqslant 1$, considering $X = (\widehat{x}; \widetilde{x}), Y = (\widehat{y}; \widetilde{y}) \in \mathcal{K_C}$, the relation $\leq$ has the following form: $X \leq Y$, i.e., $[\widehat{x} - \widetilde{x}, \widehat{x} + \widetilde{x}] \leq [\widehat{y} - \widetilde{y}, \widehat{y} + \widetilde{y}]$, if and only if either

- *$\widetilde{x} \leqslant \widetilde{y}$ and $\widehat{x} \leqslant \widehat{y} + \alpha^-(\widetilde{y} - \widetilde{x})$*

  or

- *$\widetilde{x} \leqslant \widetilde{y}$ and $\widehat{x} \leqslant \widehat{y} + \alpha^+(\widetilde{y} - \widetilde{x})$.*

Note that in this case the values α^- and α^+ are the angular coefficients of two straight lines in the half-plane $(\widetilde{x}; \widehat{x})$, i.e., an inverted representation with respect to the one used in this work; therefore, they can also be interpreted as the reciprocals of the values γ^- and γ^+ used so far.

Before proceeding, we highlight the fact that from now on the symbol $\precapprox$ will be used to indicate $\precapprox_{\gamma^-,\gamma^+}$, with $\gamma^- \leqslant 0$, $\gamma^+ \geqslant 0$.

Moreover, in addition to the $\precapprox$-order, we will consider also the strict order relation, denoted by $\precsim$, and the strong order relation, denoted by $\prec$, which stand for $\precsim_{\gamma^-,\gamma^+}$ and $\prec_{\gamma^-,\gamma^+}$ respectively.

Chapter 3

Real and complex interval-valued functions

As in Chapter 2, also in this case the contents presented are inspired by the results of the work published in [84] and [85], but this time with particular attention to the part concerning the calculus for interval-valued functions of a single real variable $F : [a, b] \to \mathcal{K}_{\mathcal{C}}$.

Indeed, in Section 3.1 the notions introduced in Section 2.2 will be applied to the analysis and calculus of interval-valued functions. Concepts related to convergence and limits, continuity, gH-differentiability and monotonicity will be introduced and analyzed in detail, as well as a discussion of extremal points, concavity and convexity of interval-valued functions will be presented, a full analysis of which will be provided, accompanied by an illustrated example. Furthermore, the periodicity of interval-valued functions will be introduced and visualized with the help of some well-known plane curves.

Afterwards, in Section 3.2 a new notation to represent complex intervals will be proposed, also showing, through examples and graphical representations, the peculiarities and advantages associated with its use. Finally, the last part of the chapter will be dedicated to the presentation of a possible example of application of the concepts seen to a topic, the q-calculus, which today is of great interest in the scientific community.

3.1 Interval-valued functions

In this Section, after having presented the notion of interval-valued function making use of different notations (endpoint and midpoint-radius), we will introduce the concept of limits, continuity and gH-derivative related to it; we will also show its connection with the comparison index, which is used extensively to discuss the monotonicity of such type of functions. Then, a discussion on extremal points, concavity and convexity of interval-valued functions with the use of gH-derivative will be presented for a complete analysis, also enriched with an illustrative example. In addition, periodicity of interval-valued functions will be outlined and illustrated with the help of some famous plane curves.

3.1.1 Interval-valued functions of a real variable

In the Subsection 1.3.6 we saw the concept of interval extension of a continuous real-valued function f of a single real variable x, by defining the function

$$F : \mathcal{K}_{\mathcal{C}} \longrightarrow \mathcal{K}_{\mathcal{C}},\ X \longmapsto F(X)$$

which sends interval X to interval $F(X)$ (see Definition 1.3.7).

In this chapter, however, we are dealing with functions that send a real variable x, i.e., defined on a subset $[a, b]$ of $\mathbb{R}$, to an interval of $\mathcal{K}_{\mathcal{C}}$ (see also [56] and [84] for details). From now on we will refer to such types of functions as *interval-valued functions* or, even more simply, *interval functions*.

Definition 3.1.1. *An interval-valued function is defined to be any*

$$F : [a, b] \longrightarrow \mathcal{K}_C$$

with

$$F(x) = [f^-(x), f^+(x)] \in \mathcal{K}_C$$

such that $f^-(x) \leqslant f^+(x)$ *for all* $x \in [a, b]$*, where the real-valued functions* $f^-(x)$ *and* $f^+(x)$ *are the so-called endpoint functions of interval* $F(x)$*.*

Otherwise, using midpoint representation, we write

$$F(x) = \left(\widehat{f}(x); \widetilde{f}(x)\right) \in \mathcal{K}_C$$

where $\widehat{f}(x) \in \mathbb{R}$ *is the midpoint value of interval* $F(x)$ *and* $\widetilde{f}(x) \in \mathbb{R}^+ \cup \{0\}$ *is the nonnegative half-length of* $F(x)$*, respectively defined as:*

$$\widehat{f}(x) = \frac{f^+(x) + f^-(x)}{2} \quad \text{and } \widetilde{f}(x) = \frac{f^+(x) - f^-(x)}{2} \geqslant 0$$

so that

$$f^-(x) = \widehat{f}(x) - \widetilde{f}(x) \quad \text{and } f^+(x) = \widehat{f}(x) + \widetilde{f}(x).$$

Graphically an interval-valued function $F(x)$ can be represented in various ways:

- by the classical graphical representation in the plane (x, y) (see Figure 3.1): in this case it is possible to define $F(x)$ in terms of endpoints functions $[f^-(x), f^+(x)]$ (thus obtaining two curves representing the extremes within which the interval-valued function is located) or in terms of midpoint-radius functions $\left(\widehat{f}(x); \widetilde{f}(x)\right)$ but in this case the two curves obtained represent respectively the midpoint function and the radius function of $F(x)$.

- by the parametric mode in the midpoint half-plane $(\widehat{z}; \widetilde{z})$, $\widetilde{z} \geqslant 0$ (see Figure 3.2) where each interval $F(x)$ is identified with the point $(\widehat{f}(x); \widetilde{f}(x))$ and the use of several arrows gives the *direction* of moving the intervals for increasing $x \in [a, b]$.

Example 3.1.1. *Let* $[a, b] = [-1.25, 2.5]$. *We consider the interval-valued function defined by* $F(x) = \left(-x^3 + 2x^2 + x - 1; 1 + \sin(\frac{\pi}{2}x)\right)$ *in midpoint notation, i.e.,* $\widehat{f}(x) = -x^3 + 2x^2 + x - 1$, $\widetilde{f}(x) = 1 + \sin(\frac{\pi}{2}x)$.
Obviously, by applying the transformations seen above we can easily go back to the endpoint notation.

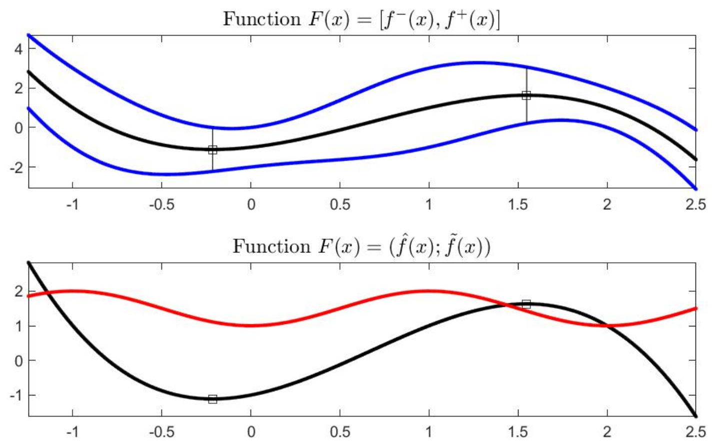


Figure 3.1: Graphical representation of F in the plane (x, y).

The graphical representation of $F(x)$ *in the plane* (x, y) *is given in Figure 3.1 where, on the top of the picture we can see the interval-valued function* $F(x)$ *in terms of endpoints functions* $f^-(x)$ *and* $f^+(x)$ *(blue color), while on the bottom part there is the same interval-valued function in terms of midpoint function* $\widehat{f}(x)$ *(black color) and radius function* $\widetilde{f}(x)$ *(red color).*

Note that for $x = a = -1.25$ *we have* $F(-1.25) = (2.828; 1.854) = [0.974, 4.682]$ *and for* $x = b = 2.5$ *it is* $F(2.5) = (-1.625; 1.5) = [-3.125, -0.125]$;

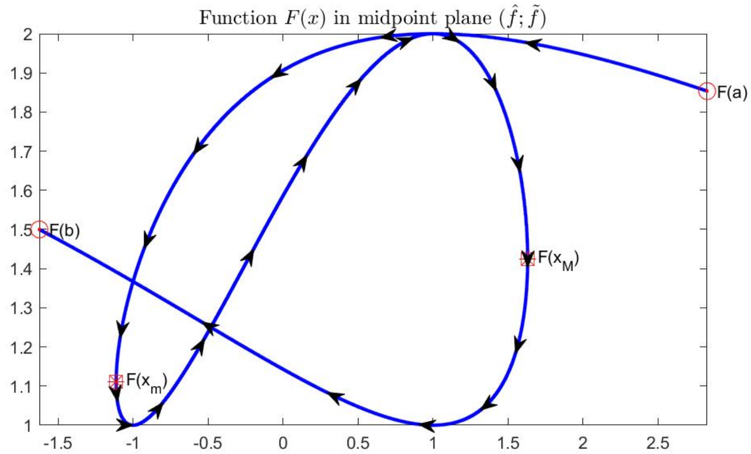


Figure 3.2: Graphical representation of F in the half-plane $(\widehat{z};\widetilde{z})$.

looking at the midpoint representation in Figure 3.2, the arrows start exactly at point $(2.828; 1.854) = [0.974, 4.682]$ *and terminate at point* $(-1.625; 1.5) = [-3.125, -0.125]$. *The values of* $x \in [-1.25, 2.5]$ *where the midpoint function* $\widehat{f}(x)$ *is minimal or maximal are, approximately:*
- $x_m = -0.215$ *with interval value* $F(x_m) = (-1.113; 1.110)$,
- $x_M = 1.549$ *with interval value* $F(x_M) = (1.631; 1.424)$.

3.1.2 Limits and continuity of interval-valued functions

As reported on [56] and [64], since $\mathcal{K}_{\mathcal{C}}$ is normed, continuity and limits of an interval-valued function are understood in the sense of such norm. According to what we have seen in Subsection 2.1.2, this obviously leads to characterize the concepts of continuity and limits in the Pompeiu–Hausdorff metric d_H for intervals (see Definition 2.1.1), which, according to (2.5), is given by the gH-difference (for details see [19] and [48]).

Therefore, it is easy to derive the definition of limit of an interval-valued function.

Definition 3.1.2. *Suppose that* $F : K \longrightarrow \mathcal{K}_C$, $K \subseteq \mathbb{R}$, *be such that* $F(x) = [f^-(x), f^+(x)] = (\widehat{f}(x); \widetilde{f}(x))$. *Let* $L = [l^-, l^+] = (\widehat{l}; \widetilde{l}) \in \mathcal{K}_C$ *and* x_0 *be an accumulation point of* K.

Then we say that L *is the limit of* F, *as* x *approaches* x_0, *and write*

$$\lim_{x \to x_0} F(x) = L$$

or alternatively: $F(x) \to L$ *as* $x \to x_0$ *(reads "*$F(x)$ *tends to* L *as* x *tends to* x_0*") if the following property holds:*

for every real $\varepsilon > 0$, there exists a real $\delta > 0$ such that for all $x \in K$, $0 < |x - x_0| < \delta$ implies that

$$\|F(x) \ominus_{gH} L\| < \varepsilon.$$

The definition of continuity of an interval-valued function is also formalized in a similar way.

Definition 3.1.3. *Given an interval-valued function $F : K \longrightarrow \mathcal{K}_C$, $K \subseteq \mathbb{R}$, we say that F is continuous in $x_0 \in K$ when the following fact holds:*

for every real $\varepsilon > 0$, there exists a real $\delta > 0$ such that for all $x \in K$, $0 < |x - x_0| < \delta$ implies that

$$\|F(x) \ominus_{gH} F(x_0)\| < \varepsilon.$$

The following result, described in [81], is well known and it follows immediately from the property

$$d_H(F(x), L) = \|F(x) \ominus_{gH} L\|\,;$$

note that the second equivalence defines the continuity at an accumulation point.

Proposition 3.1.1. *Let $F : K \longrightarrow \mathcal{K}_C$, $K \subseteq \mathbb{R}$, be such that $F(x) = [f^-(x), f^+(x)]$ and let $L = [l^-, l^+] \in \mathcal{K}_C$. Let x_0 be an accumulation point of K. Then we have*

$$\lim_{x \to x_0} F(x) = L \iff \lim_{x \to x_0} (F(x) \ominus_{gH} L) = 0$$

where the limits are in the metric d_H. If, in addition, $x_0 \in K$, we have

$$\lim_{x \to x_0} F(x) = F(x_0) \iff \lim_{x \to x_0} (F(x) \ominus_{gH} F(x_0)) = 0.$$

Furthermore, in midpoint notation, let $F(x) = (\widehat{f}(x); \widetilde{f}(x))$ and $L = (\widehat{l}; \widetilde{l})$; then the limits and continuity can be expressed, respectively, as

$$\lim_{x \to x_0} F(x) = L \iff \lim_{x \to x_0} \widehat{f}(x) = \widehat{l} \text{ and } \lim_{x \to x_0} \widetilde{f}(x) = \widetilde{l} \tag{3.1}$$

and

$$\lim_{x \to x_0} F(x) = F(x_0) \iff \lim_{x \to x_0} \widehat{f}(x) = \widehat{f}(x_0) \text{ and } \lim_{x \to x_0} \widetilde{f}(x) = \widetilde{f}(x_0).$$

The following proposition connects limits to the order of intervals; we will consider the lattice $(\mathcal{K}_C, \precapprox_{\gamma^-,\gamma^+})$ with partial order $\precapprox_{\gamma^-\gamma^+}$ defined for any fixed values of $\gamma^- \leqslant 0$ and $\gamma^+ \geqslant 0$. Analogous results can be obtained for the reverse partial order $\succapprox_{\gamma^-\gamma^+}$.

Proposition 3.1.2. *Let $F, G, H : K \longrightarrow \mathcal{K}_C$ be interval-valued functions and x_0 an accumulation point for K.*

(i) if $F(x) \precsim_{\gamma^-\gamma^+} G(x)$ for all $x \in K$ in a neighborhood of x_0 and $\lim_{x\to x_0} F(x) = L \in \mathcal{K}_C$, $\lim_{x\to x_0} G(x) = M \in \mathcal{K}_C$, then $L \precsim_{\gamma^-\gamma^+} M$;

(ii) If $F(x) \precsim_{\gamma^-\gamma^+} G(x) \precsim_{\gamma^-\gamma^+} H(x)$ for all $x \in K$ in a neighborhood of x_0 and $\lim_{x\to x_0} F(x) = \lim_{x\to x_0} H(x) = L \in \mathcal{K}_C$, then $\lim_{x\to x_0} G(x) = L$.

Proof. We will use the midpoint notation for intervals.

For the proof of *i)*, we have, according to (2.33), that $F(x) \precsim_{\gamma^-,\gamma^+} G(x)$ if and only if $\widehat{f}(x) \leqslant \widehat{g}(x)$ and $\widetilde{g}(x) + \gamma^+ \left(\widehat{f}(x) - \widehat{g}(x)\right) \leqslant \widetilde{f}(x) \leqslant \widetilde{g}(x) + \gamma^- \left(\widehat{f}(x) - \widehat{g}(x)\right)$; from (3.1) we have, at the limit, that $\widehat{l} \leqslant \widehat{m}$ and $\widetilde{m} + \gamma^+ \left(\widehat{l} - \widehat{m}\right) \leqslant \widetilde{l} \leqslant \widetilde{m} + \gamma^- \left(\widehat{l} - \widehat{m}\right)$ and this means that $L \precsim_{\gamma^-\gamma^+} M$.

For the proof of *ii)* we have $F(x) \precsim_{\gamma^-\gamma^+} G(x)$ and $G(x) \precsim_{\gamma^-\gamma^+} H(x)$ if and only if $\widehat{f}(x) \leqslant \widehat{g}(x)$, $\widetilde{g}(x) + \gamma^+ \left(\widehat{f}(x) - \widehat{g}(x)\right) \leqslant \widetilde{f}(x) \leqslant \widetilde{g}(x) + \gamma^- \left(\widehat{f}(x) - \widehat{g}(x)\right)$ and $\widehat{g}(x) \leqslant \widehat{h}(x)$, $\widetilde{h}(x) + \gamma^+ \left(\widehat{g}(x) - \widehat{h}(x)\right) \leqslant \widetilde{g}(x) \leqslant \widetilde{h}(x) + \gamma^- \left(\widehat{g}(x) - \widehat{h}(x)\right)$: from $\widehat{f}(x) \leqslant \widehat{g}(x)$, $\widehat{g}(x) \leqslant \widehat{h}(x)$, according to (3.1), we have that $\lim_{x\to x_0} \widehat{g}(x) = \widehat{l}$ exists; from $\widetilde{g}(x) + \gamma^+ \left(\widehat{f}(x) - \widehat{g}(x)\right) \leqslant \widetilde{f}(x)$ we obtain $\widetilde{g}(x) \leqslant \widetilde{f}(x) - \gamma^+ \left(\widehat{f}(x) - \widehat{g}(x)\right)$, so that we can write $\widetilde{h}(x) + \gamma^+ \left(\widehat{g}(x) - \widehat{h}(x)\right) \leqslant \widetilde{g}(x) \leqslant \widetilde{f}(x) - \gamma^+ \left(\widehat{f}(x) - \widehat{g}(x)\right)$. On the other hand, from $\lim_{x\to x_0} \widetilde{f}(x) = \widetilde{l} = \lim_{x\to x_0} \widetilde{h}(x)$, we have
$\lim_{x\to x_0} \left(\widetilde{h}(x) + \gamma^+ \left(\widehat{g}(x) - \widehat{h}(x)\right)\right) = \widetilde{l} + \gamma^+(\widehat{l} - \widehat{l}) = \widetilde{l}$
and $\lim_{x\to x_0} \left(\widetilde{f}(x) - \gamma^+ \left(\widehat{f}(x) - \widehat{g}(x)\right)\right) = \widetilde{l} - \gamma^+(\widehat{l} - \widehat{l}) = \widetilde{l}$
so that $\lim_{x\to x_0} \widetilde{g}(x) = \widetilde{l}$; the conclusion follows from (3.1) applied to G. □

Remark 3.1.1. *Similar results as in Propositions 3.1.1 and 3.1.2 are valid for the left limit with $x \longrightarrow x_0$, $x < x_0$ ($x \nearrow x_0$ for short) and for the right limit $x \longrightarrow x_0$, $x > x_0$ ($x \searrow x_0$ for short); the condition that $\lim\limits_{x\to x_0} F(x) = L$ if and only if $\lim\limits_{x\nearrow x_0} F(x) = L = \lim\limits_{x\searrow x_0} F(x)$ is obvious.*

3.1.3 The gH-derivative for interval-valued functions

The gH-derivative for an interval-valued function, expressed in terms of the difference quotient by gH-difference, has been first introduced in 1979 by S. Markov (see [56]). In the fuzzy context it has been introduced in [78]; the interval case has been analyzed in [81] and the fuzzy case again reconsidered (level wise) in [8]. Several authors have then proposed alternative equivalent definitions and studied its properties and applications; actually, it is of interest for an increasing number of researchers. A very recent and complete description of the algebraic properties of gH-derivative can be found in [14]. The following definition, based on the gH-difference, is the one that was proposed in [81] and it has the advantage of having a simpler formulation compared to other definitions.

Definition 3.1.4. *([81]) Let $x_0 \in]a, b[$ and h be such that $x_0 + h \in]a, b[$, then the gH-derivative of a function $F :]a, b[\to \mathcal{K}_C$ at x_0 is defined as*

$$F'_{gH}(x_0) = \lim_{h \to 0} \frac{1}{h} [F(x_0 + h) \ominus_{gH} F(x_0)] \tag{3.2}$$

if the limit exists and it is an element of $\mathcal{K}_{\mathcal{C}}$. The interval $F'_{gH}(x_0) \in \mathcal{K}_C$ satisfying (3.2) is called the generalized Hukuhara derivative of F (gH-derivative for short) at x_0.

Also, one-side derivatives can be considered.
The right gH-derivative of F at x_0 is

$$F'_{(r)gH}(x_0) = \lim_{h \searrow 0} \frac{1}{h} [F(x_0 + h) \ominus_{gH} F(x_0)]$$

while to the left it is defined as

$$F'_{(l)gH}(x_0) = \lim_{h \nearrow 0} \frac{1}{h} [F(x_0 + h) \ominus_{gH} F(x_0)].$$

The gH-derivative exists at x_0 if and only if the left and right derivatives at x_0 exist and are the same interval.

The following properties are indeed immediate to prove.

Proposition 3.1.3. *([84]) Let $F : [a, b] \to \mathcal{K}_C$ be given, $F(x) = \left(\widehat{f}(x); \widetilde{f}(x)\right)$. Then*

(1) $F(x)$ is left gH-differentiable at $x_0 \in]a, b]$ if and only if $\widehat{f}(x)$ and $\widetilde{f}(x)$ are left differentiable at x_0; in this case, $F'_{(l)gH}(x_0) = \left(\widehat{f}'_l(x_0); \left|\widetilde{f}'_l(x_0)\right|\right)$;

(2) $F(x)$ is right gH-differentiable at $x_0 \in [a, b[$ if and only if $\widehat{f}(x)$ and $\widetilde{f}(x)$ are right differentiable at x_0; in this case, $F'_{(r)gH}(x_0) = \left(\widehat{f}'_r(x_0); \left|\widetilde{f}'_r(x_0)\right|\right)$;

(3) $F(x)$ is gH-differentiable at $x_0 \in]a, b[$ if and only if $\widehat{f}(x)$ is differentiable and $\widetilde{f}(x)$ is left and right differentiable at x_0 with $\left|\widetilde{f}'_l(x_0)\right| = \left|\widetilde{f}'_r(x_0)\right|$ and in this case, $F'_{gH}(x_0) = \left(\widehat{f}'(x_0); \left|\widetilde{f}'_r(x_0)\right|\right) = \left(\widehat{f}'(x_0); \left|\widetilde{f}'_l(x_0)\right|\right)$; equivalently, if and only if $F'_{(l)gH}(x_0) = F'_{(r)gH}(x_0)$.

In terms of midpoint representation $F(x) = \left(\widehat{f}(x); \widetilde{f}(x)\right)$ we can write

$$\frac{F(x+h) \ominus_{gH} F(x)}{h} = \left(\frac{\widehat{f}(x+h) - \widehat{f}(x)}{h}; \left|\frac{\widetilde{f}(x+h) - \widetilde{f}(x)}{h}\right|\right)$$

and, taking the limit for $h \longrightarrow 0$, we obtain the gH-derivative of F if and only if the two limits

$$\lim_{h\longrightarrow 0} \frac{\widehat{f}(x+h) - \widehat{f}(x)}{h} \quad \text{and} \quad \lim_{h\longrightarrow 0} \left|\frac{\widetilde{f}(x+h) - \widetilde{f}(x)}{h}\right|$$

exist in $\mathbb{R}$; remark that the midpoint function $\widehat{f}$ is required to admit the ordinary derivative at x.

With respect to the existence of the second limit, the existence of the left and right derivatives $\widetilde{f}'_l(x)$ and $\widetilde{f}'_r(x)$ is required with

$$\left|\widetilde{f}'_l(x)\right| = \left|\widetilde{f}'_r(x)\right| = \widetilde{w}_F(x) \geqslant 0$$

(in particular $\widetilde{w}_F(x) = \left|\widetilde{f}'(x)\right|$ if $\widetilde{f}'(x)$ exists) so that we have

$$F'_{gH}(x) = \left(\widehat{f}'(x); \widetilde{w}_F(x)\right) \tag{3.3}$$

or, using the standard endpoint interval notation,

$$F'_{gH}(x) = \left[\widehat{f}'(x) - \widetilde{w}_F(x), \widehat{f}'(x) + \widetilde{w}_F(x)\right]. \tag{3.4}$$

Equation (3.3) is of help in the interpretation of gH-derivative; indeed, the separation of midpoint and half-length components in $F(x)$ is inherited by the gH-derivative $F'_{gH}(x)$. In particular, the correspondence

$$\begin{array}{lllll} F & = \left(\widehat{f}; \widetilde{f}\right) & : \widehat{f} & , & \widetilde{f} \\ \downarrow & & \quad\downarrow & & \downarrow \\ F'_{gH} & = \left(\widehat{f}'; \widetilde{w}_F\right) & : \widehat{f}' & , & \widetilde{w}_F = |\widetilde{f}'_l| = |\widetilde{f}'_r| \end{array} \tag{3.5}$$

shows that the midpoint derivative $\widehat{f}'$ is the derivative of the midpoint $\widehat{f}$ while the half-length derivative is the absolute value $|\widetilde{f}'_l| = |\widetilde{f}'_r|$ of the left and right derivatives of the half-length $\widetilde{f}$, with $\widetilde{f}'_l = \pm\widetilde{f}'_r$ (for details see [15] and [80]).

For the function in Example 3.1.1, we have that both $\widehat{f}(x)$ and $\widetilde{f}(x)$ are differentiable so that $F'_{gH}(x)$ exists at all internal points. Figure 3.3 shows the graphical representation of the derivatives $\widehat{f}'(x)$, $\widetilde{f}'(x)$ and $F'_{gH}(x)$ in the plane (x, y); note that the four points where $\widetilde{f}'(x)$ is zero correspond to

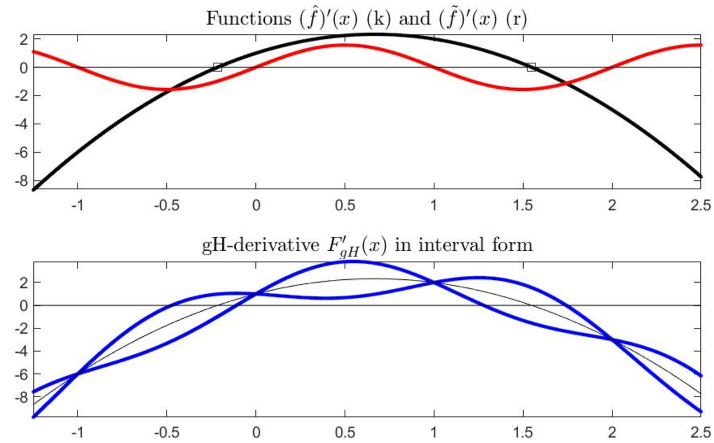


Figure 3.3: Graphical representation of the derivatives $\widehat{f}'(x)$ and $\widetilde{f}'(x)$ (top) and $F'_{gH}(x)$ (bottom) in the plane (x, y).

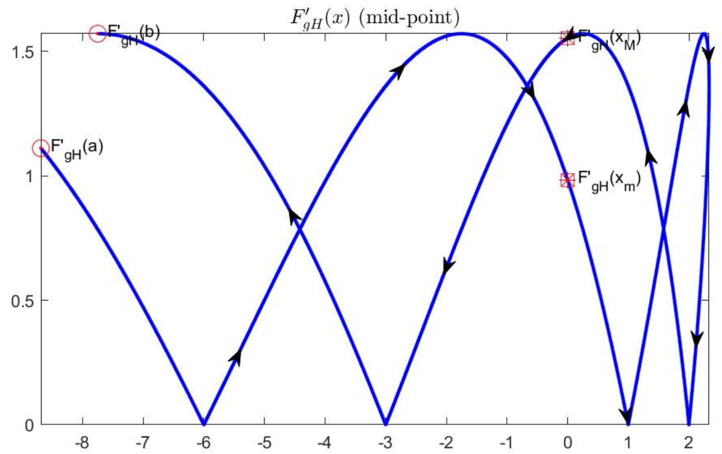


Figure 3.4: Graphical representation of $F'_{gH}(x)$ in the half-plane $(\widehat{z}; \widetilde{z})$.

a singleton gH-derivative, that is, the interval is reduced to a single point. While Figure 3.4 illustrates the graphical representation of $F'_{gH}(x)$ in the half-plane $(\widehat{z}; \widetilde{z})$; here the two marked points (red) correspond to zeros of the derivative $\widehat{f}'(x)$.

For a function $F : [a, b] \to \mathcal{K}_C$, we can define the gH-comparison index-function of $F(x)$ by

$$CI_F(x) = \frac{\widehat{f}(x)}{\|F(x_0)\|_2} = \frac{\widehat{f}(x)}{\sqrt{\left|\widehat{f}(x)\right|^2 + \left|\widetilde{f}(x)\right|^2}}.$$

If $F(x)$ has gH-derivative $F'_{gH}(x) = \left(\widehat{f}'(x); \widetilde{w}_F(x)\right)$ at x, we can consider the gH-comparison index of F'_{gH} at x, given by

$$CI_{F'_{gH}}(x) = \frac{\widehat{f}'(x)}{\left\|F'_{gH}(x)\right\|_2} = \frac{\widehat{f}'(x)}{\sqrt{\left|\widehat{f}'(x)\right|^2 + |\widetilde{w}_F(x)|^2}}$$

and if $\widehat{f}'(x) \neq 0$, the ratio

$$\gamma_{F'}(x) = \frac{\widetilde{w}_F(x)}{\widehat{f}'(x)}$$

is well defined so that,

as $\sqrt{1 + (\gamma_{F'}(x))^2} = \sqrt{1 + \left(\frac{\widetilde{w}_F(x)}{\widehat{f}'(x)}\right)^2} = \frac{\sqrt{\left|\widehat{f}'(x)\right|^2 + |\widetilde{w}_F(x)|^2}}{\left|\widehat{f}'(x)\right|}$, and then $\sqrt{\left|\widehat{f}'(x)\right|^2 + |\widetilde{w}_F(x)|^2} = \left|\widehat{f}'(x)\right| \cdot \sqrt{1 + (\gamma_{F'}(x))^2}$, we have

$$CI_{F'_{gH}}(x) = \frac{sgn(\widehat{f}'(x))}{\sqrt{1 + (\gamma_{F'}(x))^2}}$$

($sgn(z) = 1$ if $z > 0$, $sgn(z) = 0$ if $z = 0$, $sgn(z) = -1$ if $z < 0$).
We can use the index $\gamma_{F'}(x)$ extensively, to evaluate the order relations $F'_{gH}(x) \precapprox_{\gamma^-,\gamma^+} 0$ and similar.

The partial order ($\precapprox_{\gamma^-,\gamma^+}$) can be appropriately introduced for the gH-derivative by the inequality

$$\gamma^- \leqslant \gamma_{F'}(x) \leqslant \gamma^+, \text{ i.e., } \gamma^- \leqslant \frac{\widetilde{w}_F(x)}{\widehat{f}'(x)} \leqslant \gamma^+;$$

moreover, if $\widehat{f}$ and $\widetilde{f}$ are differentiable, we have $\widetilde{w}_F(x) = \left|\widetilde{f}'(x)\right|$, so

$$\gamma^- \leqslant \frac{\left|\widetilde{f}'(x)\right|}{\widehat{f}'(x)} \leqslant \gamma^+$$

which is equivalent to

$$\begin{cases} \widehat{f}'(x) \geqslant 0 \\ \widetilde{f}'(x) \leqslant \gamma^+ \widehat{f}'(x) \\ \widetilde{f}'(x) \geqslant \gamma^- \widehat{f}'(x) \end{cases} \text{ or } \begin{cases} \widehat{f}'(x) \leqslant 0 \\ \widetilde{f}'(x) \geqslant \gamma^+ \widehat{f}'(x) \\ \widetilde{f}'(x) \leqslant \gamma^- \widehat{f}'(x) \end{cases} . \tag{3.6}$$

If $\widetilde{f}$ is not differentiable or if its left and right derivatives do not have the same absolute value (i.e., $\left|\widetilde{f}_l'(x)\right| \neq \left|\widetilde{f}_r'(x)\right|$), then $F'_{gH}(x)$ does not exist, but possibly the left and right gH-derivatives $F'_{(l)gH}$, $F'_{(r)gH}$ exist and we have

$$F'_{(l)gH}(x) = (\widehat{f}_l'(x); |\widetilde{f}_l'(x)|) \text{ and } F'_{(r)gH}(x) = (\widehat{f}_r'(x); |\widetilde{f}_r'(x)|),$$

where $(\cdot)_l'$ and $(\cdot)_r'$ are the notations for left and right derivatives.

In this case, the inequalities $\gamma^- \leqslant \gamma_{F'_{(l)}}(x) \leqslant \gamma^+$ are equivalent to

$$\begin{cases} \widehat{f}_l'(x) \geqslant 0 \\ \widetilde{f}_l'(x) \leqslant \gamma^+ \widehat{f}_l'(x) \\ \widetilde{f}_l'(x) \geqslant \gamma^- \widehat{f}_l'(x) \end{cases} \text{ or } \begin{cases} \widehat{f}_l'(x) \leqslant 0 \\ \widetilde{f}_l'(x) \geqslant \gamma^+ \widehat{f}_l'(x) \\ \widetilde{f}_l'(x) \leqslant \gamma^- \widehat{f}_l'(x) \end{cases} \tag{3.7}$$

and the inequalities $\gamma^- \leqslant \gamma_{F'_{(r)}}(x) \leqslant \gamma^+$ are equivalent to

$$\begin{cases} \widehat{f}_r'(x) \geqslant 0 \\ \widetilde{f}_r'(x) \leqslant \gamma^+ \widehat{f}_r'(x) \\ \widetilde{f}_r'(x) \geqslant \gamma^- \widehat{f}_r'(x) \end{cases} \text{ or } \begin{cases} \widehat{f}_r'(x) \leqslant 0 \\ \widetilde{f}_r'(x) \geqslant \gamma^+ \widehat{f}_r'(x) \\ \widetilde{f}_r'(x) \leqslant \gamma^- \widehat{f}_r'(x). \end{cases} \tag{3.8}$$

Observe that if $\widehat{f}'(x) = 0$, then the other conditions in (3.7) and (3.8) become $\widetilde{f}_r'(x) = \widetilde{f}_l'(x) = 0$ so that $\widetilde{f}'(x) = 0$; as a consequence, if $\widehat{f}'(x) = 0$ and $\left|\widetilde{f}_l'(x)\right| = \left|\widetilde{f}_r'(x)\right| \neq 0$, then we cannot have the inequality expressed above ($\gamma^- \leqslant \gamma_{F'}(x) \leqslant \gamma^+$); therefore, we have that neither $F'_{gH}(x) \precsim_{\gamma^-,\gamma^+} 0$ nor $F'_{gH}(x) \succsim_{\gamma^-,\gamma^+} 0$, i.e., $F'_{gH}(x)$ and 0 are incomparable.

Remark 3.1.2. *As we have seen, the existence of gH-derivative $F'_{gH}(x)$ is equivalent to the existence (and their equality) of both the left and right gH-derivatives, defined as follows*

$$F'_{(l)gH}(x) = \lim_{h \nearrow 0} \frac{F(x+h) \ominus_{gH} F(x)}{h} \in \mathcal{K}_C$$

and

$$F'_{(r)gH}(x) = \lim_{h \searrow 0} \frac{F(x+h) \ominus_{gH} F(x)}{h} \in \mathcal{K}_C;$$

indeed, according to Proposition 3.1.3, we have $F'_{(l)gH}(x) = \left(\widehat{f}_l'(x); \left|\widetilde{f}_l'(x)\right|\right)$ and $F'_{(r)gH}(x) = \left(\widehat{f}_r'(x); \left|\widetilde{f}_r'(x)\right|\right)$.

In many cases, the midpoint function is defined as $\widetilde{f}(x) = |\varphi(x)|$ where $\varphi(x)$ is differentiable; then, if also $\widehat{f}'(x)$ exists, we have that $F(x)$ is gH-differentiable and $F'_{gH}(x) = \left(\widehat{f}'(x); |\varphi'(x)|\right)$.

3.1.4 Monotonicity of functions with values in $(\mathcal{K}_{\mathcal{C}}, \precapprox_{\gamma^-,\gamma^+})$

Monotonicity of interval-valued functions has not been much investigated and this is partially due to the lack of unique meaningful definition of an order for interval-valued functions. By definition of the lattice $(\mathcal{K}_C, \precapprox_{\gamma^-,\gamma^+})$, endowed with the partial order $\precapprox_{\gamma^-,\gamma^+}(\gamma^- \leqslant 0$ and $\gamma^+ \geqslant 0)$ and with use of the reverse order $\succapprox_{\gamma^-,\gamma^+}$, it is possible to analyse monotonicity and, using the gH-difference, related characteristics of inequalities for intervals.

Definition 3.1.5. *([84]) Let $F : [a,b] \to \mathcal{K}_C$ be given, $F(x) = \left(\widehat{f}(x); \widetilde{f}(x)\right)$. We say that F is*

a-i $(\precapprox_{\gamma^-,\gamma^+})$*-nondecreasing on $[a,b]$ if $x_1 < x_2$ implies $F(x_1) \precapprox_{\gamma^-,\gamma^+} F(x_2)$ for all $x_1, x_2 \in [a,b]$;*

a-ii $(\precapprox_{\gamma^-,\gamma^+})$*-nonincreasing on $[a,b]$ if $x_1 < x_2$ implies $F(x_2) \precapprox_{\gamma^-,\gamma^+} F(x_1)$ for all $x_1, x_2 \in [a,b]$;*

b-i *(strictly)* $(\precsim_{\gamma^-,\gamma^+})$*-increasing on $[a,b]$ if $x_1 < x_2$ implies $F(x_1) \precsim_{\gamma^-,\gamma^+} F(x_2)$ for all $x_1, x_2 \in [a,b]$;*

b-ii *(strictly)* $(\precsim_{\gamma^-,\gamma^+})$*-decreasing on $[a,b]$ if $x_1 < x_2$ implies $F(x_2) \precsim_{\gamma^-,\gamma^+} F(x_1)$ for all $x_1, x_2 \in [a,b]$;*

c-i *(strongly)* $(\prec_{\gamma^-,\gamma^+})$*-increasing on $[a,b]$ if $x_1 < x_2$ implies $F(x_1) \prec_{\gamma^-,\gamma^+} F(x_2)$ for all $x_1, x_2 \in [a,b]$;*

c-ii *(strongly)* $(\prec_{\gamma^-,\gamma^+})$*-decreasing on $[a,b]$ if $x_1 < x_2$ implies $F(x_2) \prec_{\gamma^-,\gamma^+} F(x_1)$ for all $x_1, x_2 \in [a,b]$.*

If one of the six conditions is satisfied, we say that F is monotonic on $[a,b]$; the monotonicity is strict if (b-i,b-ii) or strong if (c-i,c-ii) are satisfied.

The monotonicity of $F : [a,b] \to \mathcal{K}_C$ can be analyzed also locally, in a neighborhood of an internal point $x_0 \in]a,b[$, by considering condition $F(x) \precapprox_{-\gamma^-,\gamma^+} F(x_0)$ (or condition $F(x) \succapprox_{-\gamma^+,-\gamma^-} F(x_0)$) for $x \in]a,b[$ and $|x - x_0| < \delta$ with a positive small δ.

Definition 3.1.6. *([84]) Let $F : [a,b] \to \mathcal{K}_C$ be given, $F(x) = \left(\widehat{f}(x); \widetilde{f}(x)\right)$ and $x_0 \in]a,b[$. Let $U_\delta(x_0) = \{x; |x - x_0| < \delta\}$ (for positive δ) denote a neighborhood of x_0. We say that F is (locally)*

a-i $(\precapprox_{\gamma^-,\gamma^+})$*-nondecreasing in a neighborhood of x_0 if $x_1 < x_2$ implies $F(x_1) \precapprox_{\gamma^-,\gamma^+} F(x_2)$ for all $x_1, x_2 \in U_\delta(x_0) \cap [a,b]$ and some $\delta > 0$;*

a-ii $(\precapprox_{\gamma^-,\gamma^+})$*-nonicreasing in a neighborhood of x_0 if $x_1 < x_2$ implies $F(x_2) \precapprox_{\gamma^-,\gamma^+} F(x_1)$ for all $x_1, x_2 \in U_\delta(x_0) \cap [a,b]$ and some $\delta > 0$;*

b-i (strictly)$(\precsim_{\gamma^-,\gamma^+})$*-increasing in a neighborhood of* x_0 *if* $x_1 < x_2$ *implies* $F(x_1) \precsim_{\gamma^-,\gamma^+} F(x_2)$ *for all* $x_1, x_2 \in U_\delta(x_0) \cap [a,b]$ *and some* $\delta > 0$*;*

b-ii (strictly)$(\precsim_{\gamma^-,\gamma^+})$*-decreasing in a neighborhood of* x_0 *if* $x_1 < x_2$ *implies* $F(x_2) \precsim_{\gamma^-,\gamma^+} F(x_1)$ *for all* $x_1, x_2 \in U_\delta(x_0) \cap [a,b]$ *and some* $\delta > 0$*;*

c-i (strongly)$(\prec_{\gamma^-,\gamma^+})$*-increasing in a neighborhood of* x_0 *if* $x_1 < x_2$ *implies* $F(x_1) \prec_{\gamma^-,\gamma^+} F(x_2)$ *for all* $x_1, x_2 \in U_\delta(x_0) \cap [a,b]$ *and some* $\delta > 0$*;*

c-ii (strongly)$(\prec_{\gamma^-,\gamma^+})$*-decreasing in a neighborhood of* x_0 *if* $x_1 < x_2$ *implies* $F(x_2) \prec_{\gamma^-,\gamma^+} F(x_1)$ *for all* $x_1, x_2 \in U_\delta(x_0) \cap [a,b]$ *and some* $\delta > 0$*.*

Moreover, according to (2.33), we have:

$$F(x) \precapprox_{\gamma^-,\gamma^+} F(x_0) \Longleftrightarrow \begin{cases} \widehat{f}(x) \leqslant \widehat{f}(x_0) \\ \widetilde{f}(x) \geqslant \widetilde{f}(x_0) + \gamma^+ \left(\widehat{f}(x) - \widehat{f}(x_0)\right) \\ \widetilde{f}(x) \leqslant \widetilde{f}(x_0) + \gamma^- \left(\widehat{f}(x) - \widehat{f}(x_0)\right), \end{cases} \tag{3.9}$$

i.e., for increasing case,

$$x < x_0 \implies \begin{cases} \widehat{f}(x) - \widehat{f}(x_0) \leqslant 0 \\ \widetilde{f}(x) - \widetilde{f}(x_0) \geqslant \gamma^+ \left(\widehat{f}(x) - \widehat{f}(x_0)\right) \\ \widetilde{f}(x) - \widetilde{f}(x_0) \leqslant \gamma^- \left(\widehat{f}(x) - \widehat{f}(x_0)\right), \end{cases} \tag{3.10}$$

that is,

$$x < x_0 \implies \begin{cases} \widehat{f}(x) - \widehat{f}(x_0) \leqslant 0 \\ \widetilde{f}(x) - \gamma^+ \widehat{f}(x) \geqslant \widetilde{f}(x_0) - \gamma^+ \widehat{f}(x_0) \\ \widetilde{f}(x) - \gamma^- \widehat{f}(x) \leqslant \widetilde{f}(x_0) - \gamma^- \widehat{f}(x_0), \end{cases} \tag{3.11}$$

so that $F(x)$ is $(\precapprox_{\gamma^-,\gamma^+})$-monotonic at x_0 according to the monotonicity of the three functions $\widehat{f}(x)$, $\widetilde{f}(x) - \gamma^+ \widehat{f}(x)$ and $\widetilde{f}(x) - \gamma^- \widehat{f}(x)$:

Proposition 3.1.4. *([84]) Let* $F : [a,b] \to \mathcal{K}_C$ *be given,* $F(x) = \left(\widehat{f}(x); \widetilde{f}(x)\right)$ *and* $x_0 \in]a,b[$*. Then*

(i) $F(x)$ *is* $(\precapprox_{\gamma^-,\gamma^+})$*-nondecreasing in a neighborhood of* x_0 *if and only if* $\widehat{f}(x)$ *is nondecreasing,* $\widetilde{f}(x) - \gamma^+ \widehat{f}(x)$ *is nonincreasing and* $\widetilde{f}(x) - \gamma^- \widehat{f}(x)$ *is nondecreasing at* x_0*;*

(ii) $F(x)$ *is* $(\precapprox_{\gamma^-,\gamma^+})$*-nonincreasing in a neighborhood of* x_0 *if and only if* $\widehat{f}(x)$ *is nonincreasing,* $\widetilde{f}(x) - \gamma^+ \widehat{f}(x)$ *is nondecreasing and* $\widetilde{f}(x) - \gamma^- \widehat{f}(x)$ *is nonincreasing at* x_0*.*

Analogous conditions are valid for strict and strong monotonicity.

The following scheme summarizes these results:

$$F \text{ is } (\precapprox_{\gamma^-,\gamma^+}) \nearrow \quad \Longleftrightarrow \quad \begin{cases} \widehat{f} \text{ is } \nearrow \\ \widetilde{f} - \gamma^+ \widehat{f} \text{ is } \searrow \\ \widetilde{f} - \gamma^- \widehat{f} \text{ is } \nearrow, \end{cases}$$

$$F \text{ is } (\precapprox_{\gamma^-,\gamma^+}) \searrow \quad \Longleftrightarrow \quad \begin{cases} \widehat{f} \text{ is } \searrow \\ \widetilde{f} - \gamma^+ \widehat{f} \text{ is } \nearrow \\ \widetilde{f} - \gamma^- \widehat{f} \text{ is } \searrow. \end{cases}$$

Remark 3.1.3. *In terms of the endpoint functions f^- and f^+, given by $f^- = \widehat{f} - \widetilde{f}$, $f^+ = \widehat{f} + \widetilde{f}$, the conditions in (3.11), after a few simple steps, can be written as*

$$x < x_0 \implies \begin{cases} f^+(x) - f^+(x_0) + f^-(x) - f^-(x_0) \leqslant 0 \\ (1-\gamma^+)\left(f^+(x) - f^+(x_0)\right) \geqslant (1+\gamma^+)\left(f^-(x) - f^-(x_0)\right) \\ (1-\gamma^-)\left(f^+(x) - f^+(x_0)\right) \leqslant (1+\gamma^-)\left(f^-(x) - f^-(x_0)\right) \end{cases} \tag{3.12}$$

and the conditions on f^+ and f^-, for the monotonicity of F are less intuitive than the ones on $\widehat{f}$ and $\widetilde{f}$:

$$F \textit{ is } (\precapprox_{\gamma^-,\gamma^+}) \nearrow \quad \Longleftrightarrow \quad \begin{cases} f^+ + f^- \textit{ is } \nearrow \\ (1-\gamma^+)f^+ - (1+\gamma^+)f^- \textit{ is } \searrow \\ (1-\gamma^-)f^+ - (1+\gamma^-)f^- \textit{ is } \nearrow, \end{cases}$$

$$F \textit{ is } (\precapprox_{\gamma^-,\gamma^+}) \searrow \quad \Longleftrightarrow \quad \begin{cases} f^+ + f^- \textit{ is } \searrow \\ (1-\gamma^+)f^+ - (1+\gamma^+)f^- \textit{ is } \nearrow \\ (1-\gamma^-)f^+ - (1+\gamma^-)f^- \textit{ is } \searrow. \end{cases}$$

If we divide the three inequalities in (3.10) by $x - x_0 < 0$, we obtain, for F to be $(\precapprox_{\gamma^-,\gamma^+})$-nondecreasing at x_0,

$$\begin{cases} \dfrac{\widehat{f}(x) - \widehat{f}(x_0)}{x - x_0} \geqslant 0 \\ \dfrac{\widetilde{f}(x) - \widetilde{f}(x_0)}{x - x_0} \leqslant \gamma^+ \left(\dfrac{\widehat{f}(x) - \widehat{f}(x_0)}{x - x_0} \right) \\ \dfrac{\widetilde{f}(x) - \widetilde{f}(x_0)}{x - x_0} \geqslant \gamma^- \left(\dfrac{\widehat{f}(x) - \widehat{f}(x_0)}{x - x_0} \right). \end{cases} \tag{3.13}$$

Analogously, for F to be $(\precapprox_{\gamma^-,\gamma^+})$-nonincreasing at x_0, we obtain

$$\begin{cases} \dfrac{\widehat{f}(x)-\widehat{f}(x_0)}{x-x_0} \leqslant 0 \\ \dfrac{\widetilde{f}(x)-\widetilde{f}(x_0)}{x-x_0} \geqslant \gamma^+ \left(\dfrac{\widehat{f}(x)-\widehat{f}(x_0)}{x-x_0}\right) \\ \dfrac{\widetilde{f}(x)-\widetilde{f}(x_0)}{x-x_0} \leqslant \gamma^- \left(\dfrac{\widehat{f}(x)-\widehat{f}(x_0)}{x-x_0}\right). \end{cases} \tag{3.14}$$

Suppose now that $\widetilde{f}$ and $\widehat{f}$ have both left and right (finite) derivatives at x_0; denote them by $\widetilde{f}'_l(x_0)$, $\widetilde{f}'_r(x_0)$, $\widehat{f}'_l(x_0)$, $\widehat{f}'_r(x_0)$. Taking the limits in (3.13) and (3.14) with $x \nearrow x_0$ and $x \searrow x_0$, we obtain the conditions for $(\precapprox_{\gamma^-,\gamma^+})$-monotonicity of F in a neighborhood of x_0, as stated in the following proposition.

Proposition 3.1.5. *([84]) Let $F : [a,b] \to \mathcal{K}_C$ be given, $F(x) = \left(\widehat{f}(x); \widetilde{f}(x)\right)$ and assume that $\widehat{f}$ and $\widetilde{f}$ have left and right derivatives at an internal point $x_0 \in]a,b[$. The following are necessary conditions for local monotonicity:*
(i-n) if F is $(\precapprox_{\gamma^-,\gamma^+})$-nondecreasing or $(\precapprox_{\gamma^-,\gamma^+})$-increasing in a neighborhood of x_0, then

$$\begin{cases} \widehat{f}'_r(x_0) \geqslant 0 \ , \ \widehat{f}'_l(x_0) \geqslant 0 \\ \widetilde{f}'_r(x_0) \leqslant \gamma^+ \widehat{f}'_r(x_0) \ , \ \widetilde{f}'_l(x_0) \leqslant \gamma^+ \widehat{f}'_l(x_0) \\ \widetilde{f}'_r(x_0) \geqslant \gamma^- \widehat{f}'_r(x_0) \ , \ \widetilde{f}'_l(x_0) \geqslant \gamma^- \widehat{f}'_l(x_0); \end{cases} \tag{3.15}$$

(ii-n) if F is $(\precapprox_{\gamma^-,\gamma^+})$-nonincreasing or $(\precapprox_{\gamma^-,\gamma^+})$-decreasing in a neighborhood of x_0, then

$$\begin{cases} \widehat{f}'_r(x_0) \leqslant 0 \ , \ \widehat{f}'_l(x_0) \leqslant 0 \\ \widetilde{f}'_r(x_0) \geqslant \gamma^+ \widehat{f}'_r(x_0) \ , \ \widetilde{f}'_l(x_0) \geqslant \gamma^+ \widehat{f}'_l(x_0) \\ \widetilde{f}'_r(x_0) \leqslant \gamma^- \widehat{f}'_r(x_0) \ , \ \widetilde{f}'_l(x_0) \leqslant \gamma^- \widehat{f}'_l(x_0). \end{cases} \tag{3.16}$$

The following are sufficient conditions for local strong monotonicity:
(i-s) if

$$\begin{cases} \widehat{f}'_r(x_0) > 0 \ , \ \widehat{f}'_l(x_0) > 0 \\ \widetilde{f}'_r(x_0) < \gamma^+ \widehat{f}'_r(x_0) \ , \ \widetilde{f}'_l(x_0) < \gamma^+ \widehat{f}'_l(x_0) \\ \widetilde{f}'_r(x_0) > \gamma^- \widehat{f}'_r(x_0) \ , \ \widetilde{f}'_l(x_0) > \gamma^- \widehat{f}'_l(x_0), \end{cases} \tag{3.17}$$

then F is strongly $(\prec_{\gamma^-,\gamma^+})$-increasing in a neighborhood of x_0;
(ii-s) if

$$\begin{cases} \widehat{f}'_r(x_0) < 0 \ , \ \widehat{f}'_l(x_0) < 0 \\ \widetilde{f}'_r(x_0) > \gamma^+ \widehat{f}'_r(x_0) \ , \ \widetilde{f}'_l(x_0) > \gamma^+ \widehat{f}'_l(x_0) \\ \widetilde{f}'_r(x_0) < \gamma^- \widehat{f}'_r(x_0) \ , \ \widetilde{f}'_l(x_0) < \gamma^- \widehat{f}'_l(x_0), \end{cases} \tag{3.18}$$

then F is strongly $(\prec_{\gamma^-,\gamma^+})$-decreasing in a neighborhood of x_0.

If $\widehat{f}'(x)$ and $\widetilde{f}'(x)$ exist on $]a,b[$, then the conditions for monotonicity can be expressed in the obvious way as for elementary calculus, in terms of the derivatives $\widehat{f}'(x)$, $\widetilde{f}'(x) - \gamma^+\widehat{f}'(x)$ and $\widetilde{f}'(x) - \gamma^-\widehat{f}'(x)$. Therefore, the necessary conditions for nondecreasing $F(x)$ are

$$\begin{cases} \widehat{f}'(x) \geqslant 0 \\ \widetilde{f}'(x) - \gamma^+\widehat{f}'(x) \leqslant 0 \\ \widetilde{f}'(x) - \gamma^-\widehat{f}'(x) \geqslant 0 \end{cases} \tag{3.19}$$

and for nonincreasing $F(x)$ are

$$\begin{cases} \widehat{f}'(x) \leqslant 0 \\ \widetilde{f}'(x) - \gamma^+\widehat{f}'(x) \geqslant 0 \\ \widetilde{f}'(x) - \gamma^-\widehat{f}'(x) \leqslant 0. \end{cases} \tag{3.20}$$

With reference to Example 3.1.1, the functions $\widehat{f}(x)$, $\widetilde{f}(x) - \gamma^+\widehat{f}(x)$ and $\widetilde{f}(x) - \gamma^-\widehat{f}(x)$ are pictured in Figure 3.5 and their derivatives are in Figure 3.6; the partial order is fixed with $\gamma^- = -1$ and $\gamma^+ = 1$, i.e., $\precsim_{LU}$.

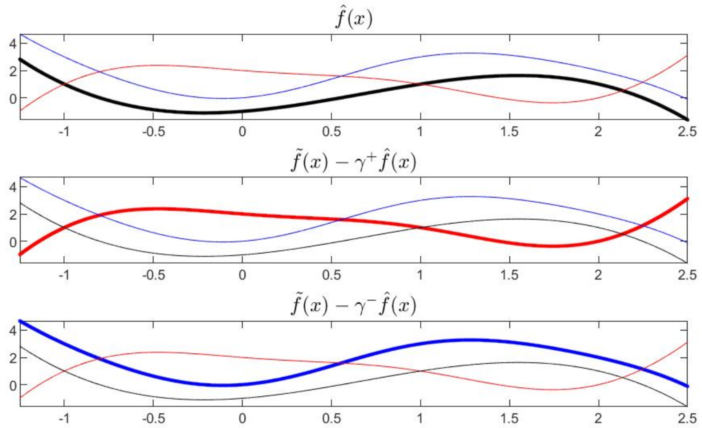


Figure 3.5: Functions $\widehat{f}(x)$, $\widetilde{f}(x) - \gamma^+\widehat{f}(x)$ and $\widetilde{f}(x) - \gamma^-\widehat{f}(x)$ in Example 3.1.1.

Now, but only for the case of a partial order $\precsim_{\gamma^-,\gamma^+}$ with the condition that $\gamma^- + \gamma^+ = 0$, i.e., $\gamma^+ = -\gamma^- = \gamma > 0$, we can establish a strong connection between the monotonicity of F and the sign of its gH-derivative $F'_{gH}(x)$. Denote the corresponding partial order $\precsim_{-\gamma,\gamma}$ simply by $\precsim_{\gamma}$.

Proposition 3.1.6. *Let $F :]a,b[\to \mathcal{K}_C$ be given, $F(x) = \left(\widehat{f}(x); \widetilde{f}(x)\right)$ and assume F has gH-derivative $F'_{gH}(x)$ at the internal points $x \in]a,b[$. Let $\gamma > 0$ be fixed. Then*

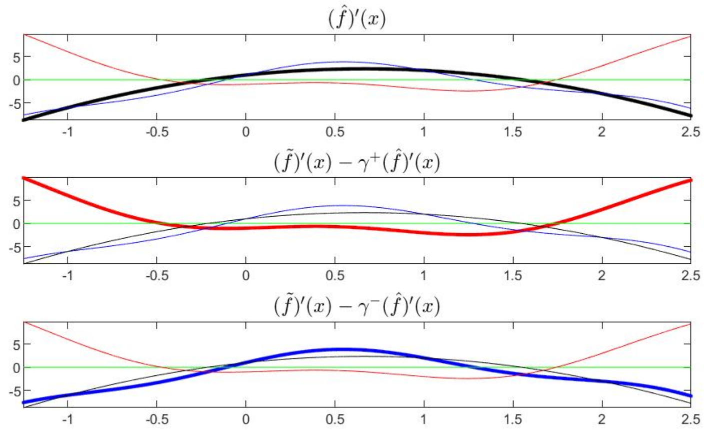


Figure 3.6: Derivatives of functions $\widehat{f}(x)$, $\widetilde{f}(x) - \gamma^{+}\widehat{f}(x)$ and $\widetilde{f}(x) - \gamma^{-}\widehat{f}(x)$ in Example 3.1.1.

(1) if F is $(\precsim_\gamma)$-nondecreasing on $]a,b[$, then $F'_{gH}(x) \succsim_\gamma 0$ for all $x \in]a,b[$;

(2) if F is $(\precsim_\gamma)$-nonincreasing on $]a,b[$, then $F'_{gH}(x) \precsim_\gamma 0$ for all $x \in]a,b[$.

Proof. We prove only (*1*).
By Definition 3.1.4, $F'_{gH}(x) = \lim\limits_{h\to 0}\frac{1}{h}[F(x+h)\ominus_{gH} F(x)]$ and F is continuous.
If F is nondecreasing, then, for sufficiently small $h > 0$, we have:
$F(x) \precsim_\gamma F(x+h)$ and so, by part (3) of Lemma 2.2.1,
$0 \precsim_\gamma F(x+h)\ominus_{gH} F(x)$ which, dividing by h, gives $0 \precsim_\gamma \dfrac{F(x+h)\ominus_{gH} F(x)}{h}$;
by taking the limit for $h \searrow 0$, according to Proposition 3.1.2, we obtain:
$\lim_{h\searrow 0} 0 \precsim_\gamma \lim_{h\searrow 0} \dfrac{F(x+h)\ominus_{gH} F(x)}{h}$, that is, $0 \precsim_\gamma F'_{gH}(x)$.
On the other hand, for $h < 0$, we have $F(x+h) \precsim_\gamma F(x)$,
i.e., $F(x+h)\ominus_{gH} F(x) \precsim_\gamma 0$ which gives $\dfrac{F(x+h)\ominus_{gH} F(x)}{-h} \precsim_\gamma 0$; by taking the limit for $h \nearrow 0$, we get $(-F'_{gH}(x)) \precsim_\gamma 0$ and, changing sign on both sides, it is $F'_{gH}(x) \succsim_\gamma 0$.
The proof of (*2*) is similar. □

An analogous result is also immediate, relating strong (local) monotonicity of F to the "sign" of its left and right derivatives

$$F'_{(l)gH}(x) = \left(\widehat{f}'_l(x); \left|\widetilde{f}'_l(x)\right|\right) \text{ and } F'_{(r)gH}(x) = \left(\widehat{f}'_r(x); \left|\widetilde{f}'_r(x)\right|\right);$$

at the extreme points of $[a,b]$, we consider only right (at a) or left (at b) monotonicity and right or left derivatives. Again we assume the condition $\gamma^{+} = -\gamma^{-} = \gamma > 0$.

Proposition 3.1.7. *Let $F : [a, b] \to \mathcal{K}_C$ be given, $F(x) = \left(\widehat{f}(x); \widetilde{f}(x)\right)$ with left and/or right gH-derivatives at a point $x_0 \in [a, b]$. Then*

i.a if $0 \prec_\gamma F'_{(l)gH}(x_0)$, then F is strongly $(\prec_\gamma)$-increasing on $[x_0 - \delta, x_0]$ for some $\delta > 0$ (here $x_0 > a$);

i.b if $0 \prec_\gamma F'_{(r)gH}(x_0)$, then F is strongly $(\prec_\gamma)$-increasing on $[x_0, x_0 + \delta]$ for some $\delta > 0$ (here $x_0 < b$);

ii.a if $0 \succ_\gamma F'_{(l)gH}(x_0)$, then F is strongly $(\prec_\gamma)$-decreasing on $[x_0 - \delta, x_0]$ for some $\delta > 0$ (here $x_0 > a$);

ii.b if $0 \succ_\gamma F'_{(r)gH}(x_0)$, then F is strongly $(\prec_\gamma)$-decreasing on $[x_0, x_0 + \delta]$ for some $\delta > 0$ (here $x_0 < b$).

Proof. We prove only $(i.a)$.
From $0 \prec_\gamma F'_{(l)gH}(x_0)$, that is, $\left(\widehat{f}'_l(x); \left|\widetilde{f}'_l(x)\right|\right) \succ_\gamma 0$, with a procedure similar to that seen in (2.36), we have: $\widehat{f}'_l(x_0) > 0$ and $\left|\widetilde{f}'_l(x_0)\right| < \gamma \widehat{f}'_l(x_0)$,

i.e., $\widehat{f}'_l(x_0) > 0$ and $-\gamma \widehat{f}'_l(x_0) < \widetilde{f}'_l(x_0) < \gamma \widehat{f}'_l(x_0)$, namely, $\begin{cases} \widehat{f}'_l(x_0) > 0 \\ \widetilde{f}'_l(x_0) > -\gamma \widehat{f}'_l(x_0) \\ \widetilde{f}'_l(x_0) < \gamma \widehat{f}'_l(x_0). \end{cases}$

From conditions (3.17), we have that F is strongly $(\prec_\gamma)$-increasing at x_0; therefore, consequently, the conclusion also follows. □

We conclude this subsection with the following

Example 3.1.2. *Function $F : [a, b] \to \mathcal{K}_C$, $F(x) = \left(\widehat{f}(x); \widetilde{f}(x)\right)$, for $x \in [a, b] = [-2, 4]$, is defined by $\widehat{f}(x) = -x^3 + 4x^2 + 3x - 1$ and $\widetilde{f}(x) = \left|x^2 - x - 2\right|$ (see Figure 3.7).*

Remark that function $\widehat{f}(x)$ is differentiable on $]a, b[$ with $\widehat{f}'(x) = -3x^2 + 8x + 3$ and $\widetilde{f}(x)$ is differentiable with $\widetilde{f}'(x) = (2x - 1)sign(x^2 - x - 2)$ for $x \neq -1$ and $x \neq 2$; at these two points the left and right derivatives exist: $\widetilde{f}'_l(-1) = -3$, $\widetilde{f}'_r(-1) = 3$, $\widetilde{f}'_l(2) = -3$, $\widetilde{f}'_r(2) = 3$.

Function $F(x)$ is gH-differentiable on $]a, b[$ (including the two points $x = -1$ and $x = 2$) and $F'_{gH}(x) = \left(-3x^2 + 8x + 3; |2x - 1|\right)$ (see Figure 3.8). Also right and left gH-derivatives exist at $a = -2$ and $b = 4$, respectively. Considering the points $a_1 = -0.527525$, $a_2 = -0.189255$, $a_3 = 2.527525$ and $a_4 = 3.522588$, the corresponding gH-derivatives are (approximately) $F'_{gH}(a_1) = [-4.11, 0]$, $F'_{gH}(a_2) = [0, 2.757]$, $F'_{gH}(a_3) = [0, 8.11]$, $F'_{gH}(a_1) = [-12.09, 0]$. Indeed, from 0 to (a_1) and from (a_4) to 4, $F'_{gH} \prec_{LU} 0$ and F is $(\precapprox_{LU}) \searrow$, while from a_2 to (a_3), $F'_{gH} \succ_{LU} 0$ and F is $(\precapprox_{LU}) \nearrow$.

Note also that when $\widetilde{f}(x)' = 0$ (see when, at the top of Figure 3.8, the red curve intersects the green line), it follows that $(f(x)')^- = (f(x)')^+ = \widehat{f}(x)'$

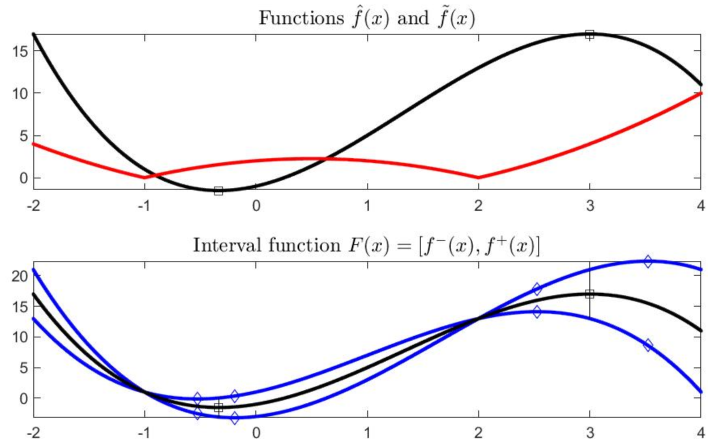


Figure 3.7: (Top) functions $\widehat{f}(x) = -x^3 + 4x^2 + 3x - 1$ and $\widetilde{f}(x) = \left|x^2 - x - 2\right|$ (respectively drawn in black and red). (Bottom) interval-valued function $F(x) = \left[\widehat{f}(x) - \widetilde{f}(x), \widehat{f}(x) + \widetilde{f}(x)\right]$.

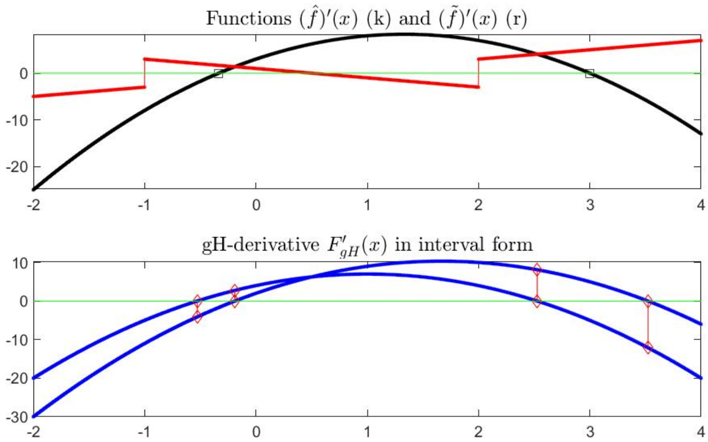


Figure 3.8: (Top) derivatives of $\widehat{f}(x)$ at all points (black) and of $\widetilde{f}(x)$ at $x \neq -1$ and $x \neq 2$ (red). (Bottom) gH-derivative of function $F(x)$; points a_i, $i = 1, ..., 4$, are marked in red.

(corresponding to the intersection of blue curves in the bottom part of the same figure).

With $\gamma^+ = -\gamma^- = 1$, i.e., with $(\precapprox_{LU})$-order, the functions $\widehat{f}(x)$, $\widetilde{f}(x) - \gamma^+\widehat{f}(x)$ and $\widetilde{f}(x) - \gamma^-\widehat{f}(x)$ are pictured in Figure 3.9.

According to Proposition 3.1.4, necessary conditions for nonincreasing $F(x)$ are satisfied on $[a, a_1]$ and $[a_4, b]$ and for nondecreasing $F(x)$ are satisfied on $[a_2, a_3]$. Corresponding necessary conditions using the sign of the

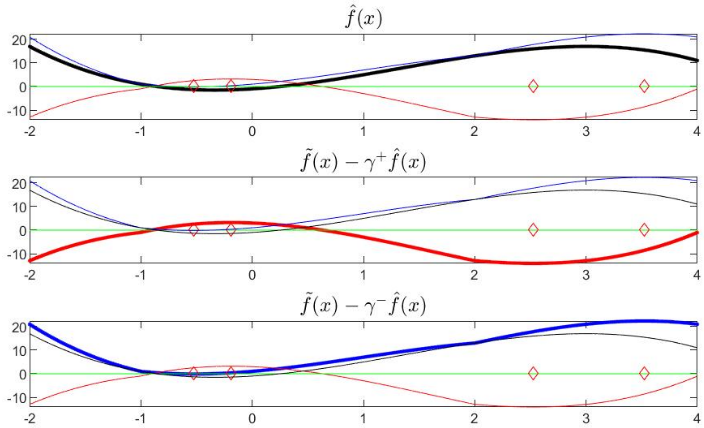


Figure 3.9: Functions $\widehat{f}(x)$, $\widetilde{f}(x) - \gamma^+ \widehat{f}(x)$ and $\widetilde{f}(x) - \gamma^- \widehat{f}(x)$ in Example 3.1.2.

derivatives of functions $\widehat{f}(x)$, $\widetilde{f}(x) - \gamma^+ \widehat{f}(x)$ and $\widetilde{f}(x) - \gamma^- \widehat{f}(x)$ can be checked in Figure 3.10; at the points $x = -1$ and $x = 2$ we can apply the conditions involving left and right derivatives of $\widetilde{f}(x)$.

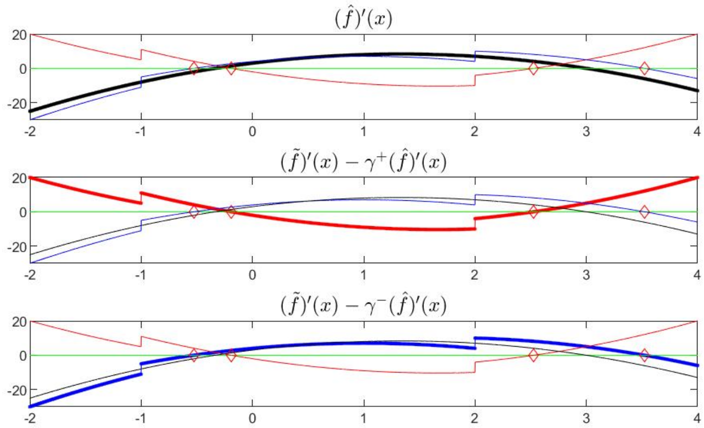


Figure 3.10: Derivatives of functions $\widehat{f}(x)$, $\widetilde{f}(x) - \gamma^+ \widehat{f}(x)$ and $\widetilde{f}(x) - \gamma^- \widehat{f}(x)$ in Example 3.1.2.

Finally, according to Proposition 3.1.7, it is easy to check that the sufficient conditions for strong $\prec_{LU}$ monotonicity are satisfied: decreasing on $[-2, a_1]$ and $[a_4, 4]$, increasing on $[a_2, a_3]$. In the remaining points $x \in]a_1, a_2[$ and $x \in]a_3, a_4[$ the sufficient conditions for strong $\prec_\gamma$ monotonicity are not satisfied (the interval-valued gH-derivatives of $F(x)$ contain zero as an interior value as also shown in Figure 3.8).

3.1.5 Extrema of interval-valued functions

The three concepts of monotonicity defined in Subsection 3.1.4 (simple, strict and strong), based on the orders $\precapprox_{\gamma^-,\gamma^+}$, $\precsim_{\gamma^-,\gamma^+}$ and $\prec_{\gamma^-,\gamma^+}$, translate into different concepts of extrema. We will adopt the following terminology.

Definition 3.1.7. *If $F(x_0) \precapprox_{\gamma^-,\gamma^+} F(x)$, we say that $F(x_0)$ dominates $F(x)$ with respect to the partial order $\precapprox_{\gamma^-,\gamma^+}$ ($F(x_0)$ ($\precapprox_{\gamma^-,\gamma^+}$)-dominates $F(x)$ for short), or equivalently that $F(x)$ is ($\precapprox_{\gamma^-,\gamma^+}$)-dominated by $F(x_0)$.*

We say that $F(x)$ and $F(x_0)$ are incomparable with respect to $\precapprox_{\gamma^-,\gamma^+}$ if both $F(x_0) \precapprox_{\gamma^-,\gamma^+} F(x)$ and $F(x) \precapprox_{\gamma^-,\gamma^+} F(x_0)$ are not valid.

Analogous domination rules are defined in terms of the strict and strong order relations $\precsim_{\gamma^-,\gamma^+}$ and $\prec_{\gamma^-,\gamma^+}$, respectively.

Remark 3.1.4. *Observe that if $F(x_0) \precapprox_{\gamma^-,\gamma^+} F(x)$ and $F(x) \precapprox_{\gamma^-,\gamma^+} F(x_0)$, i.e., if $F(x)$ and $F(x_0)$ are ($\precapprox_{\gamma^-,\gamma^+}$)-dominating each other, then $F(x) = F(x_0)$ and vice-versa, i.e., reciprocal dominance is equivalent to coincidence; the same remains true if the two orders for the dominance are obtained with different pairs (γ_1^-, γ_1^+), (γ_2^-, γ_2^+), $\gamma_i^- \leqslant 0$ and $\gamma_i^+ \geqslant 0$ ($i = 1, 2$), i.e., if $F(x_0) \precapprox_{\gamma_1^-,\gamma_1^+} F(x)$ and $F(x) \precapprox_{\gamma_2^-,\gamma_2^+} F(x_0)$, then $F(x) = F(x_0)$ and vice-versa.*

Let us now introduce the important definitions of order-based minimum and maximum points for an interval-valued function.

Definition 3.1.8. *Let $F : [a, b] \to \mathcal{K}_C$ be an interval-valued function and $x_0 \in [a, b]$. Consider the order $\precapprox_{\gamma^-,\gamma^+}$ with $\gamma^- \leqslant 0$, $\gamma^+ \geqslant 0$. We say that, with respect to $\precapprox_{\gamma^-,\gamma^+}$,*

(1) x_0 is a local lattice-minimum point of F (min-point for short) if there exists $\delta > 0$ such that $F(x_0) \precapprox_{\gamma^-,\gamma^+} F(x)$ for all $x \in]x_0-\delta, x_0+\delta[\cap [a, b]$, i.e., if all $F(x)$ around x_0 are ($\precapprox_{\gamma^-,\gamma^+}$)-dominated by $F(x_0)$;

(2) x_0 is a local lattice-maximum point of F (max-point for short) if there exists $\delta > 0$ such that $F(x) \precapprox_{\gamma^-,\gamma^+} F(x_0)$ for all $x \in]x_0-\delta, x_0+\delta[\cap [a, b]$, i.e., if all $F(x)$ around x_0 ($\precapprox_{\gamma^-,\gamma^+}$)-dominate $F(x_0)$.

*In case (1) we say that x_0 is a (γ^-, γ^+)-*min*-point for F and in case (2) we say that x_0 is a (γ^-, γ^+)-*max*-point.*

Conditions (*1*) or (*2*) in Definition 3.1.8 imply that if there exists $x' \in [a, b]$ such that $\widehat{f}(x') = \widehat{f}(x_0)$ and $\widetilde{f}(x') \neq \widetilde{f}(x_0)$, then it is impossible to have $F(x_0) \precapprox_{\gamma^-,\gamma^+} F(x')$ nor $F(x') \precapprox_{\gamma^-,\gamma^+} F(x_0)$ (unless $\gamma^- = -\infty$ and $\gamma^+ = +\infty$); this means that, except for trivial cases, if $\widehat{f}(x') = \widehat{f}(x_0)$, then $F(x')$ and $F(x_0)$ are ($\precapprox_{\gamma^-,\gamma^+}$)-incomparable or coincident.

Remark that a lattice-type extremal value corresponds, locally, to the smallest or greatest elements in the lattice $(\mathcal{K}_C, \precapprox_{\gamma^-,\gamma^+})$; it is clear that condition (*1*) implies that a min-point x_m of F is necessarily a local minimum of the midpoint function $\widehat{f}$, while condition (*2*) implies that a max-point x_M of F is a local maximum of $\widehat{f}$. It follows that a min-point or a max-point of F are to be searched, respectively, among the minimum or the maximum points of the midpoint function $\widehat{f}$. But this is not sufficient; indeed, lattice-type minimality and maximality, with respect to the partial order $\precapprox_{\gamma^-,\gamma^+}$, can be recognized exactly in terms of the three function

$$\widehat{f}\ ,\ \widetilde{f}-\gamma^+\widehat{f} \text{ and } \widetilde{f}-\gamma^-\widehat{f},$$

as we will see in this section.

It will be useful to explicitly write the conditions for ($\precapprox_{\gamma^-,\gamma^+}$)-dominance of a general interval $F(x)$, with respect to the intervals $F(x_m)$ and $F(x_M)$, that characterize the minimality and the maximality of a point x_m (for min) or a point x_M (for max).
Without explicit distinction between strict or strong dominance, from (3.9) we have

$$F(x) \precapprox_{\gamma^-,\gamma^+} F(x_M) \iff \begin{cases} \widehat{f}(x) \leqslant \widehat{f}(x_M) \\ \widetilde{f}(x) \geqslant \widetilde{f}(x_M) + \gamma^+ \left(\widehat{f}(x) - \widehat{f}(x_M)\right) \\ \widetilde{f}(x) \leqslant \widetilde{f}(x_M) + \gamma^- \left(\widehat{f}(x) - \widehat{f}(x_M)\right) \end{cases} \tag{3.21}$$

and, similarly,

$$F(x) \succapprox_{\gamma^-,\gamma^+} F(x_m) \iff \begin{cases} \widehat{f}(x) \geqslant \widehat{f}(x_m) \\ \widetilde{f}(x) \leqslant \widetilde{f}(x_m) + \gamma^+ \left(\widehat{f}(x) - \widehat{f}(x_m)\right) \\ \widetilde{f}(x) \geqslant \widetilde{f}(x_m) + \gamma^- \left(\widehat{f}(x) - \widehat{f}(x_m)\right). \end{cases} \tag{3.22}$$

With reference to the three conditions in (3.22), we have that:

- the first, $\widehat{f}(x) \geqslant \widehat{f}(x_m)$, says that x_m is a local minimum of $\widehat{f}$;
- the second, $\widetilde{f}(x) - \gamma^+\widehat{f}(x) \leqslant \widetilde{f}(x_m) - \gamma^+\widehat{f}(x_m)$, says that x_m is a local maximum of $\widetilde{f} - \gamma^+\widehat{f}$;
- the third, $\widetilde{f}(x) - \gamma^-\widehat{f}(x) \geqslant \widetilde{f}(x_m) - \gamma^-\widehat{f}(x_m)$, says that x_m is a local minimum of $\widetilde{f} - \gamma^-\widehat{f}$.

By making a similar reasoning also for the conditions of (3.21), we obtain:

- the first, $\widehat{f}(x) \leqslant \widehat{f}(x_M)$, says that x_M is a local maximum of $\widehat{f}$;
- the second, $\widetilde{f}(x) - \gamma^+\widehat{f} \geqslant \widetilde{f}(x_M) - \gamma^+\widehat{f}(x_M)$, says that x_M is a local minimum of $\widetilde{f} - \gamma^+\widehat{f}$;
- the third, $\widetilde{f}(x) - \gamma^-\widehat{f}(x) \leqslant \widetilde{f}(x_M) - \gamma^-\widehat{f}(x_M)$, says that x_M is a local maximum of $\widetilde{f} - \gamma^-\widehat{f}$.

What has just been seen can be summarized in the following result.

Proposition 3.1.8. *([85]) Let $F : [a,b] \to \mathcal{K}_C$ be an interval-valued function. Then*

(i) $x_m \in [a,b]$ *is a* min*-point of* F *if and only if it is a minimum of* $\widehat{f}$ *and* $\widetilde{f} - \gamma^- \widehat{f}$ *and it is a maximum of* $\widetilde{f} - \gamma^+ \widehat{f}$*;*

(ii) $x_M \in [a,b]$ *is a* max*-point of* F *if and only if it is a maximum of* $\widehat{f}$ *and* $\widetilde{f} - \gamma^- \widehat{f}$ *and it is a minimum of* $\widetilde{f} - \gamma^+ \widehat{f}$*.*

In particular, for the $\precsim_{LU}$ order, obtained with $\gamma^- = -1$, $\gamma^+ = 1$, we have:

$$\widetilde{f} - \gamma^+ \widehat{f} = \widetilde{f} - \widehat{f} = -(\widehat{f} - \widetilde{f}) = -f^-;$$

$$\widetilde{f} - \gamma^- \widehat{f} = \widetilde{f} + \widehat{f} = \widehat{f} + \widetilde{f} = f^+.$$

So the conditions to have a min-point are equivalent to having simultaneously a minimum for $\widehat{f}$ and f^+ as well as a maximum for $-f^-$, i.e., a minimum for f^- and f^+ (and automatically for $\widehat{f}$); on the other hand, max-point conditions are equivalent to have the same maximum points for f^- and f^+ in the ordinary sense.

The discussion above highlights the restricting notion of a lattice-extreme point, as it is not frequent that simultaneous extrema occur for the three functions $\widehat{f}$, $\widetilde{f} - \gamma^- \widehat{f}$ and $\widetilde{f} - \gamma^+ \widehat{f}$. The following definition is more general, as it considers the possibility that intervals $F(x)$ for different x are locally incomparable with respect to the actual order relation.

Definition 3.1.9. *Let $F : [a,b] \to \mathcal{K}_C$ be an interval-valued function and $x_m, x_M \in [a,b]$. We say that, with respect to the order $\precsim_{\gamma^-,\gamma^+}$ and the corresponding strict order $\precnsim_{\gamma^-,\gamma^+}$,*

(c) x_m is a local best-minimum point of F (best-min for short) if:

(c.1) it is a local minimum for the midpoint function $\widehat{f}$, and

(c.2) there exists $\delta > 0$ and no point $x \in]x_m - \delta, x_m + \delta[\cap [a,b]$ with $F(x) \neq F(x_m)$ such that $F(x) \precsim_{\gamma^-,\gamma^+} F(x_m)$;

(d) x_M is a local best-maximum point of F (best-max for short) if:

(d.1) it is a local maximum for the midpoint function $\widehat{f}$, and

(d.2) there exists $\delta > 0$ and no point $x \in]x_M - \delta, x_M + \delta[\cap [a,b]$ with $F(x) \neq F(x_M)$ such that $F(x_M) \precsim_{\gamma^-,\gamma^+} F(x)$.

Remark 3.1.5. *Definition 3.1.9 is clearly valid also for points $x_0 \in [a,b]$ coincident with one of a or b. It is also evident that a lattice-type extremum is also a best-type extremum.*

Definitions of strict and strong (local) extremal points can be given by considering the strict $\precsim_{\gamma^-,\gamma^+}$ or the strong $\prec_{\gamma^-,\gamma^+}$ orders associated to the lattice order $\precapprox_{\gamma^-,\gamma^+}$.

Definition 3.1.10. *Let $F:[a,b]\to\mathcal{K}_C$ be an interval-valued function. With respect to an order $\precapprox_{\gamma^-,\gamma^+}$ and the associated strict order $\precsim_{\gamma^-,\gamma^+}$ or strong order $\prec_{\gamma^-,\gamma^+}$, we say that*

- *a best-min point x_m is a strict (respectively strong) best-minimum point if there exists $\delta>0$ and no point $x\in]x_m-\delta,x_m+\delta[\cap[a,b]$ with $F(x)\precsim_{\gamma^-,\gamma^+}F(x_m)$ (or $F(x)\prec_{\gamma^-,\gamma^+}F(x_m)$, respectively);*
- *a best-max point x_M is a strict (respectively strong) best-maximum point if there exists $\delta>0$ and no point $x\in]x_M-\delta,x_M+\delta[\cap[a,b]$ with $F(x_M)\precsim_{\gamma^-,\gamma^+}F(x)$ (or $F(x_M)\prec_{\gamma^-,\gamma^+}F(x)$, respectively).*

Remark 3.1.6. *It is clear that the definitions of lattice-type and best-type extremality do not require any assumptions on continuity of the interval-valued function F on $[a,b]$; in the case of continuity (or left/right continuity) the existence of extreme points is also related to the local left and/or right monotonicity of F (with respect to the same partial order $\precapprox_{\gamma^-,\gamma^+}$).*

In order to illustrate basic properties of the various concepts of min/max ($\precapprox_{\gamma^-,\gamma^+}$)-extremality, we will consider a continuous function $F:[a,b]\to\mathcal{K}_C$ and will suppose that there exist two points $x_m,x_M\in[a,b]$ such that x_m is a local minimum point and x_M is a local maximum point in one of the types defined above.

In the half-plane of points $(\widehat{z};\widetilde{z})$, $\widetilde{z}\geqslant 0$, the intervals $F(x_m)$ and $F(x_M)$ have midpoint representation, respectively,

$$F(x_m)=\left(\widehat{f}(x_m);\widetilde{f}(x_m)\right)\text{ and }F(x_M)=\left(\widehat{f}(x_M);\widetilde{f}(x_M)\right). \tag{3.23}$$

It is immediate that if $x_m\in[a,b]$ is a lattice-minimum point, i.e., there exists a neighborhood of x_m such that all $F(x)$ satisfy (3.22), then none of such $F(x)$ is incomparable with $F(x_m)$; analogously, if $x_M\in[a,b]$ is a lattice-maximum point, i.e., there exists a neighborhood of x_M such that all $F(x)$ satisfy (3.21), then none of such $F(x)$ is incomparable with $F(x_M)$. We can express this fact by saying that the (local) *min-efficient frontier* for the min-point x_m is concentrated into the single interval $F(x_m)$; analogously, the (local) *max-efficient frontier* for the max-point x_M is concentrated into the single interval $F(x_M)$.
Let us remember that, as reported in [43], the concept of *efficient frontier*, also known as the “Pareto optimal set”, is defined as the set of all Pareto-efficient situations, that is when in a situation there is no scope for further improvement without making another worse off.

When instead x_m and x_M are best-type extrema and not lattice-type, then it is important to identify the intervals $F(x)$, in particular with x in a neighborhood of x_m or x_M, that are not min-dominated by $F(x_m)$ (or do not max-dominate $F(x_M)$); clearly, these $F(x)$ are necessarily ($\precapprox_{\gamma^-,\gamma^+}$)-incomparable with $F(x_m)$ (or with $F(x_M)$, respectively).

Corresponding to a minimum and to a maximum point of F, we are then interested in identifying the locally (min/max)-efficient intervals $F(x)$ and what we will call the local min or max efficient frontier for $F(x_m)$ and $F(x_M)$ around points x_m and x_M, respectively.

In the half-plane $(\widehat{z};\widetilde{z})$ the conditions to recognize the ($\precapprox_{\gamma^-,\gamma^+}$)-based dominance and incomparability, assuming $\gamma^- < 0$ and $\gamma^+ > 0$, can be written by considering the two lines through $F(x_m)$ with equations

$$F(x_m):\begin{cases}\widetilde{z}=\widetilde{f}(x_m)+\gamma^+\left(\widehat{z}-\widehat{f}(x_m)\right)\\ \widetilde{z}=\widetilde{f}(x_m)+\gamma^-\left(\widehat{z}-\widehat{f}(x_m)\right)\end{cases}$$

and the two lines through $F(x_M)$ with equations

$$F(x_M):\begin{cases}\widetilde{z}=\widetilde{f}(x_M)+\gamma^+\left(\widehat{z}-\widehat{f}(x_M)\right)\\ \widetilde{z}=\widetilde{f}(x_M)+\gamma^-\left(\widehat{z}-\widehat{f}(x_M)\right).\end{cases}$$

For any $x \in [a,b]$ define the following sets of points (sets of intervals in midpoint representation)

$$\mathbb{D}_F^-(x;\gamma^-,\gamma^+)=\left\{Z|F(x)\precapprox_{\gamma^-,\gamma^+}Z\right\},$$
$$\mathbb{D}_F^+(x;\gamma^-,\gamma^+)=\left\{Z|Z\precapprox_{\gamma^-,\gamma^+}F(x)\right\}.$$

The intervals $Z=(\widehat{z};\widetilde{z})$ belonging to $\mathbb{D}_F^-(x;\gamma^-,\gamma^+)$ are ($\precapprox_{\gamma^-,\gamma^+}$)-dominated by interval $F(x)$ (as shown in Figure 3.11) and the ones belonging to $\mathbb{D}_F^+(x;\gamma^-,\gamma^+)$ are ($\succapprox_{\gamma^-,\gamma^+}$)-dominated by interval $F(x)$.

If x_m and x_M are not lattice-type extrema of F, that is, when the efficient frontier does not simply consist of the single point x_m (respectively, x_M), then there exist some $x \in [a,b]$ around x_m (respectively, x_M) such that $F(x) \in \mathbb{D}_F^-(x_m;\gamma^-,\gamma^+)$ (respectively, $F(x) \in \mathbb{D}_F^+(x_M;\gamma^-,\gamma^+)$). See Figure 3.11 as an example of the case of minimum.

According to Definition 3.1.10 and by the fact that both sets $\mathbb{D}_F^-(x_m;\gamma^-,\gamma^+)$ and $\mathbb{D}_F^+(x_M;\gamma^-,\gamma^+)$ are indeed intervals (eventually singletons), we obtain Proposition 3.1.9, which represents the first step in finding the efficient frontier for a strict minimum and a strict maximum (for this purpose, Figure 3.12 offers a useful representation in the midpoint plane).

Proposition 3.1.9. *([85]) Let $F:[a,b]\to\mathcal{K}_C$ be an interval-valued function with values in the lattice $(\mathcal{K}_C,\precapprox_{\gamma^-,\gamma^+})$, with $\gamma^-\leqslant 0,\gamma^+\geqslant 0$. Let $x_m,x_M\in$*

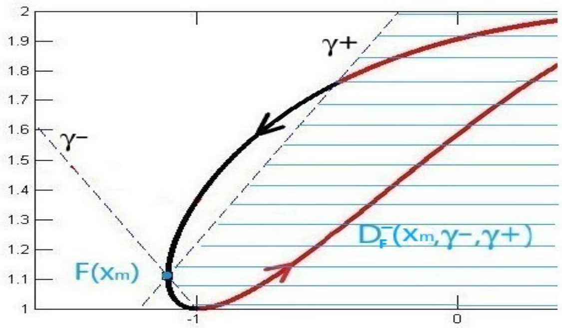


Figure 3.11: Set of intervals $\mathbb{D}_F^-(x_m;\gamma^-,\gamma^+)$ in the midpoint plane.

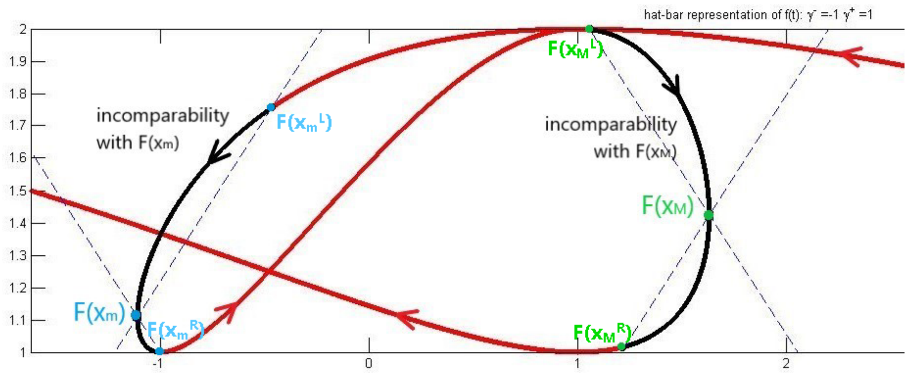


Figure 3.12: Points of $F(x)$ incomparable with $F(x_m)$ and $F(x_M)$, respectively to the left and right of the curve.

$[a,b]$ be local strict best-min and local strict best-max points of F. Then, there exist

$$x_m^L \leqslant x_m, \;\; x_m^R \geqslant x_m, \;\; x_M^L \leqslant x_M \;\; \textit{and} \;\; x_M^R \geqslant x_M$$

(all belonging to $[a,b]$) such that, respectively,

1. *$F(x)$ is incomparable with $F(x_m)$, for all $x \in [x_m^L, x_m^R]$, $x \neq x_m$;*
2. *$F(x)$ is incomparable with $F(x_M)$, for all $x \in [x_M^L, x_M^R]$, $x \neq x_M$.*

With reference to Example 3.1.1, function $\widehat{f}(x)$ has a local minimum point $x_m = -0.215$ and a local maximum point $x_M = 1.549$ as shown in Figure 3.13. Here, with the partial order $\precsim_{LU}$ (i.e., $\gamma^- = -1$ and $\gamma^+ = 1$), the locally non-dominated points corresponding to $[x_m^L, x_m^R]$ and $[x_M^L, x_M^R]$ are black colored in the top picture, which means that black intervals are not dominated by $F(x_m) = (-1.113; 1.110)$ for the local min and are not dominated by $F(x_M) = (1.631; 1.424)$ for the local max.

Note also that the domain subintervals $[x_m^L, x_m^R] = [-0.672, -0.003]$ and $[x_M^L, x_M^R] = [1.029, 1.916]$) are marked with vertical red colored lines around $x_m = -0.215$ and $x_M = 1.549$ in the bottom picture.

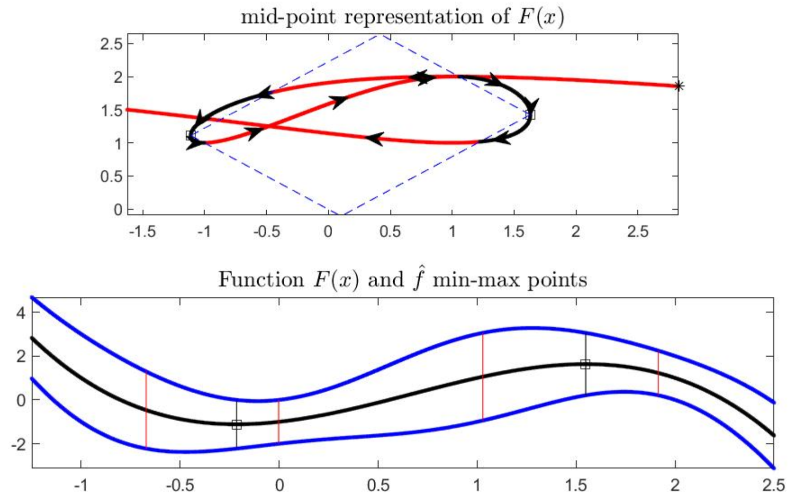


Figure 3.13: Local min and max points of $F(x)$ in Example 3.1.1 represented in the midpoint half-plane (top) and endpoint plane (bottom).

The intervals $[x_m^L, x_m^R]$ for a minimum, or $[x_M^L, x_M^R]$ for a maximum, are not difficult to determine. For example, for a minimum point x_m, a simple algorithm is to move on left and right of x_m by small steps of length $h > 0$ at points $x_m - kh$, $k = 1, 2, \ldots$ as long as $F(x_m - kh)$ is not $(\precsim_{\gamma^-,\gamma^+})$-dominated by $F(x_m)$ and at points $x_m + kh$, $k = 1, 2, \ldots$ as long as $F(x_m + kh)$ is not $(\precsim_{\gamma^-,\gamma^+})$-dominated by $F(x_m)$; if $F(x_m - k_m^L h)$ is the first dominated value on left and $F(x_m + k_m^R h)$ is the first dominated value on right, then the extremes x_m^L and x_m^R are found by appropriate bisection iterations to refine the search up to a prescribed precision.

An analogous procedure can be designed for a maximum point x_M, by moving on the left and right until points $x_M - k_M^L h$ and $x_M + k_M^R h$ with $F(x_M - k_M^L h)$ and $F(x_M + k_M^R h)$ dominate $F(x_M)$, i.e., they are $(\succsim_{\gamma^-,\gamma^+})$-dominated by $F(x_M)$; in this case the extremes x_M^L and x_M^R are found by bisections up to a prescribed precision.

A first consequence of Proposition 3.1.9 is a sufficient condition for a lattice-type extremal point.

Proposition 3.1.10. *([85]) Let $F : [a, b] \to \mathcal{K}_C$; if x_m (respectively, x_M) is a minimum point (a maximum point) of function $\widehat{f}(x)$ and $x_m^L = x_m = x_m^R$ (or $x_M^L = x_M = x_M^R$), then x_m is a lattice min-point (respectively x_M is a* *lattice max-point) of $F(x)$ and vice-versa.*

Indeed, in this case we have:

$$\mathbb{D}_F^-(x_m;\gamma^-,\gamma^+) = \{F(x_m)\} \text{ and } \mathbb{D}_F^+(x_M;\gamma^-,\gamma^+) = \{F(x_M)\}.$$

A second consequence of Proposition 3.1.9 is that the efficient interval $F(x)$, relative to the best-min point x_m or to the best-max point x_M, in the case where they are not lattice extrema, are to be searched among the points $x \in [x_m^L, x_m^R]$ and $x \in [x_M^L, x_M^R]$, respectively.

3.1.6 Efficient frontier for the extrema of an interval-valued function

The next step is now to characterize the points of the domain subintervals $[x_m^L, x_m^R]$ and $[x_M^L, x_M^R]$ that contain, respectively, a minimum point x_m and a maximum point x_M of the interval-valued function $F : [a,b] \to \mathcal{K}_C$ and which are such that all the corresponding $F(x)$ define the local efficient frontier of F around $F(x_m)$ and $F(x_M)$, respectively.

We start with a formal definition of the min/max efficient frontier (the details can be found in Figure 3.14 where an example of a min-efficient frontier and a max-efficient frontier are shown).

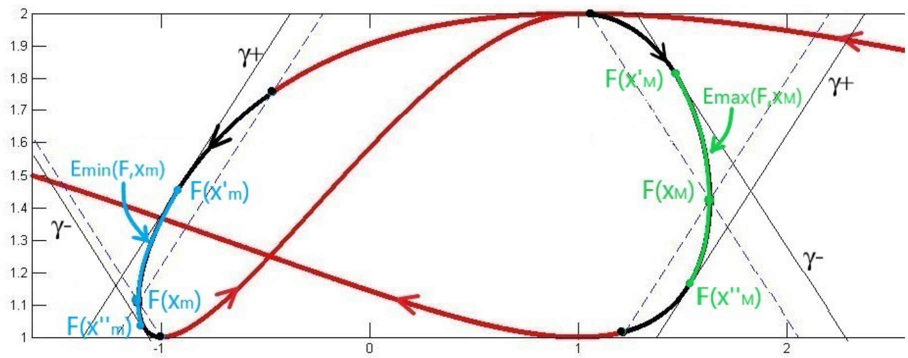


Figure 3.14: Example of $E_{\min}(F,x_m)$ and $E_{\max}(F,x_M)$: set of efficient solutions ($\precsim_{\gamma^-,\gamma^+}$)-incomparable to each other. The interval $[x'_m, x''_m]$ is the min-efficient region while the interval $[x'_M, x''_M]$ is the max-efficient region of $F(x)$.

Definition 3.1.11. *([84]) Let $F : [a,b] \to \mathcal{K}_C$ be an interval-valued function and let $x_m, x_M \in [a,b]$ be local strict best-min and local strict best-max points of F with respect to the partial order $\precsim_{\gamma^-,\gamma^+}$, $\gamma^- \leqslant 0, \gamma^+ \geqslant 0$.*

(a) *The (local) min-efficient frontier of function F associated to the best-min point x_m (or to the best-min interval-value $F(x_m)$) is the set $E_{\min}(F, x_m)$ of interval-values $F(x)$ such that:*

(a.1) $F(x_m) \in E_{\min}(F, x_m)$,

(a.2) *if $x', x'' \in [a, b]$ and $F(x'), F(x'') \in E_{\min}(F, x_m)$ then $F(x')$ and $F(x'')$ are ($\precapprox_{\gamma^-,\gamma^+}$)-incomparable,*

(a.3) *no other set E' containing $E_{\min}(F, x_m)$ has property (a.2).*

The set of points $x \in [x_m^L, x_m^R]$ such that $F(x) \in E_{\min}(F, x_m)$ are the local min-efficient points corresponding to x_m and is denoted by $eff_{\min}(F; x_m)$.

(b) *The (local) max-efficient frontier of function F associated to the best-max point x_M (or to the best-max interval-value $F(x_M)$) is the set $E_{\max}(F, x_M)$ of interval-values $F(x)$ such that:*

(b.1) $F(x_M) \in E_{\max}(F, x_M)$,

(b.2) *if $x', x'' \in [a, b]$ and $F(x'), F(x'') \in E_{\max}(F, x_M)$ then $F(x')$ and $F(x'')$ are ($\precapprox_{\gamma^-,\gamma^+}$)-incomparable,*

(b.3) *no other set E' containing $E_{\max}(F, x_M)$ has property (b.2).*

The set of points $x \in [x_M^L, x_M^R]$ such that $F(x) \in E_{\max}(F, x_M)$ are the local max-efficient points corresponding to x_M and is denoted by $eff_{\max}(F; x_M)$.

Clearly, the efficient frontiers $eff_{\min}(F; x_m)$ and $eff_{\max}(F; x_M)$ are subsets of the intervals $[x_m^L, x_m^R]$ and $[x_M^L, x_M^R]$ introduced in Proposition 3.1.9; but their characterization is not easy, as we can imagine in cases where the function $F(x)$ has possible inflexion or angular points, tangency of high order, multiple nodes, fractal-like or complex pathological patterns (see, e.g., [74]). In those cases it is not immediate to determine which points are not dominated by others of the same interval, or possibly the efficient frontiers may not be intervals.

In the case where function $F(x)$ represents locally a convex plane curve, standard results in elementary differential geometry (see, e.g., [5], chapter 2) are of help in our context.

We recall briefly some facts.

Let C_F be a curve in the half-plane $(\widehat{z}; \widetilde{z})$ with parametric equations $\widehat{z} = \widehat{f}(x)$, $\widetilde{z} = \widetilde{f}(x)$ and parameter $x \in [a, b]$ and assume that the curve is simple (no multiple points) and differentiable (i.e., both $\widehat{f}(x)$ and $\widetilde{f}(x)$ are differentiable at internal points); one says that the curve C_F has the convexity property if each of its points is such that the curve lies on one side of the tangent line to this point.

In our setting, the convexity of C_F is required only locally, by considering the restriction of $F(x)$ to points around x_m (or x_M). More precisely, let us fix the notion of local convexity of C_F by distinguishing the case of a minimum to the case of a maximum point.

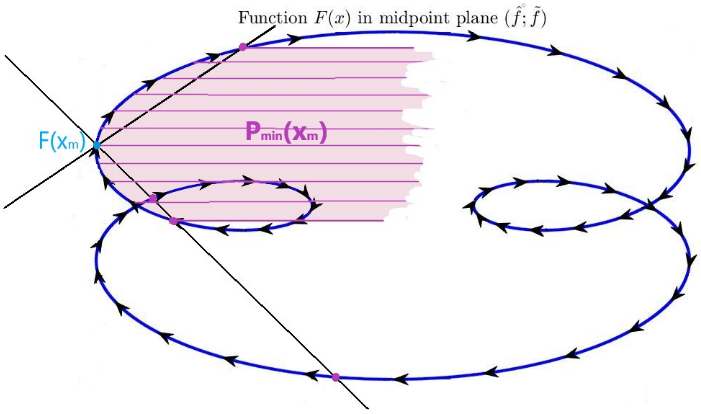


Figure 3.15: Example of the convex set $P_{min}(x_m)$ in the midpoint plane $(\widehat{z};\widetilde{z})$.

Assumption 3.1.1. *For a min point x_m (not a lattice min) we will assume that there exist $\delta_m^{'}, \delta_m^{''} \geqslant 0$ (not both equal to zero) such that the curve corresponding to the restriction of $F(x)$ to the interval $\left[x_m - \delta_m^{'}, x_m + \delta_m^{''}\right]$ is simple and convex; this happens if the portion of plane on right of the curve, i.e., the set*

$$P_{min}(x_m) = \bigcup_{x \in \left[x_m - \delta_m^{'}, x_m + \delta_m^{''}\right]} \left\{ \left(\widehat{z}; \widetilde{f}(x)\right) | \widehat{z} \geqslant \widehat{f}(x) \right\} \tag{3.24}$$

is convex (as shown in Figure 3.15); in this case, the following portion of the half-plane is convex and bounded

$$S_{min}(x_m) = P_{min}(x_m) \bigcap \{(\widehat{z}; \widetilde{z}) \, | \widehat{z}_{min} \leqslant \widehat{z} \leqslant \widehat{z}_{max} \text{ and } \widetilde{z}_{min} \leqslant \widetilde{z} \leqslant \widetilde{z}_{max}\} \tag{3.25}$$

where
$\widehat{z}_{min} = \min \left\{ \widehat{f}(x) | x \in \left[x_m - \delta_m^{'}, x_m + \delta_m^{''}\right] \right\}$,
$\widehat{z}_{max} = \max \left\{ \widehat{f}(x) | x \in \left[x_m - \delta_m^{'}, x_m + \delta_m^{''}\right] \right\}$,
$\widetilde{z}_{min} = \min \left\{ \widetilde{f}(x) | x \in \left[x_m - \delta_m^{'}, x_m + \delta_m^{''}\right] \right\}$,
$\widetilde{z}_{max} = \max \left\{ \widetilde{f}(x) | x \in \left[x_m - \delta_m^{'}, x_m + \delta_m^{''}\right] \right\}$.
It is not restrictive to assume that interval $\left[x_m - \delta_m^{'}, x_m + \delta_m^{''}\right]$ is the biggest subinterval of $[x_m^L, x_m^R]$ where the curve $F(x)$ is locally convex.

Assumption 3.1.2. *For a max point x_M (not a lattice max), assuming the existence of $\delta'_M, \delta''_M \geqslant 0$ such that the curve $F(x)$ on interval $\left[x_M - \delta'_M, x_M + \delta''_M\right]$ is simple and convex, we obtain that the portion of plane on left of the curve, i.e.,*

$$P_{max}(x_M) = \bigcup_{x \in \left[x_M - \delta'_M, x_M + \delta''_M\right]} \left\{ \left(\widehat{z}; \widetilde{f}(x) \right) | \widehat{z} \leqslant \widehat{f}(x) \right\} \tag{3.26}$$

is convex; in this case, the following set is convex and bounded

$$S_{max}(x_M) = P_{max}(x_M) \bigcap \{ (\widehat{z}; \widetilde{z}) \, | \widehat{z}_{min} \leqslant \widehat{z} \leqslant \widehat{z}_{max} \text{ and } \widetilde{z}_{min} \leqslant \widetilde{z} \leqslant \widetilde{z}_{max} \} \tag{3.27}$$

where, this time,
$\widehat{z}_{min} = \min \left\{ \widehat{f}(x) | x \in \left[x_M - \delta'_M, x_M + \delta''_M\right] \right\}$,
$\widehat{z}_{max} = \max \left\{ \widehat{f}(x) | x \in \left[x_M - \delta'_M, x_M + \delta''_M\right] \right\}$
and similarly for $\widetilde{z}_{min}$ and $\widetilde{z}_{max}$ in terms of $\widetilde{f}(x)$.
It is not restrictive to assume that interval $\left[x_M - \delta'_M, x_M + \delta''_M\right]$ is the biggest subinterval of $[x^L_M, x^R_M]$ where the curve $F(x)$ is locally convex.

Under Assumptions 3.1.1 or 3.1.2 (using the same notation) we can prove the following results:

Proposition 3.1.11. *Let $\precsim_{\gamma^-,\gamma^+}$ be a partial order on $\mathcal{K}_C$ and $F : [a, b] \to \mathcal{K}_C$ be such that $x_m \in]a, b[$ is a local min point of $\widehat{f}(x)$ and Assumption 3.1.1 is satisfied. Then there exist two points $x'_m, x''_m \in [x^L_m, x^R_m]$ with $x'_m \leqslant x_m \leqslant x''_m$ such that, for $x \in \left[x_m - \delta'_m, x_m + \delta''_m\right]$,*

(1) either x'_m maximizes $\widetilde{f}(x) - \gamma^+ \widehat{f}(x)$ and x''_m minimizes $\widetilde{f}(x) - \gamma^- \widehat{f}(x)$,

(2) or x'_m minimizes $\widetilde{f}(x) - \gamma^- \widehat{f}(x)$ and x''_m maximizes $\widetilde{f}(x) - \gamma^+ \widehat{f}(x)$.

Furthermore, interval $[x'_m, x''_m]$ is the local min-efficient frontier $eff_{min}(F; x_m)$ of Definition 3.1.11.

In particular, if x'_m and x''_m are internal to the local convexity region $\left[x_m - \delta'_m, x_m + \delta''_m\right]$ and $\widehat{f}(x)$, $\widetilde{f}(x)$ are differentiable, then

$$\begin{cases} \widetilde{f}'(x'_m) = \gamma^+ \widehat{f}'(x'_m) \\ \widetilde{f}'(x''_m) = \gamma^- \widehat{f}'(x''_m) \end{cases} \text{ or } \begin{cases} \widetilde{f}'(x''_m) = \gamma^+ \widehat{f}'(x''_m) \\ \widetilde{f}'(x'_m) = \gamma^- \widehat{f}'(x'_m). \end{cases} \tag{3.28}$$

Proof. Consider the two lines with equations $\widetilde{z} = q^+ + \gamma^+ \widehat{z}$ and $\widetilde{z} = q^- + \gamma^- \widehat{z}$; as $\widehat{z} = \widehat{f}(x)$ and $\widetilde{z} = \widetilde{f}(x)$, points of the curve $F(x)$ in common with one of the two lines will satisfy the equations

$$\varphi^+(x) = \widetilde{f}(x) - q^+ - \gamma^+ \widehat{f}(x) = 0 \text{ and } \varphi^-(x) = \widetilde{f}(x) - q^- - \gamma^- \widehat{f}(x) = 0.$$

Solving for q^+ and q^- one obtains

$$q^+ = \widetilde{f}(x) - \gamma^+ \widehat{f}(x) \text{ and } q^- = \widetilde{f}(x) - \gamma^- \widehat{f}(x)$$

and at such common points the two lines have equations

$$\widetilde{z} = \widetilde{f}(x) - \gamma^+ \widehat{f}(x) + \gamma^+ \widehat{z} \text{ and } \widetilde{z} = \widetilde{f}(x) - \gamma^- \widehat{f}(x) + \gamma^- \widehat{z}.$$

Now, by Assumption 3.1.1, the intercepts q^+ and q^-, as functions of x, are monotonic around x_m; then the maximum value q_*^+ of the $q^+(x)$ is attained at a point x_m^+ and the $q^-(x)$ has a minimum value q_*^- attained at a point x_m^-.

By taking $x'_m = \min\{x_m^-, x_m^+\}$ and $x''_m = \max\{x_m^-, x_m^+\}$, it is clear that conclusions (1) or (2) are satisfied.

If the points x_m^+ and x_m^- are internal to the local convexity region $\left[x_m - \delta'_m, x_m + \delta''_m\right]$, then the derivatives of q^+ and q^- at the attained max and min points (respectively) will be zero:

$$\widetilde{f}'(x'_m) - \gamma^+ \widehat{f}'(x'_m) = 0 \text{ and } \widetilde{f}'(x''_m) - \gamma^- \widehat{f}'(x''_m) = 0$$

or

$$\widetilde{f}'(x''_m) - \gamma^+ \widehat{f}'(x''_m) = 0 \text{ and } \widetilde{f}'(x'_m) - \gamma^- \widehat{f}'(x'_m) = 0;$$

this proves conditions (3.28).

Basically they mean that the line of equation $\widetilde{z} = q_*^+ + \gamma^+ \widehat{z}$ is tangent to the curve $F(x)$ at point $F(x_m^+)$ and the line $\widetilde{z} = q_*^- + \gamma^- \widehat{z}$ is tangent to $F(x)$ at point $F(x_m^-)$.

As shown in Figure 3.14 (left side), the proof concludes by observing that the efficient region is exactly the interval $x \in [x'_m, x''_m]$; indeed, by local convexity, we have that:

(a) no points $F(x)$ with $x \in [x'_m, x''_m]$ are dominated (or dominate) other points in the same interval, and

(b) points $F(x)$ with $x < x'_m$ and $x > x''_m$ (if any) are dominated by $F(x'_m)$ and by $F(x''_m)$, respectively.

□

Proposition 3.1.12. *Let $\precsim_{\gamma^-,\gamma^+}$ be a partial order on $\mathcal{K}_C$ and $F : [a, b] \to \mathcal{K}_C$ be such that $x_M \in]a, b[$ is a local max point of $\widehat{f}(x)$ and Assumption 3.1.2 is satisfied. Then there exist points $x'_M, x''_M \in [x_M^L, x_M^R]$ with $x'_M \leqslant x_M \leqslant x''_M$ such that, for $x \in \left[x_M - \delta'_M, x_M + \delta''_M\right]$,*

1 either x'_M minimizes $\widetilde{f}(x) - \gamma^+ \widehat{f}(x)$ and x''_M maximizes $\widetilde{f}(x) - \gamma^- \widehat{f}(x)$,

2 or x'_M maximizes $\widetilde{f}(x) - \gamma^- \widehat{f}(x)$ and x''_M minimizes $\widetilde{f}(x) - \gamma^+ \widehat{f}(x)$.

Furthermore, interval $[x'_M, x''_M]$ *is the local max-efficient frontier* $eff_{max}(F; x_M)$ *of Definition 3.1.11.*

In particular, if x'_M *and* x''_M *are internal to the local convexity region* $\left[x_M - \delta'_M, x_M + \delta''_M\right]$ *and* $\widehat{f}(x)$, $\widetilde{f}(x)$ *are differentiable, then*

$$\begin{cases} \widetilde{f}'(x'_M) = \gamma^+ \widehat{f}'(x'_M) \\ \widetilde{f}'(x''_M) = \gamma^- \widehat{f}'(x''_M) \end{cases} \quad or \quad \begin{cases} \widetilde{f}'(x''_M) = \gamma^+ \widehat{f}'(x''_M) \\ \widetilde{f}'(x'_M) = \gamma^- \widehat{f}'(x'_M). \end{cases} \tag{3.29}$$

Proof. We proceed analogously to the proof of Proposition 3.1.11; note that, in this case, under Assumption 3.1.2, two points x^-_M and x^+_M are obtained by minimizing the intercept $q^+ = \widetilde{f}(x) - \gamma^+ \widehat{f}(x)$ and by maximizing $q^- = \widetilde{f}(x) - \gamma^- \widehat{f}(x)$, respectively.

Therefore, taking $x'_M = \min\{x^-_M, x^+_M\}$ and $x''_M = \max\{x^-_M, x^+_M\}$, the tangency conditions with the curve $F(x)$ are exactly the ones in (3.29). The situation is well represented graphically in Figure 3.14 (right side). □

A procedure for the efficient frontiers corresponding to a minimum or maximum point can be obtained in a similar way as for determining the intervals $[x^L_m, x^R_m]$ or $[x^L_M, x^R_M]$; e.g., for a minimum, we move on left and right of x_m by small steps $x_m - kh$ and $x_m + kh$, $k = 1, 2, \dots$ until the monotonicity of intercepts q^+ or q^- is interrupted in two consecutive points or, equivalently, until a point is found which dominates the next one.

Also in this case, we can refine the search by appropriate bisections.

A complete example with several possible situations is presented in Section 3.1.8.

With reference to Example 3.1.1, the efficient frontiers are
$eff_{min}(F; -0.215) = [-0.473, -0.109]$ and
$eff_{max}(F; 1.549) = [1.279, 1.742]$, as pointed out in Figure 3.16.

Here, in particular, the top pictures show the efficient frontier for x_m (left) and for x_M (right) with the tangent lines to the curve F, while in the bottom picture the efficient frontiers are delimited by vertical red segments containing the min and the max points.

On the other hand, in Figure 3.17 the first derivatives of the three functions $\widehat{f}(x)$, $\widetilde{f}(x) - \gamma^+ \widehat{f}(x)$ and $\widetilde{f}(x) - \gamma^- \widehat{f}(x)$ are visualized (the second is changed in sign with respect to the notation in Assumptions 3.1.1 and 3.1.2); by checking appropriate monotonicity of the three derivatives, we see that condition 3.1.1 is satisfied in a neighborhood of x_m and condition 3.1.2 is valid around x_M.

Corresponding to the relevant points, the derivatives of the three relevant functions $\widehat{f}(x)$, $\widetilde{f}(x) - \gamma^+ \widehat{f}(x)$ and $\widetilde{f}(x) - \gamma^- \widehat{f}(x)$ (black, red and blue curve respectively) are zero, according to Propositions 3.1.11 and 3.1.12 (picture on top), while in the bottom, similarly to the previous case, the efficient frontiers are delimited by vertical red segments.

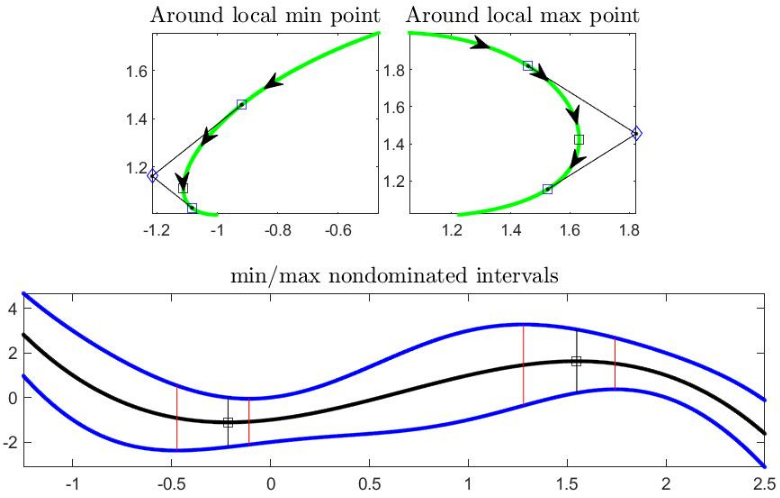


Figure 3.16: Efficient frontiers for min and max points of $F(x)$ in Example 3.1.1.

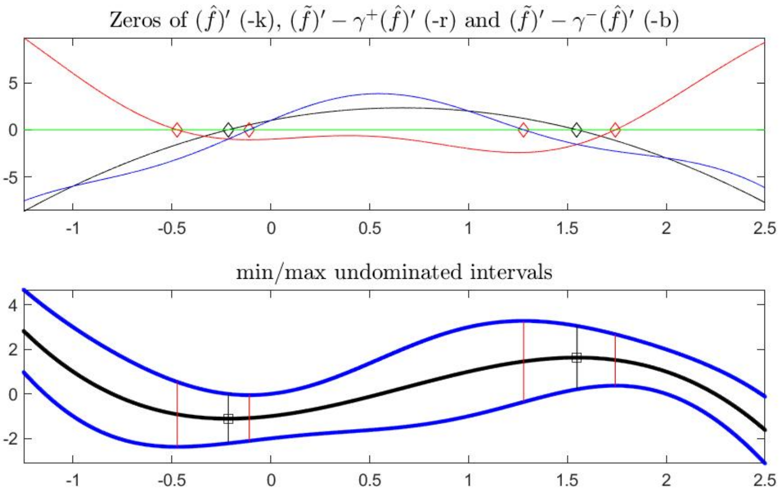


Figure 3.17: Correspondence between the derivatives of $\widehat{f}(x)$, $\widetilde{f}(x) - \gamma^+ \widehat{f}(x)$ and $\widetilde{f}(x) - \gamma^- \widehat{f}(x)$ and the efficient frontier for min and max points of $F(x)$ in Example 3.1.1.

Example 3.1.3. *Consider the function* $F(x) = \left(\cos(2\pi x); 1 + |x| \sin^2(\pi x)\right)$ *(midpoint notation) for* $x \in \left[-\frac{1}{2}, 1\right]$. *Figure 3.18 contains the graph of* $F(x) = [f^-(x), f^+(x)]$ *in interval form (top picture) and in midpoint form* $F(x) = \left(\widehat{f}(x); \widetilde{f}(x)\right)$ *(bottom).*

The point $x_M = 0$ *with* $F(x_M) = (1; 1) = [0, 2]$ *is a local maximum of* *function* $\widehat{f}$ *and the point* $x_m = \dfrac{1}{2}$ *with* $F(x_m) = \left(-1; \dfrac{3}{2}\right) = \left[-\dfrac{5}{2}, \dfrac{1}{2}\right]$ *is a local minimum of* $\widehat{f}$.

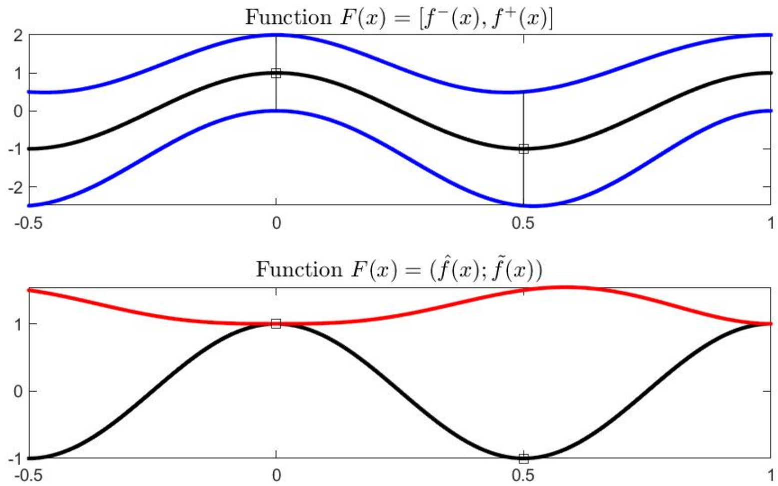


Figure 3.18: Interval (top) and midpoint (bottom) representations in the plane (x, y) of function $F(x)$ from Example 3.1.3.

Let us chose, e.g., $\gamma^- = -1$ *and* $\gamma^+ = \dfrac{1}{2}$*; Figure 3.19 shows the* $(\precsim_{-1,\frac{1}{2}})$*-dominance for the interval-values* $F(x)$ *with* x *around* x_m *and* x_M*, where function* $\widehat{f}(x)$ *has a minimum point at* $x_m = \dfrac{1}{2}$ *and a maximum at* $x_M = 0$.

In midpoint representation $F(x_m)$ *appears on the left portion of the top picture of Figure 3.19 and* $F(x_M)$ *on the right portion; the parallelogram contains the intervals* $F(x)$ *(in red color) of the graph of* F *that are* $(\precsim_{-1,\frac{1}{2}})$*-dominated by* $F(x_m)$ *and* $(\succsim_{-1,\frac{1}{2}})$*-dominated by* $F(x_M)$.

We can also note that black colored intervals in midpoint representation (top picture) are not dominated by $F(x_m)$ *for the local min and are not dominated by* $F(x_M)$ *for the local max. They correspond to the domain subintervals delimited by vertical red lines in the bottom part of the figure. Indeed, the lower part of the picture marks the intervals* $F(x_m)$ *and* $F(x_M)$ *and shows the intervals* $[x_m^L, x_m^R]$ *and* $[x_M^L, x_M^R]$ *corresponding to the points around* x_m *and* x_M *with dominated interval-values.*

Clearly, x_M *results in a local (and global on the considered domain* $\left[-\dfrac{1}{2}, 1\right]$ *of* F*) lattice-maximum point of* F*, while* x_m *is a local (and global) best-minimum point.*

Note that $[x_M^L, x_M^R]$ *reduces to the single point* x_M *while* $[x_m^L, x_m^R]$ *is the interval* $[0.4670, 0.5329]$*, approximated numerically as it is depending on the*

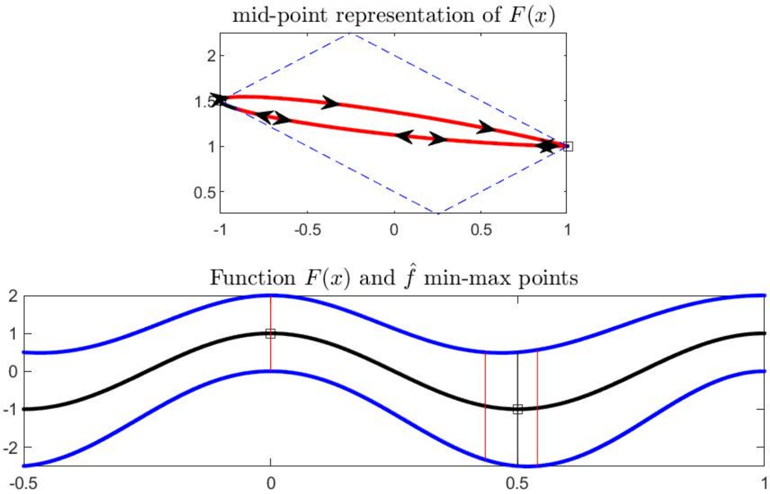


Figure 3.19: Local min and max points of $F(x)$ from Example 3.1.3 in midpoint representation in the half-plane $(\widehat{z};\widetilde{z})$ (top) and in interval representation in the plane (x,y) (bottom): black intervals in midpoint representation are not dominated by $F(x_m)$ for the local min and are not dominated by $F(x_M)$ for the local max and correspond to the domain subintervals $[x_m^L, x_m^R]$ and $[x_M^L, x_M^R]$ delimited by vertical red lines in the interval representation.

actual values of γ^- *and* γ^+.

Moreover, the local min-efficient frontier corresponding to the best-min point x_m*, i.e., the points in* $eff_{\min}(F;x_m)$*, can be easily computed (see below) and are pictured in Figure 3.20.*

The top picture shows the efficient frontier for x_m *with the tangent lines to the curve* F*: here the green points are the ones min-dominated by* x_m*, corresponding to the points in* $[x_m^L, x_m^R]$ *and the efficient frontier* $E_{\min}(F;x_m)$ *is identified in the midpoint graph of* $F(x)$ *by the green points intercepted by the lines with angular coefficients* $\gamma^+ > 0$, $\gamma^- < 0$ *and "tangent" to the graph of* F.

On the other hand, in the bottom part of the picture, the efficient frontiers are delimited by vertical red segments containing the min and max points (the max point is lattice type maximum); so the min-efficient frontier is evidenced by vertical lines around x_m*, corresponding to the points* $x \in eff_{\min}(F;x_m)$.

We conclude this section to see how local extremality of a point x_m (minimum) or x_M (maximum) is connected to the left and/or right gH-derivatives $F'_{(l)gH}(x_0) = \left(\widehat{f}'_l(x_0); \left|\widetilde{f}'_l(x_0)\right|\right)$, $F'_{(r)gH}(x_0) = \left(\widehat{f}'_r(x_0); \left|\widetilde{f}'_r(x_0)\right|\right)$ or to the gH-derivative $F'_{gH}(x_0)$ if the two are equal.

Let $F : [a,b] \to \mathcal{K}_C$ and $\precsim_{\gamma^-,\gamma^+}$ be a partial order.

Suppose first that $x_0 \in]a,b[$, $F'_{gH}(x_0) = \left(\widehat{f}'(x_0); \widetilde{w}_F(x_0)\right)$ exists (here

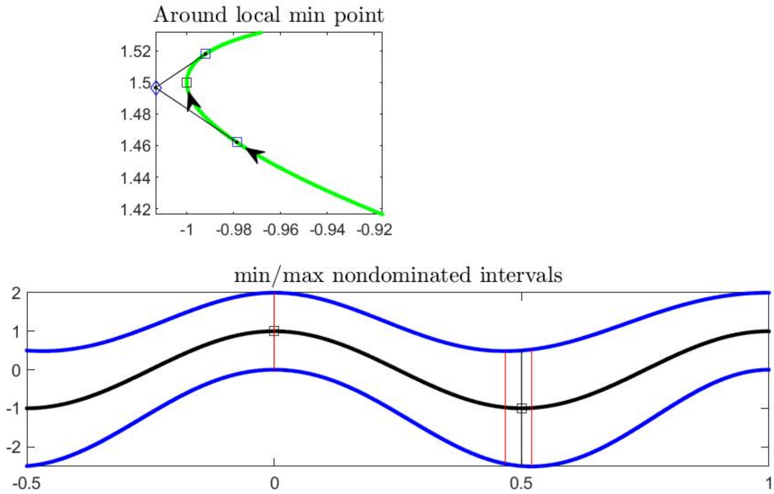


Figure 3.20: Local min-efficient frontier $E_{\min}(F; x_m)$ from Example 3.1.3 in midpoint representation in the half-plane $(\widehat{z}; \widetilde{z})$ (top) and in interval representation in the plane (x, y) (bottom): the points between the tangent lines in midpoint representation correspond to interval $[x'_m, x''_m]$, that is $eff_{\min}(F; x_m)$, delimited by vertical red lines in the second representation.

$\widetilde{w}_F(x_0) = \left|\widetilde{f}'_l(x_0)\right| = \left|\widetilde{f}'_r(x_0)\right|$, according to (3.5)).

As well represented in Figure 3.4 (by the two points highlighted in red), we have that if x_0 is a local minimum or maximum, then $\widehat{f}'(x_0) = 0$ so that

$$F'_{gH}(x_0) = (0; \widetilde{w}_F(x_0))\,;$$

as a consequence, a necessary condition for a local min or max at a point of differentiability is that $0 \in F'_{gH}(x_0)$.

If also $\widetilde{w}_F(x_0) = 0$ (i.e., if $\widetilde{f}'(x_0) = 0$), then $F'_{gH}(x_0) = (0; 0) = 0$. Otherwise, if $\widetilde{w}_F(x_0) > 0$, it follows the continuity of $\widetilde{f}(x_0)$, which in turn implies the existence of a neighborhood of x_0 where $F(x)$ is $(\precapprox_{\gamma^-,\gamma^+})$-incomparable with $F(x_0)$.

So, we have the following Fermat-like property:

Proposition 3.1.13. *Let $F :]a, b[\to \mathcal{K}_C$, $x_0 \in]a, b[$, such that F is gH-differentiable at x_0 and $\precapprox_{\gamma^-,\gamma^+}$ be a partial order on $\mathcal{K}_C$.*

1 If x_0 is a lattice extremum for F (a lattice-min or a lattice-max point), then $F'_{gH}(x_0) = 0$;

2 If x_0 is a best-extremum for F (a best-min or a best-max point), then $0 \in F'_{gH}(x_0)$.

In the cases where F has left or right gH-derivatives at x_0 (or they are not equal), necessary conditions for a lattice-min or a best-min (respectively, a lattice-max or a best-max) can be easily deduced according to (3.15) and (3.16) in Proposition 3.1.5.

Proposition 3.1.14. *Let $F : [a, b] \to \mathcal{K}_C$, $x_0 \in [a, b]$ and $\precsim_{\gamma^-,\gamma^+}$ be a partial order on $\mathcal{K}_C$. Suppose that F has left and right gH-derivatives at x_0 (if $x_0 = a$ or $x_0 = b$ we consider only the right or the left gH-derivatives, respectively).*

1.a If x_0 is a lattice minimum point for F, then $F'_{(l)gH}(x_0) \precsim_{\gamma^-,\gamma^+} 0$ and $F'_{(r)gH}(x_0) \succsim_{\gamma^-,\gamma^+} 0$.

1.b If x_0 is a lattice maximum point for F, then $F'_{(l)gH}(x_0) \succsim_{\gamma^-,\gamma^+} 0$ and $F'_{(r)gH}(x_0) \precsim_{\gamma^-,\gamma^+} 0$.

2.a If x_0 is a best-minimum point for F, then $0 \in F'_{gH}(x_0)$.

2.b If x_0 is a best-maximum point for F, then $0 \in F'_{gH}(x_0)$.

According to (3.17) and (3.18), the following are sufficient conditions based on the "sign" of left and right gH-derivatives, analogous to the well known situation for single-valued functions.

Proposition 3.1.15. *Let $F : [a, b] \to \mathcal{K}_C$, $x_0 \in [a, b]$ and $\precsim_{\gamma^-,\gamma^+}$ be a partial order on $\mathcal{K}_C$. Suppose that F has left and right gH-derivatives at x_0 (if $x_0 = a$ or $x_0 = b$ we consider only the right or the left gH-derivatives, respectively).*

(a) If $F'_{(l)gH}(x_0) \prec_{\gamma^-,\gamma^+} 0$ and $F'_{(r)gH}(x_0) \succ_{\gamma^-,\gamma^+} 0$, then x_0 is a best minimum point for F.

(b) If $F'_{(l)gH}(x_0) \succ_{\gamma^-,\gamma^+} 0$ and $F'_{(r)gH}(x_0) \prec_{\gamma^-,\gamma^+} 0$, then x_0 is a best maximum point for F.

3.1.7 Concavity and convexity of interval-valued functions

We have three types of convexity, similar to the monotonicity and local extremum concepts.

Definition 3.1.12. *Let $F : [a, b] \to \mathcal{K}_C$ be a function and let $\precsim_{\gamma^-,\gamma^+}$, $\gamma^- \leqslant 0, \gamma^+ \geqslant 0$ be a partial order for intervals. We say that*

(a-i) F is ($\precsim_{\gamma^-,\gamma^+}$)-convex on $[a, b]$ if and only if $\forall x_1, x_2 \in [a, b]$ and all $\lambda \in [0, 1]$,

$$F((1-\lambda)x_1 + \lambda x_2) \precsim_{\gamma^-,\gamma^+} (1-\lambda)F(x_1) + \lambda F(x_2);$$

(a-ii) *F is ($\precapprox_{\gamma^-,\gamma^+}$)-concave on $[a,b]$ if and only if $\forall x_1, x_2 \in [a,b]$ and all $\lambda \in [0,1]$,*

$$F((1-\lambda)x_1 + \lambda x_2) \succapprox_{\gamma^-,\gamma^+} (1-\lambda)F(x_1) + \lambda F(x_2).$$

(b-i) *F is strictly ($\precsim_{\gamma^-,\gamma^+}$)-convex on $[a,b]$ if and only if $\forall x_1, x_2 \in [a,b]$ and all $\lambda \in]0,1[$,*

$$F((1-\lambda)x_1 + \lambda x_2) \precsim_{\gamma^-,\gamma^+} (1-\lambda)F(x_1) + \lambda F(x_2);$$

(b-ii) *F is strictly ($\precsim_{\gamma^-,\gamma^+}$)-concave on $[a,b]$ if and only if $\forall x_1, x_2 \in [a,b]$ and all $\lambda \in]0,1[$,*

$$F((1-\lambda)x_1 + \lambda x_2) \succsim_{\gamma^-,\gamma^+} (1-\lambda)F(x_1) + \lambda F(x_2).$$

(c-i) *F is strongly ($\prec_{\gamma^-,\gamma^+}$)-convex on $[a,b]$ if and only if $\forall x_1, x_2 \in [a,b]$ and all $\lambda \in]0,1[$,*

$$F((1-\lambda)x_1 + \lambda x_2) \prec_{\gamma^-,\gamma^+} (1-\lambda)F(x_1) + \lambda F(x_2);$$

(c-ii) *F is strongly ($\prec_{\gamma^-,\gamma^+}$)-concave on $[a,b]$ if and only if $\forall x_1, x_2 \in [a,b]$ and all $\lambda \in]0,1[$,*

$$F((1-\lambda)x_1 + \lambda x_2) \succ_{\gamma^-,\gamma^+} (1-\lambda)F(x_1) + \lambda F(x_2).$$

The convexity of a function $F = \left(\widehat{f};\widetilde{f}\right)$ is also related to the concavity of function $-F = \left(-\widehat{f};\widetilde{f}\right)$.

Indeed, according to Proposition 2.2.11 ,we have that for intervals $A, B \in \mathcal{K}_C$, it is

$$A \precapprox_{\gamma^-,\gamma^+} B \text{ if and only if } (-B) \precapprox_{-\gamma^+,-\gamma^-} (-A);$$

note that in the last partial order, the roles of γ^- and γ^+ are interchanged by changing their sign so that $-\gamma^+ < 0$ and $-\gamma^- > 0$.

Therefore, as a consequence, we have the following property.

Proposition 3.1.16. *Let $F : [a,b] \to \mathcal{K}_C$ and $\precapprox_{\gamma^-,\gamma^+}$ be a given partial order with $\gamma^- < 0$, $\gamma^+ > 0$. Consider also the partial order $\precapprox_{-\gamma^+,-\gamma^-}$. Then $F = \left(\widehat{f};\widetilde{f}\right)$ is ($\precapprox_{\gamma^-,\gamma^+}$)-convex if and only if $-F = \left(-\widehat{f};\widetilde{f}\right)$ is ($\precapprox_{-\gamma^+,-\gamma^-}$)-concave. In particular, if $\gamma^+ = -\gamma^- = \gamma > 0$, then F is ($\precapprox_{\gamma}$)-convex if and only if $-F$ is ($\precapprox_{\gamma}$)-concave.*

Proof. According to Proposition 2.2.11, we have that for intervals $A, B \in \mathcal{K}_C$ it is $A \precsim_{\gamma^-,\gamma^+} B$ if and only if $(-B) \precsim_{-\gamma^+,-\gamma^-} (-A)$; in the last partial order the roles of γ^- and γ^+ are interchanged by changing their sign so that $-\gamma^+ < 0$ and $-\gamma^- > 0$. The proof follows immediately. □

The next result expresses the $(\precsim_{\gamma^-,\gamma^+})$-convexity of F in terms of the convexity of functions $\widehat{f}$, $\widetilde{f} - \gamma^+\widehat{f}$ and $\widetilde{f} - \gamma^-\widehat{f}$:

Proposition 3.1.17. *Let* $F = \left(\widehat{f}; \widetilde{f}\right) : [a,b] \to \mathcal{K}_C$ *and* $\precsim_{\gamma^-,\gamma^+}$ *be a given partial order; then*

1. *F is $(\precsim_{\gamma^-,\gamma^+})$-convex if and only if $\widehat{f}$ is convex, $\widetilde{f} - \gamma^+\widehat{f}$ is concave and $\widetilde{f} - \gamma^-\widehat{f}$ is convex;*
2. *F is $(\precsim_{\gamma^-,\gamma^+})$-concave if and only if $\widehat{f}$ is concave, $\widetilde{f} - \gamma^+\widehat{f}$ is convex and $\widetilde{f} - \gamma^-\widehat{f}$ is concave.*

Proof. We prove 1. Let $x_1, x_2 \in [a,b]$ and $\lambda \in [0,1]$; from the definition of convex function we have that F is $(\precsim_{\gamma^-,\gamma^+})$-convex if and only if

$$F((1-\lambda)x_1 + \lambda x_2) \precsim_{\gamma^-,\gamma^+} (1-\lambda)F(x_1) + \lambda F(x_2).$$

Therefore, denoting

$$x_\lambda = (1-\lambda)x_1 + \lambda x_2,$$

we have that F is $(\precsim_{\gamma^-,\gamma^+})$-convex if and only if

$$F(x_\lambda) \precsim_{\gamma^-,\gamma^+} (1-\lambda)F(x_1) + \lambda F(x_2)$$

which, according to (3.9), means

$$\begin{cases} \widehat{f}(x_\lambda) \leqslant (1-\lambda)\widehat{f}(x_1) + \lambda\widehat{f}(x_2) \\ \widetilde{f}(x_\lambda) \geqslant (1-\lambda)\widetilde{f}(x_1) + \lambda\widetilde{f}(x_2) + \gamma^+\left(\widehat{f}(x_\lambda) - (1-\lambda)\widehat{f}(x_1) - \lambda\widehat{f}(x_2)\right) \\ \widetilde{f}(x_\lambda) \leqslant (1-\lambda)\widetilde{f}(x_1) + \lambda\widetilde{f}(x_2) + \gamma^-\left(\widehat{f}(x_\lambda) - (1-\lambda)\widehat{f}(x_1) - \lambda\widehat{f}(x_2)\right); \end{cases}$$

From the first inequality it follows that $\widehat{f}$ is convex; instead the second inequality can also be written as

$$\widetilde{f}(x_\lambda) - \gamma^+\widehat{f}(x_\lambda) \geqslant (1-\lambda)\left(\widetilde{f}(x_1) - \gamma^+\widehat{f}(x_1)\right) + \lambda\left(\widetilde{f}(x_2) - \gamma^+\lambda\widehat{f}(x_2)\right),$$

i.e., $\widetilde{f}(x) - \gamma^+\widehat{f}(x)$ is concave. Finally, the third inequality becomes

$$\widetilde{f}(x_\lambda) - \gamma^-\widehat{f}(x_\lambda) \leqslant (1-\lambda)\left(\widetilde{f}(x_1) - \gamma^-\widehat{f}(x_1)\right) + \lambda\left(\widetilde{f}(x_2) - \gamma^-\lambda\widehat{f}(x_2)\right),$$

i.e., $\widetilde{f}(x) - \gamma^-\widehat{f}(x)$ is convex.

To prove 2., we can proceed in a similar way. □

Strict $(\precapprox_{\gamma^-,\gamma^+})$-convexity (respectively, concavity) of F in terms of functions $\widehat{f}$, $\widetilde{f}-\gamma^+\widehat{f}$ and $\widetilde{f}-\gamma^-\widehat{f}$ can be easily deduced; in this case we have:

- the midpoint function $\widehat{f}$ is strictly convex (respectively, concave);
- function $\widetilde{f}-\gamma^+\widehat{f}$ is concave (respectively, convex);
- function $\widetilde{f}-\gamma^-\widehat{f}$ is convex (respectively, concave).

Strong convexity of F corresponds to strict convexity and/or concavity of $\widehat{f}$, $\widetilde{f}-\gamma^+\widehat{f}$ and $\widetilde{f}-\gamma^-\widehat{f}$ (in the right way).

It is interesting to remark that an interval-valued function $F=(\widehat{f};\widetilde{f})$ which is both $(\precapprox_{\gamma^-,\gamma^+})$-convex and $(\precapprox_{\gamma^-,\gamma^+})$-concave on $[a,b]$ exhibits a strong linearity, in the sense that both $\widehat{f}$ and $\widetilde{f}$ are linear on $[a,b]$; indeed, from $F((1-\lambda)x_1+\lambda x_2)=(1-\lambda)F(x_1)+\lambda F(x_2)$, i.e.,

$$F(x_\lambda)=(1-\lambda)F(x_1)+\lambda F(x_2),$$

we have

$$\begin{cases} \widehat{f}(x_\lambda)=(1-\lambda)\widehat{f}(x_1)+\lambda\widehat{f}(x_2) \\ \widetilde{f}(x_\lambda)=(1-\lambda)\widetilde{f}(x_1)+\lambda\widetilde{f}(x_2)+\gamma^+\left(\widehat{f}(x_\lambda)-(1-\lambda)\widehat{f}(x_1)-\lambda\widehat{f}(x_2)\right) \\ \widetilde{f}(x_\lambda)=(1-\lambda)\widetilde{f}(x_1)+\lambda\widetilde{f}(x_2)+\gamma^-\left(\widehat{f}(x_\lambda)-(1-\lambda)\widehat{f}(x_1)-\lambda\widehat{f}(x_2)\right), \end{cases}$$

so, from the second and third equality, we obtain exactly

$$\widehat{f}((1-\lambda)x_1+\lambda x_2)=(1-\lambda)\widehat{f}(x_1)+\lambda\widehat{f}(x_2),$$
$$\widetilde{f}((1-\lambda)x_1+\lambda x_2)=(1-\lambda)\widetilde{f}(x_1)+\lambda\widetilde{f}(x_2).$$

Proposition 3.1.18. *Let $F=\left(\widehat{f};\widetilde{f}\right):[a,b]\to\mathcal{K}_C$ be $(\precapprox_{\gamma^-,\gamma^+})$-convex or $(\precapprox_{\gamma^-,\gamma^+})$-concave; then F is continuous on $]a,b[$.*

Proof. From Proposition 3.1.17, the three functions $\widehat{f}$, $g^+=\widetilde{f}-\gamma^+\widehat{f}$ and $g^-=\widetilde{f}-\gamma^-\widehat{f}$ are convex or concave, hence they are continuous on the internal points of $[a,b]$.

As $\widetilde{f}=g^++\gamma^+\widehat{f}$ and both g^+ and $\widehat{f}$ are continuous, so $\widetilde{f}$ is too, we obtain that $\widehat{f}$ and $\widetilde{f}$ are both continuous which implies that F itself is continuous. □

From Proposition 3.1.17, several ways to analyze $(\precapprox_{\gamma^-,\gamma^+})$-convexity (or concavity) in terms of the first or second derivatives of functions $\widehat{f}$, $-g^+=-(\widetilde{f}-\gamma^+\widehat{f})=\gamma^+\widehat{f}-\widetilde{f}$ and $g^-=\widetilde{f}-\gamma^-\widehat{f}$ can be easily deduced.

Proposition 3.1.19. *Let $F=\left(\widehat{f};\widetilde{f}\right):]a,b[\to\mathcal{K}_C$ with differentiable $\widehat{f}$ and $\widetilde{f}$; the following facts hold.*

1 If the first order detivatives $\widehat{f}'$ and $\widetilde{f}'$ exist, then:

1-a F is $(\precapprox_{\gamma^-,\gamma^+})$-convex on $]a,b[$ if and only if $\widehat{f}'$, $\gamma^+\widehat{f}'-\widetilde{f}'$ and $\widetilde{f}'-\gamma^-\widehat{f}'$ are increasing (nondecreasing) on $]a,b[$;

1-b F is $(\precapprox_{\gamma^-,\gamma^+})$-concave on $]a,b[$ if and only if $\widehat{f}'$, $\gamma^+\widehat{f}'-\widetilde{f}'$ and $\widetilde{f}'-\gamma^-\widehat{f}'$ are decreasing (nonincreasing) on $]a,b[$;

2 If the second order derivatives $\widehat{f}''$ and $\widetilde{f}''$ exist and are continuous, then:

2-a F is $(\precapprox_{\gamma^-,\gamma^+})$-convex on $]a,b[$ if and only if $\widehat{f}''\geqslant 0$, $\gamma^+\widehat{f}''-\widetilde{f}''\geqslant 0$ and $\widetilde{f}''-\gamma^-\widehat{f}''\geqslant 0$ on $]a,b[$;

2-b F is $(\precapprox_{\gamma^-,\gamma^+})$-concave on $]a,b[$ if and only if $\widehat{f}''\leqslant 0$, $\gamma^+\widehat{f}''-\widetilde{f}''\leqslant 0$ and $\widetilde{f}''-\gamma^-\widehat{f}''\leqslant 0$ on $]a,b[$.

Proof. The proof follows from well-known results in classical calculus. □

In order to connect concavity and convexity with the monotonicity of the gH-derivative and, for a partial order $\precapprox_\gamma$ with $\gamma>0$, with the "sign" of the second-order gH-derivative, we need the following well known result on real convex functions:

Lemma 3.1.1. *For a function $g:]a,b[\to\mathbb{R}$ to be convex on $]a,b[$, a necessary and sufficient condition is that for all $x_0\in]a,b[$ the incremental function $g_{x_0}:]a,b[\setminus\{x_0\}\to\mathbb{R}$, defined by $g_{x_0}(x)=\dfrac{g(x)-g(x_0)}{x-x_0}$, is nondecreasing for $x\in]a,b[\setminus\{x_0\}$.*
Furthermore, g admits left and right derivatives at any $x_0\in]a,b[$ and $g'_l(x_0)\leqslant g'_r(x_0)$.

Proof. Consider $x_1<x_2$ (both different from x_0); after simple manipulations, we have

$$\frac{g_{x_0}(x_1)-g_{x_0}(x_2)}{x_1-x_2}=\frac{(x_1-x_0)\left(g(x_2)-g(x_0)\right)-(x_2-x_0)\left(g(x_1)-g(x_0)\right)}{(x_2-x_1)(x_2-x_0)(x_1-x_0)} \tag{3.30}$$

$$=\frac{(x_1-x_2)g(x_0)+(x_2-x_0)g(x_1)+(x_0-x_1)g(x_2)}{(x_1-x_2)(x_2-x_0)(x_0-x_1)}.$$

On the other hand, from the convexity of g, according to definition of convex functions, for all $x\in[x_1,x_2]$ we have

$$\frac{g(x)-g(x_1)}{x-x_1}\leqslant\frac{g(x_2)-g(x_1)}{x_2-x_1},$$

that is,

$$g(x)\leqslant g(x_1)+\frac{g(x_2)-g(x_1)}{x_2-x_1}(x-x_1);$$

taking $x_1 < x_0 < x_2$ (this is not restrictive because the right-hand side in (3.30) is symmetric with respect to x_0, x_1 and x_2), we obtain

$$(x_2 - x_1)g(x_0) \leqslant (x_2 - x_0)g(x_1) + g(x_2)(x_0 - x_1)$$

and, combining with the second line in (3.30),

$$\frac{g_{x_0}(x_1) - g_{x_0}(x_2)}{x_1 - x_2} = \frac{(x_2 - x_0)g(x_1) + (x_0 - x_1)g(x_2) - (x_2 - x_1)g(x_0)}{(x_2 - x_1)(x_2 - x_0)(x_0 - x_1)} \geqslant 0.$$

To prove the last part, the incremental function $x \to g_{x_0}(x)$ is nondecreasing and admits left and right limits at x_0 with

$$G_l = \lim_{x \nearrow x_0} g_{x_0}(x) \leqslant \lim_{x \searrow x_0} g_{x_0}(x) = G_r;$$

on the other hand, clearly, $G_l = g'_l(x_0)$ and $G_r = g'_r(x_0)$ and this completes the proof. □

Proposition 3.1.20. *Let $F : [a, b] \to \mathcal{K}_C$ for all $x_0 \in]a, b[$. Consider the incremental functions of $\widehat{f}$ and $\widetilde{f}$, defined for $x \in]a, b[$, $x \neq x_0$ by*

$$\widehat{h}(x; x_0) = \frac{\widehat{f}(x) - \widehat{f}(x_0)}{x - x_0}, \ \widetilde{h}(x; x_0) = \frac{\widetilde{f}(x) - \widetilde{f}(x_0)}{x - x_0}$$

and let $\widetilde{h}^{+}(x; x_0)$ and $\widetilde{h}^{-}(x; x_0)$ be the incremental functions of $\gamma^{+}\widehat{f} - \widetilde{f}$ and $\widetilde{f} - \gamma^{-}\widehat{f}$ respectively, given by

$$\widetilde{h}^{+}(x; x_0) = \gamma^{+}\widehat{h}(x; x_0) - \widetilde{h}(x; x_0), \ \widetilde{h}^{-}(x; x_0) = \widetilde{h}(x; x_0) - \gamma^{-}\widehat{h}(x; x_0).$$

Then

1. *F is $(\precapprox_{\gamma^-,\gamma^+})$-convex on $[a, b]$ if and only if $\widehat{h}(x; x_0)$, $\widetilde{h}^{+}(x; x_0)$ and $\widetilde{h}^{-}(x; x_0)$ are nondecreasing in $]a, b[\backslash\{x_0\}$.*
2. *F is $(\precapprox_{\gamma^-,\gamma^+})$-concave on $[a, b]$ if and only if $\widehat{h}(x; x_0)$, $\widetilde{h}^{+}(x; x_0)$ and $\widetilde{h}^{-}(x; x_0)$ are nonincreasing in $]a, b[\backslash\{x_0\}$.*

Proof. From Proposition 3.1.17, the three functions $\widehat{f}$, $\gamma^{+}\widehat{f} - \widetilde{f}$ and $\widetilde{f} - \gamma^{-}\widehat{f}$ are convex (in case 1.) or concave (in case 2.); by virtue of Lemma 3.1.1, their incremental functions $\widehat{h}(x; x_0)$, $\widetilde{h}^{+}(x; x_0)$ and $\widetilde{h}^{-}(x; x_0)$ are either non decreasing (in case 1.) or non increasing (in case 2.) and the conclusion follows. □

Proposition 3.1.21. *Let $F = \left(\widehat{f}; \widetilde{f}\right) :]a, b[\to \mathcal{K}_C$ and let $\gamma^- \leqslant 0$, $\gamma^+ \geqslant 0$. If F is $(\precapprox_{\gamma^-,\gamma^+})$-convex or $(\precapprox_{\gamma^-,\gamma^+})$-concave on $]a, b[$, then $F'_{(l)gH}$ and $F'_{(r)gH}$ both exist on $]a, b[$.*
If $\widehat{f}'_l = \widehat{f}'_r$, then also $\widetilde{f}'_l = \widetilde{f}'_r$ and $F'_{(l)gH} = F'_{(r)gH} = F'_{gH}$.

Proof. We know that F is $(\precsim_{\gamma^-,\gamma^+})$-convex if and only if $\widehat{f}$, $\gamma^+\widehat{f}-\widetilde{f}$ and $\widetilde{f}-\gamma^-\widehat{f}$ are convex; from the second part of Lemma 3.1.1, their left and right derivatives exist so that also $\widetilde{f}'_l$ and $\widetilde{f}'_r$ exist.
It follows that

$$F'_{(l)gH}=\left(\widehat{f}'_l;\left|\widetilde{f}'_l\right|\right) \text{ and } F'_{(r)gH}=\left(\widehat{f}'_r;\left|\widetilde{f}'_r\right|\right)$$

with $\widehat{f}'_l\leqslant\widehat{f}'_r$, $\gamma^+\widehat{f}'_l-\widetilde{f}'_l\leqslant\gamma^+\widehat{f}'_r-\widetilde{f}'_r$ and $\widetilde{f}'_l-\gamma^-\widehat{f}'_l\leqslant\widetilde{f}'_r-\gamma^-\widehat{f}'_r$.
If $\widehat{f}'_l=\widehat{f}'_r$, then we have $-\widetilde{f}'_l\leqslant-\widetilde{f}'_r$ and $\widetilde{f}'_l\leqslant\widetilde{f}'_r$, that is, $\widetilde{f}'_l=\widetilde{f}'_r=\widetilde{f}'$; therefore, $\widetilde{f}'$ exists. □

Remark 3.1.7. *Analogously to the relationship between the sign of second derivative and convexity for ordinary point to point functions, we can establish conditions for convexity of interval-valued functions and the sign of the second order gH-derivative* $F''_{gH}(x)$*; for example, a sufficient condition for strong* $\prec_{LU}$*-convexity (i.e., with respect to the partial order* $\precsim_{\gamma^-,\gamma^+}$ *with* $\gamma^+=-\gamma^-=1$*) is the following (compare with Proposition 3.1.19):*

1. *if* $F''_{gH}(x_0)\prec_{LU}0$*, then* $F(x)$ *is strongly concave at* x_0*;*
2. *if* $F''_{gH}(x_0)\succ_{LU}0$*, then* $F(x)$ *is strongly convex at* x_0*.*

A simple case is the function $F(x)$ *of Example 3.1.2, for which the second-order gH-derivative exist for all* $x\in[-2,4]$ *with* $F''_{gH}(x)=(-6x+8;2)=[-6x+6,-6x+10]$*.*
$F(x)$ *is strongly convex for* $x\in]-2,1[$ *and strongly concave for* $x\in\left]\frac{5}{3},4\right[$*; if* $x\in\left]1,\frac{5}{3}\right[$*, we have* $0\in[-6x+6,-6x+10]$ *so that* $F(x)$ *cannot be strongly convex nor concave.*

3.1.8 Complete discussion of an example

We conclude this Section with a complete discussion of an example where all the results described so far will be applied.

Let consider function $F:\mathbb{R}\to\mathcal{K}_C$ defined by

$$\widehat{f}(x)=\cos(x)+2\cos(\frac{x}{2}) \text{ and } \widetilde{f}(x)=1.2+\sin(x)$$

for $x\in[a,b]=\left[-\frac{3}{4},\frac{3}{4}\right]$. In addition, we choose $\gamma^-=-1.2$, $\gamma^+=0.8$.

Remark that both $\widehat{f}(x)$ and $\widetilde{f}(x)$ are differentiable so that, in midpoint notation, the first order gH-derivative is $F'_{gH}(x)=(\widehat{f}'(x);|\widetilde{f}'(x)|)$ and the second order gH-derivative is $F''_{gH}(x)=(\widehat{f}''(x);|\widetilde{f}''(x)|)$ (see [80]).

All computations are performed with a precision of at least five decimal digits.

Internally to $[a, b]$, we consider eleven points where $\widehat{f}(x)$ is locally minimal or maximal (we will ignore the first and last ones as too near to boundaries a and b). They are marked in Figure 3.21 by a diamond symbol and are denoted $x_i, i = 1, 2, \ldots, 11$, corresponding to the rows in Table 3.1.

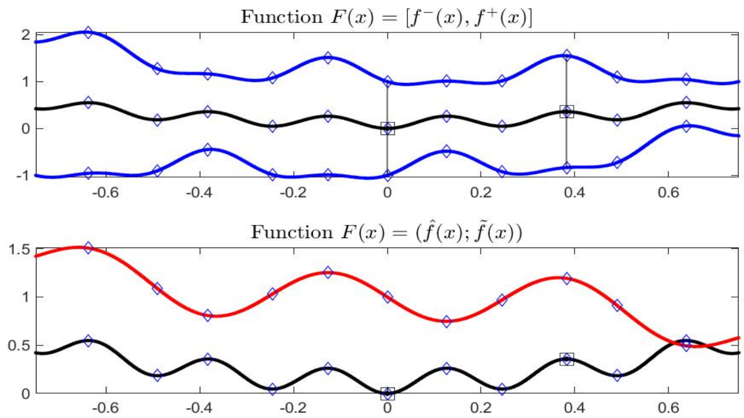


Figure 3.21: Interval-valued function $F(x)$ of the complete example (top). Bottom picture gives functions $\widehat{f}(x)$ (black color) and $\widetilde{f}(x)$ (red).

i	x_i	$F(x_i)$	$F'_{gH}(x_i)$	$F''_{gH}(x_i)$	**Type**	**Efficient Region**
1	-0.637302	$(0.549; 1.506)$	$(0.0; 0.735)$	$(-73.71; 35.18)$	max	$[-0.6433, -0.6078]$
2	-0.490592	$(0.184; 1.089)$	$(0.0; 3.842)$	$(78.26; 7.600)$	min	$[-0.5619, -0.4425]$
3	-0.382304	$(0.358; 0.807)$	$(0.0; 0.726)$	$(-76.13; 41.61)$	max	$[-0.3987, -0.3753]$
4	-0.245329	$(0.046; 1.029)$	$(0.0; 2.956)$	$(79.91; 0.844)$	min	$[-0.2755, -0.1930]$
5	-0.127422	$(0.262; 1.252)$	$(0.0; 0.0)$	$(-77.31; 38.70)$	max	$\{-0.127422\}$
6	0.0	(0.0;1.0)	(0.0;3.142)	(80.46;0.0)	min x_m	[-0.0523,0.0334]
7	0.127422	$(0.262; 0.748)$	$(0.0; 0.0)$	$(-77.31; 38.70)$	max	$\{0.127422\}$
8	0.245329	$(0.046; 0.971)$	$(0.0; 2.956)$	$(79.91; 0.844)$	min	$[0.2125, 0.2905]$
9	0.382304	$(0.358; 1.193)$	$(0.0; 0.726)$	$(-76.13; 41.61)$	max x_M	$[0.3769, 0.4174]$
10	0.490592	$(0.184; 0.911)$	$(0.0; 3.842)$	$(78.26; 7.600)$	min	$[0.4176, 0.5328]$
11	0.637302	$(0.549; 0.494)$	$(0.0; 0.735)$	$(-73.71; 35.18)$	max	$[0.6241, 0.6453]$

Table 3.1: Relevant points of $F(x)$ in the complete example.

The two points $x_m = 0$ and $x_M = 0.3823$, corresponding to a local minimum and maximum of $\widehat{f}(x)$ and marked, in the (x, y)-plane, with a square symbol, will be analyzed in detail. The vertical segments in top of Figure 3.21 represent intervals $F(x_m) = [-1, 1]$ and $F(x_M) = [-0.8356, 1.5506]$.

Figure 3.22 gives the first order gH-derivative of $F(x)$; remark that, according to fourth column in Table 3.1, we always have $0 \in F'_{gH}(x_i)$ and $F'_{gH}(x_5) = F'_{gH}(x_7) = 0$.

In Figure 3.23 the gH-derivative is pictured in midpoint half-plane $(\widehat{z}; \widetilde{z})$; the points where $\widehat{f}'(x) = 0$ are marked in correspondence with the value $\widehat{z}$ on the abscissa (compare also with Table 3.1).

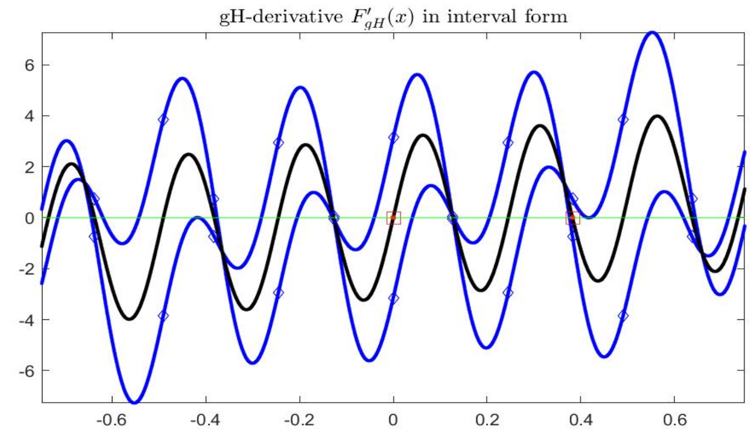


Figure 3.22: First order gH-derivative $F'_{gH}(x)$ in interval form; black curve is $\widehat{f}'(x)$.

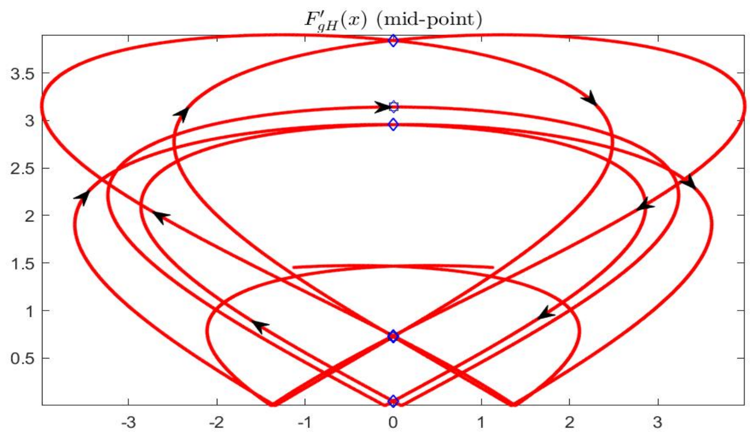


Figure 3.23: First order gH-derivative $F'_{gH}(x) = \left(\widehat{f}'(x); \left|\widetilde{f}'(x)\right|\right)$ in midpoint form.

The second order gH-derivative, represented in Figure 3.24, shows that the intervals $F''_{gH}(x)$, as expected, are entirely positive at the minima and negative at the maxima. Remark that in no points the first and second derivatives of $\widehat{f}(x)$ are simultaneously zero.

In this example, where both $\widehat{f}(x)$ and $\widetilde{f}(x)$ have continuous second

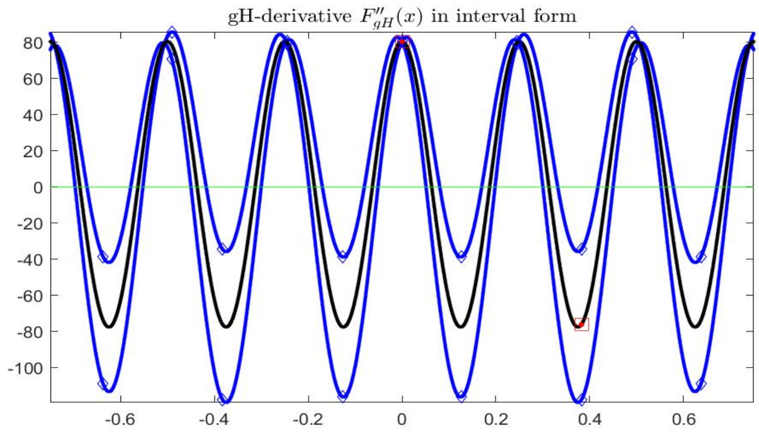


Figure 3.24: Second order gH-derivative $F''_{gH}(x)$ in interval form; the black curve is $\widehat{f}''(x)$.

order derivative, the convexity region is particularly simple to identify, due to a well known theorem based on the sign of the curvature of curve C_F associated to $F(x)$ (for details see [5]). We consider the following function $\kappa(x) = \widehat{f}'(x)\widetilde{f}''(x) - \widehat{f}''(x)\widetilde{f}'(x)$ (its sign coincides with the sign of the curvature of C_F at x). We then search for the points, on the left and right of x_m, at which the sign of $\kappa(x)$ has the same sign of $\kappa(x_m)$. Analogous result is valid for x_M.

In our example we find the interval $[-0.14535, 0.07545]$ around x_m and the interval $[0.37170, 0.41745]$ around x_M (see Figure 3.25, where the values of x corresponding to local convex portion of $F(x)$ are delimited by red vertical lines in the bottom picture).

From the local convexity of the curve C_F, the efficient regions corresponding to x_m and x_M are computed under Assumption 3.1.1 for min and Assumption 3.1.2 for max; the resulting intervals are, respectively, $[-0.05235, 0.03345]$ around x_m and $[0.37695, 0.41745]$ for x_M. In Figure 3.26 the efficient regions around the min and max points are delimited by vertical red lines.

The next Figure 3.27 gives the three functions

$$\widehat{f}(x)\ ,\ \ \gamma^{+}\widehat{f}(x) - \widetilde{f}(x)\ ,\ \ \widetilde{f}(x) - \gamma^{-}\widehat{f}(x)$$

as we have seen in Subsection 3.1.4; their sign gives information on the monotonicity of $F(x)$.

In particular, if all three functions (observe that the second function is changed in sign with respect to the properties in Subsection 3.1.4) have the same sign (hence are also not zero) at a point x, then $F(x)$ is strictly increasing or decreasing with respect to the partial order $\precsim_{\gamma^-,\gamma^+}$. The same

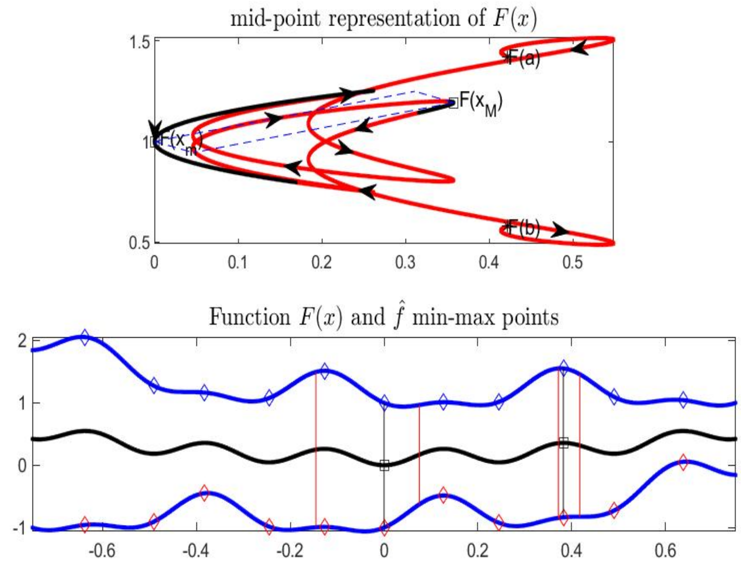


Figure 3.25: Local convexity of curve C_F corresponding to x_m and x_M.

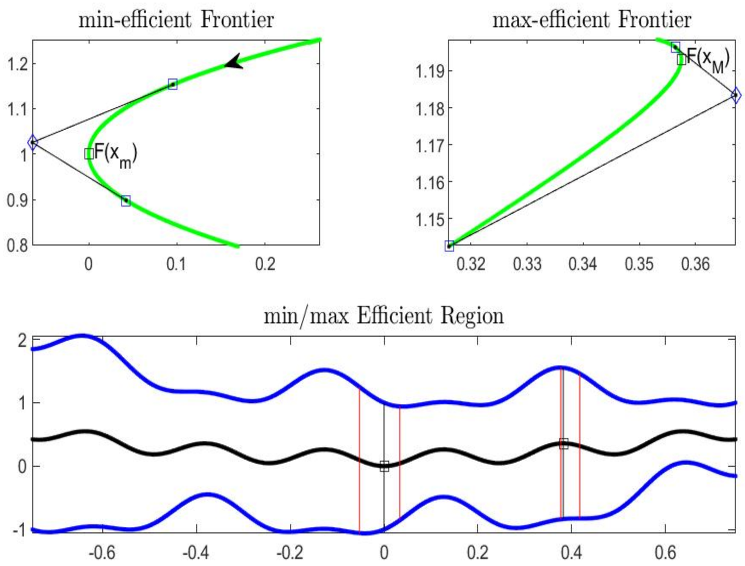


Figure 3.26: Efficient regions corresponding to x_m and x_M obtained by tangency conditions.

information can be eventually deduced from the sigh of the tree derivatives

$$\widehat{f}'(x)\ ,\ \ \gamma^{+}\widehat{f}'(x)-\widetilde{f}'(x)\ ,\ \ \widetilde{f}'(x)-\gamma^{-}\widehat{f}'(x)$$

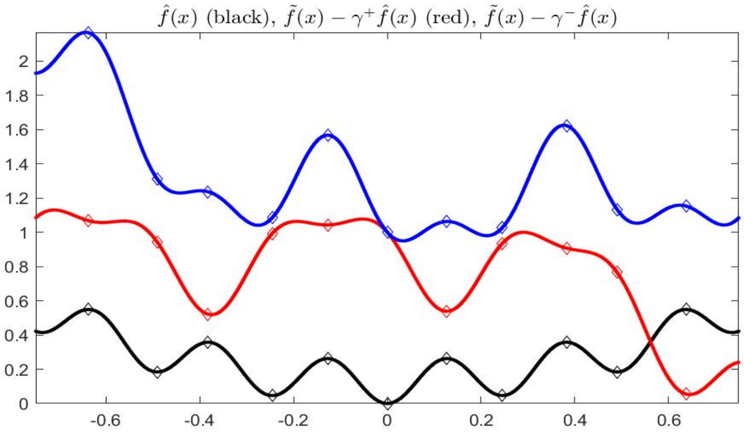


Figure 3.27: Functions $\widehat{f}(x)$, $\gamma^+\widehat{f}(x) - \widetilde{f}(x)$ and $\widetilde{f}(x) - \gamma^-\widehat{f}(x)$.

given in Figure 3.28; at points where the three derivatives have different signs, then function $F(x)$ is not $\precapprox_{\gamma^-,\gamma^+}$-monotonic.

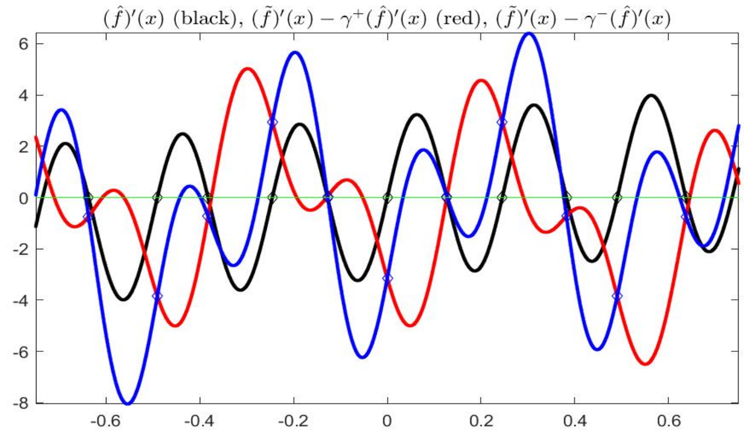


Figure 3.28: Functions $\widehat{f}'(x)$, $\gamma^+\widehat{f}'(x) - \widetilde{f}'(x)$ and $\widetilde{f}'(x) - \gamma^-\widehat{f}'(x)$.

The results of our analysis, as described for the min and max points x_m and x_M, are visualized in Figure 3.29, giving the midpoint representation of $F(x)$.

Here, we see the position of point $F(x_m)$, with the delimiters $F(x'_m)$, $F(x''_m)$ of the efficient region $E_{min}(F;x_m)$ (in green color); analogously, the position of the max point $F(x_M)$ is evidenced, with the delimiters $F(x'_M)$, $F(x''_M)$ of the efficient region $E_{max}(F;x_M)$. Clearly, the two points correspond to local best-min and best-max points (not of lattice type).

The last three Figures (3.30, 3.31 and 3.32) summarize, respectively, the computations for all the local minima and maxima considered in Table 3.1.

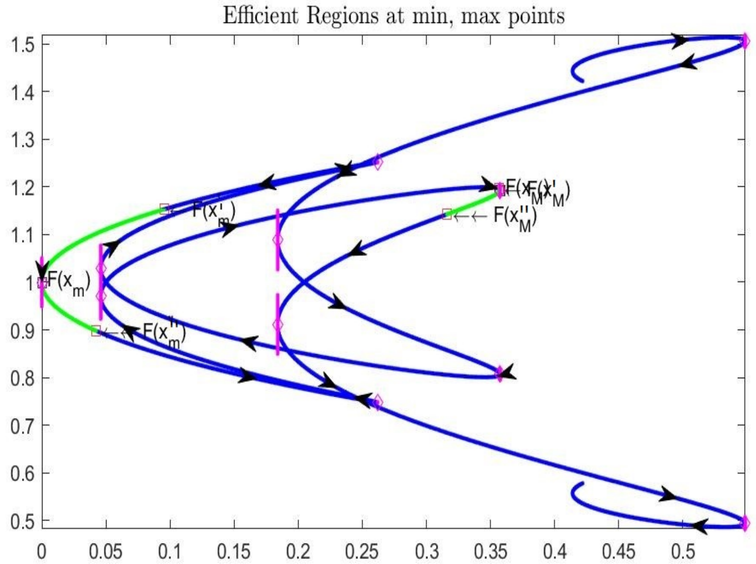


Figure 3.29: Function $F(x)$ is represented in midpoint form, together with the efficient points corresponding to x_m and x_M.

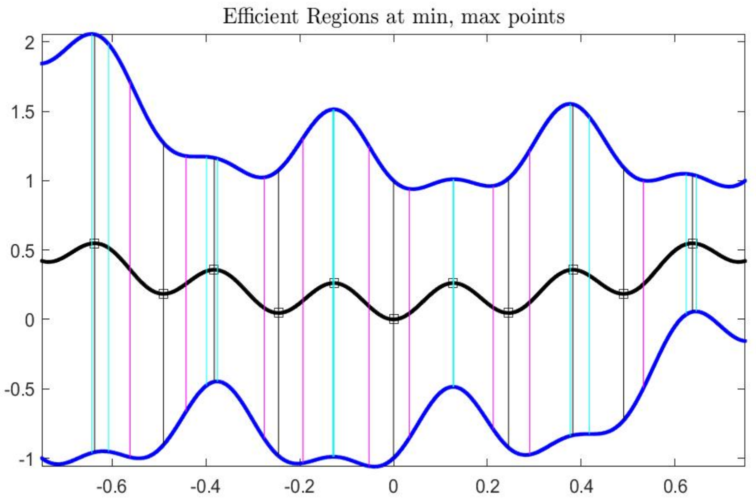


Figure 3.30: Function $F(x)$ is represented in interval form and all the zeros x_i of $\widehat{f}(x)$ are classified as min or max points.

In particular, Figure 3.30 reproduces $F(x)$ in interval form, with the visualization of the six local maxima and the five local minima, classified according to the computations.

The points x_i, $i = 1, 2, \ldots, 11$ in the first column of Table 3.1, together

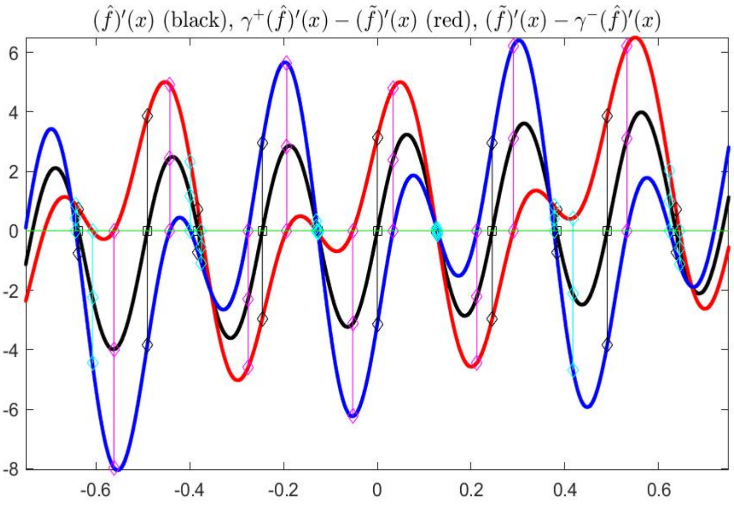


Figure 3.31: Functions $\widehat{f}'(x)$, $\gamma^{+}\widehat{f}'(x) - \widetilde{f}'(x)$ and $\widetilde{f}'(x) - \gamma^{-}\widehat{f}'(x)$ evaluated at classified min and max points.

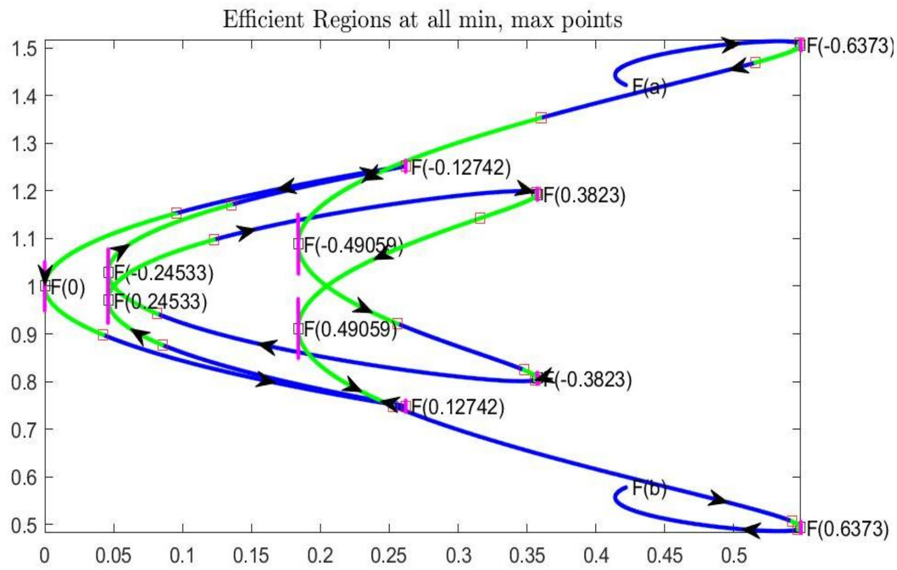


Figure 3.32: Function $F(x)$ is represented in midpoint form, with all the points $F(x_i)$ and the corresponding efficient regions.

with corresponding interval values $F(x_i)$, $i = 1, 2, \ldots, 11$ (as in the second column) are marked with a vertical segment in black color. Correspondingly, the efficient regions are delimited by vertical lines (cyan-colored for six max points and magenta for five min points). There are two local maxima, corresponding to $x_5 = -0.127422$ with $F(x_5) = (0.262; 1.252) = [-0.99, 1.514]$ and $x_7 = 0.127422$ with $F(x_7) = (0.262; 0.748) = [-0.486, 1.01]$ which are lattice max-points: the efficient frontier coincides with the point itself and the two maximal intervals dominate locally all the near intervals $F(x)$ (in the figure, the black and cyan vertical lines are coincident).

This is also visible in Figure 3.31 in terms of the values of the three derivatives $\widehat{f}'(x)$, $\gamma^+\widehat{f}'(x) - \widetilde{f}'(x)$ and $\widetilde{f}'(x) - \gamma^-\widehat{f}'(x)$ evaluated at x_i and at the points defining the efficient regions: for the two lattice maxima, the three derivative are zero, while in the other minima or maxima only $\widehat{f}'(x_i)$ is zero and the other two derivatives do not have (at least generally) the same sign (but one or both may possibly be zero). Note that the corresponding efficient frontiers are delimited by vertical lines (cyan-colored for max and magenta for min points).

Finally, Figure 3.32 summarizes all the computations by the midpoint visualization of our function $F(x)$ and all local minima (five points) and maxima (six points) are marked together with corresponding efficient regions.

3.1.9 Periodic interval-valued functions (and famous plane curves)

Before continuing, let us recall that a function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is said to be *periodic* if there exists a real number $T \neq 0$, called *period* of f, such that $D + T = D$ and $f(x + T) = f(x)$ for every $x \in D$, i.e., when f repeats its values at regular intervals. In the following the periods will be always assumed to be positive unless otherwise stated. The smallest positive period of f (if such exists) is called *fundamental* (see [61] for details).

On the other hand, geometrically, a periodic function can be defined as a function whose graph exhibits translational symmetry, i.e., a function f is periodic with period T if the graph of f is invariant under translation in the x-direction by a distance of T.

Considering points where the trajectory of an interval-valued function $F(x) = (\widehat{f}(x); \widetilde{f}(x))$ crosses itself, i.e., x_1, x_2 exist with $F(x_1) = F(x_2)$ (equivalently $\widehat{f}(x_1) = \widehat{f}(x_2)$ and $\widetilde{f}(x_1) = \widetilde{f}(x_2)$), it follows that periodicity of F is also easy to describe.

Definition 3.1.13. *A function $F : [a, b] \to \mathcal{K}_C$ is said to be periodic if, for some nonzero constant $T \in]0, b-a[$, it occurs that $F(x + T) = F(x)$ for all $x \in [a, b]$ with $x + T \in [a, b]$ (i.e., for all $x \in [a, b - T]$).*
A nonzero constant T for which this is verified, is called a period of the function and if there exists a least positive constant T with this property, it is called the fundamental period.

Clearly, if F has a period T, then this also implies that

$$(\widehat{f}(x + T); \widetilde{f}(x + T)) = (\widehat{f}(x); \widetilde{f}(x))$$

so that $\widehat{f}(x+T) = \widehat{f}(x)$ and $\widetilde{f}(x+T) = \widetilde{f}(x)$ for all $x \in [a, b-T]$, i.e., $\widehat{f}$ and $\widetilde{f}$ are periodic with period T.

Remark that if T is the fundamental period of F, this does not necessarily imply that T is fundamental period for both $\widehat{f}$ and $\widetilde{f}$. On the other hand, the periodicity of $\widehat{f}$ and $\widetilde{f}$ does not necessarily imply the periodicity of F.

Proposition 3.1.22. *Let $F : [a, b] \to \mathcal{K}_C$ be a continuous function such that $F(x) = (\widehat{f}(x); \widetilde{f}(x))$ with $\widehat{f}$ periodic of period $\widehat{T}$ and $\widetilde{f}$ periodic of period $\widetilde{T}$. Then it holds that:*

(1) if the periods $\widehat{T}$ and $\widetilde{T}$ are commensurable, i.e., $\dfrac{\widehat{T}}{\widetilde{T}} \in \mathbb{Q}$ (with $\dfrac{\widehat{T}}{\widetilde{T}} = \dfrac{p}{q}$, such that p and q are coprime), then the function F is periodic of period $T = lcm(\widehat{T}, \widetilde{T})$, i.e., T is the least common multiple between $\widehat{T}$ and $\widetilde{T}$ (that is $T = p\widetilde{T} = q\widehat{T}$) even if T does not necessarily correspond to the fundamental period of F;

(2) if the periods $\widehat{T}$ and $\widetilde{T}$ are not commensurable, i.e. $\dfrac{\widehat{T}}{\widetilde{T}} \notin \mathbb{Q}$, then function F is not periodic.

Example 3.1.4. *Consider the function $F : \mathbb{R} \to \mathcal{K}_C$ defined by periodic functions $\widehat{f}(x) = 5\sin\left(-3x + \dfrac{\pi}{3}\right)$ and $\widetilde{f}(x) = \left|cos(\dfrac{9}{4}x)\right|$. Figures 3.33 and 3.34 picture $F(x)$ for $x \in [a, b] = \left[0, \dfrac{4}{3}\pi\right]$.*

On this interval, function $\widehat{f}(x)$ has two minimal and two maximal points (see bottom picture in Figure 3.33).

We have chosen $x_m = 0.8726$ with $F(x_m) = (-5; 0.3825)$ and $x_M = 4.014$ with $F(x_M) = (5; 0.9238)$, located in Figure 3.34, where also the points corresponding to efficient regions $eff_{min}(F; x_m) = [0.8243, 0.9169]$ and $eff_{max}(F, x_M) = [3.993, 4.032]$ are given in green color. Here $\gamma^- = -1$ and $\gamma^+ = 1$ giving the ($\precapprox_{LU}$)-order.

Example 3.1.5. (**Siamese fishes**). *Function $F : \mathbb{R} \to \mathcal{K}_C$ is defined by periodic functions $\widehat{f}(x) = 5\cos(x) - (\sqrt{2}-1)\cos(5x)$ and $\widetilde{f}(x) = 1.5 + \sin(4x)$. Figures 3.35 and 3.36 picture $F(x)$ for $x \in [a, b] = [0, 2\pi]$.*

Internal to this interval, function $\widehat{f}(x)$ has two minimal and three maximal points (see bottom picture in Figure 3.35); we have chosen $x_m = 2.7489$ with $F(x_m) = (-4.7779; 0.5)$ and $x_M = 0.3911$ with $F(x_M) = (4.7779; 2.5)$, located in Figure 3.36, where also the points corresponding to efficient regions $eff_{min}(F; x_m) = [2.1828, 2.7489]$ and $eff_{max}(F, x_M) = [0, 0.3927]$ are given in green color. Here $\gamma^- = -1$ and $\gamma^+ = 0.5$.

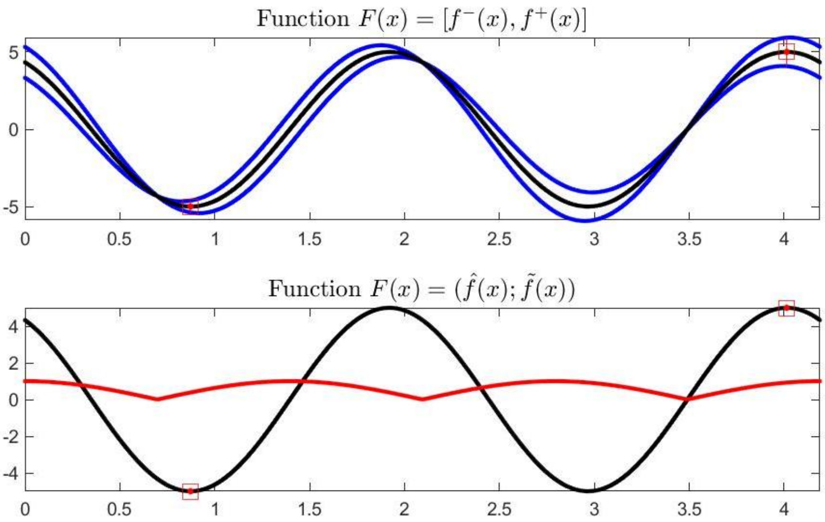


Figure 3.33: Graphical representation of periodic function $F(x)$ of Example 3.1.4 in the plane (x, y) in interval notation (top) and in midpoint notation (bottom). Marked points correspond to x_m and x_M, where the two functions are differentiable.

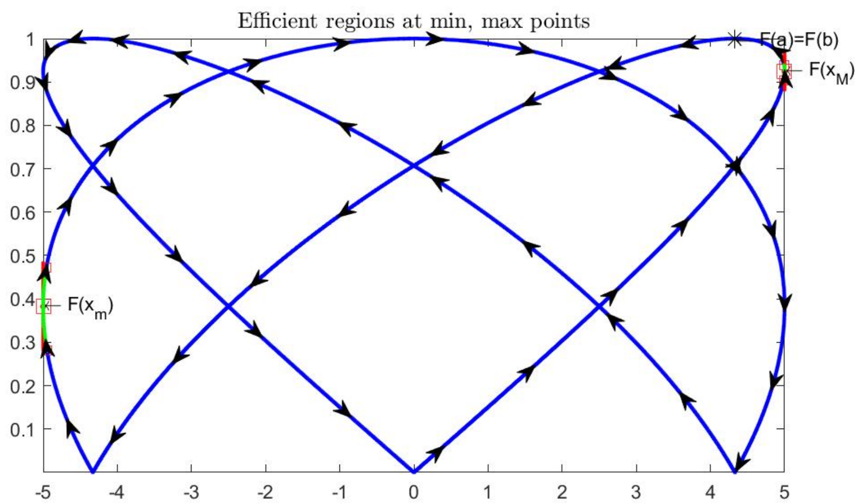


Figure 3.34: Graphical representation of periodic function $F(x)$ of Example 3.1.4 in the half-plane $(\widehat{z}; \widetilde{z})$. Marked points are $F(x_m)$ and $F(x_M)$ and the efficient regions are marked in green color.

Example 3.1.6. (**Big fish**). *Function $F : \mathbb{R} \to \mathcal{K}_C$ is defined by periodic functions $\widehat{f}(x) = \cos(x) + 2\cos(\frac{x}{2})$ and $\widetilde{f}(x) = 1.2 + \sin(x)$. Figures 3.37 and 3.38 picture $F(x)$ for $x \in [a, b] = [0, 4\pi]$.*

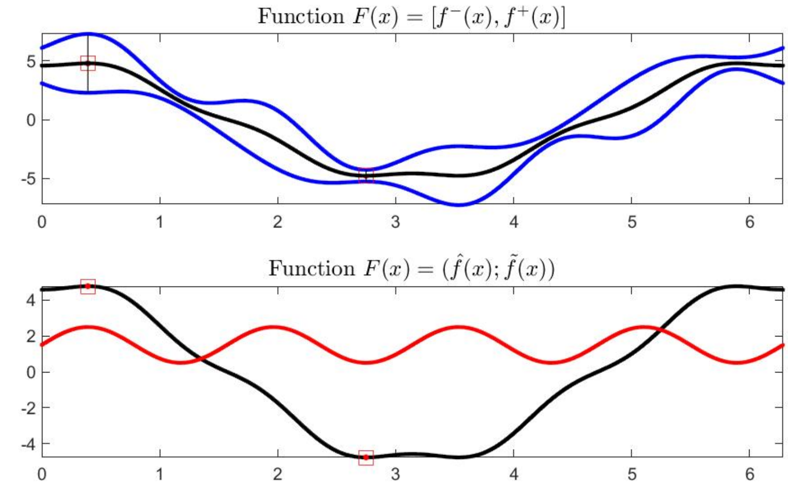


Figure 3.35: Graphical representation of periodic function $F(x)$ of Example 3.1.5 in the plane (x, y) in interval notation (top) and in midpoint notation (bottom). Marked points correspond to x_m and x_M, where the two functions are differentiable.

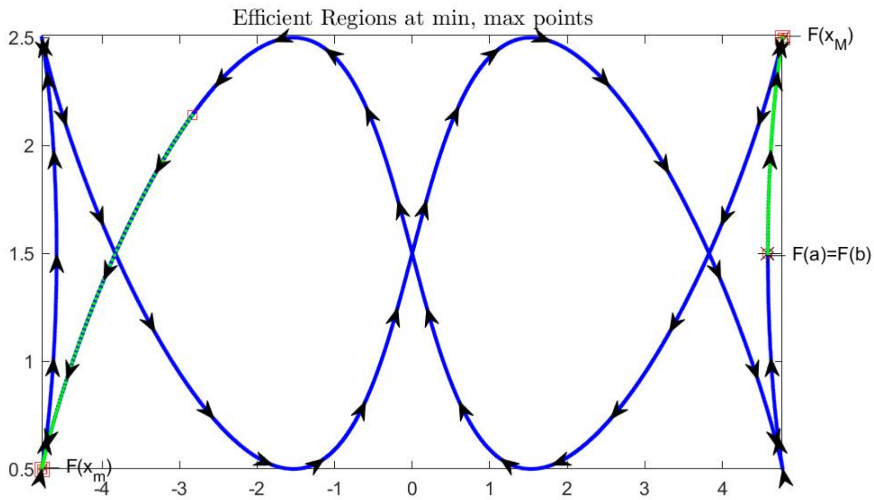


Figure 3.36: Graphical representation of periodic function $F(x)$ of Example 3.1.5 in the half-plane $(\widehat{z}; \widetilde{z})$. Marked points are $F(x_m)$ and $F(x_M)$ and the efficient regions are marked in green color.

Internal to this interval, function $\widehat{f}(x)$ has two minimal and one maximal points (see bottom picture in Figure 3.37); we have chosen $x_m = 8.378$ with $F(x_m) = (-1.5; 2.066)$ and $x_M = 2\pi$ with $F(x_M) = (-1, 1.2)$, located in Figure 3.38, where also the points corresponding to efficient regions

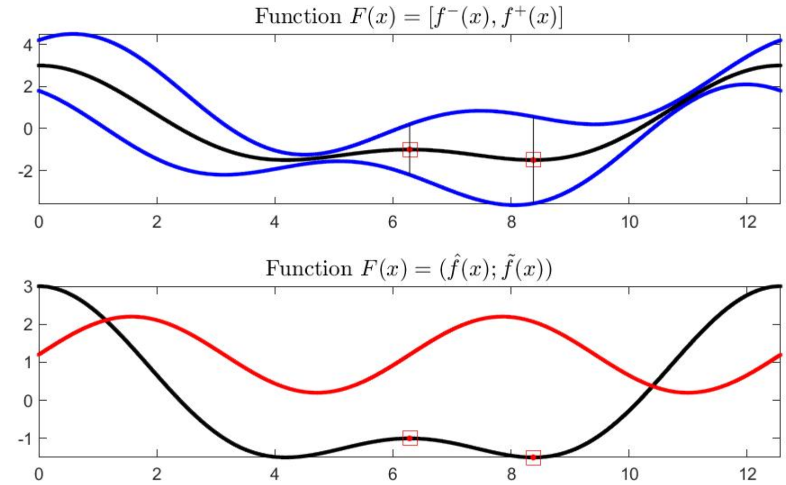


Figure 3.37: Graphical representation of periodic function $F(x)$ of Example 3.1.6 in the plane (x, y) in interval notation (top) and in midpoint notation (bottom). Marked points correspond to x_m and x_M, where the two functions are differentiable.

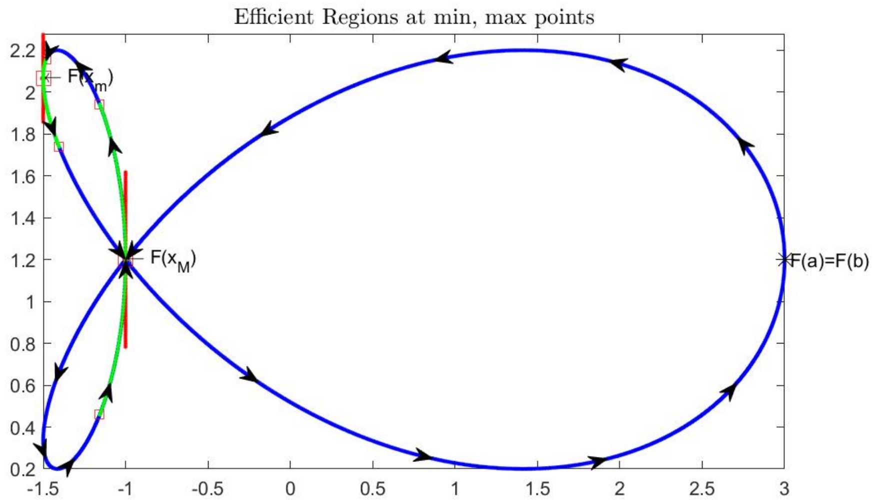


Figure 3.38: Graphical representation of periodic function $F(x)$ of Example 3.1.6 in the half-plane $(\widehat{z}; \widetilde{z})$. Marked points are $F(x_m)$ and $F(x_M)$ and the efficient regions are marked in green color.

$eff_{min}(F; x_m) = [8.1568, 8.8555]$ and $eff_{max}(F, x_M) = [5.4488, 7.1176]$ are given in green color. Here $\gamma^- = -2$ and $\gamma^+ = 2$.

3.2 Other uses of interval-valued functions

This second part of the chapter will be mainly dedicated to the presentation of new possibilities of use and application of the interval-valued functions. In particular, starting from the graphical representations outlined in Section 3.1, a new notation for representing complex intervals will be presented, the peculiarities and advantages of which will be fully exploited through an unprecedented visual approach.

Afterwards, an interesting application will be proposed concerning one of the most recent studies on the so-called q-calculus which, through an innovative approach, will be analyzed from an interval point of view.

3.2.1 A new approach for complex-valued intervals

In this section a new notation to represent complex intervals will be proposed, also showing, through examples and graphical representations, the peculiarities and benefits associated with its use.

Different notations for complex intervals

According to Subsection 1.3.5, it is possible to extend to complex numbers the same concepts of real intervals by defining a complex valued interval as the set (or box) of complex numbers corresponding to interval real part and interval imaginary part.

This means that, according to Definition 1.3.6, a complex interval is defined as

$$\mathbf{Z} = A + iB = \{\mathbf{z} = a + ib : \exists a \in A, \exists b \in B\} \subset \mathbb{C} \tag{3.31}$$

where

- $A = [a^-, a^+] = (\widehat{a}; \widetilde{a})$ is the interval real part of $\mathbf{Z}$;
- $B = [b^-, b^+] = (\widehat{b}; \widetilde{b})$ is the interval imaginary part of $\mathbf{Z}$;
- $i = [i, i] = (i; 0)$ stands for the interval imaginary unity.

We also remember that the set of complex intervals has been denoted by

$$\mathcal{K}_{\mathcal{C}}(\mathbb{C}) = \{\mathbf{z} = a + ib \ | \ a \in A, \ b \in B, \ \ A, B \in \mathcal{K}_{\mathcal{C}}\}$$

so that we can write: $\mathbf{Z} \in \mathcal{K}_C(\mathbb{C})$.

This means that, using the endpoint notation, we can define a complex valued interval as

$$\mathbf{Z} = [\mathbf{z}^-, \mathbf{z}^+] = [a^- + ib^-, a^+ + ib^+] \tag{3.32}$$

with $a^- \leqslant a^+, \ b^- \leqslant b^+$ all belonging to $\mathbb{R}$, such that

- $\mathbf{z}^- = a^- + ib^-$ represents the complex lower endpoint of $\mathbf{Z}$;
- $\mathbf{z}^+ = a^+ + ib^+$ represents the complex upper endpoint of $\mathbf{Z}$.

However, in addition to this notation, it is also possible to use the midpoint one, thanks to which we can define a complex interval as

$$\mathbf{Z} = (\widehat{\mathbf{z}}; \widetilde{\mathbf{z}}) = (\widehat{a} + i\widehat{b}; \widetilde{a} + i\widetilde{b}) \tag{3.33}$$

with $\widehat{a}, \widehat{b} \in \mathbb{R}$ and $\widetilde{a}, \widetilde{b} \geqslant 0$, such that

- $\widehat{\mathbf{z}} = \widehat{a} + i\widehat{b}$ represents the complex midpoint of $\mathbf{Z}$;
- $\widetilde{\mathbf{z}} = \widetilde{a} + i\widetilde{b}$ represents the complex radius of $\mathbf{Z}$.

Clearly the two notations are interchangeable; indeed, since via the midpoint notation we have

$$\mathbf{Z} = (\widehat{\mathbf{z}}; \widetilde{\mathbf{z}}) = (\widehat{a} + i\widehat{b}; \widetilde{a} + i\widetilde{b})$$

with $\widehat{a} = \dfrac{a^+ + a^-}{2}$ and $\widetilde{a} = \dfrac{a^+ - a^-}{2}$ (similarly $\widehat{b} = \dfrac{b^+ + b^-}{2}$, $\widetilde{b} = \dfrac{b^+ - b^-}{2}$) so that, as usual, $a^- = \widehat{a} - \widetilde{a}$ and $a^+ = \widehat{a} + \widetilde{a}$ (similarly $b^- = \widehat{b} - \widetilde{b}$, $b^+ = \widehat{b} + \widetilde{b}$); then, using endpoint notation, we clearly obtain

$$\mathbf{Z} = [\mathbf{z}^-, \mathbf{z}^+] = [a^- + ib^-, a^+ + ib^+], \text{ as it is}$$

$$z^- = \widehat{z} - \widetilde{z} = \widehat{a} + i\widehat{b} - (\widetilde{a} + i\widetilde{b}) = \widehat{a} - \widetilde{a} + i(\widehat{b} - \widetilde{b}) = a^- + ib^-;$$

$$z^+ = \widehat{z} + \widetilde{z} = \widehat{a} + i\widehat{b} + (\widetilde{a} + i\widetilde{b}) = \widehat{a} + \widetilde{a} + i(\widehat{b} + \widetilde{b}) = a^+ + ib^+.$$

In particular, while $\widehat{\mathbf{z}} = \widehat{a} + i\widehat{b} \in \mathbb{C}$ is exactly an ordinary complex number, on the other hand, $\widetilde{\mathbf{z}} = \widetilde{a} + i\widetilde{b}$ cannot be considered as such since $\widetilde{a}$ and $\widetilde{b}$ represent two widths and not two point values.

However, we could also consider $\widetilde{\mathbf{z}} = \widetilde{a} + i\widetilde{b}$ belonging to $\mathbb{C}^+$, i.e., $\widetilde{\mathbf{z}} = \widetilde{a} + i\widetilde{b}$ belongs to the first quadrant of the Gauss plane.

Taking a further step, it is even possible to consider both $\widehat{\mathbf{z}}$ and $\widetilde{\mathbf{z}}$ as complex numbers but in such a case the interval $\mathbf{Z}$ should still be understood in relation to the lattice $(\mathbb{C}, \leq)$, where, given two complex numbers $\mathbf{z_1} = a_1 + ib_1$ and $\mathbf{z_2} = a_2 + ib_2$, the order $\leq$ is defined as follows:

$$\mathbf{z_1} \leq \mathbf{z_2} \;\Leftrightarrow\; a_1 \leqslant a_2 \text{ and } b_1 \leqslant b_2.$$

So we can conclude that, given a complex number $\mathbf{z} = a + ib$ in the classical complex plane (as indicated on the right side of Figure 3.39), all the complex numbers $\mathbf{x}$ represented by the points belonging to the highlighted area to the right of $\mathbf{z}$, are such that $\mathbf{z} \leq \mathbf{x}$; while all complex numbers $\mathbf{y}$ such that $\mathbf{y} \leq \mathbf{z}$, graphically belong to the area on the opposite side with respect to $\mathbf{z}$.

Therefore, if we consider the interval (expressed in endpoint notation)

$$\mathbf{Z} = [\mathbf{z}^-, \mathbf{z}^+] = [a^- + ib^-, a^+ + ib^+] = \left\{\mathbf{z} \in \mathbb{C} \ : \mathbf{z}^- \leq \mathbf{z} \leq \mathbf{z}^+\right\}$$

or, which is the same,

$$\mathbf{Z} = A + iB = [a^-, a^+] + i[b^-, b^+] \text{ with } a^- \leqslant a^+, \ b^- \leqslant b^+,$$

then $\mathbf{Z}$ can be easily represented graphically through the rectangle shown on the left side of Figure 3.39, that is, a complex interval is a *rectangle of certainty* (see [17]).

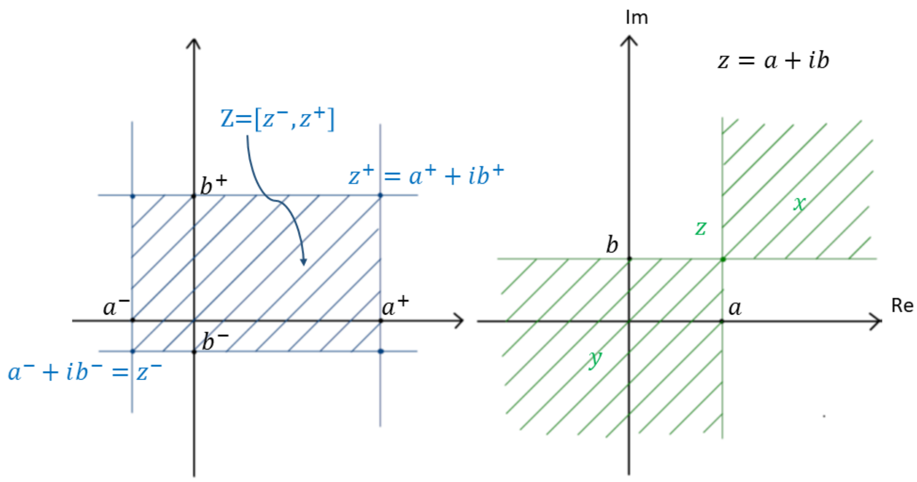


Figure 3.39: Representation in the classical complex plane of a complex number $\mathbf{z} = a + ib$, as a point (right), and of an interval complex number $\mathbf{Z} = [\mathbf{z}^-, \mathbf{z}^+]$, as a rectangle (left).

A new representation for complex intervals

As we have just seen, in general, a complex interval, expressed in endpoint notation, can be represented by a rectangle in the complex plane, that is a kind of rectangle of certainty.

On the other hand, using the midpoint notation, it is possible to define a new type of representation since the real-part intervals $(\widehat{a}; \widetilde{a})$ can be placed in the upper real half-plane as usual, while to represent the imaginary-part intervals $(\widehat{b}; \widetilde{b})$ it is possible to think of adding a new half-plane in the lower section as shown in Figure 3.40.

Therefore, the plane is thus divided into two distinct halves:

- the upper half-plane, whose points represent the real-part intervals $A = (\widehat{a}; \widetilde{a})$;

- the lower half-plane, whose points correspond to the imaginary-part intervals $B = (\widehat{b}; \widetilde{b})$.

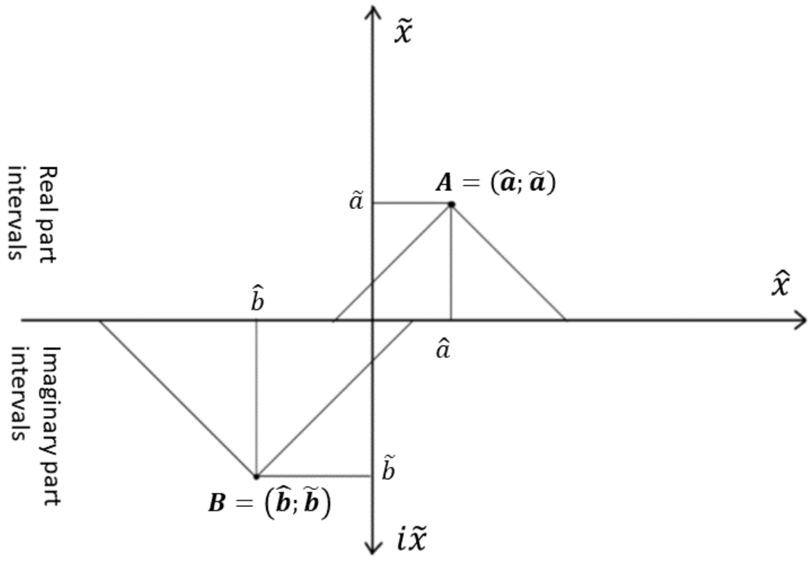


Figure 3.40: Representation of a complex interval $\mathbf{Z} = A + iB$ in midpoint notation.

The following numerical example clarifies the situation.

Example 3.2.1. *Consider the complex interval*

$$\mathbf{Z} = (\widehat{\mathbf{z}}; \widetilde{\mathbf{z}}) = (\widehat{a} + i\widehat{b}; \widetilde{a} + i\widetilde{b}) = (2 - 3i; 1 + 2i)$$

which corresponds to $\mathbf{Z} = A + iB$*, such that*

$$A = (\widehat{a}; \widetilde{a}) = (2; 1) \;\; and \;\; B = (\widehat{b}; \widetilde{b}) = (-3; 2),$$

i.e., $\mathbf{Z}$ *is an interval of complex numbers of the type*

$$\mathbf{z} = a + ib, \;\; with\, a \in (2; 1) = [1, 3] \;\; and \;\; b \in (-3; 2) = [-5, -1]$$

as shown in Figure 3.41. Here the interval is represented by two points lying respectively on the real part (upper half-plane) and on the imaginary part (lower half-plane) of the midpoint complex plane (left side of figure); however, in accordance with the classical theory (see also Subsection 1.3.5), a complex interval is also conceived as a 2-dimensional interval vector, such that it can be represented by a rectangle in the complex plane with sides parallel to the axes (right side of figure).

In particular, in the example considered we have that:

$$\mathbf{Z} = A + iB \;\; or \;\; \mathbf{Z} = (A, B)$$

where, referring to the midpoint complex plane (left side of figure):

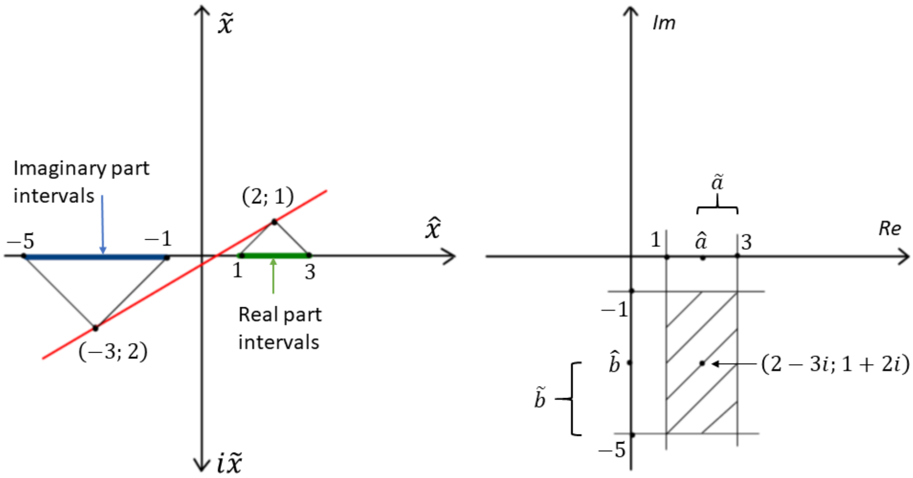


Figure 3.41: Graphic representation of the complex interval $(\widehat{a} + i\widehat{b}; \widetilde{a} + i\widetilde{b}) = (2 - 3i; 1 + 2i)$ as two different points in the midpoint complex plane (left) and as a rectangle in the classical complex plane (right).

$A = (2; 1)$ is the real-part interval, belonging to the upper half-plane,

$B = (-3; 2)$ is the imaginary-part interval, in the lower half-plane,

as well as, considering the the classic complex plane (right side of figure):

$A = [1, 3]$ is the real interval element, parallel to the horizontal axis,

$B = [-5, -1]$ is the complex interval part, parallel to the vertical axis.

Clearly, in the second (classical) case the complex interval is represented by the well known rectangle of certainty.

The complex-valued curve

Similarly to the real case (see Definition 3.1.1), we give the definition of interval-valued functions in the complex case.

Definition 3.2.1. *An interval complex-valued function (or complex interval-valued function) is defined to be any*

$$F : [a, b] \longrightarrow \mathcal{K}_C(\mathbb{C})$$

with $F(x) = [f^-(x), f^+(x)] \in \mathcal{K}_C(\mathbb{C})$ and $f^-(x) \leqslant f^+(x)$ for all $x \in [a, b] \subseteq \mathbb{R}$, where we define

$$f^-(x) = f^-_{re}(x) + i f^-_{im}(x) \;\; and \;\; f^+(x) = f^+_{re}(x) + i f^+_{im}(x).$$

Otherwise, using midpoint notation, we write $F(x) = \left(\widehat{f}(x); \widetilde{f}(x)\right)$, where:

- $\widehat{f}(x) = \widehat{f}_{re}(x) + i\widehat{f}_{im}(x) \in \mathbb{C}$ *is the midpoint value of interval* $F(x)$;
- $\widetilde{f}(x) = \widetilde{f}_{re}(x) + i\widetilde{f}_{im}(x) \in \mathbb{C}^+ \cup \{0\}$ *is the nonnegative half-length of* $F(x)$.

As in the real case, we have that

$$\widehat{f}(x) = \frac{f^+(x) + f^-(x)}{2} \quad \text{and} \quad \widetilde{f}(x) = \frac{f^+(x) - f^-(x)}{2} \geqslant 0$$

and so

$$f^-(x) = \widehat{f}(x) - \widetilde{f}(x) \quad \text{and} \quad f^+(x) = \widehat{f}(x) + \widetilde{f}(x).$$

In Section 3.1, referring to the real case, we frequently used a graphical representation of an interval-valued function

$$F : [a, b] \longrightarrow \mathcal{K}_C$$

obtained in the so-called midpoint half-plane where each interval $F(x)$ is identified with the point $(\widehat{f}(x); \widetilde{f}(x))$ and the arrows give the *direction* of moving the intervals for increasing $x \in [a, b]$.

Therefore, we are interested in verifying whether even in the complex case it is possible to carry out a similar procedure.

We can first consider the complex-valued curve defined as follows:

$$\widehat{f}(x) = \widehat{f}_{re}(x) + i\widehat{f}_{im}(x) \tag{3.34}$$

with $x \in [x_0, x_1]$.

It is interesting to observe that the real-part of function $\widehat{f}_{re}$ and the imaginary-part of function $\widehat{f}_{im}$ can be represented in the complex plane in parametric form as

$$\begin{cases} y_{re} = \widehat{f}_{re}(x) \\ y_{im} = \widehat{f}_{im}(x) \end{cases}$$

with $x \in [x_0, x_1]$ as shown in Figure 3.42.

Note that each point $(\widehat{f}_{re}(x), \widehat{f}_{im}(x))$ represent the complex midpoint $\widehat{f}(x)$ of the complex interval $F(x)$ and, as in the real case, the arrows give the direction of moving the intervals for increasing x: they started at point $\widehat{f}(x_0)$ and terminate at $\widehat{f}(x_1)$.

Moreover, as shown in Figure 3.43, for a generic x, the complex interval

$$F(x) = (\widehat{f}(x); \widetilde{f}(x))$$

is represented by a rectangle centered at the point

$$\widehat{f}(x) = \widehat{f}_{re}(x) + i\widehat{f}_{im}(x) = (\widehat{f}_{re}(x), \widehat{f}_{im}(x))$$

whose dimensions are identified by the radial component of the interval

$$\widetilde{f}(x) = \widetilde{f}_{re}(x) + i\widetilde{f}_{im}(x) = (\widetilde{f}_{re}(x), \widetilde{f}_{im}(x))$$

where:

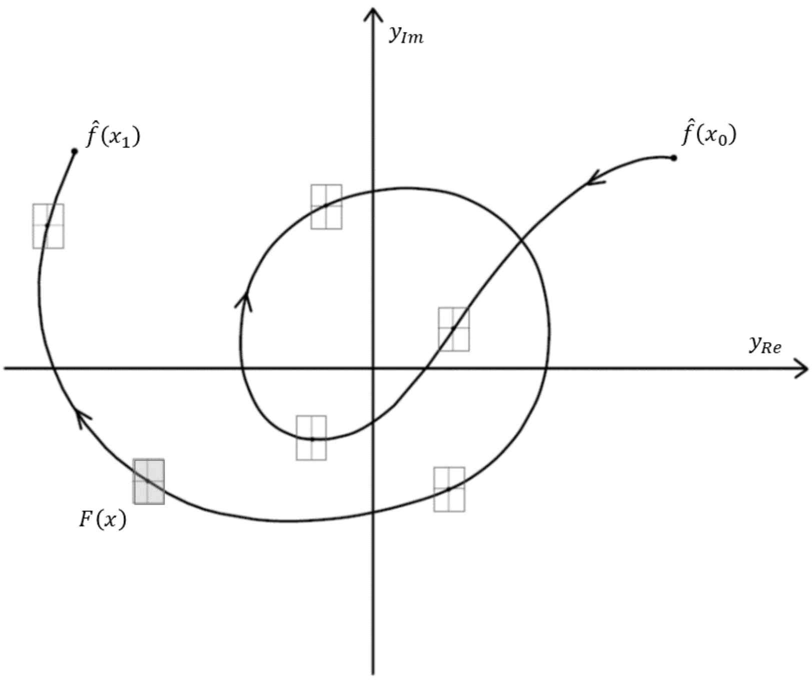


Figure 3.42: Parametric representation in the complex plane of the real-part of function $\widehat{f}_{re}$ and the imaginary-part of function $\widehat{f}_{im}$.

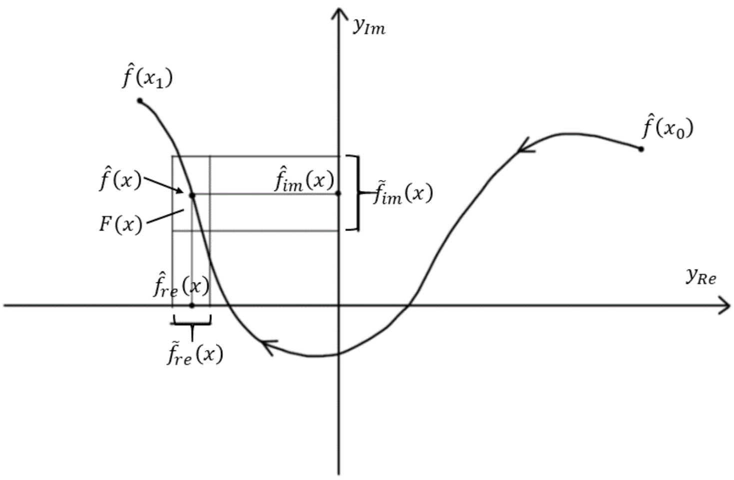


Figure 3.43: Representation of the complex interval $F(x) = (\widehat{f}(x); \widetilde{f}(x))$ for a generic x as a rectangle centred at the point $\widehat{f}(x)$.

- $\widetilde{f}_{re}(x) \geqslant 0$ is the orizontal side of the box;

- $\widetilde{f}_{im}(x) \geqslant 0$ is the vertical side of the box.

The rectangle specifically follows the movements of the arrows (as expressed in both Figures 3.42 and 3.43), thus giving a completely new graphical approach to the concept of complex-valued intervals.

3.2.2 Interval-valued q-calculus

This section suggests a possible example of application of interval analysis to a topic, the q-calculus, which nowadays holds great interest in the scientific community and, therefore, could be taken into consideration also for future research.

In particular, we will try to revisit the most recent studies on the q-calculus (where q stands for quantum) from an interval point of view, making use of the midpoint notation. To do this we will rely on the q-calculus approach outlined by Victor Kac and Pokman Cheung in [41] as well as the one used by Agnieszka B. Malinowska and Delfim F.M. Torres in [59].

We have chosen this topic as we consider it very interesting from many points of view, in particular because of its recent applications in different areas of mathematics such as orthogonal polynomials, basic hypergeometric functions, combinatorics and calculus of variations, even if, as pointed out by Thomas Ernst in [22], it seems that the majority of scientists who use q-calculus are physicists: from statictical mechanics to theory of relativity, up to the concepts of q-heat and q-wave recently introduced in [9]. Furthermore, in the last period, q-calculus is receiving significant attention from researchers who belong to the most varied fields and several new results can be found in [94] and other references cited therein.

Basic results on q-calculus

We present a brief overview of some basic concepts and definitions regarding q-calculus for real-valued functions.

Basically, the regular calculus uses limits in calculating the derivatives of real functions; nevertheless, the quantum calculus, also known as the calculus without limits, substitutes the classical derivative by a quantum difference operator which allows to deal with sets of nondifferentiable functions.

Let remember that despite, historically, in the eighteenth century Euler himself obtained the basic formulas in q-calculus, however, the first to introduce the notion of the definite q-derivative and q-integral was Frank Hilton Jackson (see [39]) in the early 1900s.

Definition 3.2.2. *([41]) Considering an arbitrary function $f(x)$, its q-differential is defined as*

$$d_q f(x) = f(qx) - f(x) \tag{3.35}$$

where q is a fixed number, $q \in]0,1[$.

So that we have in particular: $d_q x = (q-1)x$.

With the differential introduced in (3.35) it is possible to define the corresponding q-derivative.

Definition 3.2.3. *([41]) Let $f(x)$ be an arbitrary function, the expression*

$$D_q f(x) = \frac{d_q f(x)}{d_q x} = \frac{f(qx) - f(x)}{(q-1)x}, \quad x \neq 0 \tag{3.36}$$

is defined as the Jackson q-difference operator, also called the q-derivative of the function $f(x)$.

Note that if $f(x)$ is differentiable, then

$$\lim_{q \to 1} D_q f(x) = \frac{df(x)}{dx} = f'(x)$$

and the q-derivative of f at 0 is defined by $D_q f(0) = f'(0)$.

From (3.36) it follows that, from any positive integer n, the q-derivative of $f(x) = x^n$ is

$$D_q x^n = [n]_q x^{n-1} \tag{3.37}$$

where, applying the basic rules of power series, it is

$$[n]_q \stackrel{def}{=} \frac{q^n - 1}{q - 1} = 1 + ... + q^{n-1}$$

which represents the q-analogue of n and, as $q \to 1$, according to the basic rules of the limits, we have

$$\lim_{q \to 1} [n]_q = \lim_{q \to 1} \frac{q^n - 1}{q - 1} = n.$$

Furthermore, the q-factorial $[n]_q!$ of a positive integer n is given by

$$[n]_q! = \begin{cases} 1 & \text{if } n = 0 \\ [1]_q \times [2]_q \times ... \times [n]_q & \text{if } n = 1, 2, ... \end{cases}$$

This turns out to be very useful, as for example in the definition of the q-analogue of the classic exponential function e^x which, as explained in [41], becomes

$$e_q^x = \sum_{j=0}^{\infty} \frac{x^j}{[j]!}. \tag{3.38}$$

However, in addition to the one already seen, there is also another type of quantum difference operator, associated to the so-called h-calculus (see [33] and [59]).

Definition 3.2.4. *Considering an arbitrary function $f(x)$, its h-differential is defined as*

$$d_h f(x) = f(x+h) - f(x) \tag{3.39}$$

where h is a fixed number, $h > 0$.

So that we have in particular: $d_h x = h$.

With the differential introduced in (3.39) it is possible to define the corresponding h-derivative.

Definition 3.2.5. *Let $f(x)$ be an arbitrary function, the expression*

$$D_h f(x) = \frac{d_h f(x)}{d_h x} = \frac{f(x+h) - f(x)}{h}, \quad x \in \mathbb{R} \tag{3.40}$$

is the h-derivative of the function $f(x)$.

Note that if $f(x)$ is differentiable, then

$$\lim_{q \to 1} D_q f(x) = \lim_{h \to 0} D_h f(x) = \frac{df(x)}{dx}.$$

Here it clearly emerges how, while Leibniz notation $\dfrac{df(x)}{dx}$ consists of a ratio of two "infinitesimals", so rather confusing, on the other hand, the notions of q-and h-differentials are obvious as the q- and h-derivatives defined in (3.36) and (3.40) are plain ratios.

And even better, the h-derivative and the q-derivative can also be unified (in the limit) by the so-called $Hahn$ operator.

Definition 3.2.6. *Considering a real function $f(x)$, defined on an interval I containing ω_0, the expression*

$$D_{q,\omega} f(x) = \frac{f(qx + \omega) - f(x)}{(q-1)x + \omega} \tag{3.41}$$

where $x \neq \omega_0$, $\omega_0 = \dfrac{\omega}{1-q}$, $q \in]0, 1[$ and $\omega \geqslant 0$ are all real numbers, is called Hahn operator.

Remark 3.2.1. *As described in [3], [33] and [70], it is possible to introduce the foreward difference operator*

$$\Delta_{a,b}(x) = \frac{f(\sigma(x)) - f(x)}{\sigma(x) - x}$$

where $\sigma(x) = ax + b$ with $a \geqslant 1$, $b \geqslant 0$, $a + b > 1$ and $\rho(x) = \dfrac{x-b}{a}$ represents its inverse, so that

$$... \leqslant \rho^2(\alpha) \leqslant \rho(\alpha) \leqslant \alpha \leqslant \sigma(\alpha) \leqslant \sigma^2(\alpha) \leqslant ...$$

This way we can define f on a mixed-time scale

$$\{..., \rho^2(\alpha), \rho(\alpha), \alpha, \sigma(\alpha), \sigma^2(\alpha), ...\}, \quad \alpha > \frac{b}{1-a}$$

which, instead of being continuous, is a discrete subset of $\mathbb{R}$.

Now if we consider

$$\beta(x) = qx + \omega$$

we obtain that $D_{q,\omega}[f](x) = \dfrac{f(qx+\omega) - f(x)}{qx - x + \omega} = \dfrac{f(\beta(x)) - f(x)}{\beta(x) - x}$, *which means that for every x it is possible to introduce a general quantum difference operator defined by*

$$D_\beta f(x) = \frac{f(\beta(x)) - f(x)}{\beta(x) - x}$$

with $\beta(x) \neq x$ and $D_\beta f(x) = f'(x)$, such that:
if $f(x)$ is differentiable, then

$$\lim_{x \to x_0} D_\beta f(x) = x_0.$$

Interval-valued q-derivative

Making use of the concept of gH-difference, a procedure similar to the one just seen can also be applied to an interval-valued function

$$F : [a, b] \longrightarrow \mathcal{K_C} \text{ such that } F : x \longrightarrow (\widehat{f}(x); \widetilde{f}(x)).$$

Definition 3.2.7. *Considering an arbitrary interval-valued function, expressed in the midpoint notation $F(x) = (\widehat{f}(x); \widetilde{f}(x))$, its (gH, q)-differential is defined by the generalized Hukuhara difference as:*

$$d_{gH,q}F(x) = F(qx) \ominus_{gH} F(x) = (\widehat{f}(qx) - \widehat{f}(x); |\widetilde{f}(qx) - \widetilde{f}(x)|) \quad (3.42)$$

with $q \in]0, 1[$ and $x \neq 0$.

As in the classical case, with the differential introduced in (3.42) and remembering that $d_q(x) = (q-1)x$, it is also possible to establish the corresponding q-derivative for the interval case.

Definition 3.2.8. *Let $F(x) = (\widehat{f}(x); \widetilde{f}(x))$ be an arbitrary interval-valued function represented in midpoint notation, the expression*

$$D_{gH,q}F(x) = \frac{d_{gH,q}F(x)}{d_q x} = \frac{F(qt) \ominus_{gH} F(x)}{(q-1)x} \quad (3.43)$$

is called the interval-valued (gH, q)-derivative of the function $F(x)$ with $x \neq 0$.

Moreover, if the following limit exists, we define

$$D_{gH,q}F(0) = \lim_{x\to 0} D_{gH,q}F(x)$$

where, if F is gH-differentiable at x, we have

$$\lim_{q\to 1} D_{gH,q}F(x) = \frac{d_{gH}F(x)}{d_q x} = F'_{gH}(x).$$

Furthermore, the Hahn q-difference operator can be extended too.

Definition 3.2.9. *Considering the interval-valued function $F(x) = (\widehat{f}(x); \widetilde{f}(x))$, the expression*

$$D_{gH,q,\omega}F(x) = \frac{F(qx+\omega) \ominus_{gH} F(x)}{(q-1)x+\omega} \tag{3.44}$$

where $x \neq \omega_0$, $\omega_0 = \dfrac{\omega}{1-q}$, $q \in]0,1[$ and $\omega \geqslant 0$, is called Hahn (gH, q)-operator.

Note that, using midpoint notation, we always have:

$$\frac{F(qx+\omega) \ominus_{gH} F(x)}{(q-1)x+\omega} = \left(\frac{\widehat{f}(qx+\omega) - \widehat{f}(x)}{(q-1)x+\omega}; \left| \frac{\widetilde{f}(qx+\omega) - \widetilde{f}(x)}{(q-1)x+\omega} \right| \right).$$

Remark 3.2.2. *In case Minkowski-type subtraction $\ominus_M$ is used, we may consider*

$$d_{M,q}F(x) = F(qx) \ominus_M F(x) = \left(\widehat{f}(qx) - \widehat{f}(x); \widetilde{f}(qx) + \widetilde{f}(x) \right),$$

but it seems that this form of a difference is not useful as a q-differential, similarly to the fact that the M-difference $F(x+h) \ominus_M F(x)$ is not adequate as a h-differential where, according to [41], by h-differential of an arbitrary function $f(x)$ we mean (3.39) *and therefore, by h-derivative we mean* (3.40)*.*

Consider now the two interval-valued functions $F = (\widehat{f}; \widetilde{f})$ and $G = (\widehat{g}; \widetilde{g})$ on the same domain $\mathbb{X} = [x_0, x_1]$ and let $x \in \mathbb{X}$.

For $a, b \in \mathbb{R}$ the gH-linear combination of F and G is

$$(a, b) \odot_{gH} (F, G)^T = \left(a\widehat{f} + b\widehat{g}; \left| a\widetilde{f} + b\widetilde{g} \right| \right).$$

More generally, for any $\mathbf{a} = (a_1, a_2, ...a_n) \in \mathbb{R}^n$ and any vectors

$$\mathbf{F} = (F_1, ...F_n) \text{ of functions } F_i : \mathbb{R} \longrightarrow \mathcal{K}_{\mathcal{C}}(\mathbb{R}),$$

we define the gH-linear combination of $F_1, ...F_n$ to be

$$\mathbf{a} \odot_{gH} \mathbf{F}(x)^T = \left(\sum_{i=1}^{n} a_i \widehat{f}_i(x); \left| \sum_{i=1}^{n} a_i \widetilde{f}_i(x) \right| \right).$$

As a consequence we have that, applying one of the basic properties of absolute value, i.e., $||A| - |B|| \leqslant |A - B|$, to the quantities

$$A = a\widetilde{f}(qx) + b\widetilde{g}(qx) \text{ and } B = a\widetilde{f}(x) + b\widetilde{g}(x),$$

we obtain

$$d_{gH,q}((a, b) \odot_{gH} (F, G)^T) \subseteq a d_{gH,q}(F) \oplus_{gH} b d_{gH,q}(G) \tag{3.45}$$

which means that $d_{gH,q}$ is (gH, q)-subadditive with the additional property that the midpoint values of the left and the right intervals in (3.45) are the same.

We also have that, for all $k \in \mathbb{R}$,

$$d_{gH,q}(kF(x)) = k \cdot d_{gH,q}F(x)$$

and so $d_{gH,q}$ is (gH, q)-sublinear.

Indeed, we have

$$kF = \left(k\widehat{f}; |k|\widetilde{f}\right)$$

and then, according to (3.42), it is

$$d_{gH,q}(kF(x)) = kF(qx) \ominus_{gH} kF(x) = \left(k\widehat{f}(qx) - k\widehat{f}(x); \left||k|\widetilde{f}(qt) - |k|\widetilde{f}(x)\right|\right)$$

$$= k\left(\widehat{f}(qx) - \widehat{f}(x); \left|\widetilde{f}(qx) - \widetilde{f}(x)\right|\right) = k \cdot d_{gH,q}F(x).$$

Interval-valued q-integration

Considering quantum integration, we first recall the notion of q-antiderivative of a function (see [41]).

Definition 3.2.10. *Considering an arbitrary function $f(x)$, its q-antiderivative is a function, denoted by*

$$a_f(x) = \int f(x) d_q x$$

so that

$$D_q a_f(x) = f(x).$$

In order to construct this in an operational way some concepts concerning linear operators are necessary.

Indeed, we consider the linear operators M_q and n_x on the space of polynomials whose actions are respectively to bring q inside the polynomial $f(x)$ and to insert a variable x outside it, as shown below:

$$M_q[f(x)] = f(qx)$$

and

$$n_x[f(x)] = xf(x).$$

Therefore, for any polynomial $f(x)$ we have

$$M_q n_x([f(x)]) = M_q[xf(x)] = qxf(qx) = qn_x[f(qx)] = qn_x M_q[f(x)],$$

so, we obtain

$$M_q n_x = qn_x M_q. \tag{3.46}$$

Hence, as the q-antiderivative $a_f(x)$ of a given function $f(x)$ is such that

$$f(x) = D_q a_f(x) = \frac{a_f(qx) - a_f(x)}{(q-1)x},$$

this can be expressed in terms of the M_q operator as follows:

$$f(x) = \frac{a_f(qx) - a_f(x)}{(q-1)x} = \frac{M_q[a_f(x)] - a_f(x)}{(q-1)x} = \frac{1}{(q-1)x}\,(M_q - 1)\,a_f(x),$$

i.e., using the geometric series expansion,

$$a_f(x) = \frac{1}{M_q - 1}(q-1)xf(x) = (1-q)\frac{1}{1 - M_q}xf(x) = (1-q)\sum_{j=0}^{\infty} M_q^j\,(xf(x))$$

$$= (1-q)\sum_{j=0}^{\infty} q^j xf(q^j x) = (1-q)x\sum_{j=0}^{\infty} q^j f(q^j x).$$

By Definition 3.2.10, we obtain the following series which is called *Jackson* integral of $f(x)$:

$$\int f(x)d_q x = (1-q)x\sum_{j=0}^{\infty} q^j f(q^j x). \tag{3.47}$$

Supposing now $0 < a < b$, the definite q-integral is defined as

$$\int_0^b f(x)d_q x = (1-q)b\sum_{j=0}^{\infty} q^j f(q^j b)$$

and

$$\int_a^b f(x)d_q x = \int_0^b f(x)d_q x - \int_0^a f(x)d_q x.$$

At this point we are able to construct the interval-valued form of Jackson integral, using the midpoint notation of an interval-valued function

$$F(x) = \left(\widehat{f}(x); \widetilde{f}(x)\right)$$

(eventually $\widetilde{f}(x) = |\widetilde{\omega_f}(x)|$ with $\widetilde{\omega_f} : \mathbb{R} \to \mathbb{R}$) and the notion of q-differential defined by (3.42).

Considering $0 < a < b$, the definite interval-valued q-integral of F is defined by

$$\int_0^b F(x) d_q x = \left(b(1-q) \sum_{k=0}^{\infty} q^k \widehat{f}(bq^k); b(1-q) \sum_{k=0}^{\infty} q^k \widetilde{f}(bq^k) \right), \qquad (3.48)$$

i.e.,

$$\int_0^b F(x) d_q t = \left(\int_0^b \widehat{f}(x) d_q x; \int_0^b \widetilde{f}(x) d_q x \right) \qquad (3.49)$$

where the q-integrals on the right side are the standard ones for $\widehat{f}$ and $\widetilde{f}$.

Likewise, we have

$$\int_a^b F(x) d_q x = \int_0^b F(x) d_q x \ominus_{gH} \int_0^a F(x) d_q x$$

$$= \left(\int_0^b \widehat{f}(x) d_q x - \int_0^a \widehat{f}(x) d_q x; \left| \int_0^b \widetilde{f}(x) d_q x - \int_0^a \widetilde{f}(x) d_q x \right| \right). \qquad (3.50)$$

Furthermore, considering the function

$$g : \mathbb{R} \to \mathbb{R}, \text{ where } d_q g(x) = D_q g(x) d_q x,$$

applying (3.36) and (3.47), we get the following more general formula:

$$\int F(x) d_q g(x) = \int F(x) D_q g(x) d_q x = (1-q)x \sum_{j=0}^{\infty} q^j F(q^j x) D_q g(q^j x)$$

$$= (1-q)x \sum_{j=0}^{\infty} q^j F(q^j x) \frac{g(q^{j+1}x) - g(q^j x)}{(q-1)q^j x} = (1-q)x \sum_{j=0}^{\infty} q^j F(q^j x) \frac{g(q^j x) - g(q^{j+1}x)}{(1-q)q^j x}$$

$$= \sum_{j=0}^{\infty} F(q^j x) \left(g(q^j x) - g(q^{j+1}x) \right). \qquad (3.51)$$

It is assumed that the series in (3.51) converges and that the term

$$F(q^j x) \left(g(q^j x) - g(q^{j+1}x) \right)$$

corresponds to the interval

$$\left(\widehat{f}(q^j x) \left(g(q^j x) - g(q^{j+1}x) \right); \widetilde{f}(q^j x) \left(g(q^j x) - g(q^{j+1}x) \right) \right).$$

Lastly, if $\widehat{a}_f(x)$ and $\widetilde{a}_f(x)$ are antiderivatives of $\widehat{f}(x)$ and $\widetilde{f}(x)$ respectively, i.e.,

$$\widehat{a}_f(x) = \int \widehat{f}(x) d_q x = (1-q)x \sum_{j=0}^{\infty} q^j \widehat{f}(q^j x)$$

and

$$\widetilde{a}_f(x) = \int \widetilde{f}(x) d_q x = (1-q)x \sum_{j=0}^{\infty} q^j \widetilde{f}(q^j x) \geqslant 0,$$

and the two series converge for all $x \in]0,t]$, then, under the additional condition that $\widehat{f}$ and $\widetilde{f}$ are bounded on $]0,t]$, we have that

- $a_F(x) = (\widehat{a}_f(x); \widetilde{a}_f(x))$ is an interval-valued q-antiderivative of $F(x) = \left(\widehat{f}(x); \widetilde{f}(x)\right)$;
- $a_F(x)$ is continuous at $x = 0$ with $a_F(0) = 0$ (as an interval).

These results can be expressed through the following theorem whose proof is similar to that of the analogous theorem for real numbers (see [41], Chapter 19).

Theorem 3.2.1. *Let consider an interval-valued function* $F(x) = \left(\widehat{f}(x); \widetilde{f}(x)\right)$. *If* $\widehat{f}$ *and* $\widetilde{f}$ *are bounded on the interval* $]0,t]$, *then the Jackson interval-valued integral of* F *converges to a function* $a_F(x) = (\widehat{a}_f(x); \widetilde{a}_f(x))$ *which is a* q*-antiderivative of* $F(x)$.
Moreover, $a_F(x)$ *is continuous at* $x = 0$ *with* $a_F(0) = 0$.

Fundamental theorem of q-calculus for interval functions

Remember that, just as in the ordinary calculus where the concepts of derivative and definite integral are related through the Newton-Leibnitz formula, also in the q-calculus there is a similar relationship between the q-derivative and the definite q-integral, as expressed by fundamental theorem of q-calculus (see [41]).

Theorem 3.2.2 (Fundamental theorem of q-calculus)**.** *([41]) Considering a function* $f(x)$ *and its* q*-antiderivative* $a_f(x)$ *which is continuous at* $x = 0$*, then, if* $0 \leqslant x_0 < x_1 \leqslant \infty$*, we have that*

$$\int_{x_0}^{x_1} f(x) d_q x = a_f(x_1) - a_f(x_0). \tag{3.52}$$

In a very similar way it is possible to determine an analogous result for the interval case; therefore, the fundamental theorem of q-calculus for interval-valued functions can be stated as follows.

Theorem 3.2.3 (Fundamental theorem of interval-valued q-calculus)**.** *If $a_F(x) = (\widehat{a}_f(x); \widetilde{a}_f(x))$ is the interval-valued q-antiderivative of the interval-valued function $F(x) = \left(\widehat{f}(x); \widetilde{f}(x)\right)$ and $a_F(x)$ is continuous at $x = 0$ (i.e., both $\widehat{a}_f$ and $\widetilde{a}_f$ are continuoue at $x = 0$), then we have*

$$\int_{x_0}^{x_1} F(x) d_q x = a_F(x_1) \ominus_{gH} a_F(x_0) \tag{3.53}$$

with $0 \leqslant x_0 < x_1 \leqslant \infty$.

The proof follows from the definition of $\int_{x_0}^{x_1} f(x) d_q x$ and from Theorem 3.2.2 applied to $\widehat{f}$ and $\widetilde{f}$.

Remark 3.2.3. *Note that, in terms of Minkowski operations,* (3.53) *means one of the two following equalities*

$$a_F(x_1) = a_F(x_0) \oplus_M \int_{x_0}^{x_1} F(x) d_q x \quad or \quad a_F(x_0) = a_F(x_1) \ominus_M \int_{x_0}^{x_1} F(x) d_q x.$$

The complex-valued q-difference and q-derivation

As already seen in Subsection 3.2.1, each complex valued function $F(x)$ can be identified by:

- the real-part interval $F_{re}(x) = (\widehat{f}_{re}(x); \widetilde{f}_{re}(x))$;
- the imaginary-part interval $F_{im}(x) = (\widehat{f}_{im}(x); \widetilde{f}_{im}(x))$;

therefore, based on what has been introduced so far, the complex-valued q-difference and q-derivation can be defined as follows.

Definition 3.2.11. *The (gH, q)-difference on the real and imaginary parts of complex interval-valued function $F(x) \in \mathcal{K}_{\mathcal{C}}(\mathbb{C})$, are respectively:*

$$d_{gH,q} F_{re}(x) = \left(\widehat{f}_{re}(qx) - \widehat{f}_{re}(x); \left|\widetilde{f}_{re}(qx) - \widetilde{f}_{re}(x)\right|\right) \tag{3.54}$$

and

$$d_{gH,q} F_{im}(x) = \left(\widehat{f}_{im}(qx) - \widehat{f}_{im}(x); \left|\widetilde{f}_{im}(qx) - \widetilde{f}_{im}(x)\right|\right). \tag{3.55}$$

Definition 3.2.12. *The (gH, q)-derivative of a complex interval-valued function $F(x) \in \mathcal{K}_{\mathcal{C}}(\mathbb{C})$ is the complex interval identified by*

$$D_{gH,q} F(x) = \frac{d_{gH,q} F_{re}(x)}{d_q x} + i \frac{d_{gH,q} F_{im}(x)}{d_q x}. \tag{3.56}$$

Definition 3.2.12 also means that the (gH,q)-derivative of a complex interval-valued function $F(x) \in \mathcal{K}_{\mathcal{C}}(\mathbb{C})$ is the complex interval identified by the two intervals:

- $D_{gH,q}F_{re}(x) = \left(\frac{d_{gH,q}\widehat{f}_{re}(x)}{d_q x}; \left| \frac{d_{gH,q}\widetilde{f}_{re}(x)}{d_q x} \right| \right)$ for the real part;

- $D_{gH,q}F_{im}(x) = \left(\frac{d_{gH,q}\widehat{f}_{im}(x)}{d_q x}; \left| \frac{d_{gH,q}\widetilde{f}_{im}(x)}{d_q x} \right| \right)$ for the imaginary part,

also expressible as the box:

$$D_{gH,q}F(x) = D_{gH,q}F_{re}(x) + iD_{gH,q}F_{im}(x).$$

Part II

New perspectives in interval analysis

Chapter 4

An advanced algebraic setting for intervals

As well described by Svetoslav Markov in [58], on many occasions it has been useful and natural to introduce arithmetic operations and relations for intervals in the same way as we introduce operations and relations for real numbers. Indeed, in IA there is a natural tendency to follow the development of real arithmetic: from the study of algebraic properties to the classification in algebraic systems, up to axiomatization, etc. On the one hand the intervals, through addition, subtraction, multiplication and division, are treated as real numbers, on the other hand the n-dimensional intervals are added and multiplied by scalars as if they were real vectors. However, the intervals produce neither rings nor vector spaces; therefore, over the years many authors have tried their hand at studying the algebraic properties of intervals, in search of algebraic systems within which to configure them (see, e.g., [42], [88], [92]) and, more recently, Markov himself has greatly contributed with an interesting attempt to develop an axiomatic algebraic system alternative to the one based on Moore's principle of extension (see [58]).

Nevertheless, even today the algebraic structures of intervals have not been completely axiomatized. Therefore, what we intend to do is help fill this gap by introducing some innovative approaches towards the determination of interval algebraic systems.

A first step in this direction is represented by the attempt to broaden the concept of order analyzed in the first part of the work, introducing a new one with the aim of obtaining a sort of polarity between the two types of orders considered which, moreover, will also allow us to determine a very important completion of the lattice $\mathcal{K}_{\mathcal{C}}$.

4.1 Polar orders

In this section we will try to broaden the concept of γ-order $\precapprox_{\gamma^-,\gamma^+}$, introducing a new order, the γ-inclusion $\subseteqq_{\gamma^-,\gamma^+}$, capable of giving a set interpretation of ordering with the aim of obtaining a sort of polarity between the two types of orders.

Before doing this, however, some additional considerations should be made on the concept of interval of intervals (introduced in (2.29)) and on how this makes it necessary to define a polar order with respect to γ-order.

4.1.1 Bounded subsets in $\mathcal{K}_C$

Now let $A, B \in \mathcal{K}_{\mathcal{C}}$, as already mentioned in Subsection 2.2.7, according to (2.28), the "segment" joining A and B is the set of intervals

$$\mathcal{S}(A, B) = \{(1-t)A + tB \mid t \in [0, 1]\}$$

with $\mathcal{S}(A, B) = \mathcal{S}(B, A)$.

Note that this concept can also be used to define the convexity of a subset of $\mathcal{K}_{\mathcal{C}}$ like the one delimited by the three intervals A, B and C shown in Figure 4.1.

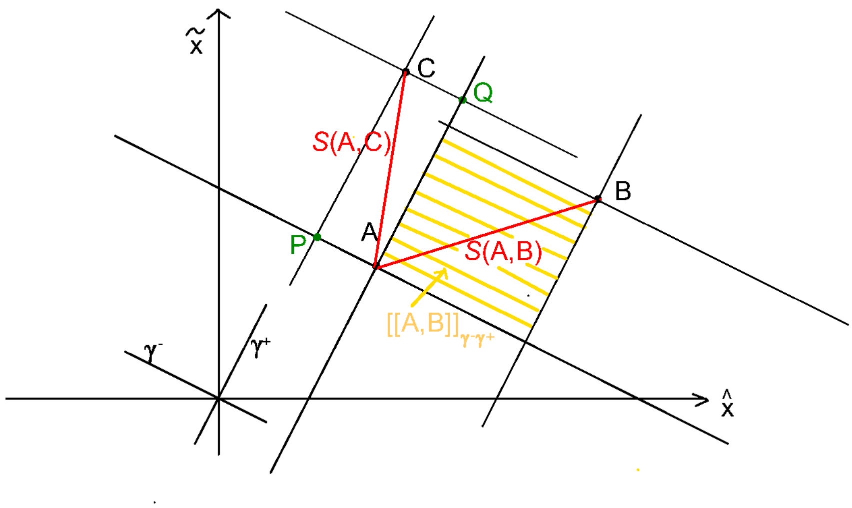


Figure 4.1: Segments that define a convex subset of $\mathcal{K}_{\mathcal{C}}$ joining comparable and incomparable intervals with respect to $\precapprox_{\gamma^-,\gamma^+}$-order.

In the same picture is also possible to see how, in case two intervals are $(\precapprox_{\gamma^-,\gamma^+}) - \textit{comparable}$, such as A and B are, we have further relevant notions.

Definition 4.1.1. *If $\mathcal{S}$ is a set of intervals in $\mathcal{K_C}$ and $A, B \in \mathcal{K_C}$ are fixed with $A \precapprox_{\gamma^-,\gamma^+} B$, we say that set $\mathcal{S}$ lies between A and B as follows:*

$$A \precapprox_{\gamma^-,\gamma^+} \mathcal{S} \precapprox_{\gamma^-,\gamma^+} B \Leftrightarrow A \precapprox_{\gamma^-,\gamma^+} S \precapprox_{\gamma^-,\gamma^+} B, \forall S \in \mathcal{S}.$$

The following property is obvious.

Proposition 4.1.1. *Let $A, B \in \mathcal{K_C}$. We have that*

$$A \precapprox_{\gamma^-,\gamma^+} B \Leftrightarrow A \precapprox_{\gamma^-,\gamma^+} \mathcal{S}(A,B) \precapprox_{\gamma^-,\gamma^+} B$$

and, assuming $A \precapprox_{\gamma^-,\gamma^+} B$, we define (with respect to the partial order $\precapprox_{\gamma^-,\gamma^+}$):

- $A = \inf_{\precapprox_{\gamma^-,\gamma^+}} \mathcal{S}(A,B)$, *also denoted as* $\min_{\precapprox_{\gamma^-,\gamma^+}} \mathcal{S}(A,B)$;
- $B = \sup_{\precapprox_{\gamma^-,\gamma^+}} \mathcal{S}(A,B)$, *also denoted as* $\max_{\precapprox_{\gamma^-,\gamma^+}} \mathcal{S}(A,B)$.

On the other hand, if two intervals are $(\precapprox_{\gamma^-,\gamma^+}) - incomparable$, such as A and C in Figure 4.1, then (and only then):

1) A and $\mathcal{S}(A,C) = \{(1-t)A + tC\}$ are incomparable $\forall t \in]0,1]$;
2) C and $\mathcal{S}(A,C) = \{(1-t)A + tC\}$ are incomparable $\forall t \in [0,1[$.

In addiction, in case of incomparability, it is possible to evidence the presence of other two important points which represent the extrema in reference to segment $\mathcal{S}(A,C)$ with respect to γ-order:

$$P = \inf_{\precapprox_{\gamma^-,\gamma^+}} \mathcal{S}(A,C) \text{ and } Q = \sup_{\precapprox_{\gamma^-,\gamma^+}} \mathcal{S}(A,C).$$

Taking into consideration (2.29), we can rephrase it with the following definition.

Definition 4.1.2. *In $(\mathcal{K_C}, \precapprox_{\gamma^-,\gamma^+})$ we define an interval (of intervals) with extreme $A, B \in \mathcal{K_C}$, assuming that $A \precapprox_{\gamma^-,\gamma^+} B$, to be the set of all intervals $X \in \mathcal{K_C}$ such that $A \precapprox_{\gamma^-,\gamma^+} X \precapprox_{\gamma^-,\gamma^+} B$, i.e.,*

$$[[A,B]]_{\precapprox_{\gamma^-,\gamma^+}} = \left\{X \in \mathcal{K_C} \mid A \precapprox_{\gamma^-,\gamma^+} X \precapprox_{\gamma^-,\gamma^+} B\right\}.$$

We simply denote it by $[[A,B]]_{\gamma^-,\gamma^+}$ in case there are no other types of orders besides $\precapprox_{\gamma^-,\gamma^+}$.

If A and B are $(\precapprox_{\gamma^-,\gamma^+}) - incomparable$, we may define

$$[[A,B]]_{\precapprox_{\gamma^-,\gamma^+}} = [[inf_{\precapprox_{\gamma^-,\gamma^+}} \mathcal{S}(A,B), sup_{\precapprox_{\gamma^-,\gamma^+}} \mathcal{S}(A,B)]]_{\precapprox_{\gamma^-,\gamma^+}}. \tag{4.1}$$

Referring again to Figure 4.1 (as well as to Figure 2.14 in Subsection 2.2.7) and assuming $A \precapprox_{\gamma^-,\gamma^+} B$, it is trivial to verify that:

$$inf_{\precapprox_{\gamma^-,\gamma^+}}[[A,B]]_{\precapprox_{\gamma^-,\gamma^+}} = A,$$

$$sup_{\precapprox_{\gamma^-,\gamma^+}}[[A,B]]_{\precapprox_{\gamma^-,\gamma^+}} = B.$$

By defining (4.1), we always have

$$\mathcal{S}(A,B) \subseteq [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}.$$

Furthermore, as it has already been introduced in Subsection 2.2.7, if $\mathcal{A} \subset \mathcal{K_C}$ is bounded (see the representation in Figure 4.2), we always have:

1) $\inf_{\precapprox_{\gamma^-,\gamma^+}} \mathcal{A} \precapprox_{\gamma^-,\gamma^+} \mathcal{A} \precapprox_{\gamma^-,\gamma^+} \sup_{\precapprox_{\gamma^-,\gamma^+}} \mathcal{A}$;

2) $\mathcal{A} \subseteq [[\inf_{\precapprox_{\gamma^-,\gamma^+}} \mathcal{A}, \sup_{\precapprox_{\gamma^-,\gamma^+}} \mathcal{A}]]_{\precapprox_{\gamma^-,\gamma^+}} \subseteq \mathcal{K_C}$.

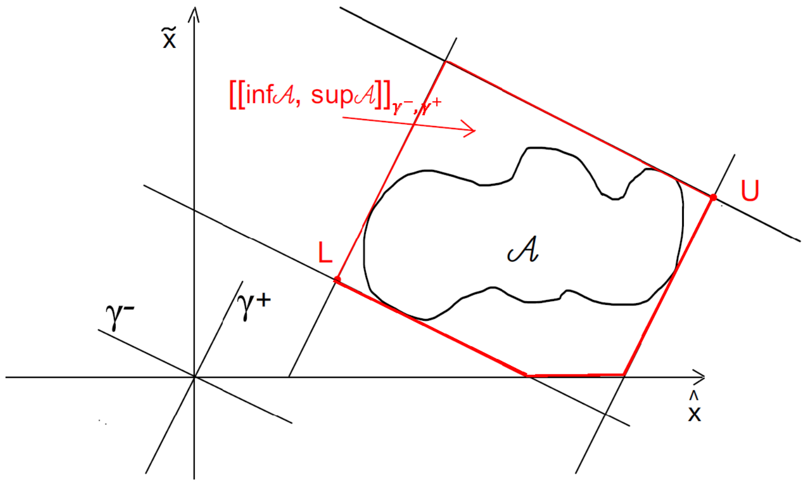


Figure 4.2: The bounded subset $\mathcal{A} \subset \mathcal{K_C}$ with $L = \inf_{\gamma^-,\gamma^+} \mathcal{A}$ and $U = \sup_{\gamma^-,\gamma^+} \mathcal{A}$.

Eventually, we can even consider a kind of lexicographic order in $\mathcal{K_C}$.

Definition 4.1.3. *Let $A, B \in \mathcal{K_C}$. For $A = (\widehat{a};\widetilde{a})$, $B = (\widehat{b};\widetilde{b})$ we define:*

1 $A \precapprox^1_{lex} B$ *iff* $\widehat{a} < \widehat{b}$ *or* $\begin{cases} \widehat{a} = \widehat{b} \\ \widetilde{a} < \widetilde{b}, \end{cases}$

2 $A \precapprox^2_{lex} B$ *iff* $\widehat{a} < \widehat{b}$ *or* $\begin{cases} \widehat{a} = \widehat{b} \\ \widetilde{a} > \widetilde{b}. \end{cases}$

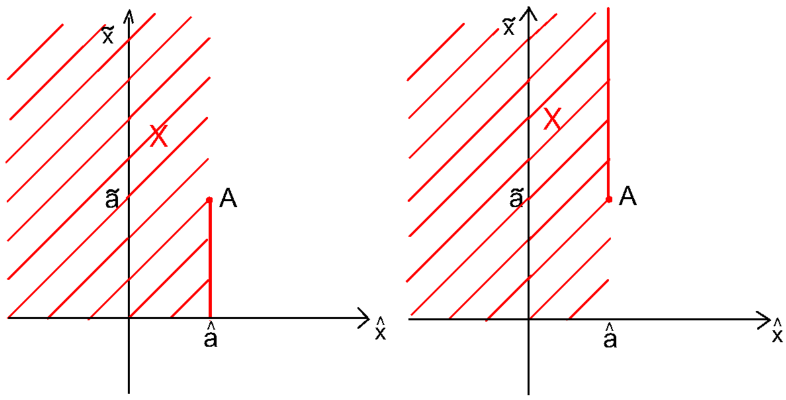


Figure 4.3: The lexicographic order in $\mathcal{K}_\mathcal{C}$: the set of intervals X such that $X \precsim^1_{lex} A$ (left) and such that $X \precsim^2_{lex} A$ (right).

Definition 4.1.3 is well represented in the Figure 4.3.

Moreover, we have that the diameter (or length) of an interval of intervals $[[A,B]]_{\precsim_{\gamma^-,\gamma^+}}$ in $(\mathcal{K}_\mathcal{C}, \precsim_{\gamma^-,\gamma^+})$ can be defined in several ways, depending on its usefulness; a simple definition, analogous to real intervals (see Subsection 2.1.2), not depending on the chosen order, is the following:

$$len([[A,B]]_{\precsim_{\gamma^-,\gamma^+}}) = d_H(A,B) = \|A \ominus_{gH} B\|$$

where, as usual, d_H stands for the Pompeiu–Hausdorff distance

$$d_H : \mathcal{K}_C \times \mathcal{K}_C \to \mathbb{R}^+ \cup \{0\}.$$

Now let $(\mathcal{K}_\mathcal{C}, \precsim_{\gamma^-,\gamma^+})$ be fixed and, for $A, B \in \mathcal{K}_\mathcal{C}$ with $A \precsim_{\gamma^-,\gamma^+} B$, consider the interval $[[A,B]]_{\precsim_{\gamma^-,\gamma^+}}$ having length (diameter) $d_H(A,B)$. As shown in Figure 4.4, a sub-interval $[[A',B']]_{\precsim_{\gamma^-,\gamma^+}}$ of $[[A,B]]_{\precsim_{\gamma^-,\gamma^+}}$ can be easily defined by taking $A', B' \in [[A,B]]_{\precsim_{\gamma^-,\gamma^+}}$ so that

$$[[A',B']]_{\precsim_{\gamma^-,\gamma^+}} \subseteq [[A,B]]_{\precsim_{\gamma^-,\gamma^+}}.$$

Note that $[[A',B']]_{\precsim_{\gamma^-,\gamma^+}}$ requires (4.1) if A' and B' are $(\precsim_{\gamma^-,\gamma^+})$-incomparable.

Definition 4.1.4. *A sequence of intervals in* $(\mathcal{K}_\mathcal{C}, \precsim_{\gamma^-,\gamma^+})$,

$$I_n = [[A_n, B_n]]_{\gamma^-,\gamma^+} \ \textit{with} \ A_n \precsim_{\gamma^-,\gamma^+} B_n,\ n \geqslant 1 \tag{4.2}$$

is said to be decreasing if $I_{n+1} \subseteq I_n, \forall n \geq 1$.
This requires that, equivalently,

$$A_1 \precsim_{\gamma^-,\gamma^+} A_2 \precsim_{\gamma^-,\gamma^+} \dots \precsim_{\gamma^-,\gamma^+} A_n \precsim_{\gamma^-,\gamma^+} B_n \precsim_{\gamma^-,\gamma^+} \dots \precsim_{\gamma^-,\gamma^+} B_1. \tag{4.3}$$

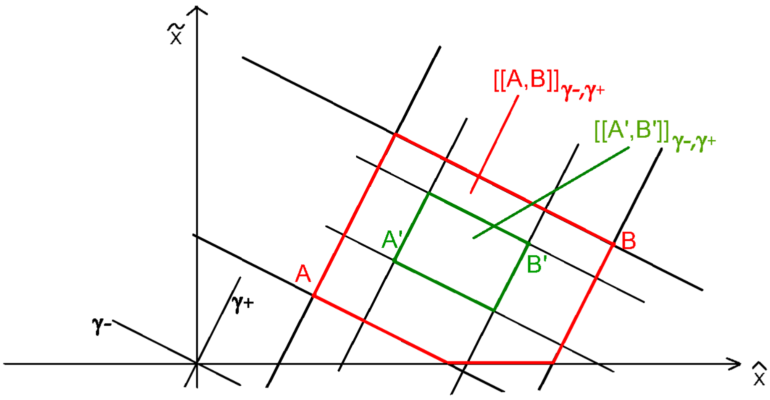


Figure 4.4: Sub-interval $[[A', B']]_{\precsim_{\gamma^-,\gamma^+}}$ of $[[A, B]]_{\precsim_{\gamma^-,\gamma^+}}$.

4.1.2 Lattices as algebraic structures

In Subsection 2.2.1 we have recalled the basic notions concerning lattices in classical theory but, before proceeding, it is advisable to remember their dual nature, logical and algebraic, as it will be of fundamental importance in order to delineate an algebraic interval theory.

Indeed, it is well known that a lattice is a non-empty ordered set $(L, \leq)$ such that, for each $x, y \in L$, the extreme lower $inf\{x, y\}$ and upper $sup\{x, y\}$ exist in L, but at the same time it is possible, in an equivalent way, to regard lattices as algebraic structures of the type $(L, \vee, \wedge)$. In fact, if $(L, \leq)$ is a lattice, we can define two lattice operations $\vee$ and $\wedge$, for each $x, y \in L$, in the following way:

$$x \vee y = sup_{\leq}\{x, y\},$$

$$x \wedge y = inf_{\leq}\{x, y\},$$

such that, as will be better explained later, the following algebraic properties hold.

(1) $\vee$ and $\wedge$ are commutative:

(i) $x \vee y = y \vee x$, $\forall x, y \in L$;

(ii) $x \wedge y = y \wedge x$, $\forall x, y \in L$.

(2) $\vee$ and $\wedge$ are associative:

(i) $x \vee (y \vee z) = (x \vee y) \vee z, \forall x, y, z \in L$;

(ii) $x \wedge (y \wedge z) = (x \wedge y) \wedge z, \forall x, y, z \in L$.

(3) The absorption laws apply:

  (i) $x \vee (x \wedge y) = x$, $\forall x, y \in L$;
  (ii) $x \wedge (x \vee y) = x$, $\forall x, y \in L$.

(4) The laws of idempotence hold (that is, every element of L is idempotent with respect to both $\vee$ and $\wedge$):

  (i) $x \vee x = x$, $\forall x \in L$;
  (ii) $x \wedge x = x$, $\forall x \in L$.

  Note that property (4) is a direct consequence of the other three.

If, conversely, $(L, \vee, \wedge)$ is an algebraic structure in which $\vee$ and $\wedge$ are two binary operations that verify (1), (2) and (3), then we can define in L a binary relation $\preceq$ so that, for each $x, y \in L$,

$$x \preceq y \Leftrightarrow x = x \wedge y$$

and we verify that $\preceq$ is an order relation that makes $(L, \preceq)$ a lattice.

Furthermore, for each $x, y \in L$ we have

$$x \vee y = sup_{\preceq}\{x, y\} \text{ and } x \wedge y = inf_{\preceq}\{x, y\}.$$

Thus, $\vee$ and $\wedge$ turn out to be the lattice operations in $(L, \preceq)$.

At the same time, if $\vee$ and $\wedge$ are the lattice operations defined in a lattice $(L, \leq)$, it is clear that the relation $\preceq$ defined above coincides with $\leq$.

In summary, given a non-empty set L, we can say that if $\mathcal{R}$ is the set of relations of order $\leq$ such that $(L, \leq)$ is a lattice and $\mathcal{O}$ is the set of pairs $(\vee, \wedge)$ of binary operations in L that verify conditions (1), (2) and (3), then we can define two different applications between $\mathcal{R}$ and $\mathcal{O}$:

1 $\alpha : \mathcal{R} \longrightarrow \mathcal{O}$, which associates the ordered pair $(\vee, \wedge) \in \mathcal{O}$ to an order relation $\leq \in \mathcal{R}$, where $\vee$ and $\wedge$ are the lattice operations of upper bound and lower bound in $(L, \leq)$;

2 $\beta : \mathcal{O} \longrightarrow \mathcal{R}$, which associates the order relation $\leq \in \mathcal{R}$, defined like above, to each $(\vee, \wedge) \in \mathcal{O}$.

This means that α and β are inverse of each other, so they are bijective.

The most obvious consequence of the existence of such bijections causes the study of lattices (considered as ordered sets) and that of the algebraic structures $(L, \vee, \wedge)$ for which conditions (1), (2) and (3) hold, is fully equivalent.

For this reason we refer to these structures by calling them "algebraic lattices". Therefore, from now on, to indicate a lattice we will indifferently refer to the ordered set structure, indicating the lattice as

$$(L \leq)$$

or to the algebraic structure, indicating the lattice as

$$(L, \vee, \wedge)$$

where the first operation represented is that of the upper bound and the second that of lower bound. It can also be convenient, and we will often do so, to designate a lattice as

$$(L, \vee, \wedge, \leq)$$

to specify in synthetic way both the order relation and the lattice operations.

Therefore, the following property holds.

Proposition 4.1.2. *An algebraic structure $(L, \vee, \wedge)$, consisting of a set L and two binary, commutative and associative operations $\vee$ and $\wedge$ on L, called join (or vel) and meet (or et), respectively, is a lattice, named algebraic lattice, if the following axiomatic identities, known as absorption laws, hold for all elements $x, y \in L$:*

- $x \vee (x \wedge y) = x$;
- $x \wedge (x \vee y) = x$.

The following two identities, called idempotent laws, are also usually regarded as axioms, even though they follow from the two absorption laws taken together:

- $x \vee x = x$, $\forall x \in L$;
- $x \wedge x = x$, $\forall x \in L$.

Recall that also the two possible notions of isomorphism for lattices as ordered sets and lattices as algebraic structures coincide.

However, it should be noted that even the notion of sublattice is algebraic, in the sense that it can only be defined in terms of lattice operations.

Indeed, if $(L, \leq)$ is a lattice, a sublattice is, by definition, a non-empty subset K of L that is closed with respect to the lattice operations $\vee$ and $\wedge$ of L. The operations induced in K by $\vee$ and $\wedge$ continue to verify conditions (1), (2) and (3) and then make K a lattice with respect to the order relation induced by $\leq$ on K (this last observation is guaranteed by the fact that the order relation of the lattice is determined by lattice operations:

$$\forall x, y \in L, \text{ we have } x \leq y \Leftrightarrow x = x \wedge y).$$

The notions of minimum and maximum also have an algebraic interpretation.

Lemma 4.1.1. *Let $(L, \vee, \wedge, \leq)$ be a lattice. An element $m \in L$ is the minimum in L if and only if m is the neutral element with respect to $\vee$, i.e., m is the identity element for the join operation:*

$$m \vee x = x \vee m = x, \forall x \in L;$$

while m is the maximum in L if and only if m is the neutral element with respect to $\wedge$, i.e., m is the identity element for the meet operation:

$$m \wedge x = x \wedge m = x, \forall x \in L.$$

In particular, we have the following statements.

Definition 4.1.5. *A bounded lattice is an algebraic structure of the form $(L, \vee, \wedge, 0, 1, \leq)$ such that $(L, \vee, \wedge)$ is a lattice, 0 (the lattice's bottom) is the identity element for the join operation $\vee$ and 1 (the lattice's top) is the identity element for the meet operation $\wedge$:*

$$x \vee 0 = 0 \vee x = x \text{ and } x \wedge 1 = 1 \wedge x = x, \text{ for every } x \in L.$$

Definition 4.1.6. *Let L be a set equipped with two binary operations, denoted by $\vee$ and $\wedge$. We define an algebraic lattice the structure $(L, \vee, \wedge)$, such that*

1) $\vee$ and $\wedge$ are commutative:

- *(i) $x \vee y = y \vee x$, $\forall x, y \in L$,*
- *(ii) $x \wedge y = y \wedge X$, $\forall x, y \in L$;*

2) $\vee$ and $\wedge$ are associative:

- *(i) $x \vee (y \vee z) = (x \vee y) \vee z$, $\forall x, y, z \in L$,*
- *(ii) $x \wedge (y \wedge z) = (x \wedge y) \wedge z, \forall x, y, z \in L$;*

3) the absorption laws hold:

- *(i) $x \vee (x \wedge y) = x, \forall x, y \in L$,*
- *(ii) $x \wedge (x \vee y) = a, \forall x, y \in L$;*

4) the idempotent laws hold:

- *(i) $x \vee x = x$, $\forall x \in L$,*
- *(ii) $x \wedge x = x$, $\forall x \in L$.*

In addition we can conclude that, since meet and join both commute and associate, a lattice can be viewed as consisting of two commutative semigroups. Moreover, for a bounded lattice (i.e., with minimum and maximum), these semigroups are in fact commutative monoids. So, the absorption law is the only defining identity that is peculiar to lattice theory.

We complete this segment with the following further definition.

Definition 4.1.7. *A (meet-)semilattice is an algebraic structure $(S, \wedge)$ consisting of a set S with a binary operation $\wedge$, such that for all members x, y and z of S, the following identities hold:*

- $x \wedge (y \wedge z) = x \wedge (y \wedge z)$ *(associativity);*

- $x \wedge y = y \wedge x$ *(commutativity);*

- $x \wedge x = x$ *(idempotency).*

A (meet-)semilattice $(S, \wedge)$ *is bounded if it includes a neutral element* i_s *such that* $x \wedge i_s = i_s \wedge x = x$ *for all* x *in* S.

Remark 4.1.1. *If the symbol* $\vee$ *replaces* $\wedge$ *in the definition just given, the structure is called a (join)-semilattice. One can be ambivalent about the particular choice of symbol for the operation, and speak simply of semilattices.*

Note that in a bounded meet-semilattice, the neutral element is the greatest element of S. Similarly, in a join-semilattice is the least element.

According to Definition 4.1.7, we can also say that a semilattice is a commutative, idempotent semigroup, i.e., a commutative *band*, as well as a bounded semilattice, being equipped with a neutral element, is an idempotent commutative monoid.

Eventually, it is possible to introduce a partial order relation on a meet-semilattice by setting $x \leq y$ whenever $x \wedge y = x$, while, for a join-semilattice, the order is induced by setting $x \leq y$ whenever $x \vee y = y$.

Finally, we conclude this subsection with some notions of the theory of distributive lattices (see [6], [13] and [67] for more information).

Definition 4.1.8. *A De Morgan Algebra is a bounded distributive lattice* $(L, \leq)$ *with a unary operation* ${}^c : L \longrightarrow L$ *such that:*

(d1) $(x^c)^c = x$, $\forall x \in L$;

(d2) $x \leq y \;\Rightarrow\; y^c \leq x^c$, $\forall x, y \in L$.

A De Morgan algebra is called a Kleene algebra if it satisfies the Klene condition:

(k) $x \wedge x^c \leq y \vee y^c$, $\forall x, y \in L$.

We will use, for a De Morgan algebra L, the notations:

- $W_0(L) = \{o \in L : o \leq o^c\}$ (weak zeros);

- $W_1(L) = \{u \in L : u \geq u^c\}$ (weak units).

It is easy to see that:

$$W_0(L) = \{x \wedge x^c : x \in L\} \tag{4.4}$$

and

$$W_1(L) = \{x \vee x^c : x \in L\}. \tag{4.5}$$

Using this notation, the Kleene condition may be reformulated:

$$\forall o \in W_0(L), \forall u \in W_1(L) : o \leq u.$$

Therefore, $W_0(L) \cap W_1(L)$ contains at most one element.
We end with the following definition.

Definition 4.1.9. *A Kleene algebra L is called a Boolean algebra iff*

$$x \wedge x^c = y \wedge y^c \text{ for all } x, y \in L, \tag{4.6}$$

i.e., iff $W_0(L) = \{0\}$.

4.1.3 Incomparability with respect to $\precapprox_{\gamma^-,\gamma^+}$-order

Let consider $A = (\widehat{a}; \widetilde{a}) \in \mathcal{K}_\mathcal{C}$. According to Definition 2.2.13, we say that:

- interval A dominates interval X (with respect to $\precapprox_{\gamma^-,\gamma^+}$) if and only if $A \precapprox_{\gamma^-,\gamma^+} X$;
- interval A is dominated by interval X (with respect to $\precapprox_{\gamma^-,\gamma^+}$) if and only if $X \precapprox_{\gamma^-,\gamma^+} A$.

This can be represented in the midpoint plane as shown in Figure 4.5.

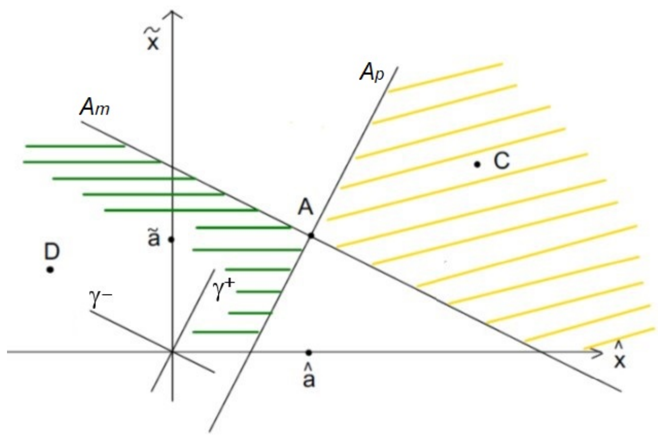


Figure 4.5: Interval A dominates interval C and is dominated by interval D with respect to order $\precapprox_{\gamma^-,\gamma^+}$.

In particular we define

$$A_m : \widetilde{x} = \widetilde{a} + \gamma^- (\widehat{x} - \widehat{a}) \text{ and } A_p : \widetilde{x} = \widetilde{a} + \gamma^+ (\widehat{x} - \widehat{a}) \tag{4.7}$$

the lines for A with angular coefficients respectively γ^-, γ^+ (see Figure 4.5).

Consequently, considering two intervals $A = (\widehat{a}; \widetilde{a})$ and $B = (\widehat{b}; \widetilde{b})$ in $\mathcal{K_C}$, with $A \neq B$, according to (4.7), we call A_m, A_p and B_m, B_p, the lines for A and B with angular coefficients respectively γ^-, γ^+. If the intersections (points) between A_m and B_p as well as between A_p and B_m exist in $\mathcal{K_C}$ (see Figure 4.6), we indicate with:

- A_mB_p (or B_pA_m) the point of intersection between A_m and B_p;
- A_pB_m (or B_mA_p) the point of intersection between A_p and B_m.

We also have the parallelism $A_m||B_m$ as well as $A_p||B_p$ (possibly $A_m \equiv B_m$ or $A_p \equiv B_p$ but not both coincidences verified at the same time).

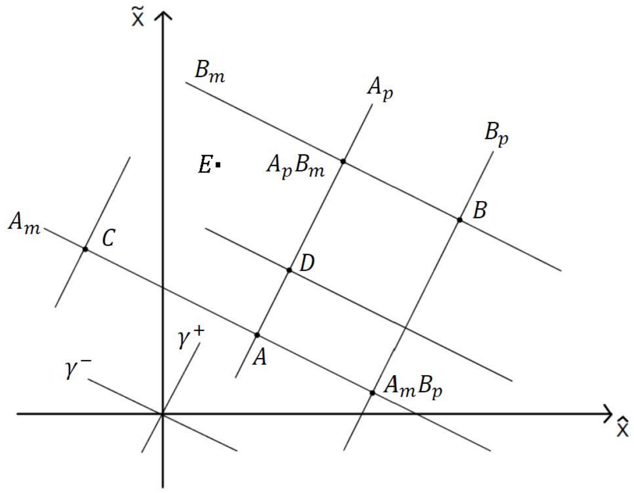


Figure 4.6: Examples of pairs of intervals aligned (A, C and A, D), unaligned (A, B and A, E), comparable ($A \precsim_{\gamma^-,\gamma^+} B$), incomparable ($A \,||_{\precsim_{\gamma^-,\gamma^+}}\, E$).

We can also introduce the following definitions.

Definition 4.1.10. *Two intervals $A, B \in \mathcal{K_C}$ are said to be incomparable with respect to $\precsim_{\gamma^-,\gamma^+}$ (or $\precsim_{\gamma^-,\gamma^+}$-incomparable) if and only if $A \neq B$ and neither $A \precsim_{\gamma^-,\gamma^+} B$ nor $B \precsim_{\gamma^-,\gamma^+} A$ are verified, which is denoted by*

$$A \,||_{\precsim_{\gamma^-,\gamma^+}}\, B.$$

Definition 4.1.11. *Let $A, B \in \mathcal{K_C}$ such that $A \neq B$ and $\gamma^-, \gamma^+ \in \mathbb{R}$:*

- *if $A_p \equiv B_p$ (or if $A_m \equiv B_m$), we say that the two points, i.e., elements of $\mathcal{K_C}$, are aligned;*
- *if $A_p \neq B_p$ and $A_m \neq B_m$, we say that the two points, i.e., elements of $\mathcal{K_C}$, are unaligned.*

From now on it will be useful to consider $\overline{\mathcal{K}_C} = \mathcal{K}_{\mathcal{C}} \cup \{-\infty, +\infty\}$, where:

$$-\infty = [-\infty, -\infty] = (-\infty; 0) \stackrel{def}{=} inf_{\precsim_{\gamma^-,\gamma^+}} \overline{\mathcal{K}_C}$$

and

$$+\infty = [+\infty, +\infty] = (+\infty; 0) \stackrel{def}{=} sup_{\precsim_{\gamma^-,\gamma^+}} \overline{\mathcal{K}_C}.$$

This means that, using endpoint notation for intervals, we have:
$A \in \overline{\mathcal{K}_C} \Leftrightarrow A = [a^-, a^+]$ with $a^-, a^+ \in \mathbb{R}$, $a^- \leqslant a^+$ or $A = [-\infty, -\infty]$ or $A = [+\infty, +\infty]$.
On the other hand, using midpoint notation for intervals, we have:
$A \in \overline{\mathcal{K}_C} \Leftrightarrow A = (\widehat{a}; \widetilde{a})$ with $\widehat{x} \in \mathbb{R}, \widetilde{a} \geqslant 0$ or $A = (-\infty; 0)$ or $A = (+\infty; 0)$.

We obtain that the poset $(\overline{\mathcal{K}_{\mathcal{C}}}, \precsim_{\gamma^-,\gamma^+})$ is a complete, bounded lattice, i.e., $\forall X, Y \in \overline{\mathcal{K}_{\mathcal{C}}}$, $\exists inf_{\precsim_{\gamma^-,\gamma^+}}\{X, Y\}, sup_{\precsim_{\gamma^-,\gamma^+}}\{X, Y\}$.
We will use the common notation:

$$X \vee_{\precsim_{\gamma^-,\gamma^+}} Y = sup_{\precsim_{\gamma^-,\gamma^+}}\{X, Y\}, \tag{4.8}$$

$$X \wedge_{\precsim_{\gamma^-,\gamma^+}} Y = inf_{\precsim_{\gamma^-,\gamma^+}}\{X, Y\}, \tag{4.9}$$

and we simplify the notation with $X \vee Y$ and $X \wedge Y$ when γ^-, γ^+ are fixed and no confusion is possible, calling, as usual, $\vee$ the *join* or *vel* and $\wedge$ the *meet* or *et*.
In addition, we know that in lattices these two operations are both binary, which means that they can be applied to a pair of elements $X, Y \in \overline{\mathcal{K}_{\mathcal{C}}}$ to yield again an element of $\overline{\mathcal{K}_{\mathcal{C}}}$.
So, we can define the following internal operations:

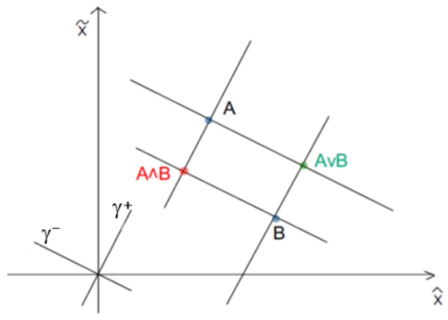
$$\vee : \overline{\mathcal{K}_{\mathcal{C}}} \times \overline{\mathcal{K}_{\mathcal{C}}} \to \overline{\mathcal{K}_{\mathcal{C}}} \text{ such that: } (X, Y) \to X \vee Y;$$

$$\wedge : \overline{\mathcal{K}_{\mathcal{C}}} \times \overline{\mathcal{K}_{\mathcal{C}}} \to \overline{\mathcal{K}_{\mathcal{C}}} \text{ such that: } (X, Y) \to X \wedge Y.$$

In particular the definition below examines all possible cases of pair of intervals in $\mathcal{K}_{\mathcal{C}}$.

Definition 4.1.12. *Let $A, B \in \mathcal{K}_{\mathcal{C}}$. We have the following cases.*

1. *If A and B are $\precsim_{\gamma^-,\gamma^+}$-comparable and $A \neq B$:*
   - *1.a if $A \precsim_{\gamma^-,\gamma^+} B$, then $A \vee B = B$ and $A \wedge B = A$;*
   - *1.b if $B \precsim_{\gamma^-,\gamma^+} A$, then $A \vee B = A$ and $A \wedge B = B$;*
2. *If A and B are $\precsim_{\gamma^-,\gamma^+}$-incomparable and $A \neq B$:*
   - *2.a if $A_pB_m \precsim_{\gamma^-,\gamma^+} A_mB_p$, then $A \vee B = A_mB_p$ and $A \wedge B = A_pB_m$;*
   - *2.b if $A_mB_p \precsim_{\gamma^-,\gamma^+} A_pB_m$, then $A \vee B = A_pB_m$ and $A \wedge B = A_mB_p$;*
3. *If $A = B$, then $A \vee B = A \wedge B = A = B$.*



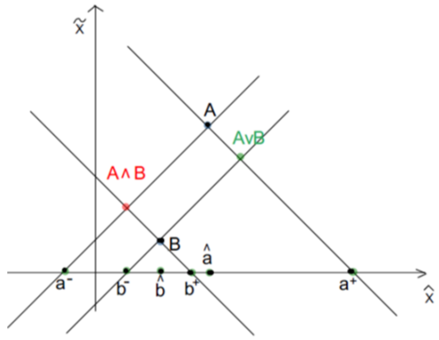


Figure 4.7: The two operations $\vee$ and $\wedge$ associated to a generic order $\precsim_{\gamma^-,\gamma^+}$ (left). The operations $\vee$ and $\wedge$ when the order associated is the LU-order $\precsim_{-1,+1}$ (right).

The two operations are shown in the left side of Figure 4.7, while, if we consider the LU-order, i.e., $(\gamma^-,\gamma^+) = (-1,+1)$, the situation is well described in the right side.

Consequently, as it is immediate to verify, we obtain that the lattices $(\overline{\mathcal{K}_\mathcal{C}},\vee,\wedge)$ and $(\overline{\mathcal{K}_\mathcal{C}},\wedge,\vee)$, considered as algebraic structures, have the following properties:

1) $\vee$ and $\wedge$ are commutative: $\forall X, Y \in \overline{\mathcal{K}_C}$

   1.a $X \vee Y = Y \vee X$,

   1.b $X \wedge Y = Y \wedge X$;

2) $\vee$ and $\wedge$ are associative: $\forall X, Y, Z \in \overline{\mathcal{K}_C}$

   2.a $(X \vee Y) \vee Z = X \vee (Y \vee Z)$,

   2.b $(X \wedge Y) \wedge Z = X \wedge (Y \wedge Z)$;

3) the absorption identities are satified: $\forall X, Y \in \overline{\mathcal{K}_C}$

   3.a $X \vee (X \wedge Y) = X$,

   3.b $X \wedge (X \vee Y) = X$;

4) the idempotency is satisfied for both $\vee$ and $\wedge$: $\forall X \in \overline{\mathcal{K}_C}$

   4.a $X \vee X = X$,

   4.b $X \wedge X = X$.

It is also easy to verify that $(\overline{\mathcal{K}_\mathcal{C}},\vee,\wedge)$ and $(\overline{\mathcal{K}_\mathcal{C}},\wedge,\vee)$ are distributive algebraic lattices:

5.a) $\vee$ is left and right distributive over $\wedge$: $\forall A, B, C \in \overline{\mathcal{K}_C}$

5.a.i $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$,

5.a.ii $(A \wedge B) \vee C = (A \vee C) \wedge (B \vee C)$;

5.b) $\wedge$ is left and right distributive over $\vee$: $\forall A, B, C \in \overline{\mathcal{K}_C}$

5.b.i $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$,

5.b.ii $(A \vee B) \wedge C = (A \wedge C) \vee (B \wedge C)$.

4.1.4 The $\subseteqq_{\gamma^-,\gamma^+}$-inclusion order

Let us now focus our attention on classical set inclusion order $\subseteq$ and make something similar to what we have just done for the partial order relation $\precapprox_{\gamma^-,\gamma^+}$, to define a corresponding partial inclusion order, denoted as

$$\subseteqq_{\gamma^-,\gamma^+}$$

with $\gamma^-, \gamma^+ \in \mathbb{R}$ such that $\gamma^- \leqslant \gamma^+$. Then we will analyze its properties with respect to intervals in detail and compare it with the partial order relation $\precapprox_{\gamma^-,\gamma^+}$.

Let consider the extended set

$$\mathcal{K}_C^{\varnothing\mathbb{R}} = \mathcal{K}_\mathcal{C} \cup \{\varnothing\} \cup \{\mathbb{R}\}$$

where, using midpoint notation, $\varnothing$ and $\mathbb{R}$ denote the two intervals $(0; -\infty)$, a notation for the empty set, and $(0; +\infty)$. We can define, with respect to the inclusion order,

$$\varnothing = (0; -\infty) \overset{def}{=} inf_{\subseteqq_{\gamma^-,\gamma^+}} \mathcal{K}_C^{\varnothing\mathbb{R}}$$

and

$$\mathbb{R} = (0; +\infty) \overset{def}{=} sup_{\subseteqq_{\gamma^-,\gamma^+}} \mathcal{K}_C^{\varnothing\mathbb{R}}.$$

Using midpoint notation for intervals, this means that:
$X \in \mathcal{K}_C^{\varnothing\mathbb{R}} \Leftrightarrow X = (\widehat{x}; \widetilde{x})$ with $\widehat{x} \in \mathbb{R}$, $\widetilde{x} \geqslant 0$ or $X = (0; -\infty)$ or $X = (0; +\infty)$.

According to (1.6) and (1.8), we also introduce the two operations:

- $A \uplus B \overset{def}{=} conv(A \cup B)$, which is the convex hull interval of $A \cup B$,

- $A \cap B$, which is the usual intersection between intervals A and B.

More generally, let γ^-, γ^+ be fixed such that $\gamma^- < 0 < \gamma^+$ and, according to (4.7), consider in the half-plane $(\widehat{x}; \widetilde{x})$ of intervals (in midpoint representation) the two lines A_m and A_p, with angular coefficients γ^- and γ^+ respectively, passing through the point $A = (\widehat{a}, \widetilde{a})$:

$$\widetilde{x} = \widetilde{a} + \gamma^- (\widehat{x} - \widehat{a}) \quad \text{and} \quad \widetilde{x} = \widetilde{a} + \gamma^+ (\widehat{x} - \widehat{a}),$$

which intersect the horizontal axis at points

$$A_{\gamma^-} = \left(\widehat{a} - \frac{\widetilde{a}}{\gamma^-}; 0\right) \quad \text{and} \quad A_{\gamma^+} = \left(\widehat{a} - \frac{\widetilde{a}}{\gamma^+}; 0\right).$$

We define the generalized ($\subseteqq_{\gamma^-,\gamma^+}$)-inclusion by saying that interval B includes interval A when the correspondent intersections with the horizontal axis do the same (see Figure 4.8):

$$\left[\widehat{a} - \frac{\widetilde{a}}{\gamma^+}, \widehat{a} - \frac{\widetilde{a}}{\gamma^-}\right] \subseteq \left[\widehat{b} - \frac{\widetilde{b}}{\gamma^+}, \widehat{b} - \frac{\widetilde{b}}{\gamma^-}\right]. \tag{4.10}$$

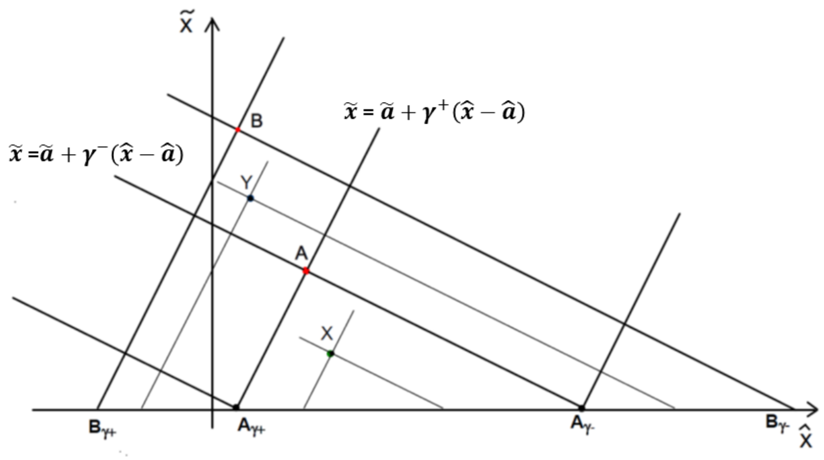


Figure 4.8: Generalized $\subseteqq_{\gamma^-,\gamma^+}$-inclusion: interval B includes interval A when the correspondent intersections with the horizontal axis do the same, just as all intervals Y include A and intervals X are included in A.

More precisely, we have the following definition.

Definition 4.1.13. *Let* $A = [a^-, a^+] = (\widehat{a}; \widetilde{a})$, $B = [b^-, b^+] = (\widehat{b}; \widetilde{b}) \in \mathcal{K}_{\mathcal{C}}$ *and* $\gamma^- < 0, \gamma^+ > 0$ *(eventually* $\gamma^- = -\infty$ *and/or* $\gamma^+ = +\infty$ *), we define the following order relation, denoted* $\subseteqq_{\gamma^-,\gamma^+}$*,*

$$A \subseteqq_{\gamma^-,\gamma^+} B \Longleftrightarrow \begin{cases} \widetilde{a} \leqslant \widetilde{b} \\ \widehat{b} - \dfrac{\widetilde{b}}{\gamma^+} \leqslant \widehat{a} - \dfrac{\widetilde{a}}{\gamma^+} \\ \widehat{a} - \dfrac{\widetilde{a}}{\gamma^-} \leqslant \widehat{b} - \dfrac{\widetilde{b}}{\gamma^-}. \end{cases} \tag{4.11}$$

It is immediate that, carrying out some passages, the (4.11) corresponds to

$$A \subseteqq_{\gamma^-,\gamma^+} B \Longleftrightarrow \begin{cases} \widetilde{a} \leqslant \widetilde{b} \\ \widetilde{a} \leqslant \widetilde{b} + \gamma^+ \left(\widehat{a} - \widehat{b}\right) \\ \widetilde{a} \leqslant \widetilde{b} + \gamma^- \left(\widehat{a} - \widehat{b}\right). \end{cases} \tag{4.12}$$

Referring again to Figure 4.8, if we consider an interval $A = (\widehat{a}; \widetilde{a}) \in \mathcal{K}_{\mathcal{C}}$ and the intersections between the midpoint axis and the two lines passing through A with coefficients γ^- and γ^+, we can say that:

- an interval $X \in \mathcal{K}_{\mathcal{C}}$, so that $X \subseteqq_{\gamma^-,\gamma^+} A$, is included in interval A with respect to the inclusion order $\subseteqq_{\gamma^-,\gamma^+}$, equivalently, we say that A is $\subseteqq_{\gamma^-,\gamma^+}$-dominated by X;
- an interval $Y \in \mathcal{K}_{\mathcal{C}}$, so that $A \subseteqq_{\gamma^-,\gamma^+} Y$, includes interval A with respect to the inclusion order $\subseteqq_{\gamma^-,\gamma^+}$, equivalently, we say that A $\subseteqq_{\gamma^-,\gamma^+}$-dominates Y.

We can also conclude that:

- interval A is dominated by interval X (with respect to $\subseteqq_{\gamma^-,\gamma^+}$) if and only if $X \subseteqq_{\gamma^-,\gamma^+} A$;
- interval A dominates interval Y (with respect to $\subseteqq_{\gamma^-,\gamma^+}$) if and only if $A \subseteqq_{\gamma^-,\gamma^+} Y$.

In general, similarly to what was done in the case of order $\precapprox_{\gamma^-,\gamma^+}$, we can provide the following definition.

Definition 4.1.14. *For a given interval $A = (\widehat{a}; \widetilde{a})$, we define the following sets of intervals X which are*

(a) ($\subseteqq_{\gamma^-,\gamma^+}$)-dominated by A:

$$\mathbb{D}_{\subset}(A; \gamma^-, \gamma^+) = \{X \in \mathcal{K}_C \mid A \subseteqq_{\gamma^-,\gamma^+} X\}; \tag{4.13}$$

(b) ($\subseteqq_{\gamma^-,\gamma^+}$)-dominating A:

$$\mathbb{D}_{\supset}(A; \gamma^-, \gamma^+) = \{X \in \mathcal{K}_C \mid X \subseteqq_{\gamma^-,\gamma^+} A\}; \tag{4.14}$$

(c) ($\subseteqq_{\gamma^-,\gamma^+}$)-incomparable with A:

$$\mathbb{I}_{\subset}(A; \gamma^-, \gamma^+) = \{X \in \mathcal{K}_C \mid X \notin \mathbb{D}_{\subset}(A; \gamma^-, \gamma^+), X \notin \mathbb{D}_{\supset}(A; \gamma^-, \gamma^+)\}. \tag{4.15}$$

In the midpoint plane this is well represented in Figure 4.9.

Similarly to Definition 4.1.10, also in this case we have the following notions.

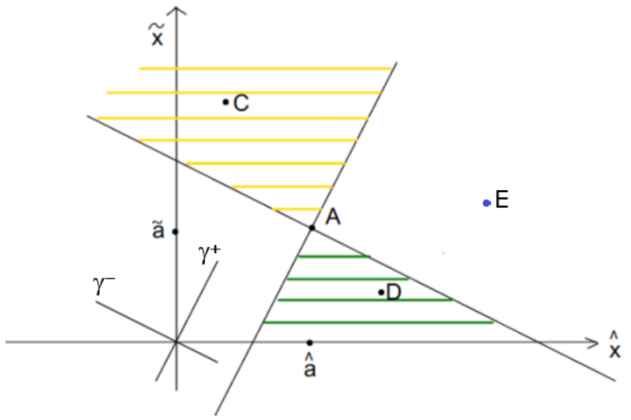


Figure 4.9: Interval A dominates interval C, is dominated by interval D and is incomparable with interval E with respect to order $\subseteqq_{\gamma^-,\gamma^+}$.

Definition 4.1.15. *Two intervals $A, B \in \mathcal{K}_{\mathcal{C}}$ are said to be incomparable with respect to $\subseteqq_{\gamma^-,\gamma^+}$ if and only if $A \neq B$ and neither $A \subseteqq_{\gamma^-,\gamma^+} B$ nor $B \subseteqq_{\gamma^-,\gamma^+} A$ are verified, which is denoted by*

$$A \,||_{\subseteqq_{\gamma^-,\gamma^+}}\, B.$$

It is immediate to prove that $\subseteqq_{\gamma^-,\gamma^+}$ represents a partial order relation (it is reflexive, antisymmetric and transitive) and, as in the case of partial order $\precapprox_{\gamma^-,\gamma^+}$, we can distinguish the three cases of lattice-order, strict-order and strong-order as follows.

Definition 4.1.16. *Let $\gamma^- < 0$ and $\gamma^+ > 0$ be fixed and consider two intervals A, B. We distinguish the following three cases.*

1. *(Lattice-order): The partial order ($\subseteqq_{\gamma^-,\gamma^+}$) will be called the lattice-order as in Definition 4.1.13 and equation* (4.12)*. This corresponds to the lattice-type concept of dominance as in points* (*a*) *and* (*b*) *of Definition 4.1.14.*

2. *(Strict-order): We say that interval A strictly dominates B, or equivalently that B is strictly dominated by A with respect to ($\subseteqq_{\gamma^-,\gamma^+}$) if and only if $A \subseteqq_{\gamma^-,\gamma^+} B$ and $A \neq B$; this means that in* (4.12) *at least one of the inequalities is strict. We write*

$$A \subseteq_{\gamma^-,\gamma^+} B \text{ if and only if } (A \subseteqq_{\gamma^-,\gamma^+} B \text{ and } A \neq B). \tag{4.16}$$

3. *(Strong-order): We say that interval A strongly dominates B, or equivalently that B is strongly dominated by A with respect to ($\subseteqq_{\gamma^-,\gamma^+}$) if*

and only if in (4.12) *the three inequalities are strict, i.e.,*

$$A \subset_{\gamma^-,\gamma^+} B \Longleftrightarrow \begin{cases} \widetilde{a} < \widetilde{b} \\ \widetilde{a} < \widetilde{b} + \gamma^+ \left(\widehat{a} - \widehat{b}\right) \\ \widetilde{a} < \widetilde{b} + \gamma^- \left(\widehat{a} - \widehat{b}\right). \end{cases} \tag{4.17}$$

The strong-order implies that interval B is not located on the two lines with angular coefficients γ^- or γ^+ and passing through A.

In particular, when $\gamma^+ = +1$ and $\gamma^- = -1$, we have that:

- the lattice-order $\subseteqq_{-1,+1}$ coincides with the standard inclusion $A \subseteq B$;
- the strict-order $\subseteq_{-1,+1}$ coincides with the standard strict inclusion $A \subset B$;
- the strong-order $A \subset_{-1,+1} B$ implies, additionally, that no endpoint of A coincides with an endpoint of B, i.e., $b^- < a^-$ and $b^+ > a^+$.

According to (4.10), is also evident that

$$A \subseteqq_{-1,1} B \Leftrightarrow [a^-, a^+] \subseteq [b^-, b^+,] \Leftrightarrow A \subseteq B$$

as shown in Figure 4.10, where $X \subseteq A$ and $A \subseteq Y$; indeed it is

$$[x^-, x^+] \subseteq [a^-, a^+] \subseteq [y^-, y^+].$$

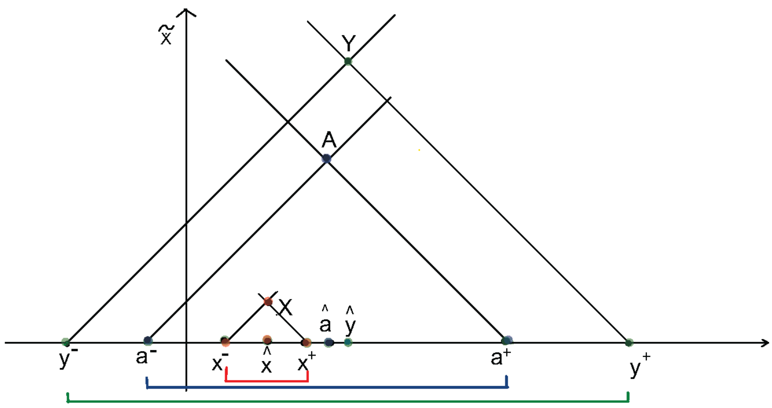


Figure 4.10: Example of inclusion order $\subseteqq_{-1,+1}$: $X \subseteq A$ and $A \subseteq Y$.

As in the case of poset $(\overline{\mathcal{K}_{\mathcal{C}}}, \precapprox_{\gamma^-,\gamma^+})$, even with generalized inclusion we see that the structure $(\mathcal{K}_C^{\varnothing\mathbb{R}}, \subseteqq_{\gamma^-,\gamma^+})$ is a complete lattice, which means that

$$\forall X, Y \in \mathcal{K}_C^{\varnothing\mathbb{R}}, \;\; \exists \, inf_{\subseteqq_{\gamma^-,\gamma^+}}\{X, Y\}, sup_{\subseteqq_{\gamma^-,\gamma^+}}\{X, Y\}.$$

We will use the notation:

$$X \uplus_{\subseteqq_{\gamma^-,\gamma^+}} Y = sup_{\subseteqq_{\gamma^-,\gamma^+}}\{X, Y\}, \tag{4.18}$$

$$X \cap_{\subseteqq_{\gamma^-,\gamma^+}} Y = inf_{\subseteqq_{\gamma^-,\gamma^+}}\{X, Y\}, \tag{4.19}$$

simplified to $X \uplus Y$ and $X \cap Y$ when no confusion is possible on $\subseteqq_{\gamma^-,\gamma^+}$.

These can be applied to a pair of elements $X, Y \in \mathcal{K}_C^{\varnothing\mathbb{R}}$ to yeld again an element of $\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}$ so that we can define the following internal operations:

$\uplus : \mathcal{K}_C^{\varnothing\mathbb{R}} \times \mathcal{K}_C^{\varnothing\mathbb{R}} \to \mathcal{K}_C^{\varnothing\mathbb{R}}$ such that: $(X, Y) \to X \uplus Y$;

$\cap : \mathcal{K}_C^{\varnothing\mathbb{R}} \times \mathcal{K}_C^{\varnothing\mathbb{R}} \to \mathcal{K}_C^{\varnothing\mathbb{R}}$ such that: $(X, Y) \to X \cap Y$.

Moreover, in order to give a correct and more precise definition of $\uplus$ and $\cap$ in $\mathcal{K}_{\mathcal{C}}$, we can use lines A_m, A_p and B_m, B_p, passing through A and B, as defined by (4.7), and we denote by

$$A_mB_p \text{ (or } B_pA_m) \text{ and } B_mA_p \text{ (or } A_pB_m)$$

the respective points of intersection of such lines.
As usual, we have that $A_m||B_m$ as well as $A_p||B_p$ (possibly $A_m \equiv B_m$ or $A_p \equiv B_p$ but not both coincidences verified at the same time).

Remark 4.1.2. *If $A_p \equiv B_p$ and $A \neq B$, then $A_m \neq B_m$ and $A_m||B_m$. So A and B are $\precapprox_{\gamma^-,\gamma^+}$-comparable, that is, $A \precapprox_{\gamma^-,\gamma^+} B$ or $B \precapprox_{\gamma^-,\gamma^+} A$.*

Similarly, if $A_m \equiv B_m$ and $A \neq B$, then $A_p \neq B_p$ and $A_p||B_p$. So, A and B are $\precapprox_{\gamma^-,\gamma^+}$-comparable, that is, $A \precapprox_{\gamma^-,\gamma^+} B$ or $B \precapprox_{\gamma^-,\gamma^+} A$.

The definition below examines all possible cases of pair of intervals in $\mathcal{K}_{\mathcal{C}}$.

Definition 4.1.17. *Let $A, B \in \mathcal{K}_{\mathcal{C}}$. We have the following different cases.*

1. *If A and B are $\precapprox_{\gamma^-,\gamma^+}$-comparable and $A \neq B$:*

   1.a *if $A \precapprox_{\gamma^-,\gamma^+} B$, then $A \uplus B = A_pB_m$ and $A \cap B = A_mB_p$;*

   1.b *if $B \precapprox_{\gamma^-,\gamma^+} A$, then $A \uplus B = A_mB_p$ and $A \cap B = A_pB_m$;*

2. *If A and B are $\precapprox_{\gamma^-,\gamma^+}$-incomparable:*

   2.a *if $A_pB_m \precapprox_{\gamma^-,\gamma^+} A_mB_p$, then $A \uplus B = A$ and $A \cap B = B$;*

   2.b *if $A_mB_p \precapprox_{\gamma^-,\gamma^+} A_pB_m$, then $A \uplus B = B$ and $A \cap B = A$;*

3. *If $A = B$, then $A \uplus B = A \cap B = A = B$.*

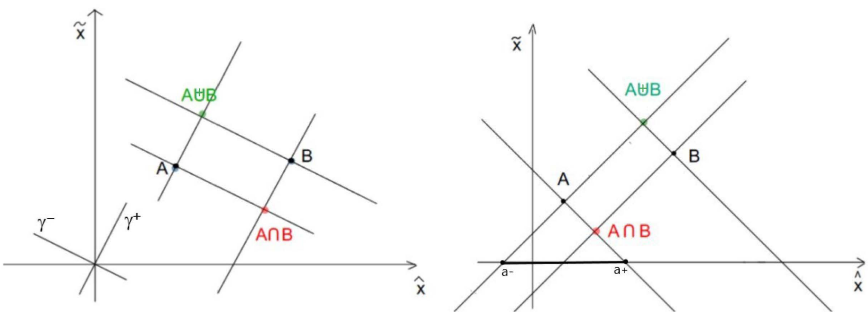


Figure 4.11: The two operations $\uplus$ and $\cap$ for a generic inclusion order $\subseteqq_{\gamma^-,\gamma^+}$ (left). The operations $\uplus$ and $\cap$ when the order associated is the LU-order $\subseteqq_{-1,+1}$ (right).

It is possible to visualize the two operations as shown in the left side of Figure 4.11, while, if we consider the LU-order, i.e., $(\gamma^-, \gamma^+) = (-1, +1)$, the situation that we obtain is well described in the right side.

In particular, in this second case, for each interval $X = (\widehat{x}; \widetilde{x})$ it is easy to visualize the corresponding intervals, expressed in endpoint notation $X = [x^-, x^+]$, whose extremes coincide exactly with the intersections between the straight lines and the horizontal axis. The same type of observation can be done for the operations of intersection and convex hull of intervals.

The algebraic structure of the lattices $(\mathcal{K}_C^{\varnothing\mathbb{R}}, \uplus, \cap)$ and $(\mathcal{K}_C^{\varnothing\mathbb{R}}, \cap, \uplus)$ are obtained from the following properties:

1) $\uplus$ and $\cap$ are commutative: $\forall X, Y \in \mathcal{K}_C^{\varnothing\mathbb{R}}$

   1.a $X \uplus Y = Y \uplus X$,

   1.b $X \cap Y = Y \cap X$;

2) $\uplus$ and $\cap$ are associative: $\forall X, Y, Z \in \mathcal{K}_C^{\varnothing\mathbb{R}}$

   2.a $(X \uplus Y) \uplus Z = X \uplus (Y \uplus Z)$,

   2.b $(X \cap Y) \cap Z = X \cap (Y \cap Z)$;

3) the absorpion identities are satified: $\forall X, Y \in \mathcal{K}_C^{\varnothing\mathbb{R}}$

   3.a $X \uplus (X \cap Y) = X$,

   3.b $X \cap (X \uplus Y) = X$;

4) the idempotency is satisfied for both $\vee$ and $\wedge$: $\forall X \in \mathcal{K}_C^{\varnothing\mathbb{R}}$

   4.a $X \uplus X = X$,

   4.b $X \cap X = X$.

It is also easy to verify that:

5.a) $\uplus$ is left and right distributive over $\cap$:

5.a.i $A \uplus (B \cap C) = (A \uplus B) \cap (A \uplus C), \forall A, B, C \in \mathcal{K}_C^{\varnothing\mathbb{R}}$,

5.a.ii $(A \cap B) \uplus C = (A \uplus C) \cap (B \uplus C), \forall A, B, C \in \mathcal{K}_C^{\varnothing\mathbb{R}}$;

5.b) $\cap$ is left and right distributive over $\uplus$:

5.b.i $A \cap (B \uplus C) = (A \cap B) \uplus (A \cap C), \forall A, B, C \in \mathcal{K}_C^{\varnothing\mathbb{R}}$,

5.b.ii $(A \uplus B) \cap C = (A \cap C) \uplus (B \cap C), \forall A, B, C \in \mathcal{K}_C^{\varnothing\mathbb{R}}$.

So $(\mathcal{K}_C^{\varnothing\mathbb{R}}, \uplus, \cap)$ and $(\mathcal{K}_C^{\varnothing\mathbb{R}}, \cap, \uplus)$ are distributive algebraic lattices.

We end this section with some additional interesting properties.

Proposition 4.1.3. *Let A and B be two intervals in $\mathcal{K}_\mathcal{C}$ such that $A \neq B$. They are $(\precapprox_{\gamma^-,\gamma^+})-$incomparable if and only if $A \subseteqq_{\gamma^-,\gamma^+} B$ or $B \subseteqq_{\gamma^-,\gamma^+} A$.*

The property rises intuitively simply by considering the situation represented in Figure 4.12, where the examples previously analyzed are taken up.

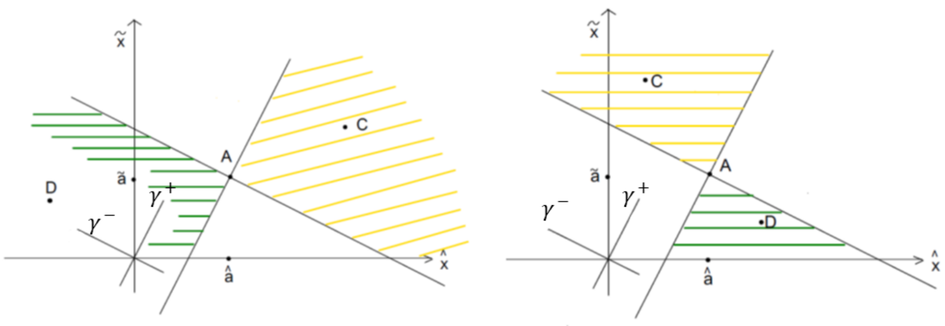


Figure 4.12: Examples of comparable and incomparable intervals with respect to the orders $\precapprox_{\gamma^-,\gamma^+}$ and $\subseteqq_{\gamma^-,\gamma^+}$, assuming that $\gamma^- < 0$ and $\gamma^+ > 0$:
$D \precapprox_{\gamma^-,\gamma^+} A \precapprox_{\gamma^-,\gamma^+} C$ as well as $A \parallel_{\subseteqq_{\gamma^-,\gamma^+}} C$ and $A \parallel_{\subseteqq_{\gamma^-,\gamma^+}} D$ (left);
$D \subseteqq_{\gamma^-,\gamma^+} A \subseteqq_{\gamma^-,\gamma^+} C$ as well as $A \parallel_{\precapprox_{\gamma^-,\gamma^+}} C$ and $A \parallel_{\precapprox_{\gamma^-,\gamma^+}} D$ (right).

Indeed, assuming that $\gamma^- < 0$ and $\gamma^+ > 0$, in the right side of the figure we can see how interval A is included in interval C (i.e., $A \subseteqq_{\gamma^-,\gamma^+} C$), that is, A dominates C with respect to the order $\subseteqq_{\gamma^-,\gamma^+}$, but they are incomparable with respect to the order $\precapprox_{\gamma^-,\gamma^+}$(i.e., $A \parallel_{\precapprox_{\gamma^-,\gamma^+}} C$) ; similarly, interval A includes interval D (i.e., $D \subseteqq_{\gamma^-,\gamma^+} A$), that is, A is dominated by D with respect to the order $\subseteqq_{\gamma^-,\gamma^+}$ and, however, also in this case there is $\precapprox_{\gamma^-,\gamma^+}$-incomparability (i.e., $A \parallel_{\precapprox_{\gamma^-,\gamma^+}} D$) .

On the contrary, in the left side of the figure we see that $A \precapprox_{\gamma^-,\gamma^+} C$, that is, A dominates C with respect to the order $\precapprox_{\gamma^-,\gamma^+}$, but clearly there is

$\subseteqq_{\gamma^-,\gamma^+}$-incomparability ($A \parallel_{\subseteqq_{\gamma^-,\gamma^+}} C$) as well as $D \precsim_{\gamma^-,\gamma^+} A$, which means that A is dominated by D with respect to the order $\precsim_{\gamma^-,\gamma^+}$, but there is an evident $\subseteqq_{\gamma^-,\gamma^+}$-incomparability ($A \parallel_{\subseteqq_{\gamma^-,\gamma^+}} D$).

In this regard, going into more detail and recalling that $A_{\gamma^+} = \left(\widehat{a} - \dfrac{\widetilde{a}}{\gamma^+}; 0\right)$ and $A_{\gamma^-} = \left(\widehat{a} - \dfrac{\widetilde{a}}{\gamma^-}; 0\right)$, we can consider the sets

$$[A_{\gamma^+}, A_{\gamma^-}] \cap [B_{\gamma^+}, B_{\gamma^-}] \text{ and } [A_{\gamma^+}, A_{\gamma^-}] \uplus [B_{\gamma^+}, B_{\gamma^-}],$$

which, as shown in Figure 4.13, correspond, respectively, to

$$A \cap B \text{ and } A \uplus B.$$

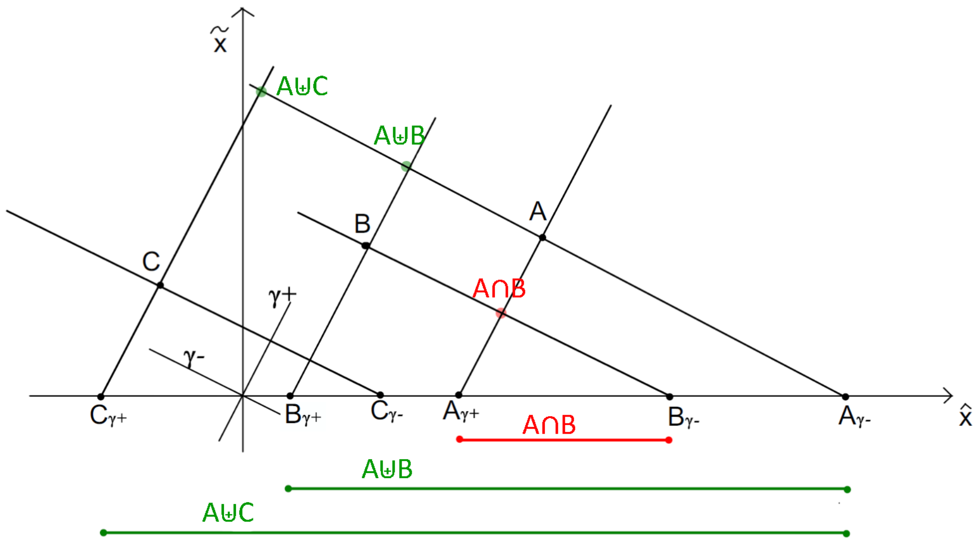


Figure 4.13: Examples of intersection and convex hull of intervals: $A \cap B \neq \varnothing$, $A \uplus B$ and $A \uplus C$.

We have the following relevant properties.

Theorem 4.1.1. *Let be A and B two unaligned intervals in $\mathcal{K}_{\mathcal{C}}$ such that $A \neq B$ and $\gamma^-, \gamma^+ \in \mathbb{R}$ with $\gamma^- < 0, \gamma^+ > 0$. Then we have:*

$$A \precsim_{\gamma^-,\gamma^+} B \text{ or } B \precsim_{\gamma^-,\gamma^+} A \Leftrightarrow B \nsubseteqq_{\gamma^-,\gamma^+} A \text{ and } A \nsubseteqq_{\gamma^-,\gamma^+} B$$

(i.e., two intervals A and B, $A \neq B$, are $(\precsim_{\gamma^-,\gamma^+})$ – comparable if and only if they are $(\subseteqq_{\gamma^-,\gamma^+})$ – incomparable ($A \,||_{\subseteqq_{\gamma^-,\gamma^+}} B$).
Vice-versa, A and B are $(\subseteqq_{\gamma^-,\gamma^+})$ – comparable if and only if they are $(\precsim_{\gamma^-,\gamma^+})$ – incomparable ($A \,||_{\precsim_{\gamma^-,\gamma^+}} B$).

Proof. Direction $\Leftarrow$ is obvious, so we will prove only direction $\Rightarrow$.

Suppose $A \precsim_{\gamma^-,\gamma^+} B$, by (2.33), this is equivalent to $\begin{cases} \widehat{a} \leqslant \widehat{b} \\ \widetilde{a} \geqslant \widetilde{b} + \gamma^+ \left(\widehat{a} - \widehat{b}\right) \\ \widetilde{a} \leqslant \widetilde{b} + \gamma^- \left(\widehat{a} - \widehat{b}\right). \end{cases}$

According to (4.12), the second inequality implies that $A \nsubseteqq_{\gamma^-,\gamma^+} B$, while the third one, developing the passages, is equivalent to $\widetilde{b} \geqslant \widetilde{a} + \gamma^-(\widehat{b} - \widehat{a})$ which implies that $B \nsubseteqq_{\gamma^-,\gamma^+} A$. □

An immediate consequence of theorem 4.1.1 is represented by the following proposition.

Proposition 4.1.4. *Let $A, B \in \mathcal{K}_{\mathcal{C}}$, such that $A \neq B$. Then $A \precsim_{\gamma^-,\gamma^+} B$ iff*

(i) $A \parallel_{\subseteqq_{\gamma^-,\gamma^+}} B$, that is, A and B are $(\subseteqq_{\gamma^-,\gamma^+})$-incomparable;

(ii) $\widehat{a} < \widehat{b}$.

4.1.5 Polarity and its main features

At this point it is time to introduce a new notion, called polarity.

Note that the term *polarity* has already been used in the past (see [72]) but with a different meaning from the one assigned to it here.

Definition 4.1.18. *We say that two orders $\leq_1$ and $\leq_2$ such that*

- *$x \leq_1 y \Leftrightarrow x$ and y are $\leq_2$-incomparable (i.e., $x \parallel_{\leq_2} y$),*
- *$x \leq_2 y \Leftrightarrow x$ and y are $\leq_1$-incomparable (i.e., $x \parallel_{\leq_1} y$),*

satisfy the polarity property (or they are "polars").

Remark 4.1.3. *The two orders $\precsim_{\gamma^-,\gamma^+}$ and $\subseteqq_{\gamma^-,\gamma^+}$ are "polars". Indeed, for a fixed $A \in \mathcal{K}_{\mathcal{C}}$, consider:*

$$\uparrow_{\precsim_{\gamma^-,\gamma^+}} A = \{X | A \precsim_{\gamma^-,\gamma^+} X\} =]A, +\infty[_{\precsim_{\gamma^-,\gamma^+}} \tag{4.20}$$

which, according to definition 2.2.13, corresponds to $\mathbb{D}_<(A; \gamma^-, \gamma^+)$;

$$\downarrow_{\precsim_{\gamma^-,\gamma^+}} A = \{Y | Y \precsim_{\gamma^-,\gamma^+} A\} =]-\infty, A[_{\precsim_{\gamma^-,\gamma^+}} \tag{4.21}$$

which, according to definition 2.2.13, corresponds to $\mathbb{D}_>(A; \gamma^-, \gamma^+)$;

$$\uparrow_{\subseteqq_{\gamma^-,\gamma^+}} A = \{T | A \subseteqq_{\gamma^-,\gamma^+} T\} =]A, +\infty[_{\subseteqq_{\gamma^-,\gamma^+}} \tag{4.22}$$

which, according to definition 4.1.14, corresponds to $\mathbb{D}_\subset(A; \gamma^-, \gamma^+)$;

$$\downarrow_{\subseteqq_{\gamma^-,\gamma^+}} A = \{Z|Z \subseteqq_{\gamma^-,\gamma^+} A\} =]-\infty, A[_{\subseteqq_{\gamma^-,\gamma^+}} \quad (4.23)$$

which, according to definition 4.1.14, corresponds to $\mathbb{D}_{\supset}(A;\gamma^-,\gamma^+)$.

So A is the separating element in all cases (see also Figure 4.14).

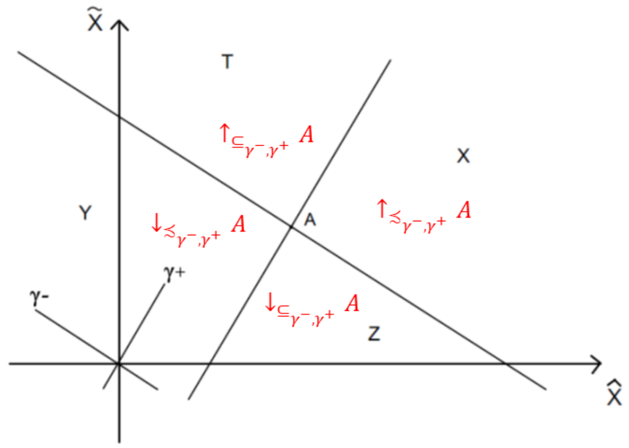


Figure 4.14: In the midpoint plane the interval A acts as separator element between the sets $\uparrow_{\precapprox_{\gamma^-,\gamma^+}} A$, $\downarrow_{\precapprox_{\gamma^-,\gamma^+}} A$, $\uparrow_{\subseteqq_{\gamma^-,\gamma^+}} A$ and $\downarrow_{\subseteqq_{\gamma^-,\gamma^+}} A$.

We can even introduce the operators that represent the boundary points as infimum and supremum (see Figure 4.15):

- $A \wedge_{\subseteqq_{\gamma^-,\gamma^+}} B = inf_{\subseteqq_{\gamma^-,\gamma^+}}\{A,B\}$, called *meet* (or *et*) with respect to $\subseteqq_{\gamma^-,\gamma^+}$, represents the infimum and we simply denote it by

$$A \cap B;$$

- $A \vee_{\subseteqq_{\gamma^-,\gamma^+}} B = conv(A \cup B) = sup_{\subseteqq_{\gamma^-,\gamma^+}}\{A,B\}$, called *join* (or *vel*) with respect to $\subseteqq_{\gamma^-,\gamma^+}$, represents the supremum, also denoted by

$$A \uplus B;$$

- $A \wedge_{\precapprox_{\gamma^-,\gamma^+}} B = inf_{\precapprox_{\gamma^-,\gamma^+}}\{A,B\}$ called *meet* (or *et*) with respect to $\precapprox_{\gamma^-,\gamma^+}$, stands for the infimum, simply denoted by

$$A \wedge B;$$

- $A \vee_{\precapprox_{\gamma^-,\gamma^+}} B = sup_{\precapprox_{\gamma^-,\gamma^+}}\{A,B\}$ called *join* (or *vel*) with respect to $\precapprox_{\gamma^-,\gamma^+}$, stands for the supremum, simply denoted by

$$A \vee B.$$

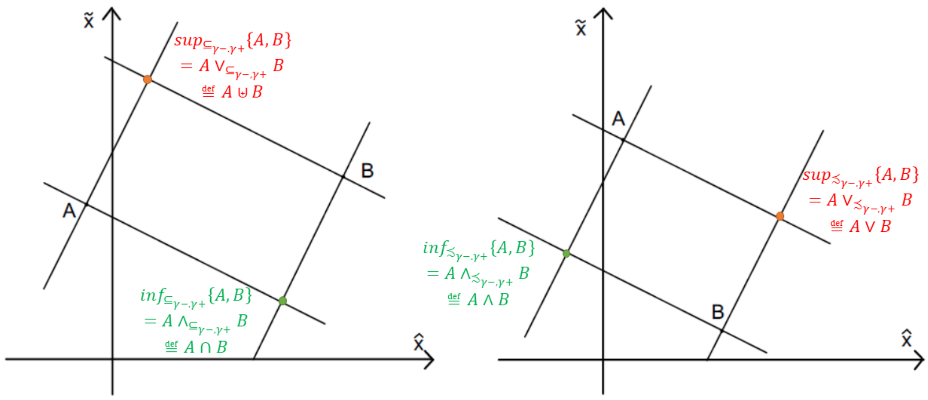


Figure 4.15: Examples of infimum and supremum with respect to inclusion order $\subseteqq_{\gamma^-,\gamma^+}$ (left) and $\precapprox_{\gamma^-,\gamma^+}$-order (right).

Finally, we can define the following sets of intervals (which are not unique, as shown in Figure 4.16):

$$A' = sup_{\precapprox_{\gamma^-,\gamma^+}}\{X | A \cap X = \varnothing, X \precapprox_{\gamma^-,\gamma^+} A\},$$

$$A'' = inf_{\precapprox_{\gamma^-,\gamma^+}}\{Y | A \cap Y = \varnothing, Y \succapprox_{\gamma^-,\gamma^+} A\}.$$

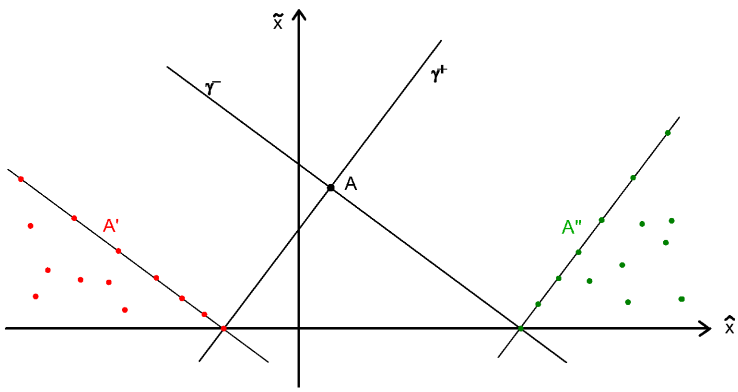


Figure 4.16: Sets A' and A'' for a generic interval A in the $(\precapprox_{\gamma^-,\gamma^+})$-case.

Note that A' and A'' do not represent only two intervals, but all the intervals lying below or exactly on the lines that intersect the midpoint axis respectively at points $\left(\widehat{a} - \dfrac{\widetilde{a}}{\gamma^+}; 0\right)$ and $\left(\widehat{a} - \dfrac{\widetilde{a}}{\gamma^-}; 0\right)$.

If we now try to consider the particular case of $(\precapprox_{LU})$-order, that is when $(\gamma^-, \gamma^+) = (-1, +1)$, and its analogous inclusion order $\subseteqq_{-1,1}$, it is possible to define the following sets:

$$\uparrow_{\precapprox_{LU}} A = \{X | A \precapprox_{LU} X\}, \quad \downarrow_{\precapprox_{LU}} A = \{Y | Y \precapprox_{LU} A\},$$

$$\uparrow_{\subseteqq_{-1,1}} A = \{T | A \subseteqq_{-1,1} T\}, \quad \downarrow_{\subseteqq_{-1,1}} A = \{Z | Z \subseteqq_{-1,1} A\},$$

which are shown in Figure 4.17.

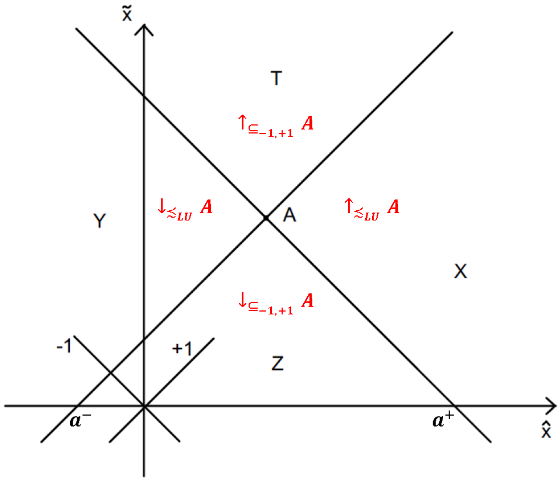


Figure 4.17: The interval A acts as the separator element between the sets $\uparrow_{\precapprox_{LU}} A$, $\downarrow_{\precapprox_{LU}} A$, $\uparrow_{\subseteqq_{-1,1}} A$ and $\downarrow_{\subseteqq_{-1,1}} A$.

Obviously it is possible to define the following operators which represent the boundary points as infimum and supremum with respect to $\precapprox_{LU}$-order and inclusion order $\subseteqq_{-1,1}$.

- $A \wedge_{\subseteqq_{-1,1}} B = inf_{\subseteqq_{-1,1}}\{A, B\}$, called *meet* (or *et*) with respect to $\subseteqq_{-1,1}$, represents the infimum, also denoted as $A \cap B$;

- $A \vee_{\subseteqq_{-1,1}} B = conv(A \cup B) = sup_{\subseteqq_{-1,1}}\{A, B\}$, called *join* (or *vel*) with respect to $\subseteqq_{-1,1}$, represents the supremum, also denoted as $A \uplus B$;

- $A \wedge_{\precapprox_{LU}} B = inf_{\precapprox_{LU}}\{A, B\}$ called *meet* (or *et*) with respect to $\precapprox_{LU}$, stands the infimum, also denoted as $A \wedge B$;

- $A \vee_{\precapprox_{LU}} B = sup_{\precapprox_{LU}}\{A, B\}$ called *join* (or *vel*) with respect to $\precapprox_{LU}$, stands for the supremum, also denoted as $A \vee B$.

Similarly to the generic case, we can define the following set of non-unique intervals:

$$A' = sup_{\precapprox_{LU}}\{X | A \cap X = \varnothing, X \precapprox_{LU} A\},$$

$$A'' = inf_{\precapprox_{LU}}\{Y | A \cap Y = \varnothing, Y \succapprox_{LU} A\}.$$

Of course, in this case A' and A'' represent all the intervals lying below or exactly on the lines that intersect the midpoint axis respectively at points $(a^-;0)$ and $(a^+;0)$.

4.1.6 Order polarity in intervals of intervals

Before concluding this section it is interesting to provide some notions which opens the way to a new algebraic interpretation of the sets of intervals. Using notations (4.8) and (4.9), we reformulate Definition 4.1.2 as follows.

Definition 4.1.19. *In* $(\mathcal{K}_\mathcal{C}, \precapprox_{\gamma^-,\gamma^+})$ *we can define an interval (of intervals) with extreme intervals* $A, B \in \mathcal{K}_\mathcal{C}$*, to be the set of all intervals* $X \in \mathcal{K}_\mathcal{C}$ *such that* $A \wedge B \precapprox_{\gamma^-,\gamma^+} X$ *and* $X \precapprox_{\gamma^-,\gamma^+} A \vee B$*, denoted by*

$$[[A,B]]_{\precapprox_{\gamma^-,\gamma^+}} = \left\{X \in \mathcal{K}_\mathcal{C} \mid A \wedge B \precapprox_{\gamma^-,\gamma^+} X \precapprox_{\gamma^-,\gamma^+} A \vee B\right\}$$

(or simply by $[[A,B]]_{\gamma^-,\gamma^+}$ *if there are no other types of orders besides* $\precapprox_{\gamma^-,\gamma^+}$*).*

More in detail, we also have the following notion.

Definition 4.1.20. *Let* $\mathcal{C} = [[C^\wedge, C^\vee]]_{\precapprox_{\gamma^-,\gamma^+}}$ *be an interval of intervals with* $C^\wedge = (\widehat{c}^\wedge; \widetilde{c}^\wedge)$ *and* $C^\vee = (\widehat{c}^\vee; \widetilde{c}^\vee)$*. We define:*

$C^\wedge \overset{def}{=} min_{\gamma^-,\gamma^+}[[C^\wedge, C^\vee]]_{\precapprox_{\gamma^-,\gamma^+}}$
as the 0-element related to the interval of intervals $\mathcal{C} = [[C^\wedge, C^\vee]]_{\precapprox_{\gamma^-,\gamma^+}}$*;*

$C^\vee \overset{def}{=} max_{\gamma^-,\gamma^+}[[C^\wedge, C^\vee]]_{\precapprox_{\gamma^-,\gamma^+}}$
as the 1-element related to the interval of intervals $\mathcal{C} = [[C^\wedge, C^\vee]]_{\precapprox_{\gamma^-,\gamma^+}}$*.*

This means that the poset

$$(\mathcal{C}, \precapprox_{\gamma^-,\gamma^+}),$$

with $\mathcal{C} = [[C^\wedge, C^\vee]]_{\precapprox_{\gamma^-,\gamma^+}}$ can be considered as a closed, bounded lattice (see left picture of Figure 4.18).

Indeed, if we consider $(\mathcal{C}, \precapprox_{\gamma^-,\gamma^+})$, we have that for all $X, Y \in \mathcal{C}$, $\exists\ X \vee Y = sup_{\precapprox_{\gamma^-,\gamma^+}}\{X,Y\}$ and $X \wedge Y = inf_{\precapprox_{\gamma^-,\gamma^+}}\{X,Y\}$.
Therefore, $\mathcal{C}$ is a lattice with maximum $C^\vee$ and minimum $C^\wedge$.

On the other hand, if we consider the algebraic structure of the form

$$(\mathcal{C}, \vee, \wedge, 0, 1)$$

such that $(\mathcal{C}, \vee, \wedge)$ is a lattice, 0 is the identity element (zero) of the *join* operation $\vee$ and 1 is the identity element (unity) of the *meet* operation $\wedge$

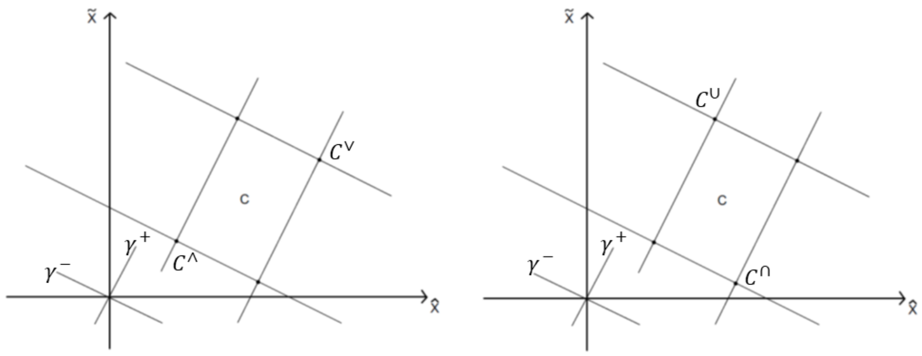


Figure 4.18: Representation of $[[C^\wedge, C^\vee]]_{\gamma^-,\gamma^+}$ (left) and $[[C^\cap, C^\cup]]_{\gamma^-,\gamma^+}$ (right) in the midpoint plane.

(i.e., $X \vee 0 = X$ and $X \wedge 1 = X$ for all $X \in \mathcal{C}$), we obtain a bounded lattice too.

The same can be done in the case of the inclusion order $\subseteqq_{\gamma^-,\gamma^+}$. In fact we have already highlighted (see Subsection 4.1.1) that also the structure $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}, \subseteqq_{\gamma^-,\gamma^+})$ is a complete lattice; therefore, we can extend to it the notion of interval of intervals too, in a way quite similar to what is described in Definition 4.1.19.

Definition 4.1.21. *In* $(\mathcal{K}_{\mathcal{C}}, \subseteqq_{\gamma^-,\gamma^+})$ *we can define an interval (of intervals) with extreme intervals* $A, B \in \mathcal{K}_{\mathcal{C}}$ *(with* $A \cap B \neq \varnothing$*), to be the set of all intervals* $X \in \mathcal{K}_{\mathcal{C}}$ *such that* $A \cap B \subseteqq_{\gamma^-,\gamma^+} X$ *and* $X \subseteqq_{\gamma^-,\gamma^+} A \cup B$*, denoted by*

$$[[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}} = \left\{X \in \mathcal{K}_{\mathcal{C}} \mid A \cap B \subseteqq_{\gamma^-,\gamma^+} X \subseteqq_{\gamma^-,\gamma^+} A \cup B\right\}$$

(or simply by $[[A, B]]_{\gamma^-,\gamma^+}$ *in case there are no other types of orders besides* $\subseteqq_{\gamma^-,\gamma^+}$*).*

Assuming $A \subseteqq_{\gamma^-,\gamma^+} B$, it is also trivial to verify that:

$$inf_{\subseteqq_{\gamma^-,\gamma^+}}[[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}} = A,$$

$$sup_{\subseteqq_{\gamma^-,\gamma^+}}[[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}} = B,$$

and

$$\mathcal{S}(A, B) \subseteq [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}.$$

Furthermore, we have the following facts.

Definition 4.1.22. *Let* $\mathcal{C} = [[C^\cap, C^\cup]]_{\subseteqq_{\gamma^-,\gamma^+}}$ *be an interval of intervals with* $C^\cap = (\hat{c}^\cap; \tilde{c}^\cap)$ *and* $C^\cup = (\hat{c}^\cup; \tilde{c}^\cup)$*. We define:*

$$C^\cap \stackrel{def}{=} min_{\gamma^-,\gamma^+}[[C^\cap, C^\cup]]_{\subseteqq_{\gamma^-,\gamma^+}}$$
as the 0-element related to the interval of intervals $\mathcal{C} = [[C^\cap, C^\cup]]_{\subseteqq_{\gamma^-,\gamma^+}}$*;*

$C^{\cup} \stackrel{def}{=} max_{\gamma^-,\gamma^+}[[C^{\cap}, C^{\cup}]]_{\subseteqq_{\gamma^-,\gamma^+}}$
as the 1-element related to the interval of intervals $\mathcal{C} = [[C^{\cap}, C^{\cup}]]_{\subseteqq_{\gamma^-,\gamma^+}}$.

This means that the poset

$$(\mathcal{C}, \subseteqq_{\gamma^-,\gamma^+}),$$

with $\mathcal{C} = [[C^{\cap}, C^{\cup}]]_{\subseteqq_{\gamma^-,\gamma^+}}$ can be considered as a closed, bounded lattice (see right picture of Figure 4.18).

Indeed, if we consider $(\mathcal{C}, \subseteqq_{\gamma^-,\gamma^+})$, we have that for all $X, Y \in \mathcal{C}$, $\exists\ X \cup Y = sup_{\subseteqq_{\gamma^-,\gamma^+}}\{X, Y\}$ and $X \cap Y = inf_{\subseteqq_{\gamma^-,\gamma^+}}\{X, Y\}$.
Therefore, $\mathcal{C}$ is a lattice with maximum $C^{\cup}$ and minimum $C^{\cap}$.

On the other hand, if we take into account the algebraic structure of the form

$$(\mathcal{C}, \cup, \cap, 0, 1)$$

such that $(\mathcal{C}, \cup, \cap)$ is a lattice, 0 is the identity element (zero) of the *join* operation $\cup$ and 1 is the identity element (unity) of the *meet* operation $\cap$ (i.e., $X \cup 0 = X$ and $X \cap 1 = X$ for all $X \in \mathcal{C}$), we obtain a bounded lattice too.

We conclude this section by noting that an interval of $\mathcal{K}_{\mathcal{C}}$ (which so far we have referred to as an interval of intervals) can be expressed indifferently using the $\precapprox_{\gamma^-,\gamma^+}$-order or the $\subseteqq_{\gamma^-,\gamma^+}$-order.

Indeed, according to Definitions 4.1.19 and 4.1.21, it is simple to verify that, for each $A, B \in \mathcal{K}_{\mathcal{C}}$ (with $A \cap B \neq \varnothing$), we have

$$[[A, B]]_{\precapprox_{\gamma^-,\gamma^+}} = [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}. \tag{4.24}$$

More in details, given two intervals A and B in $\mathcal{K}_{\mathcal{C}}$, such that $A \cap B \neq \varnothing$, then:

- if $A \precapprox_{\gamma^-,\gamma^+} B$ (which corresponds to the case where A and B are incomparable with respect to inclusion order $\subseteqq_{\gamma^-,\gamma^+}$), as

  $$A \wedge B = A \quad \text{and} \quad A \vee B = B,$$

  then, referring to Definition 4.1.2, we can simply consider

  $$[[A, B]]_{\precapprox_{\gamma^-,\gamma^+}} = \{X | A \precapprox_{\gamma^-,\gamma^+} X \precapprox_{\gamma^-,\gamma^+} B\};$$

- if $A \subseteqq_{\gamma^-,\gamma^+} B$ (which corresponds to the case where A and B are incomparable with respect to order $\precapprox_{\gamma^-,\gamma^+}$), as

  $$A \cap B = A \text{ and } A \cup B = B,$$

  then we consider

  $$[[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}} = \{X | A \subseteqq_{\gamma^-,\gamma^+} X \subseteqq_{\gamma^-,\gamma^+} B\}.$$

In such cases we have that (4.24) can also be written, respectively, as:

$$[[A, B]]_{\precapprox_{\gamma^-,\gamma^+}} = [[A \cap B, A \cup B]]_{\subseteqq_{\gamma^-,\gamma^+}} \tag{4.25}$$

and

$$[[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}} = [[A \wedge B, A \vee B]]_{\precapprox_{\gamma^-,\gamma^+}}. \tag{4.26}$$

As shown in Figures 4.19, the two areas represent the same interval of intervals.

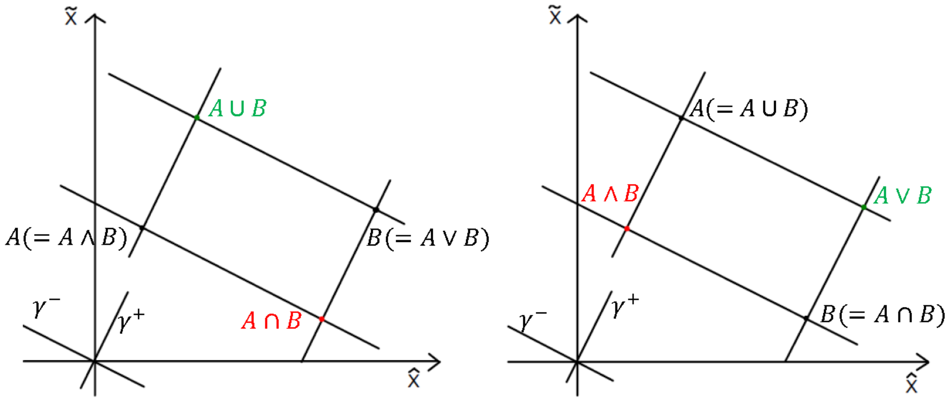


Figure 4.19: Interval of intervals in the general (γ^-, γ^+)-case:
$[[A, B]]_{\precapprox_{\gamma^-,\gamma^+}} = [[A \cap B, A \cup B]]_{\subseteqq_{\gamma^-,\gamma^+}}$ (left side)
and $[[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}} = [[A \wedge B, A \vee B]]_{\precapprox_{\gamma^-,\gamma^+}}$ (right side).

4.2 Interval lattice completion

According to [24], lattices are more interesting structures than general posets as they allow us to take the *join* (or *vel*) and the *meet* (or *et*) for any pair of elements in the set.
In particular, as mentioned in Definition 2.2.2, a complete lattice $(\mathbb{X}, \leq)$ enables us to take the *join* and the *meet* of all arbitrary subsets $\mathbb{Y} \subseteq \mathbb{X}$.

It is also well known that finite lattices are always complete while if $\mathbb{X}$ is infinite then the structure $(\mathbb{X}, \leq)$ may be a lattice but not complete. Therefore, it is interesting to analyse in detail the peculiarities of the complete lattices introduced in Section 4.1, also discussing the question of how to embed any poset into a complete lattice, thus arriving at the notion of lattice completion which is useful for both finite and infinite posets, and then apply it to the interval case. However, in order to do this we first have to recall some basic definitions of partial orders.

4.2.1 Basic definitions of partial orders applied to interval case

According to what is reported in [24], we define, for an element $x \in \mathbb{X}$, the following sets (see Figure 4.20 for the interval case).

1) Down-set of x :
$$D[x] = \{y \in \mathbb{X} \mid y \leq x\}. \tag{4.27}$$
Referring to the interval case, the down-set $D[A]$ of $A \in \mathcal{K_C}$ with respect to the orders $\precapprox_{\gamma^-,\gamma^+}$ and $\subseteqq_{\gamma^-,\gamma^+}$, according to (4.21) and (4.23), corresponds respectively to the well-known sets :
$\downarrow_{\precapprox_{\gamma^-,\gamma^+}} A = \{X | X \precapprox_{\gamma^-,\gamma^+} A\}$ and $\downarrow_{\subseteqq_{\gamma^-,\gamma^+}} A = \{X | X \subseteqq_{\gamma^-,\gamma^+} A\}$.

2) Up-set of x :
$$U[x] = \{y \in \mathbb{X} \mid x \leq y\}. \tag{4.28}$$
Referring to interval case, the up-set $U[A]$ of $A \in \mathcal{K_C}$ with respect to the orders $\precapprox_{\gamma^-,\gamma^+}$ and $\subseteqq_{\gamma^-,\gamma^+}$, according to (4.20) and (4.22), corresponds respectively to the well-known sets :
$\uparrow_{\precapprox_{\gamma^-,\gamma^+}} A = \{Y | A \precapprox_{\gamma^-,\gamma^+} Y\}$ and $\uparrow_{\subseteqq_{\gamma^-,\gamma^+}} A = \{Y | A \subseteqq_{\gamma^-,\gamma^+} Y\}$.

Moreover if $S \subseteq \mathbb{X}$ is a subset of $\mathbb{X}$, we also define the sets below.

(i) Lower bounds for S:
$$S^l = \{x \in \mathbb{X} \mid x \leq s, \forall s \in S\}. \tag{4.29}$$

(ii) Upper bounds for S:
$$S^u = \{x \in \mathbb{X} \mid s \leq x, \forall s \in S\}. \tag{4.30}$$

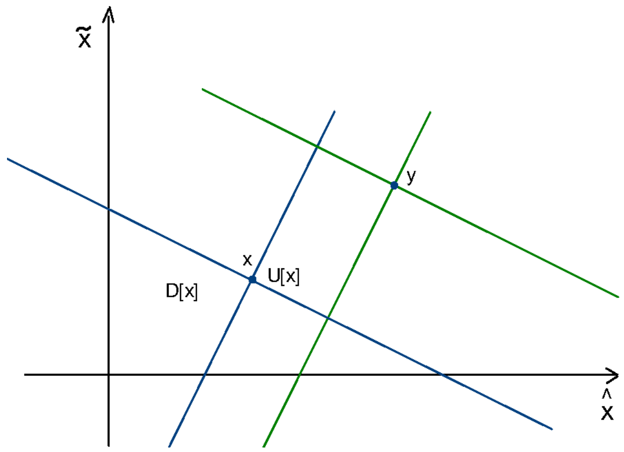


Figure 4.20: Down-set $D[X]$ and up-set $U[X]$ of interval $X \in \mathcal{K_C}$ considering the lattice $(\mathcal{K_C}, \precapprox_{\gamma^-,\gamma^+})$.

Let now recall the following definition too.

Definition 4.2.1. *([24]) A subset* $\mathbb{Y}$ *of a partial ordered set* $(\mathbb{X}, \leq)$ *is said:*

- *an order ideal if* $\forall x, y \in \mathbb{X}$, *(*$y \in \mathbb{Y}$ *and* $x \leq y$*)* $\Rightarrow x \in \mathbb{Y}$*;*

- *an order* $filter$ *if* $\forall x, y \in \mathbb{X}$, *(*$y \in \mathbb{Y}$ *and* $x \geq y$*)* $\Rightarrow x \in \mathbb{Y}$*.*

According to (4.30) and (4.29) and from Definition 4.2.1, it follows that $D[x]$ is an *order ideal* (the principal ideal of x) while $U[x]$ is an *order* $filter$ (the principal filter of x).

Similarly, in the interval case, we have the property listed below.

Proposition 4.2.1. *Considering the lattices* $(\mathcal{K_C}, \precapprox_{\gamma^-,\gamma^+})$ *and* $(\mathcal{K_C}, \subseteqq_{\gamma^-,\gamma^+})$*, we have that:*

(i) $\downarrow_{\precapprox_{\gamma^-,\gamma^+}} A$ *and* $\downarrow_{\subseteqq_{\gamma^-,\gamma^+}} A$ *are order ideals;*

(ii) $\uparrow_{\precapprox_{\gamma^-,\gamma^+}} A$ *and* $\uparrow_{\subseteqq_{\gamma^-,\gamma^+}} A$ *are order filters.*

Moreover, if x and y are elements of the poset $(\mathbb{X}, \leq)$ and $S \subseteq \mathbb{X}$ is a subset of $\mathbb{X}$, then:

(a) $x \leq y \Rightarrow U[y] \subseteq U[x]$;

(b) $x = supS \Leftrightarrow U[x] = \bigcap_{s \in S} U[s]$.

Analogously, referring to intervals, we have the following statement:

Proposition 4.2.2. *If* X, Y *are elements of* $\in \mathcal{K_C}$*, then:*

(a') $X \precapprox_{\gamma^-,\gamma^+} Y \Rightarrow (\uparrow_{\precapprox_{\gamma^-,\gamma^+}} Y) \subseteq (\uparrow_{\precapprox_{\gamma^-,\gamma^+}} X)$*;*

(b') $X = sup\mathcal{S}_\mathcal{C}$, with $\mathcal{S}_\mathcal{C} \subseteq \mathcal{K}_\mathcal{C} \Leftrightarrow (\uparrow_{\precapprox_{\gamma^-,\gamma^+}} X) = \bigcap_{S\in\mathcal{S}_\mathcal{C}}(\uparrow_{\precapprox_{\gamma^-,\gamma^+}} S)$.

Similarly:

(a") $X \subseteqq_{\gamma^-,\gamma^+} Y \Rightarrow (\uparrow_{\subseteqq_{\gamma^-,\gamma^+}} Y) \subseteq (\uparrow_{\subseteqq_{\gamma^-,\gamma^+}} X)$;

(b") $X = sup\mathcal{S}_\mathcal{C}$, with $\mathcal{S}_\mathcal{C} \subseteq \mathcal{K}_\mathcal{C} \Leftrightarrow (\uparrow_{\subseteqq_{\gamma^-,\gamma^+}} X) = \bigcap_{S\in\mathcal{S}_\mathcal{C}}(\uparrow_{\subseteqq_{\gamma^-,\gamma^+}} S)$.

4.2.2 Dedekind-McNeille completion

It is well known that good structures tend to have multiple equivalent ways of defining them, thus providing us with more possibilities to characterize them. As stated in [24], this is also the case for complete lattices since there are alternative definitions for complete lattices such as the one based on closure operators.

In this regard, let remember that, denoting by $2^{\mathbb{X}}$ the set of all subset of $\mathbb{X}$, a map

$$cl : 2^{\mathbb{X}} \longrightarrow 2^{\mathbb{X}}$$

is a closure operator if and only if:

1. $A \subseteq cl(A)$ for all $A \subseteq \mathbb{X}$ (increesing);
2. if $A \subseteq B$ then $cl(A) \subseteq cl(B)$ for all $A, B \subseteq \mathbb{X}$ (monotone);
3. $cl(cl(A)) = cl(A)$ for all $A \subseteq \mathbb{X}$ (idempotent).

For example, if we consider a poset $(\mathbb{X}, \leq)$ and any $\mathrm{Y} \subseteq \mathbb{X}$, we have that $cl(\mathrm{Y}) = \{x \in \mathbb{X} | \exists y \in \mathrm{Y} \text{ with } x \leq y\}$ is a closure operator.
We now want to find a complete lattice that embedded the poset $(\mathbb{X}, \leq)$ using the so-called *Dedekind−MacNeille completion* (DM completion for short), also called *normal completion* or *completion by cuts*. This completion is based on a closure operator and, since closure operators are equivalent to complete lattice, we obtain that applying DM completion to a specific poset will give us the desired result.

Nevertheless, before continuing, the following significant definitions and facts must be recalled.

Definition 4.2.2. *([24]) If $S \subseteq \mathbb{X}$, we say that S is a cut of $\mathbb{X}$ if and only if $(S^u)^l = S$.*

Let remember that a *cut* can be defined also in terms of a pair (S, T) with $S, T \subseteq \mathbb{X}$ such that $S^u = T$ and $T^l = S$.

Definition 4.2.3. *([24]) For a given poset $(\mathbb{X}, \leq)$ the Dedekind-MacNeille completion is the poset formed with the set of all the cut of $\mathbb{X}$ under the set of inclusion, i.e.,*

$$DM(\mathbb{X}) = \left(\{A \subseteq \mathbb{X} | (A^u)^l = A\}, \subseteq\right). \tag{4.31}$$

We have that $DM(\mathbb{X})$, with respect to inclusion $\subseteq$, is a complete lattice and the original poset $(\mathbb{X}, \leq)$ is embedded in $(DM(\mathbb{X}), \subseteq)$

Lemma 4.2.1. *([24]) If S, T are cuts, then so is $S \cap T$.*

It is interesting to note that another completion can be obtained using the ideals; the embedding by ideals yields in general a larger lattice than by DM, as $DM(\mathbb{X})$ is the smallest complete lattice that embeds $\mathbb{X}$.
If we consider $S \subseteq \mathbb{X}$, assuming $infS$ exists or $supS$ exists, we have that, $\forall x, y \in \mathbb{X}$:

- $x \leq infS \Leftrightarrow x \leq s, \forall s \in S$,
- $y \geq supS \Leftrightarrow y \geq s, \forall s \in S$,
- $x \leq y \Leftrightarrow (z \leq x \Rightarrow z \leq y), \forall z \in \mathbb{X}$.

Proposition 4.2.3. *For all $x \in \mathbb{X}$, we have $D[x] \in DM[\mathbb{X}]$, i.e.,*

$$((D[x])^u)^l = \ D[x].$$

Proof. As $D[x]^u = \{y \in \mathbb{X} | z \leq y, \forall z \in D[x]\}$, we consider $y \in D[x]^u$, i.e., by definition of upper bounds, $(z \in D[x] \Rightarrow z \leq y)$.
According to (4.27), this is equivalent to say $(z \leq x \Rightarrow z \leq y)$, which implies that $x \leq y$. This means that $y \in U[x]$, so we have $D[x]^u = U[x]$.
By duality is easy to show that $U[x]^l = D[x]$. So $((D[x])^u)^l = D[x]$. □

In particular we have that the map from $\mathbb{X}$ to $DM(\mathbb{X})$

$$x \longrightarrow D[x]$$

preserves all l.u.b. (least upper bound) and g.l.b. (greatest lower bound) in $\mathbb{X}$, i.e., if $infS$ exists, then

$$\bigcap_{s \in S} D[s] = D[infS].$$

Indeed, we have

$$x \in \bigcap_{s \in S} D[s] \ \text{ iff } \ x \in D[s], \forall s \in S, \ \ \text{that is, } \ x \leq s, \forall s \in S,$$

but this also means that $x \leq infS$; therefore, by (4.27), we have
$D[s] = \{x \in \mathbb{X} \mid x \leq s\}$ and then $x \in D[infS] = \{x \in \mathbb{X} \mid x \leq infS\}$.
This means exactly that $\bigcap_{s \in S} D[s] = D[infS]$.

The property that $DM(\mathbb{X})$ is the "smallest" completion of $\mathbb{X}$ can be proved by showing that $DM(\mathbb{X})$ contains only the elements necessary to complete $\mathbb{X}$, which is what we want to show now.

Based on what is stated in [24], we can reformulate the following definition.

Definition 4.2.4. *Considering two posets $\mathbb{X}$ and $\mathbb{Y}$, we say that:*

(a) $\mathbb{X}(\subseteq \mathbb{Y})$ *is join-dense in* $\mathbb{Y}$ *iff every* $x \in \mathbb{Y}$ *is expressible as the join of a subset* $S \subseteq \mathbb{X}$*, that is,*

$$\forall y \in \mathbb{Y} \Rightarrow \exists S \subseteq \mathbb{X} \mid y = supS;$$

(b) $\mathbb{X}(\subseteq \mathbb{Y})$ *is meet-dense in* $\mathbb{Y}$ *iff every* $x \in \mathbb{Y}$ *is expressible as the meet of a subset* $S \subseteq \mathbb{X}$*, that is,*

$$\forall y \in \mathbb{Y} \Rightarrow \exists S \subseteq \mathbb{X} \mid y = infS.$$

An important step for the construction of a completion of $\mathbb{X}$ is represented by the following theorem.

Theorem 4.2.1. *([24]) Let* $(\mathbb{X}, \leq)$ *be any poset and* $f : \mathbb{X} \to DM(\mathbb{X})$ *be defined by* $f(x) = D[x]$*. Then* $f(\mathbb{X})$ *is join-dense (and meet-dense) in* $DM(\mathbb{X})$*. Moreover, if* $\mathbb{L}$ *is a complete lattice such that* $\mathbb{X}$ *is meet-dense and join-dense in* $\mathbb{L}$*, then* $\mathbb{L}$ *is isomorphic to* $DM(\mathbb{X})$

At this point we are ready to provide a completion algorithm of a poset in the discrete case, the so called *Dedekind-MacNeille completion algorithm* (DM algorithm for short).

- When $\mathbb{X} = \{x\}$ is a singleton, then its completion is itself:

$$DM(\{x\}) = \{x\}.$$

- If we add an element y to $\mathbb{X}$, i.e., $\mathbb{X} = \{x, y\}, x \neq y$, we can obtain the completion of $\{x, y\}$ adding to $DM(\{x\})$ the elements $inf(x, y) = x \wedge y$ and $sup(x, y) = x \vee y$:

$$DM(\{x, y\}) = \{x, y, x \wedge y, x \vee y\}.$$

- If now we add an element z to $\mathbb{X}$, i.e., $\mathbb{X} = \{x, y, z\}$, $x \wedge y \leq z \leq x \vee y$, we have that:

$$DM(\{x, y, z\}) = \{x, y, x \wedge y, x \vee y, z, a, b, c, d\}$$

  where $a = z \wedge y$, $b = z \vee y$, $c = x \vee z$ and $d = x \wedge z$.

- If again $\mathbb{X} = \{x, y, z, w\}$, we have to add six new elements obtained by considering $w \wedge t$ and $w \vee t$, for all $t \in DM(\{x, y, z\})$ that are incomparable with w.

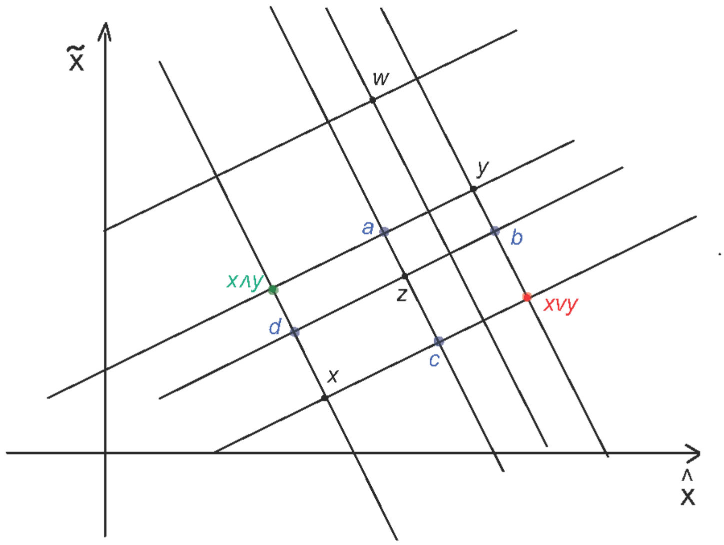


Figure 4.21: Example of the first three steps of the algorithm for Dedekind-MecNeille completion of the poset $(\mathcal{K}_\mathcal{C}, \precapprox_{\gamma^-,\gamma^+})$.

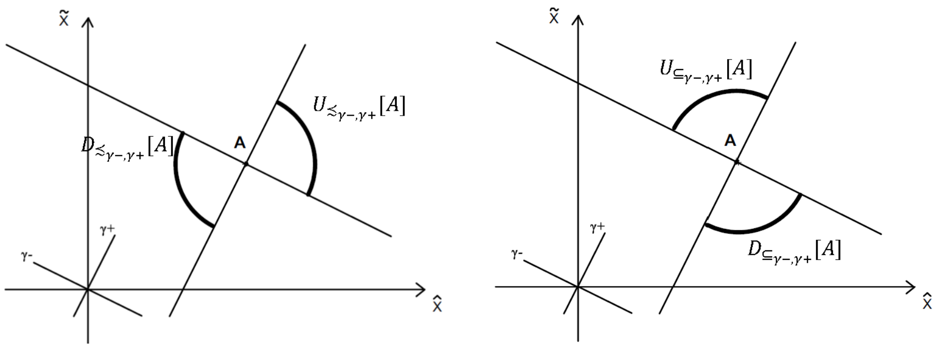


Figure 4.22: Interval A as down-set and up-set separator:
case of $(\mathcal{K}_\mathcal{C}, \precapprox_{\gamma^-,\gamma^+})$: $A = supD_{\precapprox_{\gamma^-,\gamma^+}}[A] = infU_{\precapprox_{\gamma^-,\gamma^+}}[A]$, $\forall A \in \mathcal{K}_\mathcal{C}$ (left);
case of $(\mathcal{K}_\mathcal{C}, \subseteqq_{\gamma^-,\gamma^+})$: $A = supD_{\subseteqq_{\gamma^-,\gamma^+}}[A] = infU_{\subseteqq_{\gamma^-,\gamma^+}}[A]$, $\forall A \in \mathcal{K}_\mathcal{C}$ (right).

Referring to the interval case, the algorithm for the Dedekind-MacNeille completion of the poset $(\mathcal{K}_\mathcal{C}, \precapprox_{\gamma^-,\gamma^+})$ is represented graphically in Figure 4.21, where the first three steps are clearly highlighted.

Furthermore, in Figure 4.22, it is pointed out that each interval A separates $D \precapprox_{\gamma^-,\gamma^+} [A]$ and $U \precapprox_{\gamma^-,\gamma^+} [A]$ in a unique way and we have:

$$A = supD_{\precapprox_{\gamma^-,\gamma^+}}[A] = infU_{\precapprox_{\gamma^-,\gamma^+}}[A].$$

As far as poset $(\mathcal{K}_\mathcal{C}, \subseteqq_{\gamma^-,\gamma^+})$ is concerned, we can proceed in a dual way so that we can write:

$$A = supD_{\subseteqq_{\gamma^-,\gamma^+}}[A] = infU_{\subseteqq_{\gamma^-,\gamma^+}}[A].$$

In conclusion, it is clear that the DM completion of the two algebraic lattices $(\mathcal{K}_C, \wedge, \vee, \precapprox_{\gamma^-,\gamma^+})$ and $(\mathcal{K}_C, \cap, \uplus, \subseteqq_{\gamma^-,\gamma^+})$ defined in Section 4.1 are isomorphic; this means that the two polar orders $\precapprox_{\gamma^-,\gamma^+}$ and $\subseteqq_{\gamma^-,\gamma^+}$ produce the same Dedekind-MacNeille completion.

4.2.3 Further properties related to polar orders

In this final Subsection, for semplicity and graphical needs, we will restrict our work to the particular cases of LU-order and $\subseteqq_{-1,1}$-order.

As shown in Figure 4.23, the following relations between down-set and up-set of an interval hold.

Proposition 4.2.4. *Let $A = (\widehat{a};\widetilde{a}) = [a^-, a^+] \in \mathcal{K}_{\mathcal{C}}$. Considering the $\precapprox_{LU}$-order and the $\subseteqq_{-1,1}$-order, we have that:*

(1) $D_{\precapprox_{LU}}[a^-] \subseteq D_{\precapprox_{LU}}[A] \subseteq D_{\precapprox_{LU}}[a^+]$,
$U_{\precapprox_{LU}}[a^+] \subseteq U_{\precapprox_{LU}}[A] \subseteq U_{\precapprox_{LU}}[a^-]$;

(2) $\begin{cases} U_{\subseteqq_{-1,1}}[A] \subseteq U_{\subseteqq_{-1,1}}[a^-] \\ U_{\subseteqq_{-1,1}}[A] \subseteq U_{\subseteqq_{-1,1}}[a^+] \end{cases}$ *and* $\begin{cases} D_{\subseteqq_{-1,1}}[a^-] \subseteq D_{\subseteqq_{-1,1}}[A] \\ D_{\subseteqq_{-1,1}}[a^+] \subseteq D_{\subseteqq_{-1,1}}[A]. \end{cases}$

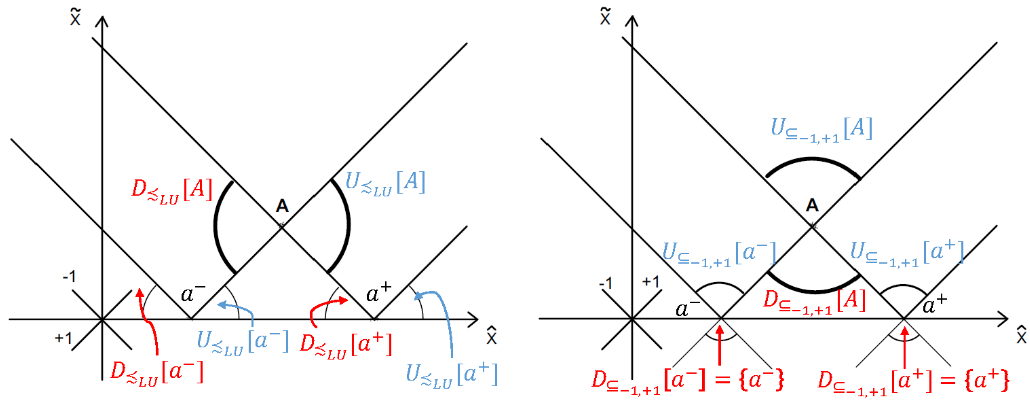


Figure 4.23: Relations between down-set and up-set of interval A with respect to $\precapprox_{LU}$-order (left) and to $\subseteqq_{-1,1}$-order (right).

It is also easy to verify that:

(a) $D_{\precapprox_{LU}}[A] = D_{\precapprox_{LU}}[a^-] \cup (U_{\subseteqq_{-1,1}}[a^-] \setminus U_{\subseteqq_{-1,1}}[A])$,

(b) $U_{\precapprox_{LU}}[A] = U_{\precapprox_{LU}}[a^+] \cup (U_{\subseteqq_{-1,1}}[a^+] \setminus U_{\subseteqq_{-1,1}}[A])$,

(c) $D_{\subseteqq_{-1,1}}[A] = D_{\precapprox_{LU}}[a^+] \setminus D_{\precapprox_{LU}}[A] = U_{\precapprox_{LU}}[a^-] \setminus U_{\precapprox_{LU}}[A]$.

Moreover, as shown in Figure 4.24, we even highlight that all the intervals $B = (\widehat{b}; \widetilde{b}) = [b^-, b^+]$, having an extreme in common with $A = (\widehat{a}; \widetilde{a}) = [a^-, a^+]$, belong to the intersections between down-set and up-set of X.

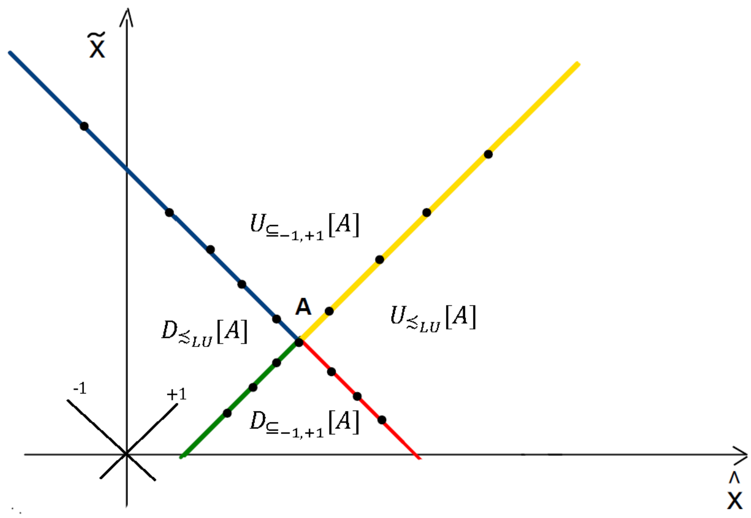


Figure 4.24: Intervals (having an extreme in common with interval A) lying on the intersections between down-set and up-set of interval A.

More specifically, if B is such that:

- $b^- = a^-$ and $b^+ > a^+$, then $B \in U_{\subseteqq_{-1,1}}[A] \cap U_{\precapprox_{LU}}[A]$ (yellow line);
- $b^- = a^-$ and $b^+ < a^+$, then $B \in D_{\subseteqq_{-1,1}}[A] \cap D_{\precapprox_{LU}}[A]$ (green line);
- $b^+ = a^+$ and $b^- < a^-$, then $B \in U_{\subseteqq_{-1,1}}[A] \cap D_{\precapprox_{LU}}[A]$ (blue line);
- $b^+ = a^+$ and $b^- > a^-$, then $B \in D_{\subseteqq_{-1,1}}[A] \cap U_{\precapprox_{LU}}[A]$ (red line).

Eventually, we can also define up-sets and down-sets in a "strict" sense as follow:

$$U_{\subset}[A] = \{Z | Z \in U_{\subseteq}[A],\ z^-, z^+ \notin \{a^-, a^+\}\},$$
$$U_{\prec}[A] = \{Z | Z \in U_{\precapprox}[A],\ z^-, z^+ \notin \{a^-, a^+\}\}.$$

Similarly:

$$D_{\subset}[A] = \{Z | Z \in D_{\subseteq}[A],\ z^-, z^+ \notin \{a^-, a^+\}\},$$
$$D_{\prec}[A] = \{Z | Z \in D_{\precapprox}[A],\ z^-, z^+ \notin \{a^-, a^+\}\}.$$

Chapter 5

Interval algebraic structures

In this chapter the concepts introduced in the previous ones will be applied in order to outline some aspects of the basic algebraic structures, in an attempt to enrich the theory, thus overcoming the limitation and narrowness that up to now has been found in the literature whenever one has tried to introduce and define non-trivial interval algebraic structures.

Furthermore, in the attempt, on the one hand to maintain the validity of important properties, on the other to explore new ones, more types of approaches will be proposed, from which as many interval algebraic structures will arise.

In particular, thanks to the concept of polarity between orders, it will be possible to determine algebraic structures hitherto unexplored in interval theory, some quite well-known, such as semirings or pre-semirings, others more unusual, ranging from lattice-ordered semigroups to the so-called clodum.

5.1 Interval semirings

We will begin our process of building and redefining algebraic interval structures starting from what was introduced in Section 4.1, but, before going on, it is necessary to recall some classical definitions. In particular we will refer to those used in [25] and [26].

5.1.1 Semirings in classic algebra

Definition 5.1.1. *We define semigroup $(G, *)$ a nonempty set G equipped with an associative operation $*$.*
A semigroup $(G, *)$ is a monoid with neutral element (or identity) e if

$$\forall x \in G,\ \exists e \in G,\ \textit{such that}\ \ x * e = x.$$

A monoid $(G, *)$ is a group if each of its elements has an inverse, that is,

$$\forall x \in G,\ \exists x',\ \textit{such that}\ \ x * x' = e.$$

If $*$ is commutative (i.e., $x * y = y * x,\ \forall x, y \in G$), then the structure is said to be commutative (or abelian, in case it is a group).

Definition 5.1.2. *Let PS be a nonempty set equipped with two binary operations, denoted by $+$ and $\cdot$ (called addition and multiplication). We define a pre-semiring the structure $(PS, +, \cdot)$, such that the following conditions are satisfied.*

1) $(PS, +)$ is a commutative semigroup:

- *(i) operation $+$ is associative: $(x+y)+z = x+(y+z), \forall x, y, z \in PS$;*
- *(ii) operation $+$ is commutative: $x + y = y + x, \forall x, y \in PS$.*

2) $(PS, \cdot)$ is a semigroup:

operation $\cdot$ is associative: $(x \cdot y) \cdot z = x \cdot (y \cdot z), \forall x, y, z \in PS$.

3) Multiplication $\cdot$ is left and right distributive over addition $+$:

- *(i) $x \cdot (y + z) = (x \cdot y) + (x \cdot z), \forall x, y, z \in PS$;*
- *(ii) $(x + y) \cdot z = (x \cdot z) + (y \cdot z), \forall x, y, z \in PS$.*

If also multiplication $\cdot$ is commutative (i.e., $x \cdot y = y \cdot z, \forall x, y \in PS$), then the pre-semiring is said to be commutative.

Remark 5.1.1. *We observe that in the above definition, we do not assume the existence of neutral elements. If it does exist, then, depending on the case, we will talk about:*

- *pre-semiring with zero (which includes the existence of the neutral element with respect to addition* $+$*);*

- *pre-semiring with unity (which includes the existence of the neutral element with respect to multiplication* $\cdot$*);*

- *pre-semiring with zero and unity (which includes the existence of both neutral elements with respect to addition* $+$ *and multiplication* $\cdot$*).*

In the latter case, according to [26], we can also define a pre-semiring as a tuple $(PS, +, \cdot, 0, 1)$ *where* $+$ *and* $\cdot$ *are binary operators on* PS *for which* $(PS, +, 0)$ *is a commutative monoid,* $(PS, \cdot, 1)$ *is a monoid, and* $\cdot$ *distributes over* $+$*.*

Definition 5.1.3. *Let* H *be a nonempty set equipped with two binary operations, denoted by* $+$ *and* $\cdot$ *(called addition and multiplication). We define a hemiring the structure* $(H, +, \cdot)$*, such that the following conditions are satisfied.*

1) $(H, +)$ *is a commutative monoid with neutral element 0:*

  (i) operation $+$ *is associative:* $(x + y) + z = x + (y + z), \forall x, y, z \in H$*;*

  (ii) operation $+$ *has a neutral element* $0 \in S$*:* $0+x = x+0 = x, \forall x \in H$ *(called additive identity or zero or 0-element of the hemiring);*

  (iii) operation $+$ *is commutative:* $x + y = y + x, \forall x, y \in H$*.*

2) $(H, \cdot)$ *is a semigroup:*

  operation $\cdot$ *is associative:* $(x \cdot y) \cdot z = x \cdot (y \cdot z), \forall x, y, z \in H$*.*

3) Multiplication $\cdot$ *is left and right distributive over addition:*

  (i) $x \cdot (y + z) = (x \cdot y) + (x \cdot z), \forall x, y, z \in H$*;*

  (ii) $(x + y) \cdot z = (x \cdot z) + (y \cdot z), \forall x, y, z \in H$*.*

4) 0 is the absorbing element for multiplication:

  $0 \cdot x = x \cdot 0 = 0, \forall x \in H$*.*

If also multiplication $\cdot$ *is commutative (i.e.,* $x \cdot y = y \cdot x, \forall x, y \in H$*), then the hemiring is said to be commutative.*

Definition 5.1.4. *Let* P *be a set equipped with two binary operations, denoted by* $+$ *and* $\cdot$ *(called addition and multiplication). We define a pseudoring or non-unitary ring, the structure* $(P, +, \cdot)$*, such that the following conditions are satisfied.*

1) $(P, +)$ *is an abelian group with neutral element 0:*

(i) *operation + is associative:* $(x+y)+z = x+(y+z), \forall x, y, z \in P$*;*

(ii) *operation + has a neutral element* $0 \in P : 0+x = x+0 = x, \forall x \in P$ *(called additive identity or zero or 0-element of the pseudoring);*

(iii) *existence of the opposite or symmetric element with respect to +:* $\forall x \in P$, $\exists x'(= -x)$, *such that* $x + x' = 0$ *(called additive inverse of the pseudoring);*

(iv) *operation + is commutative:* $x + y = y + x, \forall x, y \in P$.

2) $(P, \cdot)$ *is a semigroup:*

operation $\cdot$ *is associative:* $(x \cdot y) \cdot z = x \cdot (y \cdot z), \forall x, y, z \in P$.

3) *Multiplication* $\cdot$ *is left and right distributive over addition +:*

(i) $x \cdot (y+z) = (x \cdot y) + (x \cdot z), \forall x, y, z \in P$*;*

(ii) $(x+y) \cdot z = (x \cdot z) + (y \cdot z), \forall x, y, z \in P$.

4) *0 is the absorbing element for multiplication:*

$$0 \cdot x = x \cdot 0 = 0, \forall x \in P.$$

If multiplication $\cdot$ *is commutative (i.e.,* $x \cdot y = y \cdot x, \forall x, y \in P$*), then the pseudoring is said to be commutative.*

Definition 5.1.5. *Let* S *be a set equipped with two binary operations, denoted by + and* $\cdot$ *(called addition and multiplication). We define a semiring the structure* $(S, +, \cdot)$*, such that the following conditions are satisfied.*

1) $(S, +)$ *is a commutative monoid with neutral element 0:*

(i) *operation + is associative:* $(x+y)+z = x+(y+z), \forall x, y, z \in S$*;*

(ii) *operation + has a neutral element* $0 \in S : 0+x = x+0 = x, \forall x \in S$ *(called additive identity or zero or 0-element of the semiring);*

(iii) *operation + is commutative:* $x + y = y + x, \forall x, y \in S$.

2) $(S, \cdot)$ *is a monoid with neutral element 1:*

(i) *operation* $\cdot$ *is associative:* $(x \cdot y) \cdot z = x \cdot (y \cdot z), \forall x, y, z \in S$*;*

(ii) *operation* $\cdot$ *has a neutral element* $1 \in S$*:* $1 \cdot x = x \cdot 1 = x, \forall x \in S$ *(called multiplicative identity or unity or 1-element of the semiring).*

3) *Multiplication* $\cdot$ *is left and right distributive over addition +:*

(i) $x \cdot (y+z) = (x \cdot y) + (x \cdot z), \forall x, y, z \in S$*;*

(ii) $(x+y) \cdot z = (x \cdot z) + (y \cdot z), \forall x, y, z \in S$.

$(A,+,\cdot)$	**pre-semiring (com.)**	**hemiring (com.)**	**pseudoring (com.)**	**semiring (com.)**	**ring (com.)**
$(A,+)$	semigroup com.	monoid com.	group ab.	monoid com.	group ab.
$+$ associative	x	x	x	x	x
$+$ commutative	x	x	x	x	x
neutral element 0		x	x	x	x
additive inverse			x		x
$(A,\cdot)$	semigroup (com.)	semigroup (com.)	semigroup (com.)	monoid (com.)	monoid (com.)
$\cdot$ associative	x	x	x	x	x
neutral element 1				x	x
($\cdot$ commutative)	(x)	(x)	(x)	(x)	(x)
$\cdot$ distributive wrt $+$	x	x	x	x	x
0 absorbing for $\cdot$		x	x	x	x

Table 5.1: Classification of the main algebraic structures with their associated properties. ab. = abelian, com. = commutative.

4) 0 is the absorbing element for multiplication:

$$0 \cdot x = x \cdot 0 = 0, \forall x \in S.$$

If also multiplication $\cdot$ is commutative (i.e., $x \cdot y = y \cdot x, \forall x, y \in S$), then the semiring is said to be commutative.

Remark 5.1.2. *Note that:*

(i) in order for a pre-semiring (with zero and unity) to be a semiring, 0 *(the neutral element for +) must be absorbing for $\cdot$;*

(ii) in order for a hemiring to be a semiring, the multiplicative identity 1 *(i.e., neutral element for $\cdot$) must exist;*

(iii) the well known ring structure $(R, +, \cdot)$ results to be given by a combination of the pseudoring and semiring structures as it is defined by the following conditions:

- *$(R, +)$ is an additive abelian group;*
- *$(R, \cdot)$ is a multiplicative monoid;*
- *multiplication $\cdot$ is left and right distributive over addition +;*
- 0 *is the absorbing element for multiplication.*

Table 5.1 provides a classification of the algebraic structures defined in this subsection and the properties associated with.

We now denote the set of all additively-idempotent elements of a semiring S by

$$I^{+}(S) = \{i \in S \mid i + i = i\}.$$

This set is nonempty since it contains the 0-element of the semiring. We also denote the set of all multiplicatively-idempotent elements of S by

$$I^{\times}(S) = \{i \in S \mid i \cdot i = i\}.$$

This set is nonempty too since it contains the 1-element of the semiring.

Definition 5.1.6. *The semiring $(S, +, \cdot)$ is said to be doubly-idempotent (or simply idempotent) if and only if is both additively and multiplicatively idempotent (i.e., if and only if $S = I^{+}(S) \cap I^{\times}(S)$).*

Remark 5.1.3. *Note that if the zero element and the unity element of a semiring S coincide, i.e., $1 = 0$, then $s = s \cdot 1 = s \cdot 0 = 0$ for each element s of S and so $S = O$. In order to avoid this trivial case, we will assume that all semirings under consideration are nontrivial, i.e., $1 \neq 0$.*

Definition 5.1.7. *A semiring $(S, +, \cdot)$ is said to be zero-sum-free or antinegative if and only if, given $x, y \in S$, we have*

$$x + y = 0 \Rightarrow x = y = 0. \tag{5.1}$$

Antinegative semirings are also called antirings.

Definition 5.1.8. *A semiring $(S, +, \cdot)$ is said to be zero-divisor-free or entire if and only if, given $x, y \in S$, we have*

$$x \cdot y = 0 \Rightarrow x = 0 \text{ or } y = 0. \tag{5.2}$$

We can also redefine some general notations as follows.

Definition 5.1.9. *Let S be a set equipped with two binary operations, $\circ_1$ and $\circ_2$. We define a semiring the structure $(S, \circ_1, \circ_2)$, such that the following conditions are satisfied.*

1) $(S, \circ_1)$ is a commutative monoid with neutral element i_1:

- *(i) operation $\circ_1$ is associative;*
- *(ii) operation $\circ_1$ has the neutral element i_1 (called 0-element of the semiring);*
- *(iii) operation $\circ_1$ is commutative.*

2) $(S, \circ_2)$ is a monoid with neutral element i_2:

- *(i) operation$\circ_2$ is associative;*

(ii) operation $\circ_2$ *has the neutral element* i_2 *(called 1-element of the semiring).*

3) Operation $\circ_2$ *is left and right distributive over* $\circ_1$*:*

(i) $x \circ_2 (y \circ_1 z) = (x \circ_2 y) \circ_1 (x \circ_2 z), \forall x, y, z \in S;$

(ii) $(x \circ_1 y) \circ_2 z = (x \circ_2 z) \circ_1 (y \circ_2 z), \forall x, y, z \in S.$

4) i_1 *is the absorbing element for* $\circ_2$ *:*

$$i_1 \circ_2 x = x \circ_2 i_1 = i_1, \forall x \in S.$$

If also $\circ_2$ *is commutative, we say that the semiring* $(S, \circ_1, \circ_2)$ *is commutative.*

Definition 5.1.10. *An element* x *of a semiring* $(S, \circ_1, \circ_2)$ *is defined:*

- *additively idempotent (or* $\circ_1$*-idempotent) if and only if*

$$x \circ_1 x = x \text{ in } S;$$

- *multiplicatively idempotent (or* $\circ_2$*-idempotent) if and only if*

$$x \circ_2 x = x \text{ in } S.$$

We denote the set of all additively-idempotent (or $\circ_1$-idempotent) elements of S by

$$I^{\circ_1}(S) = \{x \in S \mid x \circ_1 x = x\}.$$

This set is nonempty since it contains the 0-element of the semiring S. We also denote the set of all multiplicatively-idempotent (or $\circ_2$-idempotent) elements of S by

$$I^{\circ_2}(S) = \{x \in S \mid x \circ_2 x = x\}.$$

This set is nonempty too since it contains the 1-element of the semiring S.

Definition 5.1.11. *The semiring* $(S, \circ_1, \circ_2)$ *is idempotent if and only if is both additively and multiplicatively idempotent (i.e., if and only if* $S = I^{\circ_1}(S) \cap I^{\circ_2}(S)$ *).*

Definition 5.1.12. *A semiring* $(S, \circ_1, \circ_2)$ *is said to be zero-sum-free or antinegative if and only if, given* $x, y \in S$*, we have*

$$x \circ_1 y = i_1 \Rightarrow x = y = i_1. \tag{5.3}$$

Antinegative semirings are also called antirings.

Definition 5.1.13. *A semiring* $(S, \circ_1, \circ_2)$ *is said to be zero-divisor-free or entire if and only if, given* $x, y \in S$*, we have*

$$x \circ_2 y = i_1 \Rightarrow x = i_1 \text{ or } y = i_1. \tag{5.4}$$

5.1.2 Interval structures as semirings

As already done in Section 4.1, we consider

$$\overline{\mathcal{K}_C} = \mathcal{K}_\mathcal{C} \cup \{-\infty, +\infty\}$$

where, for all $\gamma^-, \gamma^+ \in \mathbb{R}$ such that $\gamma^- \leqslant \gamma^+$, we define:

$$-\infty = [-\infty, -\infty] = (-\infty; 0) = inf_{\precsim_{\gamma^-,\gamma^+}} \overline{\mathcal{K}_C};$$

$$+\infty = [+\infty, +\infty] = (+\infty; 0) = sup_{\precsim_{\gamma^-,\gamma^+}} \overline{\mathcal{K}_C},$$

that is, in midpoint notation:

$$A \in \overline{\mathcal{K}_C} \Leftrightarrow A = (\widehat{a}; \widetilde{a}) \text{ with } \widehat{a} \in \mathbb{R}, \widetilde{a} \geqslant 0 \ \text{ or } \ A = (-\infty; 0) \ \text{ or } \ A = (+\infty; 0).$$

Then we proceed by considering $\overline{\mathcal{K}_C}$ associated with the well known partial order $\precsim_{\gamma^-,\gamma^+}$; in particular, when $\gamma^- = -1$ and $\gamma^+ = +1$, we can denote the $\precsim_{LU}$-order (where $\precsim_{LU} = \precsim_{-1,1}$) simply with $\precsim$.

It is well known (see Subsection 2.2.7) that $(\overline{\mathcal{K}_C}, \precsim_{\gamma^-,\gamma^+})$, or more specifically $(\overline{\mathcal{K}_C}, \vee, \wedge, \precsim_{\gamma^-,\gamma^+})$, is a complete lattice with:

(i) $-\infty = (-\infty; 0)$ as the $0_{\precsim_{\gamma^-,\gamma^+}}$-element;

(ii) $+\infty = (+\infty; 0)$ as the $1_{\precsim_{\gamma^-,\gamma^+}}$-element;

(iii) operations $\vee$ and $\wedge$ are defined, according to (4.8) and (4.9), for all A, B in $\overline{\mathcal{K}_C}$ (see also Figure 4.15), by:

$$A \vee B \stackrel{def}{=} sup_{\precsim_{\gamma^-,\gamma^+}} \{A, B\};$$

$$A \wedge B \stackrel{def}{=} inf_{\precsim_{\gamma^-,\gamma^+}} \{A, B\}.$$

It follows immediately that:

a1) $A \wedge (+\infty; 0) = A, \forall A \in \overline{\mathcal{K}_C}$ (this means that $(+\infty; 0)$ can be considerated the neutral element for $\wedge$);

a2) $A \vee (-\infty; 0) = A, \forall A \in \overline{\mathcal{K}_C}$ (this means that $(-\infty; 0)$ can be considerated the neutral element for $\vee$);

b1) $A \wedge (-\infty; 0) = (-\infty; 0), \forall A \in \overline{\mathcal{K}_C}$ (so $(-\infty; 0)$ is the absorbing element for $\wedge$);

b2) $A \vee (+\infty; 0) = (+\infty; 0), \forall A \in \overline{\mathcal{K}_C}$ (so $(+\infty; 0)$ is the absorbing element for $\vee$);

c) $\wedge$ and $\vee$ are associative, commutative and distributive (left and right and each-other).

Therefore, we obtain the following proposition.

Proposition 5.1.1. *$(\overline{\mathcal{K}_C}, \vee, \wedge)$ and $(\overline{\mathcal{K}_C}, \wedge, \vee)$ are commutative, idempotent semirings.*

Proof. $(\overline{\mathcal{K}_C}, \vee, \wedge)$ is a commutative, idempotent semiring, as:

1a) $(\overline{\mathcal{K}_C}, \vee)$ is a commutative, idempotent monoid with neutral element $(-\infty; 0)$:

(i) $\vee$ is associative: $(A \vee B) \vee C = A \vee (B \vee C), \forall A, B, C \in \overline{\mathcal{K}_C}$;

(ii) $\vee$ has a neutral element $i_\vee = (-\infty; 0) \in \overline{\mathcal{K}_C}$: $A \vee (-\infty; 0) = (-\infty; 0) \vee A = A, \forall A \in \overline{\mathcal{K}_C}$ (so, $(-\infty; 0)$ is the 0-element of the semiring $(\overline{\mathcal{K}_C}, \vee, \wedge)$);

(iii) $\vee$ is commutative: $A \vee B = B \vee A, \forall A, B \in \overline{\mathcal{K}_C}$;

(iv) $\vee$ is idempotent: $A \vee A = A, \forall A \in \overline{\mathcal{K}_C}$.

2a) $(\overline{\mathcal{K}_C}, \wedge)$ is a commutative, idempotent monoid with neutral element $(+\infty; 0)$:

(i) $\wedge$ is associative: $(A \wedge B) \wedge C = A \wedge (B \wedge C), \forall A, B, C \in \overline{\mathcal{K}_C}$;

(ii) $\wedge$ has a neutral element $i_\wedge = (+\infty; 0) \in \overline{\mathcal{K}_C}$: $A \wedge (+\infty; 0) = (+\infty; 0) \wedge A = A, \forall A \in \overline{\mathcal{K}_C}$ (so, $(+\infty; 0)$ is the 1-element of the semiring $(\overline{\mathcal{K}_C}, \vee, \wedge)$);

(iii) $\wedge$ is commutative: $A \wedge B = B \wedge A, \forall A, B \in \overline{\mathcal{K}_C}$;

(iv) $\wedge$ is idempotent : $A \wedge A = A, \forall A \in \overline{\mathcal{K}_C}$.

3a) $\wedge$ is left and right distributive over $\vee$:

(i) $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C), \forall A, B, C \in \overline{\mathcal{K}_C}$;

(ii) $(A \vee B) \wedge C = (A \wedge C) \vee (B \wedge C), \forall A, B, C \in \overline{\mathcal{K}_C}$.

4a) $(-\infty; 0)$ is the absorbing element for $\wedge$:

$$A \wedge (-\infty; 0) = (-\infty; 0) \wedge A = (-\infty; 0), \forall A \in \overline{\mathcal{K}_C}.$$

Analogously, we have that also $(\overline{\mathcal{K}_C}, \wedge, \vee)$ is a commutative, idempotent semiring, as:

1b) $(\overline{\mathcal{K}_C}, \wedge)$ is a commutative, idempotent monoid with neutral element $(+\infty; 0)$:

(i) $\wedge$ is associative: $(A \wedge B) \wedge C = A \wedge (B \wedge C), \forall A, B, C \in \overline{\mathcal{K}_C}$;

(ii) $\wedge$ has a neutral element $i_\wedge = (+\infty; 0) \in \overline{\mathcal{K}_C}$: $A \wedge (+\infty; 0) = (+\infty; 0) \wedge A = A, \forall A \in \overline{\mathcal{K}_C}$ (so, $(+\infty; 0)$ is the 0-element of the semiring $(\overline{\mathcal{K}_C}, \wedge, \vee)$);

(iii) $\wedge$ is commutative: $A \wedge B = B \wedge A, \forall A, B \in \overline{\mathcal{K}_C}$;

(iv) $\wedge$ is idempotent: $A \wedge A = A, \forall A \in \overline{\mathcal{K}_C}$.

2b) $(\overline{\mathcal{K}_C}, \vee)$ is a commutative, idempotent monoid with neutral element $(-\infty; 0)$:

(i) $\vee$ is associative: $(A \vee B) \vee C = A \vee (B \vee C), \forall A, B, C \in \overline{\mathcal{K}_C}$;

(ii) $\vee$ has a neutral element $i_\vee = (-\infty; 0) \in \overline{\mathcal{K}_C}$: $A \vee (-\infty; 0) = (-\infty; 0) \vee A = A, \forall A \in \overline{\mathcal{K}_C}$ (so, $(-\infty; 0)$ is the 1-element of the semiring $(\overline{\mathcal{K}_C}, \wedge, \vee)$);

(iii) $\vee$ is commutative: $A \vee B = B \vee A, \forall A, B \in \overline{\mathcal{K}_C}$;

(iv) $\vee$ is idempotent : $A \vee A = A, \forall A \in \overline{\mathcal{K}_C}$.

3b) $\vee$ is left and right distributive over $\wedge$:

(i) $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C), \forall A, B, C \in \overline{\mathcal{K}_C}$;

(ii) $(A \wedge B) \vee C = (A \vee C) \wedge (B \vee C), \forall A, B, C \in \overline{\mathcal{K}_C}$.

4b) $(+\infty; 0)$ is the absorbing element for $\vee$:

$$A \vee (+\infty; 0) = (+\infty; 0) \vee A = (+\infty; 0), \forall A \in \overline{\mathcal{K}_C}.$$

□

In some cases it could be interesting to define $(\overline{\mathcal{K}_C}, \vee, \wedge)$ and $(\overline{\mathcal{K}_C}, \wedge, \vee)$ just as pre-semirings without assuming $-\infty$ and $+\infty$ as the 0-element and 1-element for the first structure and, vice-versa, as the 1-element and 0-element for the second.
Indeed, $(\overline{\mathcal{K}_C}, \vee, \wedge)$ is a commutative, idempotent pre-semiring, as:

1a) $(\overline{\mathcal{K}_C}, \vee)$ is a commutative, idempotent semigroup:

(i) $\vee$ is associative: $(A \vee B) \vee C = A \vee (B \vee C), \forall A, B, C \in \overline{\mathcal{K}_C}$;

(ii) $\vee$ is commutative: $A \vee B = B \vee A, \forall A, B \in \overline{\mathcal{K}_C}$;

(iii) $\vee$ is idempotent: $A \vee A = A, \forall A \in \overline{\mathcal{K}_C}$.

2a) $(\overline{\mathcal{K}_C}, \wedge)$ is a commutative, idempotent semigroup:

(i) $\wedge$ is associative: $(A \wedge B) \wedge C = A \wedge (B \wedge C), \forall A, B, C \in \overline{\mathcal{K}_C}$;

(ii) $\wedge$ is commutative: $A \wedge B = B \wedge A, \forall A, B \in \overline{\mathcal{K}_C}$;

(iii) $\wedge$ is idempotent: $A \wedge A = A, \forall A \in \overline{\mathcal{K}_C}$.

3a) $\wedge$ is left and right distributive over $\vee$:

(i) $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C), \forall A, B, C \in \overline{\mathcal{K}_C}$;

(ii) $(A \vee B) \wedge C = (A \wedge C) \vee (B \wedge C), \forall A, B, C \in \overline{\mathcal{K}_C}$.

Likewise, $(\overline{\mathcal{K}_C}, \wedge, \vee)$ is a commutative, idempotent pre-semiring, as:

1b) $(\overline{\mathcal{K}_C}, \wedge)$ is a commutative, idempotent semigroup:

(i) $\wedge$ is associative: $(A \wedge B) \wedge C = A \wedge (B \wedge C), \forall A, B, C \in \overline{\mathcal{K}_C}$;
(ii) $\wedge$ is commutative: $A \wedge B = B \wedge A, \forall A, B \in \overline{\mathcal{K}_C}$;
(iii) $\wedge$ is idempotent: $A \wedge A = A, \forall A \in \overline{\mathcal{K}_C}$.

2b) $(\overline{\mathcal{K}_C}, \vee)$ is a commutative, idempotent semigroup:

(i) $\vee$ is associative: $(A \vee B) \vee C = A \vee (B \vee C), \forall A, B, C \in \overline{\mathcal{K}_C}$;
(ii) $\vee$ is commutative: $A \vee B = B \vee A, \forall A, B \in \overline{\mathcal{K}_C}$;
(iii) $\vee$ is idempotent: $A \vee A = A, \forall A \in \overline{\mathcal{K}_C}$.

3b) $\vee$ is left and right distributive over $\wedge$:

(i) $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C), \forall A, B, C \in \overline{\mathcal{K}_C}$;
(ii) $(A \wedge B) \vee C = (A \vee C) \wedge (B \vee C), \forall A, B, C \in \overline{\mathcal{K}_C}$.

Similarly, it might also be interesting to consider $(\overline{\mathcal{K}_C}, \vee, \wedge)$ and $(\overline{\mathcal{K}_C}, \wedge, \vee)$ just as hemirings without assuming, respectively, $+\infty$ and $-\infty$ as the 1-elements of the structures.

Now, according to what was introduced in Section 4.1, we repeat the reasoning for the case of inclusion order $\subseteqq_{\gamma^-,\gamma^+}$.

In this regard, let consider

$$\mathcal{K}_C^{\varnothing\mathbb{R}} = \mathcal{K}_{\mathcal{C}} \cup \{\varnothing\} \cup \{\mathbb{R}\}$$

where, for all $\gamma^-, \gamma^+ \in \mathbb{R}$ such that $\gamma^- \leqslant \gamma^+$, we define:

$$\varnothing = (0; -\infty) = inf_{\subseteqq_{\gamma^-,\gamma^+}} \mathcal{K}_C^{\varnothing\mathbb{R}};$$

$$\mathbb{R} = (0; +\infty) = sup_{\subseteqq_{\gamma^-,\gamma^+}} \mathcal{K}_C^{\varnothing\mathbb{R}},$$

that is, in midpoint notation:

$$A \in \mathcal{K}_C^{\varnothing\mathbb{R}} \Leftrightarrow A = (\widehat{a}; \widetilde{a}) \text{ with } \widehat{a} \in \mathbb{R}, \widetilde{a} \geqslant 0 \;\text{ or }\; A = (0; -\infty) \;\text{ or }\; A = (0; +\infty).$$

Then we consider $\mathcal{K}_C^{\varnothing\mathbb{R}}$ associated with the well known partial order $\subseteqq_{\gamma^-,\gamma^+}$; in particular, when $\gamma^- = -1$ and $\gamma^+ = +1$, we can denote the $\subseteqq_{-1,+1}$-order simply with $\subseteq$.

It is well known (see Section 4.1) that $(\mathcal{K}_C^{\varnothing\mathbb{R}}, \subseteqq_{\gamma^-,\gamma^+})$, or more specifically $(\mathcal{K}_C^{\varnothing\mathbb{R}}, \uplus, \cap, \subseteqq_{\gamma^-,\gamma^+})$, is a complete lattice with:

(i) $\varnothing = (0; -\infty)$ as the $0_{\subseteqq_{\gamma^-,\gamma^+}}$-element;

(ii) $\mathbb{R} = (0; +\infty)$ as the $1_{\subseteqq_{\gamma^-,\gamma^+}}$-element;

(iii) operations $\uplus$ and $\cap$ are defined, according to (4.18) and (4.19), for all A, B in $\mathcal{K}_C^{\varnothing\mathbb{R}}$ (see Figure 4.15), by:

$A \uplus B \stackrel{def}{=} sup_{\subseteq_{\equiv\gamma^-,\gamma^+}}\{A, B\} = conv(A \cup B)$ is the convex hull interval of $A \cup B$;

$A \cap B \stackrel{def}{=} inf_{\subseteq_{\equiv\gamma^-,\gamma^+}}\{A, B\}$ stands for the usual intersection of intervals.

It follows that:

a1) $A \cap (0; +\infty) = A, \forall A \in \mathcal{K}_C^{\varnothing\mathbb{R}}$ (this means that $(0; +\infty)$ can be considered the neutral element for $\cap$);

a2) $A \uplus (0; -\infty) = A, \forall A \in \mathcal{K}_C^{\varnothing\mathbb{R}}$ (this means that $(0; -\infty)$ can be considered the neutral element for $\uplus$);

b1) $A \cap (0; -\infty) = (0; -\infty), \forall A \in \mathcal{K}_C^{\varnothing\mathbb{R}}$ (so $(0; -\infty)$ is the absorbing element for $\cap$);

b2) $A \uplus (0; +\infty) = (0; +\infty), \forall A \in \mathcal{K}_C^{\varnothing\mathbb{R}}$ (so $(0; +\infty)$ is the absorbing element for $\uplus$);

c) $\cap$ and $\uplus$ are associative, commutative and distributive (left and right and each-other).

Therefore, we obtain the following proposition.

Proposition 5.1.2. *$(\mathcal{K}_C^{\varnothing\mathbb{R}}, \uplus, \cap)$ and $(\mathcal{K}_C^{\varnothing\mathbb{R}}, \cap, \uplus)$ are commutative, idempotent semirings.*

Proof. $(\mathcal{K}_C^{\varnothing\mathbb{R}}, \uplus, \cap)$ is a commutative, idempotent semiring, as:

1a) $(\mathcal{K}_C^{\varnothing\mathbb{R}}, \uplus)$ is a commutative, idempotent monoid with neutral element $\varnothing = (0; -\infty)$:

(i) $\uplus$ is associative: $(A \uplus B) \uplus C = A \uplus (B \uplus C), \forall A, B, C \in \mathcal{K}_C^{\varnothing\mathbb{R}}$;

(ii) $\uplus$ has a neutral element $i_\uplus = \varnothing \in \mathcal{K}_C^{\varnothing\mathbb{R}}$: $\varnothing \uplus A = A \uplus \varnothing = A, \forall A \in \mathcal{K}_C^{\varnothing\mathbb{R}}$ (so, $\varnothing$ is the 0-element of the semiring);

(iii) $\uplus$ is commutative: $A \uplus B = B \uplus A, \forall A, B \in \mathcal{K}_C^{\varnothing\mathbb{R}}$;

(iv) $\uplus$ is idempotent: $A \uplus A = A, \forall A \in \mathcal{K}_C^{\varnothing\mathbb{R}}$.

2a) $(\mathcal{K}_C^{\varnothing\mathbb{R}}, \cap)$ is a commutative, idempotent monoid with neutral element $\mathbb{R} = (0; +\infty)$:

(i) $\cap$ is associative: $(A \cap B) \cap C = A \cap (B \cap C), \forall A, B, C \in \mathcal{K}_C^{\varnothing\mathbb{R}}$;

(ii) $\cap$ has a neutral element $i_\cap = \mathbb{R} \in \mathcal{K}_C^{\varnothing\mathbb{R}}$: $\mathbb{R} \cap A = A \cap \mathbb{R} = A, \forall A \in \mathcal{K}_C^{\varnothing\mathbb{R}}$ (so, $\mathbb{R}$ is the 1-element of the semiring);

(iii) $\cap$ is commutative: $A \cap B = B \cap A, \forall A, B \in \mathcal{K}_C^{\varnothing\mathbb{R}}$;

(iv) $\cap$ is idempotent: $A \cap A = A, \forall A \in \mathcal{K}_C^{\varnothing\mathbb{R}}$.

3a) $\cap$ is left and right distributive over $\uplus$:

(i) $A \cap (B \uplus C) = (A \cap B) \uplus (A \cap C), \forall A, B, C \in \mathcal{K}_C^{\varnothing\mathbb{R}}$;

(ii) $(A \uplus B) \cap C = (A \cap C) \uplus (B \cap C), \forall A, B, C \in \mathcal{K}_C^{\varnothing\mathbb{R}}$.

4a) $\varnothing$ is the absorbing element for $\cap$:

$$\varnothing \cap A = A \cap \varnothing = \varnothing, \forall A \in \mathcal{K}_C^{\varnothing\mathbb{R}}.$$

In an analogous way we have that $(\mathcal{K}_C^{\varnothing\mathbb{R}}, \cap, \uplus,)$ is a commutative, idempotent semiring, as:

1b) $(\mathcal{K}_C^{\varnothing\mathbb{R}}, \cap)$ is a commutative, idempotent monoid with neutral element $\mathbb{R}$:

(i) $\cap$ is associative: $(A \cap B) \cap C = A \cap (B \cap C), \forall A, B, C \in \mathcal{K}_C^{\varnothing\mathbb{R}}$;

(ii) $\cap$ has a neutral element $i_\cap = \mathbb{R} \in \mathcal{K}_C^{\varnothing\mathbb{R}}$: $\mathbb{R} \cap A = A \cap \mathbb{R} = A, \forall A \in \mathcal{K}_C^{\varnothing\mathbb{R}}$ (so, $\mathbb{R}$ is the 0-element of the semiring);

(iii) $\cap$ is commutative: $A \cap B = B \cap A, \forall A, B \in \mathcal{K}_C^{\varnothing\mathbb{R}}$;

(iv) $\cap$ is idempotent: $A \cap A = A, \forall A \in \mathcal{K}_C^{\varnothing\mathbb{R}}$.

2b) $(\mathcal{K}_C^{\varnothing\mathbb{R}}, \uplus)$ is a commutative, idempotent monoid with neutral element $\varnothing$:

(i) $\uplus$ is associative: $(A \uplus B) \uplus C = A \uplus (B \uplus C), \forall A, B, C \in \mathcal{K}_C^{\varnothing\mathbb{R}}$;

(ii) $\uplus$ has a neutral element $i_\uplus = \varnothing \in \mathcal{K}_C^{\varnothing\mathbb{R}}$: $\varnothing \uplus A = A \uplus \varnothing = A, \forall A \in \mathcal{K}_C^{\varnothing\mathbb{R}}$ (so, $\varnothing$ is the 1-element of the semiring);

(iii) $\uplus$ is commutative: $A \uplus B = B \uplus A, \forall A, B \in \mathcal{K}_C^{\varnothing\mathbb{R}}$;

(iv) $\uplus$ is idempotent: $A \uplus A = A, \forall A \in \mathcal{K}_C^{\varnothing\mathbb{R}}$.

3b) $\uplus$ is left and right distributive over $\cap$:

(i) $A \uplus (B \cap C) = (A \uplus B) \cap (A \cap C), \forall A, B, C \in \mathcal{K}_C^{\varnothing\mathbb{R}}$;

(ii) $(A \cap B) \uplus C = (A \uplus C) \cap (B \uplus C), \forall A, B, C \in \mathcal{K}_C^{\varnothing\mathbb{R}}$.

4b) $\mathbb{R}$ is the absorbing element for $\uplus$:

$$\mathbb{R} \uplus A = A \uplus \mathbb{R} = \mathbb{R}, \forall A \in \mathcal{K}_C^{\varnothing\mathbb{R}}.$$

□

Also in this case it may be useful to consider $(\mathcal{K}_C^{\varnothing\mathbb{R}}, \uplus, \cap)$ and $(\mathcal{K}_C^{\varnothing\mathbb{R}}, \cap, \uplus)$ just as pre-semirings, without assuming $\varnothing$ and $\mathbb{R}$ as the 0-element and 1-element for the first structure and, vice-versa, as the 1-element and 0-element for the second.

Indeed, $(\mathcal{K}_C^{\varnothing\mathbb{R}}, \uplus, \cap)$ is a commutative, idempotent pre-semiring, as:

1a) $(\mathcal{K}_C^{\varnothing\mathbb{R}}, \uplus)$ is a commutative, idempotent semigroup:

(i) $\uplus$ is associative: $(A \uplus B) \uplus C = A \uplus (B \uplus C), \forall A, B, C \in \mathcal{K}_C^{\varnothing\mathbb{R}}$;

(ii) $\uplus$ is commutative: $A \uplus B = B \uplus A, \forall A, B \in \mathcal{K}_C^{\varnothing\mathbb{R}}$;

(iii) $\uplus$ is idempotent: $A \uplus A = A, \forall A \in \mathcal{K}_C^{\varnothing\mathbb{R}}$.

2a) $(\mathcal{K}_C^{\varnothing\mathbb{R}}, \cap)$ is a commutative, idempotent semigroup:

(i) $\cap$ is associative: $(A \cap B) \cap C = A \cap (B \cap C), \forall A, B, C \in \mathcal{K}_C^{\varnothing\mathbb{R}}$;

(ii) $\cap$ is commutative: $A \cap B = B \cap A, \forall A, B \in \mathcal{K}_C^{\varnothing\mathbb{R}}$;

(iii) $\cap$ is idempotent: $A \cap A = A, \forall A \in \mathcal{K}_C^{\varnothing\mathbb{R}}$.

3a) $\cap$ is left and right distributive over $\uplus$:

(i) $A \cap (B \uplus C) = (A \cap B) \uplus (A \cap C), \forall A, B, C \in \mathcal{K}_C^{\varnothing\mathbb{R}}$;

(ii) $(A \uplus B) \cap C = (A \cap C) \uplus (B \cap C), \forall A, B, C \in \mathcal{K}_C^{\varnothing\mathbb{R}}$.

Similarly, $(\mathcal{K}_C^{\varnothing\mathbb{R}}, \cap, \uplus)$ is a commutative, idempotent pre-semiring, as:

1b) $(\mathcal{K}_C^{\varnothing\mathbb{R}}, \cap)$ is a commutative, idempotent semigroup:

(i) $\cap$ is associative: $(A \cap B) \cap C = A \cap (B \cap C), \forall A, B, C \in \mathcal{K}_C^{\varnothing\mathbb{R}}$;

(ii) $\cap$ is commutative: $A \cap B = B \cap A, \forall A, B \in \mathcal{K}_C^{\varnothing\mathbb{R}}$;

(iii) $\cap$ is idempotent: $A \cap A = A, \forall A \in \mathcal{K}_C^{\varnothing\mathbb{R}}$.

2b) $(\mathcal{K}_C^{\varnothing\mathbb{R}}, \uplus)$ is a commutative, idempotent semigroup:

(i) $\uplus$ is associative: $(A \uplus B) \uplus C = A \uplus (B \uplus C), \forall A, B, C \in \mathcal{K}_C^{\varnothing\mathbb{R}}$;

(ii) $\uplus$ is commutative: $A \uplus B = B \uplus A, \forall A, B \in \mathcal{K}_C^{\varnothing\mathbb{R}}$;

(iii) $\uplus$ is idempotent: $A \uplus A = A, \forall A \in \mathcal{K}_C^{\varnothing\mathbb{R}}$.

3b) $\uplus$ is left and right distributive over $\cap$:

(i) $A \uplus (B \cap C) = (A \uplus B) \cap (A \uplus C), \forall A, B, C \in \mathcal{K}_C^{\varnothing\mathbb{R}}$;

(ii) $(A \cap B) \uplus C = (A \uplus C) \cap (B \uplus C), \forall A, B, C \in \mathcal{K}_C^{\varnothing\mathbb{R}}$.

Moreover, even in this case it can also be interesting to consider $(\mathcal{K}_C^{\varnothing\mathbb{R}}, \uplus, \cap)$ and $(\mathcal{K}_C^{\varnothing\mathbb{R}}, \cap, \uplus)$ just as hemiring without assuming, respectively, $\mathbb{R}$ and $\varnothing$ as the 1-elements of the structures.

Eventually, it is possible to take into account the addition in $\mathcal{K}_C$, defined as the classical Minkowski operation for intervals (defined in Subsection 2.1.1):

$$X \oplus Y = (\widehat{x} + \widehat{y}; \widetilde{x} + \widetilde{y}) \tag{5.5}$$

(with $X = (\widehat{x}; \widetilde{x}), Y = (\widehat{y}; \widetilde{y}) \in \mathcal{K}_{\mathcal{C}}$) and extend it to the sets $\overline{\mathcal{K}_C}$ and $\mathcal{K}_C^{\varnothing\mathbb{R}}$ as it follows:

$$\oplus : \overline{\mathcal{K}_C} \times \overline{\mathcal{K}_C} \longrightarrow \overline{\mathcal{K}_C} \text{ such that: } (X, Y) \longrightarrow X \oplus Y;$$

$$\oplus : \mathcal{K}_C^{\varnothing\mathbb{R}} \times \mathcal{K}_C^{\varnothing\mathbb{R}} \longrightarrow \mathcal{K}_C^{\varnothing\mathbb{R}} \text{ such that: } (X,Y) \longrightarrow X \oplus Y.$$

However, establishing some conventions:

1) $\widehat{x} + (+\infty) = +\infty$ and $\widehat{x} + (-\infty) = -\infty, \forall \widehat{x} \in \mathbb{R}$,

2) $(-\infty) + (+\infty) = 0$, $0 \cdot (-\infty) = 0$ and $0 \cdot (+\infty) = 0$,

the result is that $(\overline{\mathcal{K}_C}, \oplus)$ and $(\mathcal{K}_C^{\varnothing\mathbb{R}}, \oplus)$ are both commutative monoids, as they are associative, commutative and $(0;0)$ can be interpreted as the neutral element.

Moreover, if we consider the $\precapprox_{\gamma^-,\gamma^+}$-order associated to $\overline{\mathcal{K}_\mathcal{C}}$, it is easy to verify, even graphically, that, for all $A, B, C \in \overline{\mathcal{K}_\mathcal{C}}$, we have:

- $\oplus$ is distributive with respect to $\vee : (A \vee B) \oplus C = (A \oplus C) \vee (B \oplus C)$;

- $\oplus$ is distributive with respect to $\wedge : (A \wedge B) \oplus C = (A \oplus C) \wedge (B \oplus C)$.

On the other hand, it is also trivial to prove that the vice-versa is not valid as $\exists A, B, C \in \overline{\mathcal{K}_\mathcal{C}}$ such that:

- $(A \oplus B) \wedge C \neq (A \wedge C) \oplus (B \wedge C)$;

- $(A \oplus B) \vee C \neq (A \vee C) \oplus (B \vee C)$.

In a similar way, considering the $\subseteqq_{\gamma^-,\gamma^+}$-order associated to $\mathcal{K_C}^{\varnothing\mathbb{R}}$, we have that, for all $A, B, C \in \mathcal{K_C}^{\varnothing\mathbb{R}}$:

- $\oplus$ is distributive with respect to $\uplus : (A \uplus B) \oplus C = (A \oplus C) \uplus (B \oplus C)$;

- $\oplus$ is sub-distributive with respect to $\cap : (A \cap B) \oplus C \subseteq (A \oplus C) \cap (B \oplus C)$.

Remark 5.1.4. *Note that only if we consider intervals X, Y, Z that are not mutually disjoint, then the addition $\oplus$ is distributive with respect to $\cap$.*

Lastly, we conclude by pointing out that also in this case the vice-versa is not true, as $\exists A, B, C \in \mathcal{K_C}^{\varnothing\mathbb{R}}$, such that:

- $(A \oplus B) \uplus C \neq (A \uplus C) \oplus (B \uplus C)$;

- $(A \oplus B) \cap C \neq (A \cap C) \oplus (B \cap C)$.

To complete the discussion, we try to further refine the structures by defining, in addition to the well-known set $\overline{\mathcal{K}_\mathcal{C}} = \mathcal{K}_\mathcal{C} \cup \{\pm\infty\}$, also the following two sets:

$$\mathcal{K}_\mathcal{C}^{+\infty} \overset{def}{=} \mathcal{K}_\mathcal{C} \cup \{+\infty\} \quad \text{and} \quad \mathcal{K}_\mathcal{C}^{-\infty} \overset{def}{=} \mathcal{K}_\mathcal{C} \cup \{-\infty\}.$$

We can even suppose that

$+\infty = [+\infty, +\infty] = (+\infty; 0) \stackrel{def}{=} sup_{\precsim_{\gamma^-,\gamma^+}} \overline{\mathcal{K}_{\mathcal{C}}}$ coincides with

$$(+\infty; 0) = (+\infty; \alpha),\ \forall \alpha \geqslant 0$$

while
$-\infty = [-\infty, -\infty] = (-\infty; 0) \stackrel{def}{=} inf_{\precsim_{\gamma^-,\gamma^+}} \overline{\mathcal{K}_{\mathcal{C}}}$ coincides with

$$(-\infty; 0) = (-\infty; \alpha),\ \forall \alpha \geqslant 0.$$

Therefore, for all $A = (\widehat{a}; \widetilde{a}), B = (\widehat{b}; \widetilde{b}) \in \overline{\mathcal{K}_{\mathcal{C}}}$, considering the operations defined so far, i.e.,

- $A \vee B = sup_{\precsim_{\gamma^-,\gamma^+}}(A, B)$;
- $A \wedge B = inf_{\precsim_{\gamma^-,\gamma^+}}(A, B)$;
- $A \oplus B = (\widehat{a} + \widehat{b}; \widetilde{a} + \widetilde{b})$,

we have that the following facts hold.

Proposition 5.1.3. *the structures* $(K_C^{-\infty}, \vee, \oplus)$ *and* $(K_C^{+\infty}, \wedge, \oplus)$ *are commutative semirings.*

Proof. $(K_C^{-\infty}, \vee, \oplus)$ is a commutative semiring, as:

1a) $(\mathcal{K}_{\mathcal{C}}^{-\infty}, \vee)$ is a commutative monoid with neutral element $i_\vee = (-\infty; 0)$:

  (i) $\vee$ is associative;

  (ii) $\vee$ has the neutral element $i_\vee = (-\infty; 0) \in \mathcal{K}_{\mathcal{C}}^{-\infty} :\ A \vee (-\infty; 0) = (-\infty; 0) \vee A = A$, $\forall A \in \mathcal{K}_{\mathcal{C}}^{-\infty}$ (so, $i_\vee = (-\infty; 0)$ is the 0-element of the semiring);

  (iii) $\vee$ is commutative.

2a) $(\mathcal{K}_{\mathcal{C}}^{-\infty}, \oplus)$ is a commutative monoid with neutral element $i_\oplus = (0; 0)$:

  (i) $\oplus$ is associative;

  (ii) $\oplus$ has the neutral element $i_\oplus = (0; 0) \in \mathcal{K}_{\mathcal{C}}^{-\infty}$: $X \oplus (0; 0) = (0; 0) \oplus A = A$, $\forall A \in \mathcal{K}_{\mathcal{C}}^{-\infty}$ (so, $i_\oplus = (0; 0)$ is the 1-element of the semiring);

  (iii) $\oplus$ is commutative.

3a) $\oplus$ is left and right distributive over $\vee$:

  (i) $A \oplus (B \vee C) = (A \oplus B) \vee (A \oplus C), \forall A, B, C \in K_C^{-\infty}$;

  (ii) $(A \vee B) \oplus C = (A \oplus C) \vee (B \oplus C), \forall A, B, C \in K_C^{-\infty}$.

4a) $i_\vee = (-\infty; 0)$ is the absorbing element for $\oplus$:

$$(-\infty;0) \oplus A = A \oplus (-\infty;0) = (-\infty;0), \forall A \in K_C^{-\infty}.$$

Analogously, also the structure $(K_C^{+\infty}, \wedge, \oplus)$ is a commutative semiring as:

1b) $(\mathcal{K}_\mathcal{C}^{+\infty}, \wedge)$ is a commutative monoid with neutral element $i_\wedge = (+\infty;0)$:

   (i) $\wedge$ is associative;

   (ii) $\wedge$ has the neutral element $i_\wedge = (+\infty;0) \in \mathcal{K}_\mathcal{C}^{+\infty}$: $A \wedge (+\infty;0) = (+\infty;0) \wedge A = A$, $\forall A \in \mathcal{K}_\mathcal{C}^{+\infty}$ (so, $i_\wedge = (+\infty;0)$ is the 0-element of the semiring);

   (iii) $\wedge$ is commutative.

2b) $(\mathcal{K}_\mathcal{C}^{+\infty}, \oplus)$ is a commutative monoid with neutral element $i_\oplus = (0;0)$:

   (i) $\oplus$ is associative;

   (ii) $\oplus$ has the neutral element $i_\oplus = (0;0) \in \mathcal{K}_\mathcal{C}^{+\infty}$: $A \oplus (0;0) = (0;0) \oplus A = A$, $\forall A \in \mathcal{K}_\mathcal{C}^{+\infty}$ (so, $i_\oplus = (0;0)$ is the 1-element of the semiring);

   (iii) $\oplus$ is commutative.

3b) $\oplus$ is left and right distributive over $\wedge$:

   (i) $A \oplus (B \wedge C) = (A \oplus B) \wedge (A \oplus C), \forall A, B, C \in K_C^{+\infty}$;

   (ii) $(A \wedge B) \oplus C = (A \oplus C) \wedge (B \oplus C), \forall A, B, C \in K_C^{+\infty}$.

4b) $i_\wedge = (+\infty;0)$ is the absorbing element for $\oplus$:

$$(+\infty;0) \oplus A = A \oplus (+\infty;0) = (+\infty;0), \forall A \in K_C^{+\infty}.$$

□

Thus, starting with the lattice $(\mathcal{K}_\mathcal{C}, \precsim_{\gamma^-,\gamma^+})$, we can obtain two commutative semiring structures:

a) $(\mathcal{K}_\mathcal{C}^{-\infty}, \vee, \oplus)$ where $\mathcal{K}_\mathcal{C}^{-\infty} = \mathcal{K}_\mathcal{C} \cup \{-\infty\}$, with:

   - 0-element $(-\infty;0) \equiv -\infty$ (neutral element for $\vee$ and absorbing element for $\oplus$);
   - 1-element $(0;0) \equiv 0$ (neutral element for $\oplus$).

b) $(\mathcal{K}_\mathcal{C}^{+\infty}, \wedge, \oplus)$ where $\mathcal{K}_\mathcal{C}^{+\infty} = \mathcal{K}_\mathcal{C} \cup \{+\infty\}$, with:

   - 0-element $(+\infty;0) \equiv +\infty$ (neutral element for $\wedge$ and absorbing element for $\oplus$);
   - 1-element $(0;0) \equiv 0$ (neutral element for $\oplus$).

It is easy to verify that the same type of costruction can be repeated in the case of inclusion order, by defining, in addition to the well-known set $\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}} = \mathcal{K}_{\mathcal{C}} \cup \{\varnothing\} \cup \{\mathbb{R}\}$, also the following ones:

$$\mathcal{K}_{\mathcal{C}}^{\varnothing} \stackrel{def}{=} \mathcal{K}_{\mathcal{C}} \cup \{\varnothing\} \quad \text{and} \quad \mathcal{K}_{\mathcal{C}}^{\mathbb{R}} \stackrel{def}{=} \mathcal{K}_{\mathcal{C}} \cup \{\mathbb{R}\}.$$

Eventually, we can also suppose that:

$\varnothing = (0; -\infty) \stackrel{def}{=} inf_{\subseteq_{\equiv\gamma^-,\gamma^+}} \mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}$ coincides with

$$(0; -\infty) = (\alpha; -\infty),\ \forall \alpha \in \mathbb{R}$$

and

$\mathbb{R} = (0; +\infty) \stackrel{def}{=} sup_{\subseteq_{\equiv\gamma^-,\gamma^+}} \mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}$ coincides with

$$(0 + \infty) = (\alpha; +\infty),\ \forall \alpha \in \mathbb{R}.$$

Therefore, for all $A = (\widehat{a}; \widetilde{a}), B = (\widehat{b}; \widetilde{b}) \in \mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}$, considering the operations below:

- $A \uplus B = sup_{\subseteq_{\equiv\gamma^-,\gamma^+}}(A, B) = conv(A \cup B)$;
- $A \cap B = inf_{\subseteq_{\equiv\gamma^-,\gamma^+}}(A, B)$;
- $A \oplus B = (\widehat{a} + \widehat{b}; \widetilde{a} + \widetilde{b})$,

we have that the following statements hold.

Proposition 5.1.4. *The structure* $(K_C^{\varnothing}, \uplus, \oplus)$ *is a commutative semiring.*

Proof. $(K_C^{\varnothing}, \uplus, \oplus)$ is a commutative semiring, as:

1) $(\mathcal{K}_{\mathcal{C}}^{\varnothing}, \uplus)$ is a commutative monoid with neutral element $i_{\uplus} = (0; -\infty) = \varnothing$:
   (i) $\uplus$ is associative;
   (ii) $\uplus$ has the neutral element $i_{\uplus} = (0; -\infty) \in \mathcal{K}_{\mathcal{C}}^{\varnothing} : A \uplus (0; -\infty) = (0; -\infty) \uplus A = A,\ \forall A \in \mathcal{K}_{\mathcal{C}}^{\varnothing}$ (so, $i_{\uplus} = (0; -\infty)$ is the 0-element of the semiring);
   (iii) $\uplus$ is commutative.
2) $(\mathcal{K}_{\mathcal{C}}^{\varnothing}, \oplus)$ is a commutative monoid with neutral element $i_{\oplus} = (0; 0) = 0$:
   (i) $\oplus$ is associative;
   (ii) $\oplus$ has the neutral element $i_{\oplus} = (0; 0) \in \mathcal{K}_{\mathcal{C}}^{\varnothing}$: $A \oplus (0; 0) = (0; 0) \oplus A = A,\ \forall A \in \mathcal{K}_{\mathcal{C}}^{\varnothing}$ (so, $i_{\oplus} = (0; 0)$ is the 1-element of the semiring);
   (iii) $\oplus$ is commutative.
3) $\oplus$ is left and right distributive over $\uplus$:

(i) $A \oplus (B \uplus C) = (A \oplus B) \uplus (A \oplus C), \forall A, B, C \in K_C^{\varnothing}$;
(ii) $(A \uplus B) \oplus C = (A \oplus C) \uplus (Y \oplus Z), \forall A, B, C \in K_C^{\varnothing}$.

4) $i_{\uplus} = (0; -\infty)$ is the absorbing element for $\oplus$:

$$(0; -\infty) \oplus A = A \oplus (0; -\infty) = (0; -\infty), \forall A \in K_C^{\varnothing}.$$

□

Thus, in this case too, starting with the lattice $(\mathcal{K}_\mathcal{C}, \subseteq_{\equiv_{\gamma^-,\gamma^+}})$, we can obtain a commutative semiring structure:

$(\mathcal{K}_\mathcal{C}^{\varnothing}, \uplus, \oplus)$ where $\mathcal{K}_\mathcal{C}^{\varnothing} = \mathcal{K}_\mathcal{C} \cup \{\varnothing\}$, with:

- 0-element $(0; -\infty) \equiv \varnothing$ (neutral element for $\uplus$ and absorbing element for $\oplus$);
- 1-element $(0; 0) \equiv 0$ (neutral element for $\oplus$).

Remark 5.1.5. *We note that it is not possible to do the same for the structure $(K_C^{\mathbb{R}}, \cap, \oplus)$ which does not originate a semiring due to the subdistributivity of $\oplus$ with respect to $\cap$. Indeed, as already mentioned in Remark 5.1.4, just in case the intervals are not mutually disjoint the addition $\oplus$ is distributive with respect to $\cap$.*

At this point, using Definitions 5.1.12 and 5.1.13, we can further improve the concepts introduced so far giving the following properties.

Proposition 5.1.5. *$(\overline{\mathcal{K}_\mathcal{C}}, \vee, \wedge)$ and $(\overline{\mathcal{K}_\mathcal{C}}, \wedge, \vee)$ are zero-sum-free semirings (or antirings).*

Proof. The proof is immediate since, for definition, we have
$\forall A, B \in \overline{\mathcal{K}_\mathcal{C}},\ A \vee B = sup_{\precsim_{\approx_{\gamma^-,\gamma^+}}}(A, B)$, it follows that:
$A \vee B = (-\infty; 0) = -\infty \Rightarrow A = B = (-\infty; 0) = -\infty$.
Similarly, as $\forall A, B \in \overline{\mathcal{K}_\mathcal{C}},\ A \wedge B = inf_{\precsim_{\approx_{\gamma^-,\gamma^+}}}(A, B)$, it follows that:
$A \wedge B = (+\infty; 0) = +\infty \Rightarrow A = B = (+\infty; 0) = +\infty$. □

Proposition 5.1.6. *$(\mathcal{K}_\mathcal{C}^{\varnothing\mathbb{R}}, \uplus, \cap)$ and $(\mathcal{K}_\mathcal{C}^{\varnothing\mathbb{R}}, \cap, \uplus)$ are zero-sum-free semirings (or antirings).*

Proof. Proceeding in the same way as in Proposition 5.1.5, we have that, since $\forall A, B \in \mathcal{K}_\mathcal{C}^{\varnothing\mathbb{R}},\ A \uplus B = sup_{\subseteq_{\equiv_{\gamma^-,\gamma^+}}}(A, B)$, it follows that:
$A \uplus B = (0; -\infty) = \varnothing \Rightarrow A = B = (0; -\infty) = \varnothing$.
Similarly, as $\forall A, B \in \mathcal{K}_\mathcal{C}^{\varnothing\mathbb{R}},\ A \cap B = inf_{\subseteq_{\equiv_{\gamma^-,\gamma^+}}}(A, B)$, it follows that:
$A \cap B = (0; +\infty) = \mathbb{R} \Rightarrow A = B = (0; +\infty) = \mathbb{R}$. □

Proposition 5.1.7. *$(K_C^{-\infty}, \vee, \oplus)$, $(K_C^{+\infty}, \wedge, \oplus)$ and $(K_C^{\varnothing}, \uplus, \oplus)$ are zero-sum-free semirings (or antirings).*

Proof. See the proofs of Proposition 5.1.5 and Proposition 5.1.6 (only first part). □

Proposition 5.1.8. *$(\overline{\mathcal{K}_{\mathcal{C}}}, \vee, \wedge)$ and $(\overline{\mathcal{K}_{\mathcal{C}}}, \wedge, \vee)$ are zero-divisor-free (or entire) semirings.*

Proof. The proof is immediate since, by definition, we have
$\forall A, B \in \overline{\mathcal{K}_{\mathcal{C}}},\ A \wedge B = inf_{\precapprox_{\gamma^-,\gamma^+}}(A, B)$, it follows that:
$A \wedge B = (-\infty; 0) = -\infty \Rightarrow A = (-\infty; 0) = -\infty$ or $B = (-\infty; 0) = -\infty$.
Similarly, as $\forall A, B \in \overline{\mathcal{K}_{\mathcal{C}}},\ A \vee B = sup_{\precapprox_{\gamma^-,\gamma^+}}(A, B)$, it follows that:
$A \vee B = (+\infty; 0) = +\infty \Rightarrow A = (+\infty; 0) = +\infty$ or $B = (+\infty; 0) = +\infty$. □

Proposition 5.1.9. *$(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}, \cap, \uplus)$ is a zero-divisor-free (or entire) semiring.*

Proof. Proceeding in the same way as Proposition 5.1.8, we have that, since $\forall A, B \in \mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}},\ A \uplus B = sup_{\subseteq_{\gamma^-,\gamma^+}}(A, B)$, it follows that:
$A \uplus B = (0; +\infty) = \mathbb{R} \Rightarrow A = (0; +\infty; 0) = \mathbb{R}$ or $B = (0; +\infty; 0) = \mathbb{R}$. □

Remark 5.1.6. *Note that $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}, \uplus, \cap)$ is not a zero-divisor-free structure since $\forall A, B \in \mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}$, $A \cap B = (0; -\infty) = \varnothing$ does not necessarily imply that one of the two intervals is equal to $\varnothing$.*

Proposition 5.1.10. *$(\mathcal{K}_{\mathcal{C}}^{-\infty}, \vee, \oplus)$ and $(\mathcal{K}_{\mathcal{C}}^{+\infty}, \wedge, \oplus)$ are zero-divisor-free (or entire) semirings.*

Proof. The proof is immediate since, by definition, we have
$A \oplus B = (\widehat{a} + \widehat{b}; \widetilde{a} + \widetilde{b})\ \forall A, B \in \mathcal{K}_{\mathcal{C}}^{-\infty}$, it follows that:
$A \oplus B = (-\infty; 0) = -\infty \Rightarrow A = (-\infty; 0) = -\infty$ or $B = (-\infty; 0) = -\infty$.
Similarly, $A \oplus B = (+\infty; 0) = +\infty \Rightarrow A = (+\infty; 0) = +\infty$ or $B = (+\infty; 0) = +\infty$, $\forall A, B \in \mathcal{K}_{\mathcal{C}}^{+\infty}$. □

Proposition 5.1.11. *$(\mathcal{K}_{\mathcal{C}}^{\varnothing}, \uplus, \oplus)$ is a zero-divisor-free (or entire) semiring.*

Proof. Proceeding in the same way as proposition (5.1.10), we have that: $A \oplus B = (0; -\infty) = \varnothing \Rightarrow A = (0; -\infty) = \varnothing$ or $B = (0; -\infty) = \varnothing$, $\forall A, B \in \mathcal{K}_{\mathcal{C}}^{\varnothing}$. □

We also add that the semirings $(\overline{\mathcal{K}_{\mathcal{C}}}, \vee, \wedge)$, $(\overline{\mathcal{K}_{\mathcal{C}}}, \wedge, \vee)$ and $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}, \cap, \uplus)$ as well as $(\mathcal{K}_{\mathcal{C}}^{-\infty}, \vee, \oplus)$, $(\mathcal{K}_{\mathcal{C}}^{+\infty}, \wedge, \oplus)$ and $(\mathcal{K}_{\mathcal{C}}^{\varnothing}, \uplus, \oplus)$, being zero-sum-free and zero-divisor-free, are said to be *information algebras* (see [47]).

Moreover, $\mathcal{K}_{\mathcal{C}}^{-\infty}$, $\mathcal{K}_{\mathcal{C}}^{+\infty}$, $\mathcal{K}_{\mathcal{C}}^{\varnothing}$ and $\mathcal{K}_{\mathcal{C}}^{\mathbb{R}}$ are also additively-idempotent; specifically, we have that:

- $\mathcal{K}_{\mathcal{C}}^{-\infty}$ is $\vee$-idempotent, as $\forall A \in \mathcal{K}_{\mathcal{C}}^{-\infty}$, it is $A \vee A = A$;
- $\mathcal{K}_{\mathcal{C}}^{+\infty}$ is $\wedge$-idempotent, as $\forall A \in \mathcal{K}_{\mathcal{C}}^{+\infty}$, it is $A \wedge A = A$;
- $\mathcal{K}_{\mathcal{C}}^{\varnothing}$ is $\uplus$-idempotent, as $\forall A \in \mathcal{K}_{\mathcal{C}}^{\varnothing}$, it is $A \uplus A = A$;

- $\mathcal{K}_{\mathcal{C}}^{\mathbb{R}}$ is $\cap$-idempotent, as $\forall A \in \mathcal{K}_{\mathcal{C}}^{\mathbb{R}}$, it is $A \cap A = A$.

On the other hand, some of the semirings described above (specifically $(\mathcal{K}_{\mathcal{C}}^{-\infty}, \vee, \oplus)$, $(\mathcal{K}_{\mathcal{C}}^{+\infty}, \wedge, \oplus)$ and $(\mathcal{K}_{\mathcal{C}}^{\varnothing}, \uplus, \oplus)$) are all not multiplicatively-idempotent (or $\oplus$-idempotent); indeed 0 is the only $\oplus$-idempotent element as

$$0 \oplus 0 = 0 \;\text{ and }\; A \oplus A = A \Leftrightarrow A = 0.$$

However, the following statements can still be made.

1) $(\mathcal{K}_{\mathcal{C}}^{-\infty}, \vee)$ is an idempotent, commutative monoid with neutral element $-\infty$ (called $\vee$-identity), as:

   (1a) $\vee$ is associative;

   (1b) $\vee$ has the neutral element (the so-called $\vee$-identity): $i_\vee = (-\infty; 0) \equiv -\infty \in \mathcal{K}_{\mathcal{C}}^{-\infty}$ — $A \vee i_\vee = i_\vee \vee A = A$, $\forall A \in \mathcal{K}_{\mathcal{C}}^{-\infty}$;

   (1c) $\vee$ is commutative;

   (1d) $\vee$ is idempotent, as: $\forall A \in \mathcal{K}_{\mathcal{C}}^{-\infty}$, $A \vee A = A$.

2) $(\mathcal{K}_{\mathcal{C}}^{+\infty}, \wedge)$ is an idempotent, commutative monoid with neutral element $+\infty$ (called $\wedge$-identity), as:

   (2a) $\wedge$ is associative;

   (2b) $\wedge$ has the neutral element (the so-called $\wedge$-identity): $i_\wedge = (+\infty; 0) \equiv +\infty \in \mathcal{K}_{\mathcal{C}}^{+\infty} \mid A \wedge i_\wedge = i_\wedge \wedge A = A$, $\forall A \in \mathcal{K}_{\mathcal{C}}^{+\infty}$;

   (2c) $\wedge$ is commutative;

   (2d) $\wedge$ is idempotent, as: $\forall A \in \mathcal{K}_{\mathcal{C}}^{+\infty}$, $A \wedge A = A$.

3) $(\mathcal{K}_{\mathcal{C}}^{\varnothing}, \uplus)$ is an idempotent, commutative monoid with neutral element $\varnothing$ (called $\uplus$-identity), as:

   (3a) $\uplus$ is associative;

   (3b) $\uplus$ has the neutral element (the so-called $\uplus$-identity): $i_\uplus = (0; -\infty) \equiv \varnothing \in \mathcal{K}_{\mathcal{C}}^{\varnothing} \mid A \uplus i_\uplus = i_\uplus \uplus A = A$, $\forall A \in \mathcal{K}_{\mathcal{C}}^{\varnothing}$;

   (3c) $\uplus$ is commutative;

   (3d) $\uplus$ is idempotent, as: $\forall A \in \mathcal{K}_{\mathcal{C}}^{\varnothing}$, $A \uplus A = A$.

4) $(\mathcal{K}_{\mathcal{C}}^{\mathbb{R}}, \cap)$ is an idempotent, commutative monoid with neutral element $\mathbb{R}$ (called $\cap$-identity), as:

   (4a) $\cap$ is associative;

   (4b) $\cap$ has the neutral element (the so-called $\cap$-identity): $i_\cap = (0; +\infty) \equiv \mathbb{R} \in \mathcal{K}_{\mathcal{C}}^{\mathbb{R}} \mid A \cap i_\cap = i_\cap \cap A =$, $\forall A \in \mathcal{K}_{\mathcal{C}}^{\mathbb{R}}$;

   (4c) $\cap$ is commutative;

(4d) $\cap$ is idempotent, as: $\forall A \in \mathcal{K}_{\mathcal{C}}^{\mathbb{R}}$, $A \cap A = A$.

Finally, we consider the possibility of infinite (or at least countably-infinite) sums in such types of semirings as it would be very important in applications (see [25]).

We remember, in accordance with [44], that if T is a non-empty finite subset of an idempotent semiring S, the sum of its elements is its supremum. By analogy, if T is any non empty subset of S, we denote by

$$\sum_{t \in T} t$$

the supremum of T, if it exists. This notation is justified since, in particular, the supremum of $T \cup T'$ is the sum of the suprema of T and T', where $T, T' \subseteq S$ are non empty.

Therefore, remembering that an ordered set is complete if each of its subsets has a supremum, we give the follow definition.

Definition 5.1.14. *A semiring $(S, +, \cdot)$ is said to be complete if it is complete as an ordered set and satisfies the following distributivity conditions:*

$$\left(\sum_{t \in T} t\right) \cdot s = \sum_{t \in T} (t \cdot s) \quad \textit{and} \quad s \cdot \left(\sum_{t \in T} t\right) = \sum_{t \in T} (s \cdot t),$$

for any $T \subseteq S$, $s \in S$.

In other words an idempotent semiring $(S, +, \cdot)$ is complete if it is closed for infinite sums (i.e., if the sum of infinite numbers of terms is always defined) and if the product distributes over infinite sums too.

Hence, considering the semiring $(\mathcal{K}_{\mathcal{C}}^{+\infty}, \wedge, \oplus)$, we know that:

- the corresponding ordered set $(\mathcal{K}_{\mathcal{C}}^{+\infty}, \precapprox_{\gamma^-,\gamma^+})$ is a complete lattice as each of its subsets has a supremum;

- for any non empty subset $T^{+\infty} \subseteq \mathcal{K}_{\mathcal{C}}^{+\infty}$, the operation:

$$\bigwedge_{A \in T^{+\infty}} A = \bigwedge \{A | A \in T^{+\infty} \subseteq \mathcal{K}_{\mathcal{C}}^{+\infty}\}$$

  is such that, for all $B \in \mathcal{K}_{\mathcal{C}}^{+\infty}$, we have

$$\left(\bigwedge_{A \in T^{+\infty}} A\right) \oplus B = \bigwedge_{A \in T^{+\infty}} (A \oplus B) \text{ and } B \oplus \left(\bigwedge_{A \in T^{+\infty}} A\right) = \bigwedge_{A \in T^{+\infty}} (B \oplus A).$$

Analogously, considering $(\mathcal{K}_{\mathcal{C}}^{-\infty}, \vee, \oplus)$, we still have that:

- the corresponding ordered set $(\mathcal{K}_{\mathcal{C}}^{-\infty}, \precapprox_{\gamma^-,\gamma^+})$ is a complete lattice;

- for any non empty subset $T^{-\infty} \subseteq \mathcal{K}_{\mathcal{C}}^{-\infty}$, the operation:

$$\bigvee_{A \in T^{-\infty}} A = \bigvee \{A | A \in T^{-\infty} \subseteq \mathcal{K}_{\mathcal{C}}^{-\infty}\}$$

is such that, for all $B \in \mathcal{K}_{\mathcal{C}}^{-\infty}$, we have

$$\left(\bigvee_{A \in T^{-\infty}} A \right) \oplus B = \bigvee_{A \in T^{-\infty}} (A \oplus B) \text{ and } B \oplus \left(\bigvee_{A \in T^{-\infty}} A \right) = \bigvee_{A \in T^{-\infty}} (B \oplus A).$$

We reason in a similar way for the semiring $(\mathcal{K}_{\mathcal{C}}^{\varnothing}, \uplus, \oplus)$ as well as for the semirings $(\overline{\mathcal{K}_{\mathcal{C}}}, \vee, \wedge)$, $(\overline{\mathcal{K}_{\mathcal{C}}}, \wedge, \vee)$, $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}, \uplus, \cap)$ and $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}, \cap, \uplus)$.

As a consequence of this we obtain that the following proposition holds.

Proposition 5.1.12. *$(\overline{\mathcal{K}_{\mathcal{C}}}, \vee, \wedge)$, $(\overline{\mathcal{K}_{\mathcal{C}}}, \wedge, \vee)$, $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}, \uplus, \cap)$, $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}, \cap, \uplus)$ $(\mathcal{K}_{\mathcal{C}}^{-\infty}, \vee, \oplus)$, $(\mathcal{K}_{\mathcal{C}}^{+\infty}, \wedge, \oplus)$ and $(\mathcal{K}_{\mathcal{C}}^{\varnothing}, \uplus, \oplus)$ are complete semirings.*

To sum up, if we analyze in detail all the structures identified in this section, we have that:

1) $(\overline{\mathcal{K}_{\mathcal{C}}}, \vee, \wedge; -\infty, +\infty, \precapprox_{\gamma^-,\gamma^+})$ is a commutative, idempotent, zero-sum-free, zero-divisor free and complete semiring, as:

   1.1 $\vee$ is associative;

   1.2 $\vee$ is commutative;

   1.3 $\vee$ has the neutral element: $-\infty$ (zero of the semiring);

   1.4 $\vee$ is idempotent, as: $\forall A \in \overline{\mathcal{K}_{\mathcal{C}}}$, $A \vee A = A$;

   [so $(\overline{\mathcal{K}_{\mathcal{C}}}, \vee)$ is a commutative, idempotent monoid]

   1.5 $\wedge$ is associative;

   1.6 $\wedge$ is commutative;

   1.7 $\wedge$ has the neutral element: $+\infty$ (unity of the semiring);

   1.8 $\wedge$ is idempotent, as: $\forall A \in \overline{\mathcal{K}_{\mathcal{C}}}$, $A \wedge A = A$;

   [so $(\overline{\mathcal{K}_{\mathcal{C}}}, \wedge)$ is a commutative, idempotent monoid]

   1.9 $\wedge$ is distributive with respect to $\vee$;

   1.10 $-\infty$ is the absorbing element for $\wedge$

   [so $(\overline{\mathcal{K}_{\mathcal{C}}}, \vee, \wedge)$ is a commutative, idempotent semiring]

   1.11 $(\overline{\mathcal{K}_{\mathcal{C}}}, \vee, \wedge)$ is zero-sum-free: $A \vee B = -\infty \Leftrightarrow A = B = -\infty$;

   1.12 $(\overline{\mathcal{K}_{\mathcal{C}}}, \vee, \wedge)$ is zero-divisor-free: $A \wedge B \neq -\infty \Leftrightarrow A \neq -\infty \neq B$;

   1.13 $(\overline{\mathcal{K}_{\mathcal{C}}}, \vee, \wedge)$ is complete: $\vee$ distributes over infinite $\wedge$;

   [so $(\overline{\mathcal{K}_{\mathcal{C}}}, \vee, \wedge)$ is a zero-sum-free, zero-divisor-free, complete semiring]

2) $(\overline{\mathcal{K}_{\mathcal{C}}}, \wedge, \vee; +\infty, -\infty, \precsim_{\gamma^-,\gamma^+})$ is a commutative, idempotent, zero-sum-free, zero-divisor free and complete semiring, as:

2.1 $\wedge$ is associative;

2.2 $\wedge$ is commutative;

2.3 $\wedge$ has the neutral element: $+\infty$ (zero of the semiring);

2.4 $\wedge$ is idempotent, as: $\forall A \in \overline{\mathcal{K}_{\mathcal{C}}}$, $A \wedge A = A$;

[so $(\overline{\mathcal{K}_{\mathcal{C}}}, \wedge)$ is a commutative, idempotent monoid]

2.5 $\vee$ is associative;

2.6 $\vee$ is commutative;

2.7 $\vee$ has the neutral element: $-\infty$ (unity of the semiring);

2.8 $\vee$ is idempotent, as: $\forall A \in \overline{\mathcal{K}_{\mathcal{C}}}$, $A \vee A = A$;

[so $(\overline{\mathcal{K}_{\mathcal{C}}}, \vee)$ is a commutative, idempotent monoid]

2.9 $\vee$ is distributive with respect to $\wedge$;

2.10 $+\infty$ is the absorbing element for $\vee$

[so $(\overline{\mathcal{K}_{\mathcal{C}}}, \wedge, \vee)$ is a commutative, idempotent semiring]

2.11 $(\overline{\mathcal{K}_{\mathcal{C}}}, \wedge, \vee)$ is zero-sum-free: $A \wedge B = +\infty \Leftrightarrow A = B = +\infty$;

2.12 $(\overline{\mathcal{K}_{\mathcal{C}}}, \wedge, \vee)$ is zero-divisor-free: $A \vee B \neq +\infty \Leftrightarrow A \neq +\infty \neq B$;

2.13 $(\overline{\mathcal{K}_{\mathcal{C}}}, \wedge, \vee)$ is complete: $\wedge$ distributes over infinite $\vee$;

[so $(\overline{\mathcal{K}_{\mathcal{C}}}, \wedge, \vee)$ is a zero-sum-free, zero-divisor-free, complete semiring]

3) $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}, \uplus, \cap; \varnothing, \mathbb{R}, \subseteqq_{\gamma^-,\gamma^+})$ is a commutative, idempotent, zero-sum-free and complete semiring, as:

3.1 $\uplus$ is associative;

3.2 $\uplus$ is commutative;

3.3 $\uplus$ has the neutral element: $\varnothing$ (zero of the semiring);

3.4 $\uplus$ is idempotent, as: $\forall A \in \mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}$, $A \uplus A = A$;

[so $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}, \uplus)$ is a commutative, idempotent monoid]

3.5 $\cap$ is associative;

3.6 $\cap$ is commutative;

3.7 $\cap$ has the neutral element: $\mathbb{R}$ (unity of the semiring);

3.8 $\cap$ is idempotent, as: $\forall A \in \mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}$, $A \cap A = A$;

[so $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}, \wedge)$ is a commutative, idempotent monoid]

3.9 $\cap$ is distributive with respect to $\uplus$;

3.10 $\varnothing$ is the absorbing element for $\cap$

[so $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}, \uplus, \cap)$ is a commutative, idempotent semiring]

3.11 $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}, \uplus, \cap)$ is zero-sum-free: $A \uplus B = \varnothing \Leftrightarrow A = B = \varnothing$;

3.12 $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}, \uplus, \cap)$ is complete: $\uplus$ distributes over infinite $\cap$;

[so $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}, \uplus, \cap)$ is a zero-sum-free complete semiring]

4) $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}, \cap, \uplus; \mathbb{R}, \varnothing, \subseteqq_{\gamma^-,\gamma^+})$ is a commutative, idempotent, zero-sum-free, zero-divisor free and complete semiring, as:

4.1 $\cap$ is associative;

4.2 $\cap$ is commutative;

4.3 $\cap$ has the neutral element: $\mathbb{R}$ (zero of the semiring);

4.4 $\cap$ is idempotent, as: $\forall A \in \mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}$, $A \cap A = A$;

[so $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}, \cap)$ is a commutative, idempotent monoid]

4.5 $\uplus$ is associative;

4.6 $\uplus$ is commutative;

4.7 $\uplus$ has the neutral element: $\varnothing$ (unity of the semiring);

4.8 $\uplus$ is idempotent, as: $\forall A \in \mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}$, $A \uplus A = A$;

[so $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}, \uplus)$ is a commutative, idempotent monoid]

4.9 $\uplus$ is distributive with respect to $\cap$;

4.10 $\mathbb{R}$ is the absorbing element for $\uplus$;

[so $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}, \cap, \uplus)$ is a commutative, idempotent semiring]

4.11 $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}, \cap, \uplus)$ is zero-sum-free: $A \cap B = \mathbb{R} \Leftrightarrow A = B = \mathbb{R}$;

4.12 $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}, \cap, \uplus)$ is zero-divisor-free: $A \uplus B \neq \mathbb{R} \Leftrightarrow A \neq \mathbb{R} \neq B$;

4.13 $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}, \cap, \uplus)$ is complete: $\cap$ distributes over infinite $\uplus$;

[so $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}, \cap, \uplus)$ is a zero-sum-free, zero-divisor-free, complete semiring]

5) $(\mathcal{K}_{\mathcal{C}}^{-\infty}, \vee, \oplus; -\infty, 0, \precsim_{\gamma^-,\gamma^+})$ is a commutative, zero-sum-free, zero-divisor-free and complete semiring, as:

5.1 $\vee$ is associative;

5.2 $\vee$ is commutative;

5.3 $\vee$ has the neutral element: $-\infty$ (zero of the semiring);

5.4 $\vee$ is idempotent, as: $\forall A \in \mathcal{K}_{\mathcal{C}}^{-\infty}$, $A \vee A = A$;

[so $(\mathcal{K}_{\mathcal{C}}^{-\infty}, \vee)$ is a commutative, idempotent monoid]

5.5 $\oplus$ is associative;

5.6 $\oplus$ is commutative;

5.7 $\oplus$ has the neutral element: 0 (unity of the semiring);

[so $(\mathcal{K}_{\mathcal{C}}^{-\infty}, \oplus)$ is a commutative monoid]

5.8 $\oplus$ is distributive with respect to $\vee$;

5.9 $-\infty$ is the absorbing element for $\oplus$;

[so $(\mathcal{K}_{\mathcal{C}}^{-\infty}, \vee, \oplus)$ is a commutative semiring]

5.10 $(\mathcal{K}_{\mathcal{C}}^{-\infty}, \vee, \oplus)$ is zero-sum-free: $A \vee B = -\infty \Leftrightarrow A = B = -\infty$;

5.11 $(\mathcal{K}_{\mathcal{C}}^{-\infty}, \vee, \oplus)$ is zero-divisor-free: $A \oplus B \neq -\infty \Leftrightarrow A \neq -\infty \neq B$;

5.12 $(\mathcal{K}_{\mathcal{C}}^{-\infty}, \vee, \oplus)$ is complete: it is closed for infinite sums and $\vee$ distributes over infinite sums;

[so $(\mathcal{K}_{\mathcal{C}}^{-\infty}, \vee, \oplus)$ is a zero-sum-free, zero-divisor-free, complete semiring]

6) $(\mathcal{K}_{\mathcal{C}}^{+\infty}, \wedge, \oplus; +\infty, 0, \precapprox_{\gamma^-,\gamma^+})$ is a commutative, zero-sum-free, zero-divisor-free and complete semiring, as:

6.1 $\wedge$ is associative;

6.2 $\wedge$ is commutative;

6.3 $\wedge$ has the neutral element: $+\infty$ (zero of the semiring);

6.4 $\wedge$ is idempotent, as: $\forall A \in \mathcal{K}_{\mathcal{C}}^{+\infty}$, $A \wedge A = A$;

[so $(\mathcal{K}_{\mathcal{C}}^{+\infty}, \wedge)$ is a commutative, idempotent monoid]

6.5 $\oplus$ is associative;

6.6 $\oplus$ is commutative;

6.7 $\oplus$ has the neutral element: 0 (unity of the semiring);

[so $(\mathcal{K}_{\mathcal{C}}^{+\infty}, \oplus)$ is a commutative monoid]

6.8 $\oplus$ is distributive with respect to $\wedge$;

6.9 $+\infty$ is the absorbing element for $\oplus$;

[so $(\mathcal{K}_{\mathcal{C}}^{+\infty}, \wedge, \oplus)$ is a commutative semiring]

6.10 $(\mathcal{K}_{\mathcal{C}}^{+\infty}, \wedge, \oplus)$ is zero-sum-free: $A \wedge B = +\infty \Leftrightarrow A = B = +\infty$;

6.11 $(\mathcal{K}_{\mathcal{C}}^{+\infty}, \wedge, \oplus)$ is zero-divisor-free: $A \oplus B \neq +\infty \Leftrightarrow A \neq +\infty \neq B$;

6.12 $(\mathcal{K}_{\mathcal{C}}^{+\infty}, \wedge, \oplus)$ is complete: it is closed for infinite sums and $\wedge$ distributes over infinite sums;

[so $(\mathcal{K}_{\mathcal{C}}^{+\infty}, \wedge, \oplus)$ is a zero-sum-free, zero-divisor-free, complete semiring]

7) $(\mathcal{K}_{\mathcal{C}}^{\varnothing}, \uplus, \oplus; \varnothing, 0, \subseteqq_{\gamma^-,\gamma^+})$ is a commutative, zero-sum-free, zero-divisor-free and complete semiring, as:

7.1 $\uplus$ is associative;

7.2 $\uplus$ is commutative;

7.3 $\uplus$ has the neutral element: $\varnothing$ (zero of the semiring);

Semiring	**0 – element**	**1 – element**	**Properties**
$(\overline{\mathcal{K}_{\mathcal{C}}}, \vee, \wedge)$	$-\infty = (-\infty; 0)$	$+\infty = (+\infty; 0)$	C, ZS, ZD, I, E
$(\overline{\mathcal{K}_{\mathcal{C}}}, \wedge, \vee)$	$+\infty = (+\infty; 0)$	$-\infty = (-\infty; 0)$	C, ZS, ZD, I, E
$(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}, \uplus, \cap)$	$\varnothing = (0; -\infty)$	$\mathbb{R} = (0; +\infty)$	C, ZS, I, E
$(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}, \cap, \uplus)$	$\mathbb{R} = (0; +\infty)$	$\varnothing = (0; -\infty)$	C, ZS, ZD, I, E
$(\mathcal{K}_{\mathcal{C}}^{-\infty}, \vee, \oplus)$	$-\infty = (-\infty; 0)$	$0 = (0; 0)$	C, ZS, ZD, E
$(\mathcal{K}_{\mathcal{C}}^{+\infty}, \wedge, \oplus)$	$+\infty = (+\infty; 0)$	$0 = (0; 0)$	C, ZS, ZD, E
$(\mathcal{K}_{\mathcal{C}}^{\varnothing}, \uplus, \oplus)$	$\varnothing = (0; -\infty)$	$0 = (0; 0)$	C, ZS, ZD, E

Table 5.2: Classification of interval semirings. C = commutative, ZS= zero-sum-free (or antinegative), ZD= zero-divisor-free (or entire), I = idempotent, E=complete.

7.4 $\uplus$ is idempotent, as: $\forall A \in \mathcal{K}_{\mathcal{C}}^{\varnothing}$, $A \uplus A = A$;

[so $(\mathcal{K}_{\mathcal{C}}^{\varnothing}, \uplus)$ is a commutative, idempotent monoid]

7.5 $\oplus$ is associative;

7.6 $\oplus$ is commutative;

7.7 $\oplus$ has the neutral element: 0 (unity of the semiring);

[so $(\mathcal{K}_{\mathcal{C}}^{\varnothing}, \oplus)$ is a commutative monoid]

7.8 $\oplus$ is distributive with respect to $\uplus$;

7.9 $\varnothing$ is the absorbing element for $\oplus$;

[so $(\mathcal{K}_{\mathcal{C}}^{\varnothing}, \uplus, \oplus)$ is a commutative semiring]

7.10 $(\mathcal{K}_{\mathcal{C}}^{\varnothing}, \uplus, \oplus)$ is zero-sum-free: $A \uplus B = \varnothing \Leftrightarrow A = B = \varnothing$;

7.11 $(\mathcal{K}_{\mathcal{C}}^{\varnothing}, \uplus, \oplus)$ is zero-divisor-free: $A \oplus B \neq \varnothing \Leftrightarrow A \neq \varnothing \neq B$;

7.12 $(\mathcal{K}_{\mathcal{C}}^{\varnothing}, \uplus, \oplus)$ is complete: it is closed for infinite sums and $\uplus$ distributes over infinite sums;

[so $(\mathcal{K}_{\mathcal{C}}^{\varnothing}, \uplus, \oplus)$ is a zero-sum-free, zero-divisor-free, complete semiring]

Table 5.2 summarizes the different types of interval semirings we have defined in this section and the properties associated with.

5.2 Alternative approaches to interval semirings

The problems encountered in Subsection 5.1.2, many of which arise when the intersection of intervals is an empty set, i.e., when intervals are disjointed, led to the development of alternative interpretative approaches; these in turn give rise to further types of algebraic structures, semirings and more, which will be extensively analyzed in this Section.

In fact, different approaches will be introduced both to overcome the aforementioned problems but also to search for structures with further interesting properties in addition to those already seen.

5.2.1 The dual approach

As regards the search for structures with additional properties compared to those analyzed so far, a first attempt is represented by an extension of the set of intervals $\mathcal{K}_\mathcal{C}$, introducing a sort of dual structure and redefining the set itself as indicated below:

$$\mathcal{K}_\mathcal{C}^+ = \mathcal{K}_\mathcal{C} = \{(\widehat{a};\widetilde{a}) \mid \widehat{a},\widetilde{a} \in \mathbb{R}, \widetilde{a} \geqslant 0\}$$

which represents the classic set $\mathcal{K}_\mathcal{C}$ of *proper* intervals considered so far

$$A = (\widehat{a};\widetilde{a}) \text{ with } \widetilde{a} \geqslant 0 \text{ or } A = [a^-, a^+] \text{ with } a^- \leqslant a^+;$$

similarly, we define

$$\mathcal{K}_\mathcal{C}^- = \{(\widehat{a};-\widetilde{a}) \mid \widehat{a},\widetilde{a} \in \mathbb{R}, \widetilde{a} \geqslant 0\}$$

which stands for the set of what we call the *dual* intervals, denoted by

$$A = (\widehat{a};-\widetilde{a}) \text{ with } \widetilde{a} \geqslant 0 \text{ or } A = [a^+, a^-] \text{ with } a^- \leqslant a^+,$$

where $A = (\widehat{a};\widetilde{a}) = [a^-, a^+] \in \mathcal{K}_\mathcal{C}^+$.

Finally, we consider the union

$$\mathcal{K}_\mathcal{C}^\pm = \mathcal{K}_\mathcal{C}^+ \cup \mathcal{K}_\mathcal{C}^- \tag{5.6}$$

where $\mathcal{K}_\mathcal{C}^+ \cap \mathcal{K}_\mathcal{C}^- = \mathbb{R}$ and $(a;0) = [a,a] = \{a\}$.

For the elements $A \in \mathcal{K}_\mathcal{C}^\pm$ we denote by $A^* \in \mathcal{K}_\mathcal{C}^\pm$ the *dual of A*, defined (in endpoint notation) by

$$A^* = [a^+, a^-] \text{ with } A = [a^-, a^+] \tag{5.7}$$

or (in midpoint notation) by

$$A^* = (\widehat{a};-\widetilde{a}) \text{ with } A = (\widehat{a};\widetilde{a}) \tag{5.8}$$

so that

- if $A = (\widehat{a}; \widetilde{a}) = [a^-, a^+] \in \mathcal{K}_{\mathcal{C}}^+$, that is, $a^- \leqslant a^+$ and $\widetilde{a} \geqslant 0$, then $A^* = (\widehat{a}; -\widetilde{a}) = [a^+, a^-] \in \mathcal{K}_{\mathcal{C}}^-$ (see Figure 5.1);
- if $A = (\widehat{a}; \widetilde{a}) = [a^-, a^+] \in \mathcal{K}_{\mathcal{C}}^-$, that is, $a^- \geqslant a^+$ and $\widetilde{a} \leqslant 0$, then $A^* = (\widehat{a}; -\widetilde{a}) = [a^+, a^-] \in \mathcal{K}_{\mathcal{C}}^+$.

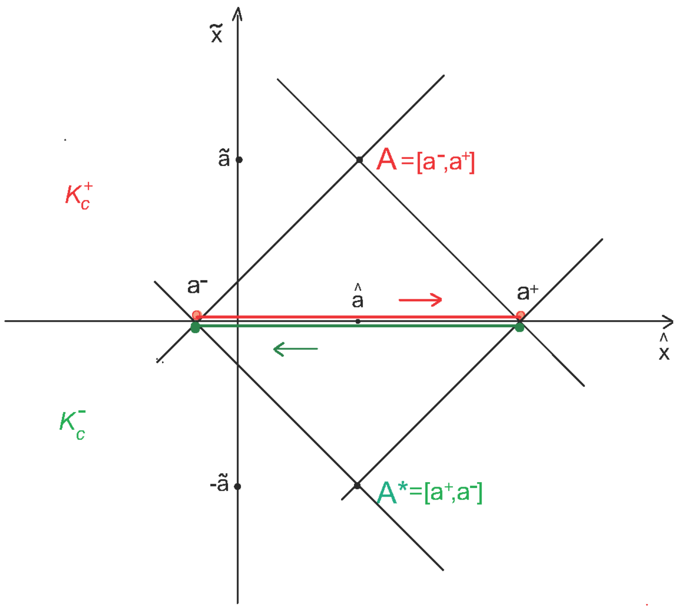


Figure 5.1: Interval $A = [a^-, a^+] = (\widehat{a}; \widetilde{a}) \in \mathcal{K}_{\mathcal{C}}^+$ and its dual $A^* = [a^+, a^-] = (\widehat{a}; -\widetilde{a}) \in \mathcal{K}_{\mathcal{C}}^-$ in the midpoint plane $(\widehat{x}; \widetilde{x})$.

Remark 5.2.1. *According to* (5.7) *and* (5.8), *interval* A^* *cannot be interpreted as complementary to* A *as can be easily deduced from Figure 5.1.*

Since we are interested in defining an order in $\mathcal{K}_{\mathcal{C}}^{\pm}$, the most logical choice is to try to consider some sort of extension of the gamma order $\precapprox_{\gamma^-,\gamma^+}$ with $\gamma^- < 0$ and $\gamma^+ > 0$ fixed.

Let $A = (\widehat{a}; \widetilde{a})$ and $B = (\widehat{b}; \widetilde{b})$ be two intervals in $\mathcal{K}_{\mathcal{C}}^{\pm}$, with $A \neq B$; according to (4.7), we define as usual A_m, A_p and B_m, B_p, the lines for A and B, with angular coefficients respectively γ^-, γ^+, i.e.,

$$A_m : \widetilde{x} = \widetilde{a} + \gamma^- (\widehat{x} - \widehat{a}) \quad \text{and} \quad A_p : \widetilde{x} = \widetilde{a} + \gamma^+ (\widehat{x} - \widehat{a})$$

as well as

$$B_m : \widetilde{x} = \widetilde{b} + \gamma^- \left(\widehat{x} - \widehat{b}\right) \quad \text{and} \quad B_p : \widetilde{x} = \widetilde{b} + \gamma^+ \left(\widehat{x} - \widehat{b}\right),$$

where $\widehat{a}, \widetilde{a}, \widehat{b}, \widetilde{b} \in \mathbb{R}$ (see Figure 5.2).

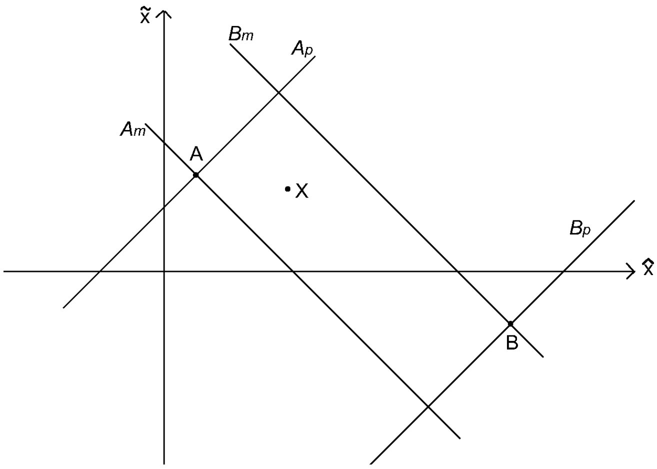


Figure 5.2: The lines for A and B, with angular coefficients respectively γ^-, γ^+: A_m, A_p and B_m, B_p in $\mathcal{K}_{\mathcal{C}}^{\pm}$.

Before proceeding, it should be noted that, considering a pair of intervals A and B in $\mathcal{K}_{\mathcal{C}}^{\pm}$, there are four different cases which correspond to the four different positions, as represented in Figure 5.3, in which the two intervals can occur:

(1) $A, B \in \mathcal{K}_{\mathcal{C}}^{+}$;

(2) $A, B \in \mathcal{K}_{\mathcal{C}}^{-}$;

(3) $A \in \mathcal{K}_{\mathcal{C}}^{+}$ and $B \in \mathcal{K}_{\mathcal{C}}^{-}$;

(4) $A \in \mathcal{K}_{\mathcal{C}}^{-}$ and $B \in \mathcal{K}_{\mathcal{C}}^{+}$.

Let us start by considering case (1): $A = (\widehat{a}; \widetilde{a}), B = (\widehat{b}; \widetilde{b}) \in \mathcal{K}_{\mathcal{C}}^{+}$ (top left of the picture of Figure 5.3).

We can define the order $\lessapprox^*_{\gamma^-,\gamma^+}$ as follows:

$$A \lessapprox^*_{\gamma^-,\gamma^+} B \Longleftrightarrow A \precapprox_{\gamma^-,\gamma^+} B \tag{5.9}$$

where, according to (2.33), it is

$$A \precapprox_{\gamma^-,\gamma^+} B \Longleftrightarrow \begin{cases} \widetilde{a} \geqslant \widetilde{b} + \gamma^+ \left(\widehat{a} - \widehat{b}\right) \\ \widetilde{a} \leqslant \widetilde{b} + \gamma^- \left(\widehat{a} - \widehat{b}\right) \end{cases} \Longleftrightarrow \widetilde{b} + \gamma^+ \left(\widehat{a} - \widehat{b}\right) \leqslant \widetilde{a} \leqslant \widetilde{b} + \gamma^- \left(\widehat{a} - \widehat{b}\right)$$

or, which is the same,

$$A \precapprox_{\gamma^-,\gamma^+} B \Longleftrightarrow \begin{cases} \widetilde{b} \leqslant \widetilde{a} + \gamma^+ \left(\widehat{b} - \widehat{a}\right) \\ \widetilde{b} \geqslant \widetilde{a} + \gamma^- \left(\widehat{b} - \widehat{a}\right) \end{cases} \Longleftrightarrow \widetilde{a} + \gamma^- \left(\widehat{b} - \widehat{a}\right) \leqslant \widetilde{b} \leqslant \widetilde{a} + \gamma^+ \left(\widehat{b} - \widehat{a}\right)$$

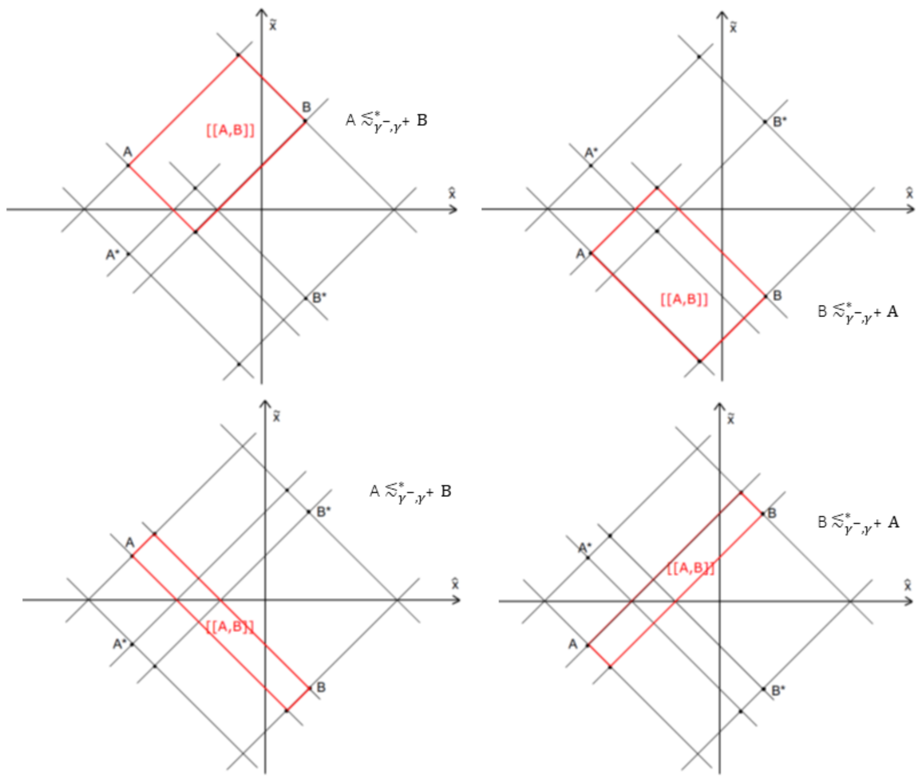


Figure 5.3: The four different cases in which a pair of intervals A and B can lie relative to each other in $\mathcal{K}_{\mathcal{C}}^{\pm}$ with respect to order $A \lessapprox^*_{\gamma^-,\gamma^+} B$.

so that we obtain

$$A \lessapprox^*_{\gamma^-,\gamma^+} X \Longleftrightarrow \begin{cases} \widetilde{a} \geqslant \widetilde{x} + \gamma^+ (\widehat{a} - \widehat{x}) \\ \widetilde{a} \leqslant \widetilde{x} + \gamma^- (\widehat{a} - \widehat{x}) \end{cases} \Longleftrightarrow \begin{cases} \widetilde{x} \leqslant \widetilde{a} + \gamma^+ (\widehat{x} - \widehat{a}) \\ \widetilde{x} \geqslant \widetilde{a} + \gamma^- (\widehat{x} - \widehat{a}) \end{cases}$$

and

$$X \lessapprox^*_{\gamma^-,\gamma^+} B \Longleftrightarrow \begin{cases} \widetilde{x} \geqslant \widetilde{b} + \gamma^+ \left(\widehat{x} - \widehat{b}\right) \\ \widetilde{x} \leqslant \widetilde{b} + \gamma^- \left(\widehat{x} - \widehat{b}\right) \end{cases} \Longleftrightarrow \begin{cases} \widetilde{b} \leqslant \widetilde{x} + \gamma^+ \left(\widehat{b} - \widehat{x}\right) \\ \widetilde{b} \geqslant \widetilde{x} + \gamma^- \left(\widehat{b} - \widehat{x}\right) \end{cases};$$

therefore, we can write

$$\begin{cases} \widetilde{a} + \gamma^- (\widehat{x} - \widehat{a}) \leqslant \widetilde{x} \leqslant \widetilde{b} + \gamma^- \left(\widehat{x} - \widehat{b}\right) \\ \widetilde{b} + \gamma^+ \left(\widehat{x} - \widehat{b}\right) \leqslant \widetilde{x} \leqslant \widetilde{a} + \gamma^+ (\widehat{x} - \widehat{a}) . \end{cases}$$

Referring now to the other three cases identified, we have that the situation is analogous.

Specifically, as shown in Figure 5.3, considering any $A = (\widehat{a};\widetilde{a})$ and $B = (\widehat{b};\widetilde{b})$ in $\mathcal{K}_{\mathcal{C}}^{\pm}$ with $A \lesseqqgtr^{*}_{\gamma^-,\gamma^+} B$ it follows that, as $A^* = (\widehat{a};-\widetilde{a})$ and $B^* = (\widehat{b};-\widetilde{b})$, it is trivial to verify that

$$A \lessapprox^{*}_{\gamma^-,\gamma^+} B \Leftrightarrow B^* \lessapprox^{*}_{\gamma^-,\gamma^+} A^*;$$

therefore, the next proposition follows.

Proposition 5.2.1. *Let* $A, B \in \mathcal{K}_{\mathcal{C}}^{\pm}$*, then*

$$A \lessapprox^{*}_{\gamma^-,\gamma^+} B \Leftrightarrow B^* \lessapprox^{*}_{\gamma^-,\gamma^+} A^*.$$

The following statement summarizes the situation.

Proposition 5.2.2. *let* A *and* B *be any two intervals of* $\mathcal{K}_{\mathcal{C}}^{\pm}$.

(1) If $A, B \in \mathcal{K}_{\mathcal{C}}^{+}$*, then* (5.9) *holds;*

(2) If $A, B \in \mathcal{K}_{\mathcal{C}}^{-}$*, then* $A \lessapprox^{*}_{\gamma^-,\gamma^+} B \Leftrightarrow B^* \lessapprox^{*}_{\gamma^-,\gamma^+} A^*$*;*

(3) if $A \in \mathcal{K}_{\mathcal{C}}^{+}$ *and* $B \in \mathcal{K}_{\mathcal{C}}^{-}$*, then* $A \lessapprox^{*}_{\gamma^-,\gamma^+} B$ *always holds;*

(4) if $A \in \mathcal{K}_{\mathcal{C}}^{-}$ *and* $B \in \mathcal{K}_{\mathcal{C}}^{+}$*, then* $B \lessapprox^{*}_{\gamma^-,\gamma^+} A$ *always holds.*

Thanks to what has just been seen, we have that the order $\lessapprox^{*}_{\gamma^-,\gamma^+}$ turns out to be a partial order as it satisfies the reflexive, the antisymmetric and the transitive properties; therefore, the set $\mathcal{K}_{\mathcal{C}}^{\pm}$ endowed with the partial order $\lessapprox^{*}_{\gamma^-,\gamma^+}$ is a poset.

Moreover, according to Definition 2.2.1, the structure $(\mathcal{K}_{\mathcal{C}}^{\pm}, \lessapprox^{*}_{\gamma^-,\gamma^+})$ is also a lattice, as any of its elements A and B have a supremum $sup_{\lessapprox^{*}_{\gamma^-,\gamma^+}} \{A, B\}$ and an infimum $inf_{\lessapprox^{*}_{\gamma^-,\gamma^+}} \{A, B\}$ in $\mathcal{K}_{\mathcal{C}}^{\pm}$.

At this point, as occurred in the case of $\mathcal{K}_{\mathcal{C}}$, it becomes necessary to introduce the following set:

$$\overline{\mathcal{K}_{\mathcal{C}}^{\pm}} \stackrel{def}{=} \mathcal{K}_{\mathcal{C}}^{\pm} \cup \{\pm\infty\},$$

where, as usual, $-\infty = (-\infty; 0)$ and $+\infty = (+\infty; 0)$. We also define

$$-\infty = (-\infty; 0) = inf_{\lessapprox^{*}_{\gamma^-,\gamma^+}} \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$$

as well as

$$(-\infty)^* = (-\infty; 0)^* = sup_{\lessapprox^{*}_{\gamma^-,\gamma^+}} \overline{\mathcal{K}_{\mathcal{C}}^{\pm}},$$

so that, for all $\gamma^- < 0$ and $\gamma^+ > 0$, it is:

$$-\infty \lessapprox^{*}_{\gamma^-,\gamma^+} X \lessapprox^{*}_{\gamma^-,\gamma^+} (-\infty)^*, \ \forall X \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$$

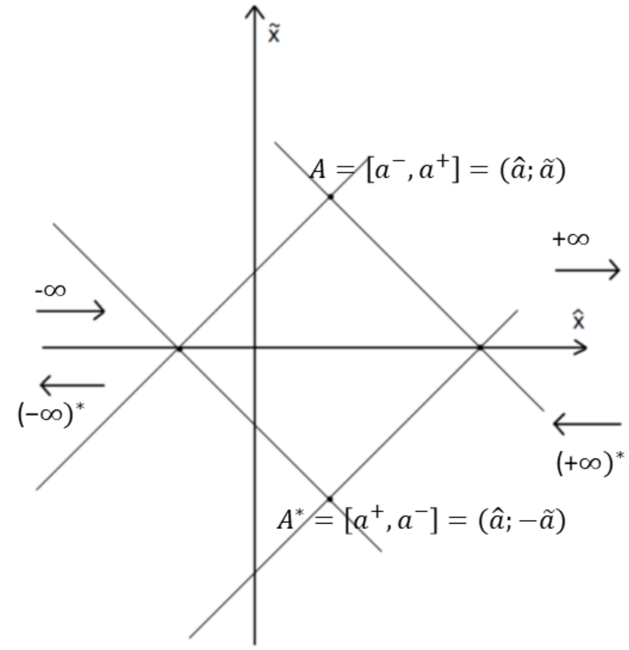


Figure 5.4: Representation of the elements A, A^*, $-\infty$ and $(-\infty)^*$ in $\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$.

(see Figure 5.4).

What we have just seen means that the lattice $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \lessapprox^*_{\gamma^-,\gamma^+})$ has a minimum, denoted by $-\infty$, and a maximum, denoted by $(-\infty)^*$, which satisfy the inequality $-\infty \lessapprox^*_{\gamma^-,\gamma^+} X \lessapprox^*_{\gamma^-,\gamma^+} (-\infty)^*$ for every $X \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$, i.e., $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \lessapprox^*_{\gamma^-,\gamma^+})$ is a bounded lattice.

Furthermore, according to Definition 2.2.2, we can also add that the structure $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \lessapprox^*_{\gamma^-,\gamma^+})$ is a complete lattice, as the following proposition holds.

Proposition 5.2.3. *Consider a partial order* $\lessapprox^*_{\gamma^-,\gamma^+}$ *on* $\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$ *and let* $\mathbb{S} \subset \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$ *be any nonempty bounded subset of intervals. Then, there exist both* $\inf(\mathbb{S})$, $\sup(\mathbb{S}) \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$ *such that for all* $X \in \mathbb{S}$

$$\inf(\mathbb{S}) \lessapprox^*_{\gamma^-,\gamma^+} X \lessapprox^*_{\gamma^-,\gamma^+} \sup(\mathbb{S}).$$

Proof. Similar to the proof of Proposition 2.2.10. □

On the other side, since it is possible to consider lattices also as algebraic structures, we can look at $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \lessapprox^*_{\gamma^-,\gamma^+})$ as a structure of the type $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \wedge)$ where the lattice operations $\vee$ and $\wedge$ stand for supremum and infimum of two elements $X, Y \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$:

$$X \vee Y = sup_{\lessapprox^*_{\gamma^-,\gamma^+}}\{X, Y\},$$

$X \wedge Y = inf_{\lessapprox^*_{\gamma^-,\gamma^+}}\{X, Y\}.$

This means that we have the following binary functions:

$\vee : \overline{\mathcal{K}^{\pm}_{\mathcal{C}}} \times \overline{\mathcal{K}^{\pm}_{\mathcal{C}}} \to \overline{\mathcal{K}^{\pm}_{\mathcal{C}}}$ such that: $(X, Y) \to X \vee Y = sup_{\lessapprox^*_{\gamma^-,\gamma^+}}\{X, Y\};$

$\wedge : \overline{\mathcal{K}^{\pm}_{\mathcal{C}}} \times \overline{\mathcal{K}^{\pm}_{\mathcal{C}}} \to \overline{\mathcal{K}^{\pm}_{\mathcal{C}}}$ such that: $(X, Y) \to X \wedge Y = inf_{\lessapprox^*_{\gamma^-,\gamma^+}}\{X, Y\}.$

In particular, the definition below examines all possible cases of pairs of intervals in $\mathcal{K}^{\pm}_{\mathcal{C}}$.

Definition 5.2.1. *Let $A, B \in \mathcal{K}^{\pm}_{\mathcal{C}}$. Let us consider the following cases.*

1. *If $A, B \in \mathcal{K}^{+}_{\mathcal{C}}$ or $A, B \in \mathcal{K}^{-}_{\mathcal{C}}$ we have:*

    1.a *if $A \lessapprox^*_{\gamma^-,\gamma^+} B$, then $A \vee B = B$ and $A \wedge B = A$;*

    1.b *if $B \lessapprox^*_{\gamma^-,\gamma^+} A$, then $A \vee B = A$ and $A \wedge B = B$;*

    1.c *if $A||_{\lessapprox^*_{\gamma^-,\gamma^+}} B$ and $A_pB_m \lessapprox^*_{\gamma^-,\gamma^+} A_mB_p$, then $A \vee B = A_mB_p$ and $A \wedge B = A_pB_m$;*

    1.d *if $A||_{\lessapprox^*_{\gamma^-,\gamma^+}} B$ and $A_mB_p \lessapprox^*_{\gamma^-,\gamma^+} A_pB_m$, then $A \vee B = A_pB_m$ and $A \wedge B = A_mB_p$.*

2. *If $A \in \mathcal{K}^{+}_{\mathcal{C}}$ and $B \in \mathcal{K}^{-}_{\mathcal{C}}$ (therefore $A \lessapprox^*_{\gamma^-,\gamma^+} B$), we have $A \vee B = B$ and $A \wedge B = A$.*

3. *If $A \in \mathcal{K}^{-}_{\mathcal{C}}$ and $B \in \mathcal{K}^{+}_{\mathcal{C}}$ (therefore $B \lessapprox^*_{\gamma^-,\gamma^+} A$), we have $A \vee B = A$ and $A \wedge B = B$.*

4. *If $A = B$ then $A \vee B = A \wedge B = A = B$.*

Regarding the two operations introduced above, it is easy to verify that the following properties hold:

1 $\vee$ and $\wedge$ are commutative: $\forall A, B \in \overline{\mathcal{K}^{\pm}_{\mathcal{C}}}$

1.a $A \vee B = B \vee A$,

1.b $A \wedge B = B \wedge A$;

2 $\vee$ and $\wedge$ are associative: $\forall A, B, C \in \overline{\mathcal{K}^{\pm}_{\mathcal{C}}}$

2.a $(A \vee B) \vee C = A \vee (B \vee C)$,

2.b $(A \wedge B) \wedge C = A \wedge (B \wedge C)$;

3 the absorption laws apply: $\forall A, B \in \overline{\mathcal{K}^{\pm}_{\mathcal{C}}}$

3.a $A \vee (A \wedge B) = A$,

3.b $A \wedge (A \vee B) = A$;

4 the idempotency is satisfied for both $\vee$ and $\wedge$: $\forall A \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$

4.a $A \vee A = A$,

4.b $A \wedge A = A$.

It is also easy to verify that:

5.a $\vee$ is left and right distributive over $\wedge$:

5.a.i $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C), \forall A, B, C \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$,

5.a.ii $(A \wedge B) \vee C = (A \vee C) \wedge (B \vee C), \forall A, B, C \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$;

5.b $\wedge$ is left and right distributive over $\vee$:

5.b.i $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C), \forall A, B, C \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$,

5.b.ii $(A \vee B) \wedge C = (A \wedge C) \vee (B \wedge C), \forall A, B, C \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$.

So, according to Definition 4.1.6, the structure $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \wedge, \lessapprox^*_{\gamma^-,\gamma^+})$ is an algebraic distributive lattice and, having $-\infty$ and $(-\infty)^*$ as minimum and maximum, it is also bounded.

In this regard we can also add that:

6.a $-\infty$ (the lattice's bottom) is the neutral element for the join operation $\vee$: $A \vee (-\infty) = A$, $\forall A \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$;

6.b $(-\infty)^*$ (the lattice's top) is the neutral element for the meet operation $\wedge$: $A \wedge (-\infty)^* = A$, $\forall A \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$;

6.c $-\infty$ is the absorbing element for the meet operation $\wedge$: $A \wedge (-\infty) = (-\infty)$, $\forall A \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$;

6.d $(-\infty)^*$ is the absorbing element for the join operation $\vee$: $A \vee (-\infty)^* = (-\infty)^*$, $\forall A \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$.

All this allow us to state the following proposition.

Proposition 5.2.4. *The structure* $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \wedge, (-\infty), (-\infty)^*, \lessapprox^*_{\gamma^-,\gamma^+})$ *as well as* $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge, \vee, (-\infty)^*, (-\infty), \lessapprox^*_{\gamma^-,\gamma^+})$ *are commutative, idempotent semirings.*

Proof. $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \wedge)$ is a commutative, idempotent semiring, as:

1a) $(\overline{\mathcal{K}_{C}^{\pm}}, \vee)$ is a commutative, idempotent monoid with neutral element $(-\infty)$:

(i) $\vee$ is associative: $(A \vee B) \vee C = A \vee (B \vee C), \forall A, B, C \in \overline{\mathcal{K}_{C}^{\pm}}$;

(ii) $\vee$ has a neutral element $i_\vee = (-\infty) \in \overline{\mathcal{K}_C^\pm} : A \vee (-\infty) = (-\infty) \vee A = A, \forall A \in \overline{\mathcal{K}_C^\pm}$ (so, $(-\infty)$ is the 0-element of the semiring $(\overline{\mathcal{K}_C^\pm}, \vee, \wedge)$;

(iii) $\vee$ is commutative: $A \vee B = B \vee A, \forall A, B \in \overline{\mathcal{K}_C^\pm}$;

(iv) $\vee$ is idempotent: $A \vee A = A, \forall A \in \overline{\mathcal{K}_C^\pm}$.

2a) $(\overline{\mathcal{K}_C^\pm}, \wedge)$ is a commutative, idempotent monoid with neutral element $(-\infty)^*$:

(i) $\wedge$ is associative: $(A \wedge B) \wedge C = A \wedge (B \wedge C), \forall A, B, C \in \overline{\mathcal{K}_C^\pm}$;

(ii) $\wedge$ has a neutral element $i_\wedge = (-\infty)^* \in \overline{\mathcal{K}_C^\pm} : A \wedge (-\infty)^* = (-\infty)^* \wedge A = A, \forall A \in \overline{\mathcal{K}_C^\pm}$ (so, $(-\infty)^*$ is the 1-element of the semiring $(\overline{\mathcal{K}_C^\pm}, \vee, \wedge)$;

(iii) $\wedge$ is commutative: $A \wedge B = B \wedge A, \forall A, B \in \overline{\mathcal{K}_C^\pm}$;

(iv) $\wedge$ is idempotent: $A \wedge A = A, \forall A \in \overline{\mathcal{K}_C^\pm}$.

3a) $\wedge$ is left and right distributive over $\vee$:

(i) $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C), \forall A, B, C \in \overline{\mathcal{K}_C^\pm}$;

(ii) $(A \vee B) \wedge C = (A \wedge C) \vee (B \wedge C), \forall A, B, C \in \overline{\mathcal{K}_C^\pm}$.

4a) $(-\infty)$ is the absorbing element for $\wedge$:

$$A \wedge (-\infty) = (-\infty) \wedge A = (-\infty), \forall A \in \overline{\mathcal{K}_C^\pm}.$$

Analogously, we have that also $(\overline{\mathcal{K}_C^\pm}, \wedge, \vee)$ is a commutative, idempotent semiring, as:

1b) $(\overline{\mathcal{K}_C^\pm}, \wedge)$ is a commutative, idempotent monoid with neutral element $(-\infty)^*$:

(i) $\wedge$ is associative: $(A \wedge B) \wedge C = A \wedge (B \wedge C), \forall A, B, C \in \overline{\mathcal{K}_C^\pm}$;

(ii) $\wedge$ has a neutral element $i_\wedge = (-\infty)^* \in \overline{\mathcal{K}_C^\pm} : A \wedge (-\infty)^* = (-\infty)^* \wedge A = A, \forall A \in \overline{\mathcal{K}_C^\pm}$ (so, $(-\infty)^*$ is the 0-element of the semiring $(\overline{\mathcal{K}_C^\pm}, \wedge, \vee)$);

(iii) $\wedge$ is commutative: $A \wedge B = B \wedge A, \forall A, B \in \overline{\mathcal{K}_C^\pm}$;

(iv) $\wedge$ is idempotent: $A \wedge A = A, \forall A \in \overline{\mathcal{K}_C^\pm}$.

2b) $(\overline{\mathcal{K}_C^\pm}, \vee)$ is a commutative, idempotent monoid with neutral element $(-\infty)$:

(i) $\vee$ is associative: $(A \vee B) \vee C = A \vee (B \vee C), \forall A, B, C \in \overline{\mathcal{K}_C^{\pm}}$;

(ii) $\vee$ has a neutral element $i_\vee = (-\infty) \in \overline{\mathcal{K}_C^{\pm}}$: $A \vee (-\infty) = (-\infty; 0) \vee A = A, \forall A \in \overline{\mathcal{K}_C^{\pm}}$ (so, $(-\infty)$ is the 1-element of the semiring $(\overline{\mathcal{K}_C^{\pm}}, \wedge, \vee)$);

(iii) $\vee$ is commutative: $A \vee B = B \vee A, \forall A, B \in \overline{\mathcal{K}_C}$;

(iv) $\vee$ is idempotent: $A \vee A = A, \forall A \in \overline{\mathcal{K}_C^{\pm}}$.

3b) $\vee$ is left and right distributive over $\wedge$:

(i) $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C), \forall A, B, C \in \overline{\mathcal{K}_C^{\pm}}$;

(ii) $(A \wedge B) \vee C = (A \vee C) \wedge (B \vee C), \forall A, B, C \in \overline{\mathcal{K}_C^{\pm}}$.

4b) $(-\infty)^*$ is the absorbing element for $\vee$:

$$A \vee (-\infty)^* = (-\infty)^* \vee A = (-\infty)^*, \forall A \in \overline{\mathcal{K}_C^{\pm}}.$$

□

Finally, thanks to the particular construction of the above structures and to the Proposition 5.2.1, it is easy to verify that other properties are also valid, such as:

(1) $(A^*)^* = A, \ \forall A \in \overline{\mathcal{K}_C^{\pm}}$;

(2) $A \lessapprox^*_{\gamma^-,\gamma^+} B \Rightarrow B^* \lessapprox^*_{\gamma^-,\gamma^+} A^*, \ \forall A, B \in \overline{\mathcal{K}_C^{\pm}}$.

From these properties, according to Definition 4.1.8, we obtain the next proposition.

Proposition 5.2.5. *The structures* $(\overline{\mathcal{K}_C^{\pm}}, \vee, \wedge, \lessapprox^*_{\gamma^-,\gamma^+})$ *and* $(\overline{\mathcal{K}_C^{\pm}}, \wedge, \vee, \lessapprox^*_{\gamma^-,\gamma^+})$ *are De Morgan Algebras.*

As a consequence of Proposition 5.2.5, the following well-known laws also hold:

(3.a) $(A \vee B)^* = A^* \wedge B^*, \ \forall A, B \in \overline{\mathcal{K}_C^{\pm}}$;

(3.b) $(A \wedge B)^* = A^* \vee B^*, \ \forall A, B \in \overline{\mathcal{K}_C^{\pm}}$;

Besides, it's easy to check that the Kleene conditions are also valid, i.e.:

(4) $A \wedge A^* \lessapprox^*_{\gamma^-,\gamma^+} B \vee B^*, \ \forall A, B \in \overline{\mathcal{K}_C^{\pm}}$.

This means that, again according to Definition 4.1.8, the following statement holds.

Kleene algebra	**0 − element**	**1 − element**
$(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \wedge, \lessapprox^*_{\gamma^-,\gamma^+})$	$-\infty = (-\infty; 0)$	$(-\infty)^* = (-\infty; 0)^*$
$(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge, \vee, \lessapprox^*_{\gamma^-,\gamma^+})$	$(-\infty)^* = (-\infty; 0)^*$	$-\infty = (-\infty; 0)$

Table 5.3: Classification of interval Kleene algebras with zero and unity.

Proposition 5.2.6. *The structures* $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \wedge, \lessapprox^*_{\gamma^-,\gamma^+})$ *and* $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge, \vee, \lessapprox^*_{\gamma^-,\gamma^+})$ *are Kleene algebras.*

Therefore, based on (4.4) and (4.5), according to Definition 5.2.1, it is possible to define:

- $W_0(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}) = \{X \wedge X^* : X \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}\} = \overline{\mathcal{K}_{\mathcal{C}}^{+}} = \mathcal{K}_{\mathcal{C}}^{+} \cup \{-\infty, +\infty\}$;
- $W_1(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}) = \{X \vee X^* : X \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}\} = \overline{\mathcal{K}_{\mathcal{C}}^{-}} = \mathcal{K}_{\mathcal{C}}^{-} \cup \{(-\infty)^*, (+\infty)^*\}$.

Hence, the Kleene condition can be riformulated as

$$\forall X \in W_0(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}),\ \forall Y \in W_1(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}):\ X \lessapprox^*_{\gamma^-,\gamma^+} Y,$$

that is,

$$\forall X \in \overline{\mathcal{K}_{\mathcal{C}}^{+}},\ \forall Y \in \overline{\mathcal{K}_{\mathcal{C}}^{-}}:\ X \lessapprox^*_{\gamma^-,\gamma^+} Y.$$

Accordingly,

$$W_0(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}) \cap W_1(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}) = \overline{\mathcal{K}_{\mathcal{C}}^{+}} \cap \overline{\mathcal{K}_{\mathcal{C}}^{-}} = [-\infty, +\infty] = \overline{\mathbb{R}}.$$

Table 5.3 summarizes the structures defined in this subsection.

Lastly, similarly to (2.28), it is also possible to consider the "segment" in $\mathcal{K}_{\mathcal{C}}^{\pm}$ defined as

$$S(A, B) = \{X_t | X_t = (1-t)A \oplus tB,\ t \in [0, 1]\},$$

where $\oplus$ stands for Minkowski's addition.

In other words, we have:

$$X_t = \left((1-t)\widehat{a} + t\widehat{b}; (1-t)\widetilde{a} + t\widetilde{b}\right).$$

Definition 5.2.2. *A subset* $\mathbb{X}$ *of* $\mathcal{K}_{\mathcal{C}}^{\pm}$ *is said to be convex if and only if*

$$S(A, B) \subseteq \mathbb{X},\ \ \forall A, B \in \mathbb{X}.$$

Remark 5.2.2. *It should be noted that the structures outlined here undoubtedly possess interesting properties but, at the same time, it is impossible to succeed in defining any other type of structure with "polar" characteristics with respect to them, as the order considered here does not lend itself to this type of formulation.*

5.2.2 The polar approach

What we are interested in doing now is trying to overcome the problems caused by disjoint intervals, which emerged in Subsection 5.1.2, also keeping the characteristics of polarity between different orders valid, in all possible cases previously analyzed; it is therefore a question of considering the structures examined in Subsection 5.1.2 and expanding them to the set $\mathcal{K}_{\mathcal{C}}^{\pm}$.

Let us first consider the set $\mathcal{K}_{\mathcal{C}}^{\pm}$ to which, however, unlike the one introduced in (5.6), the following notations are associated:

$$\mathcal{K}_{\mathcal{C}}^{+} = \mathcal{K}_{\mathcal{C}} = \{(\widehat{a}; \widetilde{a}) \mid \widehat{a}, \widetilde{a} \in \mathbb{R}, \widetilde{a} \geqslant 0\}$$

which represents the set of intervals denoted by

$$A = (\widehat{a}; \widetilde{a}) \text{ with } \widetilde{a} \geqslant 0 \text{ or } A = [a^-, a^+]^+;$$

similarly, we define

$$\mathcal{K}_{\mathcal{C}}^{-} = \{(\widehat{a}; -\widetilde{a}) \mid \widehat{a}, \widetilde{a} \in \mathbb{R}, \widetilde{a} \geqslant 0\}$$

which stands for the set of intervals denoted by

$$A = (\widehat{a}; -\widetilde{a}) \text{ with } \widetilde{a} \geqslant 0 \text{ or } A = [a^-, a^+]^-.$$

As usual, for each element $A = (\widehat{a}; \widetilde{a}) \in \mathcal{K}_{\mathcal{C}}^{\pm}$ we denote its "dual" with $A^* = (\widehat{a}; -\widetilde{a}) \in \mathcal{K}_{\mathcal{C}}^{\pm}$.

After that, for all $A = (\widehat{a}; \widetilde{a})$ and $B = (\widehat{b}; \widetilde{b})$ in $\mathcal{K}_{\mathcal{C}}^{\pm}$, we define the order $\lessapprox_{\gamma^-,\gamma^+}$ as follows:

$$A \lessapprox_{\gamma^-,\gamma^+} B \Longleftrightarrow A \precapprox_{\gamma^-,\gamma^+} B$$

so, according to (2.33), it is

$$A \lessapprox_{\gamma^-,\gamma^+} B \Longleftrightarrow \begin{cases} \widetilde{a} \geqslant \widetilde{b} + \gamma^+ \left(\widehat{a} - \widehat{b}\right) \\ \widetilde{a} \leqslant \widetilde{b} + \gamma^- \left(\widehat{a} - \widehat{b}\right) \end{cases}$$

or, which is the same,

$$A \lessapprox_{\gamma^-,\gamma^+} B \Longleftrightarrow \begin{cases} \widetilde{b} \leqslant \widetilde{a} + \gamma^+ \left(\widehat{b} - \widehat{a}\right) \\ \widetilde{b} \geqslant \widetilde{a} + \gamma^- \left(\widehat{b} - \widehat{a}\right). \end{cases}$$

It is simple to verify that the order $\lessapprox_{\gamma^-,\gamma^+}$ turns out to be a partial order as it satisfies the reflexive, the antisymmetric and the transitive properties; therefore, the set $\mathcal{K}_{\mathcal{C}}^{\pm}$ endowed with the partial order $\lessapprox_{\gamma^-,\gamma^+}$ is a poset.

Moreover, according to Definition 2.2.1, the structure $(\mathcal{K}_{\mathcal{C}}^{\pm}, \lessapprox_{\gamma^-,\gamma^+})$ is also a lattice, as any of its elements A and B have a supremum $sup_{\lessapprox_{\gamma^-,\gamma^+}} \{A, B\}$ and an infimum $inf_{\lessapprox_{\gamma^-,\gamma^+}} \{A, B\}$ in $\mathcal{K}_{\mathcal{C}}^{\pm}$.

At this point, as happened in the other cases considered up to now, it is useful to define the following set:

$$\overline{\mathcal{K}_{\mathcal{C}}^{\pm}} \overset{def}{=} \mathcal{K}_{\mathcal{C}}^{\pm} \cup \{\pm\infty\},$$

where, as usual, we define

$$-\infty = (-\infty; 0) = inf_{\lessapprox_{\gamma^-,\gamma^+}} \overline{\mathcal{K}_{\mathcal{C}}^{\pm}} \text{ and } +\infty = (+\infty; 0) = sup_{\lessapprox_{\gamma^-,\gamma^+}} \overline{\mathcal{K}_{\mathcal{C}}^{\pm}},$$

so that, for all $\gamma^- < 0$ and $\gamma^+ > 0$, it is:

$$-\infty \lessapprox_{\gamma^-,\gamma^+} X \lessapprox_{\gamma^-,\gamma^+} +\infty, \ \forall X \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$$

(see Figure 5.5).

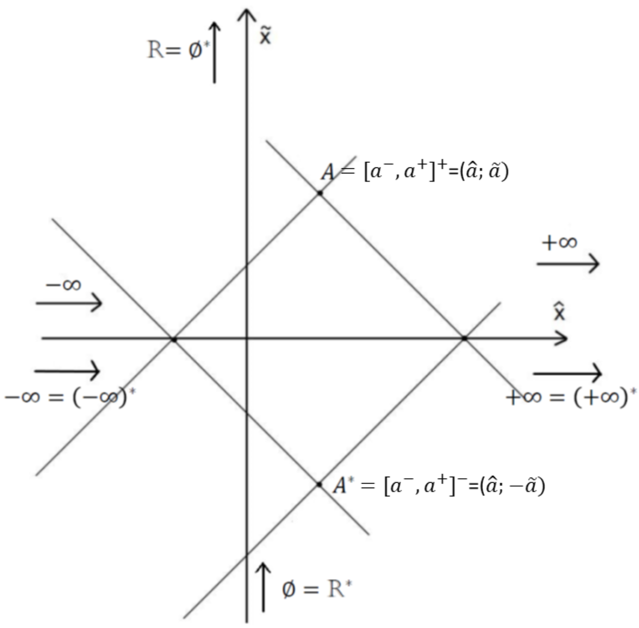


Figure 5.5: Representation of the elements $-\infty$ and $+\infty$ in $\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$ as well as $\varnothing$ and $\mathbb{R}$ in $\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$.

What we have just seen means that the lattice $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \lessapprox_{\gamma^-,\gamma^+})$ has a minimum, denoted by $-\infty$, and a maximum, denoted by $+\infty$, which satisfy the inequality $-\infty \lessapprox_{\gamma^-,\gamma^+} X \lessapprox_{\gamma^-,\gamma^+} +\infty$ for every $X \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$, i.e., $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \lessapprox_{\gamma^-,\gamma^+})$ is a bounded lattice.

Furthermore, according to Definition 2.2.2, we can also add that the structure $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \lessapprox_{\gamma^-,\gamma^+})$ is a complete lattice, as the following proposition holds.

Proposition 5.2.7. *Consider a partial order* $\lessapprox_{\gamma^-,\gamma^+}$ *on* $\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$ *and let* $\mathbb{S} \subset \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$ *be any nonempty bounded subset of intervals. Then, there exist both* $\inf(\mathbb{S})$, $\sup(\mathbb{S}) \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$ *such that for all* $X \in \mathbb{S}$

$$\inf(\mathbb{S}) \lessapprox_{\gamma^-,\gamma^+} X \lessapprox_{\gamma^-,\gamma^+} \sup(\mathbb{S}).$$

Proof. Similar to the proof of Proposition 2.2.10. □

On the other side, since it is possible to consider lattices also as algebraic structures, we can look at $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \lessapprox_{\gamma^-,\gamma^+})$ as a structure of the type $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \wedge)$ where, in this case, the lattice operations $\vee$ and $\wedge$ stand for supremum and infimum of two elements $X, Y \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$:

$$X \vee Y = sup_{\lessapprox_{\gamma^-,\gamma^+}}\{X, Y\},$$

$$X \wedge Y = inf_{\lessapprox_{\gamma^-,\gamma^+}}\{X, Y\}.$$

This means that we have the following binary functions:

$$\vee : \overline{\mathcal{K}_{\mathcal{C}}^{\pm}} \times \overline{\mathcal{K}_{\mathcal{C}}^{\pm}} \to \overline{\mathcal{K}_{\mathcal{C}}^{\pm}} \text{ such that: } (X, Y) \to X \vee Y = sup_{\lessapprox_{\gamma^-,\gamma^+}}\{X, Y\};$$

$$\wedge : \overline{\mathcal{K}_{\mathcal{C}}^{\pm}} \times \overline{\mathcal{K}_{\mathcal{C}}^{\pm}} \to \overline{\mathcal{K}_{\mathcal{C}}^{\pm}} \text{ such that: } (X, Y) \to X \wedge Y = inf_{\lessapprox_{\gamma^-,\gamma^+}}\{X, Y\}.$$

Regarding the two operations introduced above, it is easy to verify that the following properties hold:

1 $\vee$ and $\wedge$ are commutative: $\forall A, B \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$

- 1.a $A \vee B = B \vee A$,
- 1.b $A \wedge B = B \wedge A$;

2 $\vee$ and $\wedge$ are associative: $\forall A, B, C \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$

- 2.a $(A \vee B) \vee C = A \vee (B \vee C)$,
- 2.b $(A \wedge B) \wedge C = A \wedge (B \wedge C)$;

3 the absorption laws apply: $\forall A, B \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$

- 3.a $A \vee (A \wedge B) = A$,
- 3.b $A \wedge (A \vee B) = A$;

4 the idempotency is satisfied for both $\vee$ and $\wedge$: $\forall A \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$

- 4.a $A \vee A = A$,
- 4.b $A \wedge A = A$.

It is also easy to verify that:

5.a $\vee$ is left and right distributive over $\wedge$:

5.a.i $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C), \forall A, B, C \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$,

5.a.ii $(A \wedge B) \vee C = (A \vee C) \wedge (B \vee C), \forall A, B, C \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$;

5.b $\wedge$ is left and right distributive over $\vee$:

5.b.i $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C), \forall A, B, C \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$,

5.b.ii $(A \vee B) \wedge C = (A \wedge C) \vee (B \wedge C), \forall A, B, C \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$.

So, according to Definition 4.1.6, the structure $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \wedge, \lessapprox_{\gamma^-,\gamma^+})$ is an algebraic distributive lattice and, having $-\infty$ and $+\infty$ as minimum and maximum, it is also bounded.

In this regard we can also add that:

6.a $-\infty$ (the lattice's bottom) is the neutral element for the join operation $\vee$: $A \vee (-\infty) = A,\ \forall A \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$;

6.b $+\infty$ (the lattice's top) is the neutral element for the meet operation $\wedge$: $A \wedge (+\infty) = A,\ \forall A \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$;

6.c $-\infty$ is the absorbing element for the meet operation $\wedge$: $A \wedge (-\infty) = (-\infty),\ \forall A \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$;

6.d $+\infty$ is the absorbing element for the join operation $\vee$: $A \vee (+\infty) = (+\infty),\ \forall A \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$.

All this allow us to state the following proposition.

Proposition 5.2.8. *The structure* $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \wedge, (-\infty), (+\infty), \lessapprox_{\gamma^-,\gamma^+})$ *as well as* $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge, \vee, (+\infty), (-\infty), \lessapprox_{\gamma^-,\gamma^+})$ *are commutative, idempotent semirings.*

Proof. $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \wedge)$ is a commutative, idempotent semiring, as:

1a) $(\overline{\mathcal{K}_{C}^{\pm}}, \vee)$ is a commutative, idempotent monoid with neutral element $(-\infty)$:

(i) $\vee$ is associative: $(A \vee B) \vee C = A \vee (B \vee C), \forall A, B, C \in \overline{\mathcal{K}_{C}^{\pm}}$;

(ii) $\vee$ has a neutral element $i_{\vee} = (-\infty) \in \overline{\mathcal{K}_{C}^{\pm}} : A \vee (-\infty) = (-\infty) \vee A = A, \forall A \in \overline{\mathcal{K}_{C}^{\pm}}$ (so, $(-\infty)$ is the 0-element of the semiring $(\overline{\mathcal{K}_{C}^{\pm}}, \vee, \wedge)$;

(iii) $\vee$ is commutative: $A \vee B = B \vee A, \forall A, B \in \overline{\mathcal{K}_{C}^{\pm}}$;

(iv) $\vee$ is idempotent: $A \vee A = A, \forall A \in \overline{\mathcal{K}_{C}^{\pm}}$.

2a) $(\overline{\mathcal{K}_C^{\pm}}, \wedge)$ is a commutative, idempotent monoid with neutral element $(+\infty)$:

(i) $\wedge$ is associative: $(A \wedge B) \wedge C = A \wedge (B \wedge C), \forall A, B, C \in \overline{\mathcal{K}_C^{\pm}}$;

(ii) $\wedge$ has a neutral element $i_\wedge = (+\infty) \in \overline{\mathcal{K}_C^{\pm}} : A \wedge (+\infty) = (+\infty) \wedge A = A, \forall A \in \overline{\mathcal{K}_C^{\pm}}$ (so, $(+\infty)$ is the 1-element of the semiring $(\overline{\mathcal{K}_C^{\pm}}, \vee, \wedge)$;

(iii) $\wedge$ is commutative: $A \wedge B = B \wedge A, \forall A, B \in \overline{\mathcal{K}_C^{\pm}}$;

(iv) $\wedge$ is idempotent: $A \wedge A = A, \forall A \in \overline{\mathcal{K}_C^{\pm}}$.

3a) $\wedge$ is left and right distributive over $\vee$:

(i) $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C), \forall A, B, C \in \overline{\mathcal{K}_C^{\pm}}$;

(ii) $(A \vee B) \wedge C = (A \wedge C) \vee (B \wedge C), \forall A, B, C \in \overline{\mathcal{K}_C^{\pm}}$.

4a) $(-\infty)$ is the absorbing element for $\wedge$:

$$A \wedge (-\infty) = (-\infty) \wedge A = (-\infty), \forall A \in \overline{\mathcal{K}_C^{\pm}}.$$

Analogously, we have that also $(\overline{\mathcal{K}_C^{\pm}}, \wedge, \vee)$ is a commutative, idempotent semiring, as:

1b) $(\overline{\mathcal{K}_C^{\pm}}, \wedge)$ is a commutative, idempotent monoid with neutral element $(+\infty)$:

(i) $\wedge$ is associative: $(A \wedge B) \wedge C = A \wedge (B \wedge C), \forall A, B, C \in \overline{\mathcal{K}_C^{\pm}}$;

(ii) $\wedge$ has a neutral element $i_\wedge = (+\infty) \in \overline{\mathcal{K}_C^{\pm}} : A \wedge (+\infty) = (+\infty) \wedge A = A, \forall A \in \overline{\mathcal{K}_C^{\pm}}$ (so, $(+\infty)$ is the 0-element of the semiring $(\overline{\mathcal{K}_C^{\pm}}, \wedge, \vee)$);

(iii) $\wedge$ is commutative: $A \wedge B = B \wedge A, \forall A, B \in \overline{\mathcal{K}_C^{\pm}}$;

(iv) $\wedge$ is idempotent: $A \wedge A = A, \forall A \in \overline{\mathcal{K}_C^{\pm}}$.

2b) $(\overline{\mathcal{K}_C^{\pm}}, \vee)$ is a commutative, idempotent monoid with neutral element $(-\infty)$:

(i) $\vee$ is associative: $(A \vee B) \vee C = A \vee (B \vee C), \forall A, B, C \in \overline{\mathcal{K}_C^{\pm}}$;

(ii) $\vee$ has a neutral element $i_\vee = (-\infty) \in \overline{\mathcal{K}_C^{\pm}} : A \vee (-\infty) = (-\infty; 0) \vee A = A, \forall A \in \overline{\mathcal{K}_C^{\pm}}$ (so, $(-\infty)$ is the 1-element of the semiring $(\overline{\mathcal{K}_C^{\pm}}, \wedge, \vee)$);

(iii) $\vee$ is commutative: $A \vee B = B \vee A, \forall A, B \in \overline{\mathcal{K}_C}$;

(iv) $\vee$ is idempotent: $A \vee A = A, \forall A \in \overline{\mathcal{K}_C^{\pm}}$.

3b) $\vee$ is left and right distributive over $\wedge$:

(i) $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C), \forall A, B, C \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$;

(ii) $(A \wedge B) \vee C = (A \vee C) \wedge (B \vee C), \forall A, B, C \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$.

4b) $(+\infty)$ is the absorbing element for $\vee$:

$$A \vee (+\infty) = (+\infty) \vee A = (+\infty), \forall A \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}.$$

□

It is interesting to note how, in this case, it is possible to introduce a concept of polarity similar to the one analysed in Section 4.1.

In addition, we can even define two new operations of union and intersection, now denoted by $\sqcup$ and $\sqcap$, that extend those examined in Section 4.1 as shown in Figure 5.6.

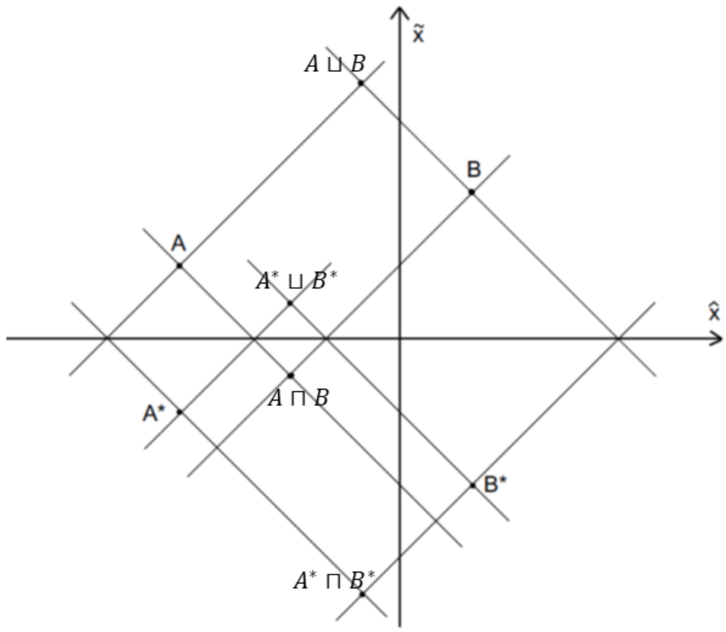


Figure 5.6: Example of union $\sqcup$ and intersection $\sqcap$ in $\mathcal{K}_{\mathcal{C}}^{\pm}$.

Let $A, B \in \mathcal{K}_{\mathcal{C}}$, then:

- if $A \sqcap B \in \mathcal{K}_{\mathcal{C}}^{+}$, then it represents the proper intersection;

- if $A \sqcap B \in \mathcal{K}_{\mathcal{C}}^{-}$, then it represents the "dual" interval that convexifies $A \cup B$, i.e., $A \sqcup B = (A \cup B) \cup (A \sqcap B)^*$.

Moreover, it should be remembered that, considering $A \lessapprox_{\gamma^-,\gamma^+} B$, there are four distinct possible positions that A can assume with respect to B in $\mathcal{K}_{\mathcal{C}}^{\pm}$, as shown in the example represented in Figure 5.7, where, more in detail, it is possible to identify the following situations:

- if $A = [a^-, a^+]^+, B = [b^-, b^+]^+ \in \mathcal{K}_{\mathcal{C}}^+$ (top left of the picture), then $A \sqcup B = [a^-, b^+]^+ \in \mathcal{K}_{\mathcal{C}}^+$ and $A \sqcap B = [b^-, a^+]^+ \in \mathcal{K}_{\mathcal{C}}^+$ or $A \sqcap B = [a^+, b^-]^- \in \mathcal{K}_{\mathcal{C}}^-$;
- if $A = [a^-, a^+]^-, B = [b^-, b^+]^- \in \mathcal{K}_{\mathcal{C}}^-$ (top right of the picture), then $A \sqcup B = [a^+, b^-]^+ \in \mathcal{K}_{\mathcal{C}}^+$ or $A \sqcup B = [b^-, a^+]^- \in \mathcal{K}_{\mathcal{C}}^-$ and $A \sqcap B = [a^-, b^+]^- \in \mathcal{K}_{\mathcal{C}}^-$;
- if $A = [a^-, a^+]^+ \in \mathcal{K}_{\mathcal{C}}^+$ and $B = [b^-, b^+]^- \in \mathcal{K}_{\mathcal{C}}^-$ (bottom left), then $A \sqcup B = [a^-, b^-]^+ \in \mathcal{K}_{\mathcal{C}}^+$ and $A \sqcap B = [a^+, b^+]^- \in \mathcal{K}_{\mathcal{C}}^-$;
- if $A = [a^-, a^+]^- \in \mathcal{K}_{\mathcal{C}}^-$ and $B = [b^-, b^+]^+ \in \mathcal{K}_{\mathcal{C}}^+$ (bottom right), then $A \sqcup B = [a^+, b^+]^+ \in \mathcal{K}_{\mathcal{C}}^+$ and $A \sqcap B = [a^-, b^-]^- \in \mathcal{K}_{\mathcal{C}}^-$.

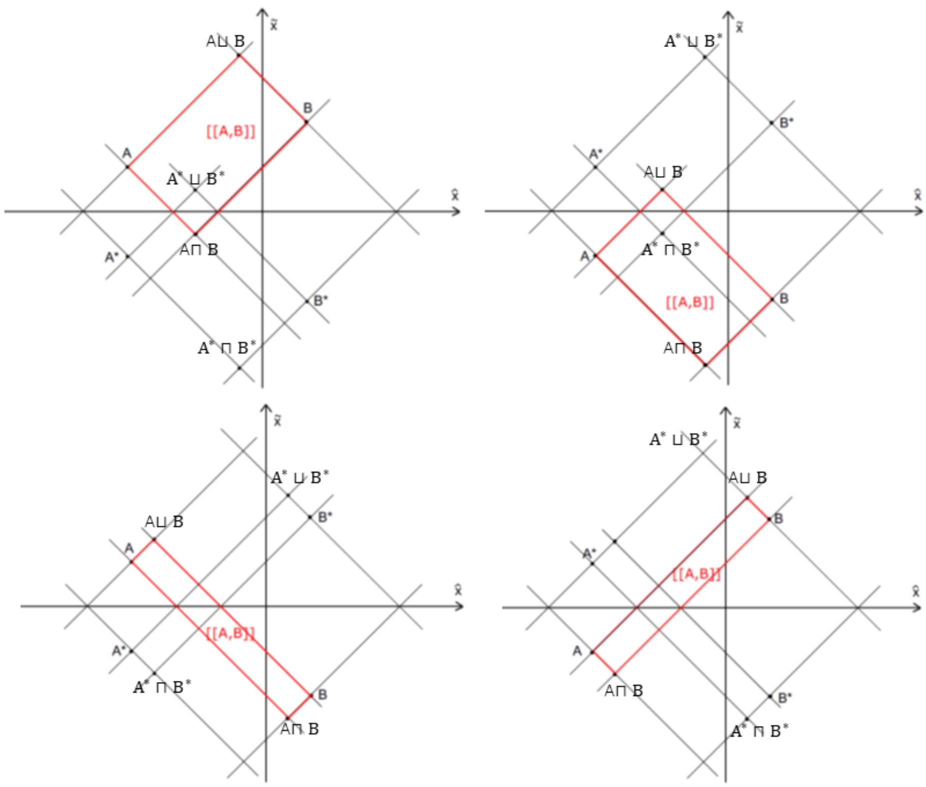


Figure 5.7: Example of the four different positions of A and B in $\mathcal{K}_{\mathcal{C}}^{\pm}$ (with $A \lessapprox_{\gamma^-,\gamma^+} B$) and the respective operations of union $\sqcup$ and intersection $\sqcap$.

In order to give a correct definition of union and intersection in $\mathcal{K}_{\mathcal{C}}^{\pm}$ we can consider, as usual, lines A_m, A_p and B_m, B_p, passing through A and B, as defined by (4.7), with $A_m||B_m$ and $A_p||B_p$ (possibly $A_m \equiv B_m$ or $A_p \equiv B_p$ but not both coincidences verified at the same time).

Remark 5.2.3. *If $A_p \equiv B_p$ and $A \neq B$, then $A_m \neq B_m$ and $A_m||B_m$. So A and B are $\lessapprox_{\gamma^-,\gamma^+}$-comparable, that is $A \lessapprox_{\gamma^-,\gamma^+} B$ or $B \lessapprox_{\gamma^-,\gamma^+} A$.*

Similarly, if $A_m \equiv B_m$ and $A \neq B$, then $A_p \neq B_p$ and $A_p||B_p$. So A and B are $\lessapprox_{\gamma^-,\gamma^+}$-comparable, that is $A \lessapprox_{\gamma^-,\gamma^+} B$ or $B \lessapprox_{\gamma^-,\gamma^+} A$.

In any case, the intersections (points) between the two lines A_m and B_p as well as between A_p and B_m are always well defined; therefore, as stated in Section 4.1, we indicate with:

- A_mB_p (or B_pA_m) the point of intersection between A_m and B_p;
- A_pB_m (or B_mA_p) the point of intersection between A_p and B_m.

Be that as it may, we have A_mB_p, $A_pB_m \in \mathcal{K}_{\mathcal{C}}^{\pm}$. Also, analogously to Definition 4.1.11, we have the notion below.

Definition 5.2.3. *Let $A, B \in \mathcal{K}_{\mathcal{C}}^{\pm}$ such that $A \neq B$ and $\gamma^-, \gamma^+ \in \mathbb{R}$:*

- *if $A_p \equiv B_p$ (or if $A_m \equiv B_m$), we say that the two points, i.e., elements of $\mathcal{K}_{\mathcal{C}}^{\pm}$, are aligned;*
- *if $A_p \neq B_p$ and $A_m \neq B_m$, we say that the two elements of $\mathcal{K}_{\mathcal{C}}^{\pm}$ are unaligned.*

Lemma 5.2.1. *Let $A, B \in \mathcal{K}_{\mathcal{C}}^{\pm}$ such that $A \neq B$ and $\gamma^-, \gamma^+ \in \mathbb{R}$ with $\gamma^- < 0, \gamma^+ > 0$, then, considering the order $\lessapprox_{\gamma^-,\gamma^+}$, we have that:*
A and B are $\lessapprox_{\gamma^-,\gamma^+}$-incomparable $\Leftrightarrow$ A_mB_p and A_pB_m are $\lessapprox_{\gamma^-,\gamma^+}$-comparable.

Proof. Similar to theorem 4.1.1. □

Now, we can give the following definition.

Definition 5.2.4. *Let $A, B \in \mathcal{K}_{\mathcal{C}}^{\pm}$. There are several different cases.*

1. *If A and B are $\lessapprox_{\gamma^-,\gamma^+}$-comparable and $A \neq B$, we have:*
   - *1.a if $A \lessapprox_{\gamma^-,\gamma^+} B$, then $A \sqcup B = A_pB_m$ and $A \sqcap B = A_mB_p$;*
   - *1.b if $B \lessapprox_{\gamma^-,\gamma^+} A$, then $A \sqcup B = A_mB_p$ and $A \sqcap B = A_pB_m$.*
2. *If A and B are $\lessapprox_{\gamma^-,\gamma^+}$-incomparable, we have:*
   - *2.a if $A_pB_m \lessapprox_{\gamma^-,\gamma^+} A_mB_p$, then $A \sqcup B = A$ and $A \sqcap B = B$;*
   - *2.b if $A_mB_p \lessapprox_{\gamma^-,\gamma^+} A_pB_m$, then $A \sqcup B = B$ and $A \sqcap B = A$.*
3. *If $A = B$ then $A \sqcup B = A \sqcap B = A = B$.*

Note that the operations $\sqcup$ and $\sqcap$ of Definition 5.2.4 depend on the order $\lessapprox_{\gamma^-,\gamma^+}$ chosen for $\mathcal{K}_{\mathcal{C}}^{\pm}$ and give rise to the polarity between $\lessapprox_{\gamma^-,\gamma^+}$ and the inclusion order $\sqsubseteq_{\gamma^-,\gamma^+}$ defined by:

$$A \sqsubseteq_{\gamma^-,\gamma^+} B \;\Leftrightarrow A \sqcap B = A \text{ and } A \sqcup B = B. \tag{5.10}$$

As an obvious consequence of (5.10) and Definition 5.2.4 , we also have that:

$$A \sqcup B = A \sqcap B \;\Leftrightarrow\; A = B.$$

As shown in Figure 5.8, the above definitions are also valid when $A = \{\widehat{a}\}$ and/or $B = \left\{\widehat{b}\right\}$. In this case we have $\{\widehat{a}\}^* = \{\widehat{a}\}$, $\forall \widehat{a} \in \mathbb{R}$.

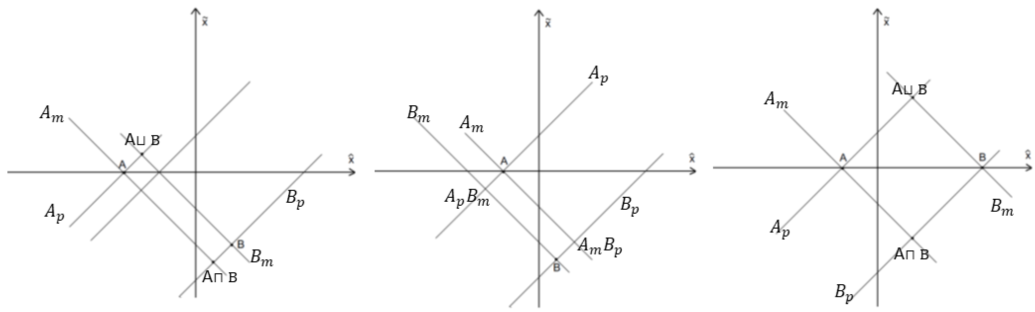


Figure 5.8: Examples of special cases in which $A = \{\widehat{a}\}$ with $A \lessapprox_{\gamma^-,\gamma^+} B$ (left), $A = \{\widehat{a}\}$ and $A_pB_m \lessapprox_{\gamma^-,\gamma^+} A_mB_p$ (center), $A = \{\widehat{a}\}$ and $B = \left\{\widehat{b}\right\}$ (right).

Definitely, for every $A, B \in \mathcal{K}_{\mathcal{C}}^{\pm}$, we define the order $\sqsubseteq_{\gamma^-,\gamma^+}$ as follows:

$$A \sqsubseteq_{\gamma^-,\gamma^+} B \Longleftrightarrow A \subseteqq_{\gamma^-,\gamma^+} B$$

so, according to (4.12), it is

$$A \sqsubseteq_{\gamma^-,\gamma^+} B \Longleftrightarrow \begin{cases} \widetilde{a} \leqslant \widetilde{b} + \gamma^+ \left(\widehat{a} - \widehat{b}\right) \\ \widetilde{a} \leqslant \widetilde{b} + \gamma^- \left(\widehat{a} - \widehat{b}\right). \end{cases} \tag{5.11}$$

This means that:

- an interval $X \in \mathcal{K}_{\mathcal{C}}^{\pm}$ such that $X \sqsubseteq_{\gamma^-,\gamma^+} A$, is included in A with respect to inclusion order $\sqsubseteq_{\gamma^-,\gamma^+}$;
- an interval $Y \in \mathcal{K}_{\mathcal{C}}^{\pm}$ such that $A \sqsubseteq_{\gamma^-,\gamma^+} Y$, includes interval A with respect to inclusion order $\sqsubseteq_{\gamma^-,\gamma^+}$.

Note that Figure 5.9 offers an example of operations $\vee, \wedge$ and $\sqcup, \sqcap$ applied to intervals A and B in $\mathcal{K}_{\mathcal{C}}^{\pm}$ in the particular LU-order case where $\gamma^+ = +1$ and $\gamma^- = -1$.

More in details we have that, given $A = (\widehat{a};\widetilde{a})$, $B = (\widehat{b};\widetilde{b}) \in \mathcal{K}_{\mathcal{C}}^{\pm}$ and $A^* = (\widehat{a};-\widetilde{a})$, $B^* = (\widehat{b};-\widetilde{b}) \in \mathcal{K}_{\mathcal{C}}^{\pm}$, the following properties are valid, as will be better explained later on.

- $a^- = A \wedge A^*$ (as well as $b^- = B \wedge B^*$);
- $a^+ = A \vee A^*$ (as well as $b^+ = B \vee B^*$);
- $A^* \sqcap B = (A \sqcup B^*)^*$ and $A^* \sqcup B = (A \sqcap B^*)^*$ (De Morgan rules).

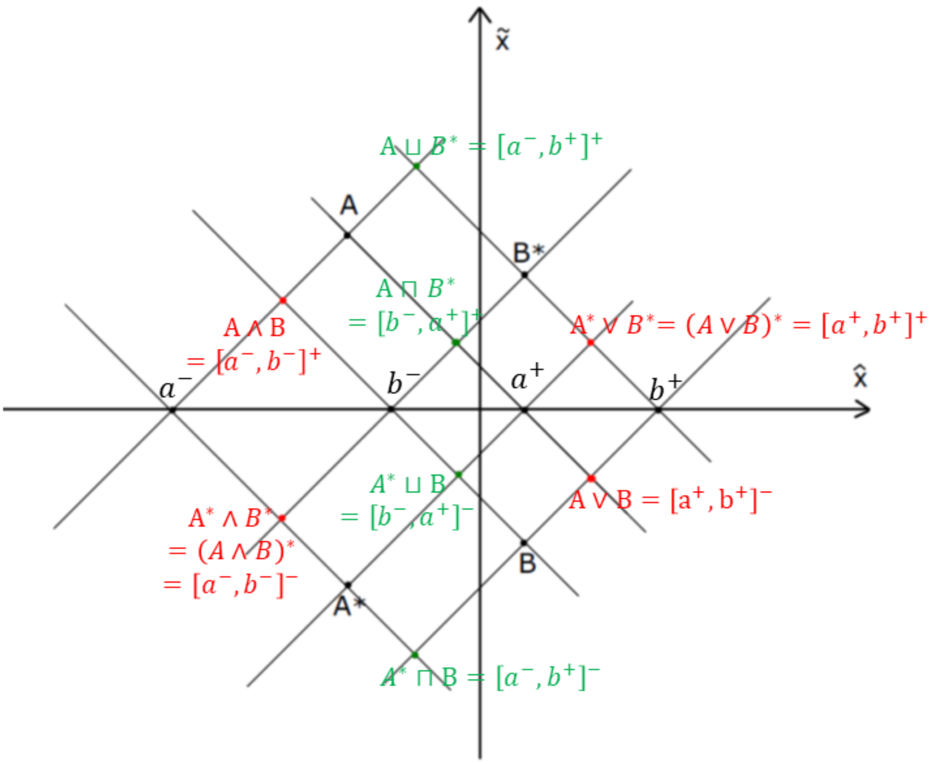


Figure 5.9: Examples of operations $\vee$, $\wedge$ and $\sqcup$, $\sqcap$ with respect to the orders $\lessapprox_{LU}$ and $\sqsubseteqq_{-1,+1}$.

Similarly, the following facts are easily obtained from Figure 5.10, which offers another interesting example of how $\vee$ and $\wedge$, as well as $\sqcup$ and $\sqcap$, operate on $X \in \mathcal{K}_{\mathcal{C}}^{\pm}$. Specifically, for the points $B_1, ...B_5$ we have:

- $A \sqcup B_2 = A$, $A \sqcap B_2 = B_2$, $A \sqcup B_3 = B_3$, $A \sqcap B_3 = A$ (they are incomparable with respect to $\lessapprox_{\gamma^-,\gamma^+}$);
- $B_1 \lessapprox A$, $A \lessapprox B_4$, $B_5 \lessapprox A$ (they are $\lessapprox_{\gamma^-,\gamma^+}$-comparable).

In addition, we also have

$$\begin{cases} A \sqcap B_1 \in \mathcal{K}_{\mathcal{C}}^{+} \\ B_1 \lessapprox A \sqcap B_1 \lessapprox A, \end{cases} \quad \begin{cases} A \sqcap B_4 \in \mathcal{K}_{\mathcal{C}}^{-} \\ A \lessapprox A \sqcap B_4 \lessapprox B_4, \end{cases} \quad \begin{cases} A \sqcap B_5 \in \mathcal{K}_{\mathcal{C}}^{-} \\ B_5 \lessapprox A \sqcap B_5 \lessapprox A. \end{cases}$$

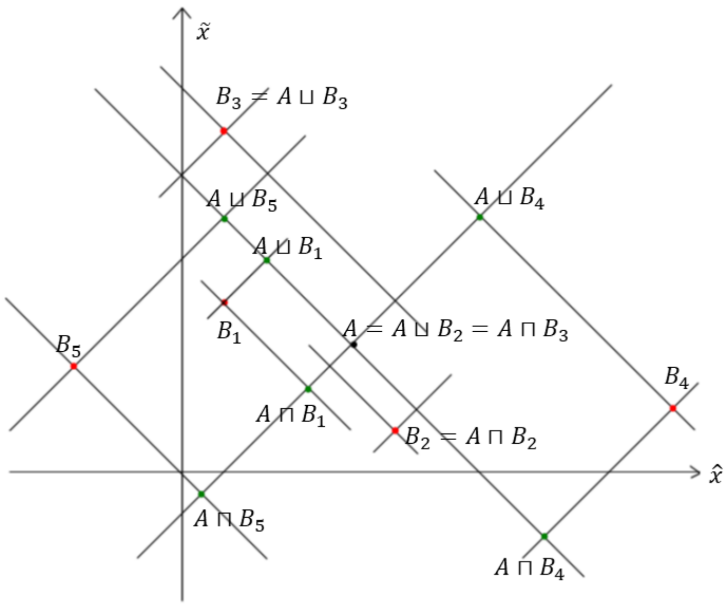


Figure 5.10: Examples of operations $\sqcup$ and $\sqcap$ in case intersection belongs to $\mathcal{K}_{\mathcal{C}}^{+}$ or to $\mathcal{K}_{\mathcal{C}}^{-}$.

Moreover, according to Definition 4.1.10, the two orders satisfy the polarity property.

Proposition 5.2.9. *Two unaligned intervals $A, B \in \mathcal{K}_{\mathcal{C}}^{\pm}$ are incomparable with respect to $\sqsubseteq_{\gamma^-,\gamma^+}$ if and only if they are comparable with respect to $\lessapprox_{\gamma^-,\gamma^+}$. This is denoted by*

$$A \,||_{\sqsubseteq_{\gamma^-,\gamma^+}}\, B.$$

Vice-versa, they are incomparable with respect to $\lessapprox_{\gamma^-,\gamma^+}$ if and only if they are comparable with respect to $\sqsubseteq_{\gamma^-,\gamma^+}$. This is denoted by

$$A \,||_{\lessapprox_{\gamma^-,\gamma^+}}\, B.$$

So, also in this case we have that the order $\sqsubseteq_{\gamma^-,\gamma^+}$ turns out to be a partial order as it satisfied the reflexive, the antisymmetric and the transitive properties; therefore, the set $\mathcal{K}_{\mathcal{C}}^{\pm}$ endowed with the partial order $\sqsubseteq_{\gamma^-,\gamma^+}$ is a poset.

Furthermore, according to Definition 2.2.1, the structure $(\mathcal{K}_{\mathcal{C}}^{\pm}, \sqsubseteq_{\gamma^-,\gamma^+})$ is also a lattice, as any of its elements A and B have a supremum $sup_{\sqsubseteq_{\gamma^-,\gamma^+}}\{A, B\}$ and an infimum $inf_{\sqsubseteq_{\gamma^-,\gamma^+}}\{A, B\}$ in $\mathcal{K}_{\mathcal{C}}^{\pm}$.

At this point, also in this case, it becomes necessary to define the following set:

$$\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}} \overset{def}{=} \mathcal{K}_{\mathcal{C}}^{\pm} \cup \{\varnothing, \mathbb{R}\},$$

where $\varnothing = (0; -\infty) = inf_{\sqsubseteq_{\gamma^-,\gamma^+}} \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$ and $\mathbb{R} = (0; +\infty) = sup_{\sqsubseteq_{\gamma^-,\gamma^+}} \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$ with

$$\mathbb{R} = \varnothing^* \text{ and } \varnothing = \mathbb{R}^*$$

(see Figure 5.5) so that, for all $\gamma^- < 0$ and $\gamma^+ > 0$, it is:

$$\varnothing \sqsubseteq_{\gamma^-,\gamma^+} X \sqsubseteq_{\gamma^-,\gamma^+} \mathbb{R}, \ \forall X \in \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}.$$

This means that the lattice $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqsubseteq_{\gamma^-,\gamma^+})$ has a minimum, denoted by $\varnothing$, and a maximum, denoted by $\mathbb{R}$, which satisfy the inequality $\varnothing \sqsubseteq_{\gamma^-,\gamma^+} X \sqsubseteq_{\gamma^-,\gamma^+} \mathbb{R}$ for every $X \in \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$, i.e., $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqsubseteq_{\gamma^-,\gamma^+})$ is a bounded lattice.

Again, according to Definition 2.2.2, we can also add that the structure $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqsubseteq_{\gamma^-,\gamma^+})$ is a complete lattice as, similarly to Proposition 5.2.3, the following statement holds.

Proposition 5.2.10. *Consider a partial order* $\sqsubseteq_{\gamma^-,\gamma^+}$ *on* $\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$ *and let* $\mathbb{S} \subset \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$ *be any nonempty bounded subset of intervals. Then, there exist both* $\inf(\mathbb{S}), \ \sup(\mathbb{S}) \in \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$ *such that for all* $X \in \mathbb{S}$

$$\inf(\mathbb{S}) \sqsubseteq_{\gamma^-,\gamma^+} X \sqsubseteq_{\gamma^-,\gamma^+} \sup(\mathbb{S}).$$

Moreover, since we can also consider lattices as algebraic structures, it is possible to look at $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqsubseteq_{\gamma^-,\gamma^+})$ as a structure of the type $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup, \sqcap)$ where the lattice operations $\sqcup$ and $\sqcap$ stand for supremum and infimum of two elements $X, Y \in \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$:

$$X \sqcup Y = sup_{\sqsubseteq_{\gamma^-,\gamma^+}}\{X, Y\},$$

$$X \sqcap Y = inf_{\sqsubseteq_{\gamma^-,\gamma^+}}\{X, Y\}.$$

This means that we can define two binary functions:

$$\sqcup : \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}} \times \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}} \to \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}} \text{ such that: } (X, Y) \to X \sqcup Y = sup_{\sqsubseteq_{\gamma^-,\gamma^+}}\{X, Y\};$$

$$\sqcap : \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}} \times \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}} \to \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}} \text{ such that: } (X, Y) \to X \sqcap Y = inf_{\sqsubseteq_{\gamma^-,\gamma^+}}\{X, Y\}.$$

It is immediate to verify that the following properties hold:

1 $\sqcup$ and $\sqcap$ are commutative: $\forall A, B \in \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$

- 1.a $A \sqcup B = B \sqcup A$,
- 1.b $A \sqcap B = B \sqcap A$;

2 $\sqcup$ and $\sqcap$ are associative: $\forall A, B, C \in \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$

- 2.a $(A \sqcup B) \sqcup C = A \sqcup (B \sqcup C)$, 2.b $(A \sqcap B) \sqcap C = A \sqcap (B \sqcap C)$;

3 the absorption laws apply: $\forall A, B \in \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$

3.a $A \sqcup (A \sqcap B) = A$,

3.b $A \sqcap (A \sqcup B) = A$;

4 the idempotency is satisfied for both $\sqcup$ and $\sqcap$: $\forall A \in \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$

4.a $A \sqcup A = A$,

4.b $A \sqcap A = A$.

It is also easy to verify that:

5.a $\sqcup$ is left and right distributive over $\sqcap$:

5.a.i $A \sqcup (B \sqcap C) = (A \sqcup B) \sqcap (A \sqcup C), \forall A, B, C \in \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$,

5.a.ii $(A \sqcap B) \sqcup C = (A \sqcup C) \sqcap (B \sqcup C), \forall A, B, C \in \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$;

5.b $\sqcap$ is left and right distributive over $\sqcup$:

5.b.i $A \sqcap (B \sqcup C) = (A \sqcap B) \sqcup (A \sqcap C), \forall A, B, C \in \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$,

5.b.ii $(A \sqcup B) \sqcap C = (A \sqcap C) \sqcup (B \sqcap C), \forall A, B, C \in \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$.

So, according to Definition 4.1.6, the structure $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup, \sqcap, \sqsubseteq_{\gamma^-,\gamma^+})$ is an algebraic distributive lattice and, having $\varnothing$ and $\mathbb{R}$ as minimum and maximum, it is also bounded.

Lastly, we can also add that:

6.a $\varnothing$ (the lattice's bottom) is the neutral element for the join operation $\sqcup$: $A \sqcup \varnothing = A$, $\forall A \in \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$;

6.b $\mathbb{R}$ (the lattice's top) is the neutral element for the meet operation $\sqcap$: $A \sqcap \mathbb{R} = A$, $\forall A \in \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$;

6.c $\varnothing$ is the absorbing element for the meet operation $\sqcap$: $A \sqcap \varnothing = \varnothing$, $\forall A \in \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$;

6.d $\mathbb{R}$ is the absorbing element for the join operation $\sqcup$: $A \sqcup \mathbb{R} = \mathbb{R}$, $\forall A \in \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$.

All this allow us to state the following.

Proposition 5.2.11. *The two structures* $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup, \sqcap, \varnothing, \mathbb{R}, \sqsubseteq_{\gamma^-,\gamma^+})$ *and* $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcap, \sqcup, \mathbb{R}, \varnothing, \sqsubseteq_{\gamma^-,\gamma^+})$ *are commutative, idempotent semirings.*

Proof. $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup, \sqcap)$ is a commutative, idempotent semiring, as:

1a) $(\mathcal{K}_C^{\pm\varnothing\mathbb{R}}, \sqcup)$ is a commutative, idempotent monoid with neutral element $\varnothing$:

(i) $\sqcup$ is associative: $(A \sqcup B) \sqcup C = A \sqcup (B \sqcup C), \forall A, B, C \in \mathcal{K}_C^{\pm\varnothing\mathbb{R}}$;

(ii) $\sqcup$ has a neutral element $i_\sqcup = \varnothing \in \mathcal{K}_C^{\pm\varnothing\mathbb{R}}$: $A \sqcup \varnothing = \varnothing \sqcup A = A, \forall A \in \mathcal{K}_C^{\pm\varnothing\mathbb{R}}$ (so, $\varnothing$ is the 0-element of the semiring $(\mathcal{K}_C^{\pm\varnothing\mathbb{R}}, \sqcup, \sqcap)$);

(iii) $\sqcup$ is commutative: $A \sqcup B = B \sqcup A, \forall A, B \in \mathcal{K}_C^{\pm\varnothing\mathbb{R}}$;

(iv) $\sqcup$ is idempotent: $A \sqcup A = A, \forall A \in \mathcal{K}_C^{\pm\varnothing\mathbb{R}}$.

2a) $(\mathcal{K}_C^{\pm\varnothing\mathbb{R}}, \sqcap)$ is a commutative, idempotent monoid with neutral element $\mathbb{R}$:

(i) $\sqcap$ is associative: $(A \sqcap B) \sqcap C = A \sqcap (B \sqcap C), \forall A, B, C \in \mathcal{K}_C^{\pm\varnothing\mathbb{R}}$;

(ii) $\sqcap$ has a neutral element $i_\sqcap = \mathbb{R} \in \mathcal{K}_C^{\pm\varnothing\mathbb{R}}$: $A \sqcap \mathbb{R} = \mathbb{R} \sqcap A = A, \forall A \in \mathcal{K}_C^{\pm\varnothing\mathbb{R}}$ (so, $\mathbb{R}$ is the 1-element of the semiring $(\mathcal{K}_C^{\pm\varnothing\mathbb{R}}, \sqcup, \sqcap)$);

(iii) $\sqcap$ is commutative: $A \sqcap B = B \sqcap A, \forall A, B \in \mathcal{K}_C^{\pm\varnothing\mathbb{R}}$;

(iv) $\sqcap$ is idempotent: $A \sqcap A = A, \forall A \in \mathcal{K}_C^{\pm\varnothing\mathbb{R}}$.

3a) $\sqcap$ is left and right distributive over $\sqcup$:

(i) $A \sqcap (B \sqcup C) = (A \sqcap B) \sqcup (A \sqcap C), \forall A, B, C \in \mathcal{K}_C^{\pm\varnothing\mathbb{R}}$;

(ii) $(A \sqcup B) \sqcap C = (A \sqcap C) \sqcup (B \sqcap C), \forall A, B, C \in \mathcal{K}_C^{\pm\varnothing\mathbb{R}}$.

4a) $\varnothing$ is the absorbing element for $\sqcap$:

$$A \sqcap \varnothing = \varnothing \sqcap A = \varnothing, \forall A \in \mathcal{K}_C^{\pm\varnothing\mathbb{R}}.$$

Analogously, we have that also $(\mathcal{K}_C^{\pm\varnothing}, \sqcap, \sqcup)$ is a commutative, idempotent semiring, as:

1b) $(\mathcal{K}_C^{\pm\varnothing\mathbb{R}}, \sqcap)$ is a commutative, idempotent monoid with neutral element $\mathbb{R}$:

(i) $\sqcap$ is associative: $(A \sqcap B) \sqcap C = A \sqcap (B \sqcap C), \forall A, B, C \in \mathcal{K}_C^{\pm\varnothing\mathbb{R}}$;

(ii) $\sqcap$ has a neutral element $i_\sqcap = \mathbb{R} \in \mathcal{K}_C^{\pm\varnothing\mathbb{R}}$: $A \sqcap \mathbb{R} = \mathbb{R} \sqcap A = A, \forall A \in \mathcal{K}_C^{\pm\varnothing\mathbb{R}}$ (so, $\mathbb{R}$ is the 0-element of the semiring $(\mathcal{K}_C^{\pm\varnothing\mathbb{R}}, \sqcap, \sqcup)$);

(iii) $\sqcap$ is commutative: $A \sqcap B = B \sqcap A, \forall A, B \in \mathcal{K}_C^{\pm\varnothing\mathbb{R}}$;

(iv) $\sqcap$ is idempotent: $A \sqcap A = A, \forall A \in \mathcal{K}_C^{\pm\varnothing\mathbb{R}}$.

2b) $(\mathcal{K}_C^{\pm\varnothing\mathbb{R}}, \sqcup)$ is a commutative, idempotent monoid with neutral element $\varnothing$:

(i) $\sqcup$ is associative: $(A \sqcup B) \sqcup C = A \sqcup (B \sqcup C), \forall A, B, C \in \varnothing\mathbb{R}\mathcal{K}_{\mathbb{C}}^{\pm}$;

(ii) $\sqcup$ has a neutral element $i_\sqcup = \varnothing \in \mathcal{K}_C^{\pm\varnothing\mathbb{R}}$: $A \sqcup \varnothing = \varnothing \sqcup A = A, \forall A \in \mathcal{K}_C^{\pm\varnothing\mathbb{R}}$ (so, $\varnothing$ is the 1-element of the semiring $(\mathcal{K}_C^{\pm\varnothing\mathbb{R}}, \sqcap, \sqcup)$);

(iii) $\sqcup$ is commutative: $A \sqcup B = B \sqcup A, \forall A, B \in \mathcal{K}_C^{\pm\varnothing\mathbb{R}}$;
(iv) $\sqcup$ is idempotent: $A \sqcup A = A, \forall A \in \mathcal{K}_C^{\pm\varnothing\mathbb{R}}$.

3b) $\sqcup$ is left and right distributive over $\sqcap$:

(i) $A \sqcup (B \sqcap C) = (A \sqcup B) \sqcap (A \sqcup C), \forall A, B, C \in \mathcal{K}_C^{\pm\varnothing\mathbb{R}}$;
(ii) $(A \sqcap B) \sqcup C = (A \sqcup C) \sqcap (B \sqcup C), \forall A, B, C \in \mathcal{K}_C^{\pm\varnothing\mathbb{R}}$.

4b) $\mathbb{R}$ is the absorbing element for $\sqcup$:

$$A \sqcup \mathbb{R} = \mathbb{R} \sqcup A = \mathbb{R}, \forall A \in \mathcal{K}_C^{\pm\varnothing\mathbb{R}}.$$

□

Finally we note that other properties are also involved:

(1) $(A^*)^* = A,\ \forall A \in \mathcal{K}_C^{\pm\varnothing\mathbb{R}}$;

(2) $A \sqsubseteqq_{\gamma^-,\gamma^+} B \Longrightarrow B^* \sqsubseteqq_{\gamma^-,\gamma^+} A^*,\ \forall A, B \in \mathcal{K}_C^{\pm\varnothing\mathbb{R}}$.

From these properties, according to Definition 4.1.8, we have the next proposition.

Proposition 5.2.12. *The structures* $(\mathcal{K}_\mathcal{C}^{\pm\varnothing\mathbb{R}}, \sqcup, \sqcap, \sqsubseteqq_{\gamma^-,\gamma^+})$ *and* $(\mathcal{K}_\mathcal{C}^{\pm\varnothing\mathbb{R}}, \sqcap, \sqcup, \sqsubseteqq_{\gamma^-,\gamma^+})$ *are De Morgan Algebras.*

As a consequence of Proposition 5.2.12, the following well-known laws also hold (as it can be clearly seen in the example shown in Figure 5.9).

(3.a) $(A \sqcup B)^* = A^* \sqcap B^*,\ \forall A, B \in \mathcal{K}_C^{\pm\varnothing\mathbb{R}}$;

(3.b) $(A \sqcap B)^* = A^* \sqcup B^*,\ \forall A, B \in \mathcal{K}_C^{\pm\varnothing\mathbb{R}}$.

Remark 5.2.4. *Note how in the case of structures* $(\overline{\mathcal{K}_\mathcal{C}^{\pm}}, \vee, \wedge, \lessapprox_{\gamma^-,\gamma^+})$ *and* $(\overline{\mathcal{K}_\mathcal{C}^{\pm}}, \wedge, \vee, \lessapprox_{\gamma^-,\gamma^+})$ *we do not obtain De Morgan algebras since the condition* (*d*2) *of Definition 4.1.8 is not satisfied. Indeed, in this case we have:*

$$A \lessapprox_{\gamma^-,\gamma^+} B \Longleftrightarrow A^* \lessapprox_{\gamma^-,\gamma^+} B^*$$

Lastly, just like in Subsection 5.1.2, let consider again Minkowsky addition, defined in (5.5), extending it to sets $\overline{\mathcal{K}_\mathcal{C}^{\pm}}$ and $\mathcal{K}_\mathcal{C}^{\pm\varnothing\mathbb{R}}$ as it follows:

$$\oplus : \overline{\mathcal{K}_C^{\pm}} \times \overline{\mathcal{K}_C^{\pm}} \longrightarrow \overline{\mathcal{K}_C^{\pm}} \quad \text{such that: } (X, Y) \longrightarrow X \oplus Y;$$
$$\oplus : \mathcal{K}_C^{\pm\varnothing\mathbb{R}} \times \mathcal{K}_C^{\pm\varnothing\mathbb{R}} \longrightarrow \mathcal{K}_C^{\pm\varnothing\mathbb{R}} \ \text{such that: } (X, Y) \longrightarrow X \oplus Y.$$

with the usual conventions.

The result is that $(\overline{\mathcal{K}_C^{\pm}}, \oplus)$ and $(\mathcal{K}_C^{\pm\varnothing\mathbb{R}}, \oplus)$ are both commutative monoids, as they are associative, commutative and $(0; 0)$ can be interpreted as the neutral element.

Moreover, if we take into account the $\lessapprox_{\gamma^-,\gamma^+}$-order associated to $\overline{\mathcal{K}_\mathcal{C}^{\pm}}$, it is easy to verify, even graphically, that, for all $A, B, C \in \overline{\mathcal{K}_\mathcal{C}^{\pm}}$, we have:

- $\oplus$ is distributive with respect to $\vee$: $(A \vee B) \oplus C = (A \oplus C) \vee (B \oplus C)$;

- $\oplus$ is distributive with respect to $\wedge$: $(A \wedge B) \oplus C = (A \oplus C) \wedge (B \oplus C)$.

In a similar way, it is possible to consider the $\sqsubseteq_{\gamma^-,\gamma^+}$-order associated to ${\mathcal{K}_\mathcal{C}}^{\pm\varnothing\mathbb{R}}$. Also in this case for all $A, B, C \in {\mathcal{K}_\mathcal{C}}^{\pm\varnothing\mathbb{R}}$, we have:

- $\oplus$ is distributive with respect to $\sqcup$: $(A \sqcup B) \oplus C = (A \oplus C) \sqcup (B \oplus C)$;

- $\oplus$ is distributive with respect to $\sqcap$: $(A \sqcap B) \oplus C = (A \oplus C) \sqcap (B \oplus C)$.

Remark 5.2.5. *Note that, unlike what happened with ${\mathcal{K}_\mathcal{C}}^{\varnothing\mathbb{R}}$ where the addition $\oplus$ is distributive with respect to $\cap$ only if the intervals are not mutually disjointed (as described in Subsection 5.1.2), considering ${\mathcal{K}_\mathcal{C}}^{\pm\varnothing\mathbb{R}}$, we have that distributivity is always valid for both orders, $\lessapprox_{\gamma^-,\gamma^+}$ and $\sqsubseteq_{\gamma^-,\gamma^+}$. This depends on the fact that, considering the whole plan, there are no problems of disjunction between intervals.*

Proposition 5.2.13. *The structures $(\overline{\mathcal{K}_\mathcal{C}^{\pm}}, \vee, \oplus, \lessapprox_{\gamma^-,\gamma^+})$ and $(\overline{\mathcal{K}_\mathcal{C}^{\pm}}, \wedge, \oplus, \lessapprox_{\gamma^-,\gamma^+})$ are commutative semirings.*

Proof. $(\overline{\mathcal{K}_\mathcal{C}^{\pm}}, \vee, \oplus)$ is a commutative semiring, as:

1a) $(\overline{\mathcal{K}_\mathcal{C}^{\pm}}, \vee)$ is a commutative monoid with neutral element $i_\vee = (-\infty; 0)$:

  (i) $\vee$ is associative;

  (ii) $\vee$ has the neutral element $i_\vee = (-\infty; 0) \in \overline{\mathcal{K}_\mathcal{C}^{\pm}}$:
  $A \vee (-\infty; 0) = (-\infty; 0) \vee A = A$, $\forall A \in \overline{\mathcal{K}_\mathcal{C}^{\pm}}$
  (so, $i_\vee = (-\infty; 0)$ is the 0-element of the semiring);

  (iii) $\vee$ is commutative.

2a) $(\overline{\mathcal{K}_\mathcal{C}^{\pm}}, \oplus)$ is a commutative monoid with neutral element $i_\oplus = (0; 0)$:

  (i) $\oplus$ is associative;

  (ii) $\oplus$ has the neutral element $i_\oplus = (0; 0) \in \overline{\mathcal{K}_\mathcal{C}^{\pm}}$:
  $X \oplus (0; 0) = (0; 0) \oplus A = A$, $\forall A \in \overline{\mathcal{K}_\mathcal{C}^{\pm}}$
  (so, $i_\oplus = (0; 0)$ is the 1-element of the semiring);

  (iii) $\oplus$ is commutative.

3a) $\oplus$ is left and right distributive over $\vee$:

  (i) $A \oplus (B \vee C) = (A \oplus B) \vee (A \oplus C), \forall A, B, C \in \overline{\mathcal{K}_\mathcal{C}^{\pm}}$;

  (ii) $(A \vee B) \oplus C = (A \oplus C) \vee (B \oplus C), \forall A, B, C \in \overline{\mathcal{K}_\mathcal{C}^{\pm}}$.

4a) $i_\vee = (-\infty; 0)$ is the absorbing element for $\oplus$:

$$(-\infty;0) \oplus A = A \oplus (-\infty;0) = (-\infty;0), \forall A \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}.$$

Analogously, also the structure $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge, \oplus)$ is a commutative semiring as:

1b) $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge)$ is a commutative monoid with neutral element $i_{\wedge} = (+\infty;0)$:

   (i) $\wedge$ is associative;

   (ii) $\wedge$ has the neutral element $i_{\wedge} = (+\infty;0) \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$:
   $A \wedge (+\infty;0) = (+\infty;0) \wedge A = A$, $\forall A \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$
   (so, $i_{\wedge} = (+\infty;0)$ is the 0-element of the semiring);

   (iii) $\wedge$ is commutative.

2b) $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \oplus)$ is a commutative monoid with neutral element $i_{\oplus} = (0;0)$:

   (i) $\oplus$ is associative;

   (ii) $\oplus$ has the neutral element $i_{\oplus} = (0;0) \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$:
   $A \oplus (0;0) = (0;0) \oplus A = A$, $\forall A \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$
   (so, $i_{\oplus} = (0;0)$ is the 1-element of the semiring);

   (iii) $\oplus$ is commutative.

3b) $\oplus$ is left and right distributive over $\wedge$:

   (i) $A \oplus (B \wedge C) = (A \oplus B) \wedge (A \oplus C), \forall A, B, C \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$;

   (ii) $(A \wedge B) \oplus C = (A \oplus C) \wedge (B \oplus C), \forall A, B, C \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$.

4b) $i_{\wedge} = (+\infty;0)$ is the absorbing element for $\oplus$:

$$(+\infty;0) \oplus A = A \oplus (+\infty;0) = (+\infty;0), \forall A \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}.$$

□

Proposition 5.2.14. *The structures* $(K_C^{\pm\varnothing\mathbb{R}}, \sqcup, \oplus, \sqsubseteq_{\gamma^-,\gamma^+})$ *and* $(K_C^{\pm\varnothing\mathbb{R}}, \sqcap, \oplus, \sqsubseteq_{\gamma^-,\gamma^+})$ *are commutative semirings.*

Proof. $(K_C^{\pm\varnothing\mathbb{R}}, \sqcup, \oplus)$ is a commutative semiring, as:

1) $(K_C^{\pm\varnothing\mathbb{R}}, \sqcup)$ is a commutative monoid with neutral element $i_{\sqcup} = (0;-\infty) = \varnothing$:

   (i) $\sqcup$ is associative;

   (ii) $\sqcup$ has the neutral element $i_{\sqcup} = (0;-\infty) \in K_C^{\pm\varnothing\mathbb{R}}$: $A \sqcup (0;-\infty) = (0;-\infty) \sqcup A = A$, $\forall A \in K_C^{\pm\varnothing\mathbb{R}}$ (so, $i_{\sqcup} = (0;-\infty)$ is the 0-element of the semiring);

   (iii) $\sqcup$ is commutative.

2) $(K_C^{\pm\varnothing\mathbb{R}}, \oplus)$ is a commutative monoid with neutral element $i_{\oplus} = (0;0) = 0$:

(i) $\oplus$ is associative;

(ii) $\oplus$ has the neutral element $i_\oplus = (0;0) \in K_C^{\pm\varnothing\mathbb{R}}$: $A \oplus (0;0) = (0;0) \oplus A = A$, $\forall A \in K_C^{\pm\varnothing\mathbb{R}}$ (so, $i_\oplus = (0;0)$ is the 1-element of the semiring);

(iii) $\oplus$ is commutative.

3) $\oplus$ is left and right distributive over $\sqcup$:

(i) $A \oplus (B \sqcup C) = (A \oplus B) \sqcup (A \oplus C), \forall A, B, C \in K_C^{\pm\varnothing\mathbb{R}}$;

(ii) $(A \sqcup B) \oplus C = (A \oplus C) \sqcup (Y \oplus Z), \forall A, B, C \in K_C^{\pm\varnothing\mathbb{R}}$.

4) $i_\sqcup = (0; -\infty)$ is the absorbing element for $\oplus$:

$$(0; -\infty) \oplus A = A \oplus (0; -\infty) = (0; -\infty), \forall A \in K_C^{\pm\varnothing\mathbb{R}}.$$

Analogously, also $(K_C^{\pm\varnothing\mathbb{R}}, \sqcap, \oplus)$ is a commutative semiring, as:

1) $(K_C^{\pm\varnothing\mathbb{R}}, \sqcap)$ is a commutative monoid with neutral element $i_\sqcap = (0; +\infty) = \mathbb{R}$:

(i) $\sqcap$ is associative;

(ii) $\sqcap$ has the neutral element $i_\sqcap = (0; +\infty) \in K_C^{\pm\varnothing\mathbb{R}}$: $A \sqcap (0; +\infty) = (0; +\infty) \sqcap A = A$, $\forall A \in K_C^{\pm\varnothing\mathbb{R}}$ (so, $i_\sqcap = (0; +\infty)$ is the 0-element of the semiring);

(iii) $\sqcup$ is commutative.

2) $(K_C^{\pm\varnothing\mathbb{R}}, \oplus)$ is a commutative monoid with neutral element $i_\oplus = (0;0) = 0$:

(i) $\oplus$ is associative;

(ii) $\oplus$ has the neutral element $i_\oplus = (0;0) \in K_C^{\pm\varnothing\mathbb{R}}$: $A \oplus (0;0) = (0;0) \oplus A = A$, $\forall A \in K_C^{\pm\varnothing\mathbb{R}}$ (so, $i_\oplus = (0;0)$ is the 1-element of the semiring);

(iii) $\oplus$ is commutative.

3) $\oplus$ is left and right distributive over $\sqcap$:

(i) $A \oplus (B \sqcap C) = (A \oplus B) \sqcap (A \oplus C), \forall A, B, C \in K_C^{\pm\varnothing\mathbb{R}}$;

(ii) $(A \sqcap B) \oplus C = (A \oplus C) \sqcap (Y \oplus Z), \forall A, B, C \in K_C^{\pm\varnothing\mathbb{R}}$.

4) $i_\sqcap = (0; +\infty)$ is the absorbing element for $\oplus$:

$$(0; +\infty) \oplus A = A \oplus (0; +\infty) = (0; +\infty), \forall A \in K_C^{\pm\varnothing\mathbb{R}}.$$

□

Now, according to Definitions 5.1.12 and 5.1.13, we have the following properties.

Proposition 5.2.15. *$(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \wedge, \lessapprox_{\gamma^-,\gamma^+})$ and $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge, \vee, \lessapprox_{\gamma^-,\gamma^+})$ are zero-sum-free semirings (or antirings).*

Proof. the proof is immediate since, for definition, we have
$\forall A, B \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}},\ A \vee B = sup_{\lessapprox_{\gamma^-,\gamma^+}}(A, B)$, it follows that:
$A \vee B = (-\infty; 0) = -\infty \Rightarrow A = B = (-\infty; 0) = -\infty$.
Similarly, as $\forall A, B \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}},\ A \wedge B = inf_{\lessapprox_{\gamma^-,\gamma^+}}(A, B)$, it follows that:
$A \wedge B = (+\infty; 0) = (+\infty) \Rightarrow A = B = (+\infty; 0) = (+\infty)$. □

Proposition 5.2.16. *$(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup, \sqcap, \sqsubseteq_{\gamma^-,\gamma^+})$ and $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcap, \sqcup, \sqsubseteq_{\gamma^-,\gamma^+})$ are zero-sum-free semirings (or antirings).*

Proof. Proceeding in the same way as Proposition 5.2.15, we have that, since $\forall A, B \in \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}},\ A \sqcup B = sup_{\sqsubseteq_{\gamma^-,\gamma^+}}(A, B)$, it follows that:
$A \sqcup B = (0; -\infty) = \varnothing \Rightarrow A = B = (0; -\infty) = \varnothing$.
Similarly, as $\forall A, B \in \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}},\ A \sqcap B = inf_{\sqsubseteq_{\gamma^-,\gamma^+}}(A, B)$, it follows that:
$A \sqcap B = (0; +\infty) = \mathbb{R} \Rightarrow A = B = (0; +\infty) = \mathbb{R}$. □

Proposition 5.2.17. *$(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \oplus, \lessapprox_{\gamma^-,\gamma^+})$ and $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge, \oplus, \lessapprox_{\gamma^-,\gamma^+})$ as well as $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup, \oplus, \sqsubseteq_{\gamma^-,\gamma^+})$ and $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcap, \oplus, \sqsubseteq_{\gamma^-,\gamma^+})$ are zero-sum-free semirings (or antirings).*

Proof. As the proof of Proposition 5.2.15 and Proposition 5.2.16. □

Proposition 5.2.18. *$(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \wedge, \lessapprox_{\gamma^-,\gamma^+})$ and $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge, \vee, \lessapprox_{\gamma^-,\gamma^+})$ are zero-divisor-free (or entire) semirings.*

Proof. The proof is immediate since, for definition, we have
$\forall A, B \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}},\ A \wedge B = inf_{\lessapprox_{\gamma^-,\gamma^+}}(A, B)$, it follows that:
$A \wedge B = (-\infty; 0) = -\infty \Rightarrow A = (-\infty; 0) = -\infty$ or $B = (-\infty; 0) = -\infty$.
Similarly, as $\forall A, B \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}},\ A \vee B = sup_{\lessapprox_{\gamma^-,\gamma^+}}(A, B)$, it follows that:
$A \vee B = (+\infty; 0) = +\infty \Rightarrow A = (+\infty; 0) = +\infty$ or $B = (+\infty; 0) = +\infty$. □

Proposition 5.2.19. *$(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup, \sqcap, \sqsubseteq_{\gamma^-,\gamma^+})$ and $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcap, \sqcup, \sqsubseteq_{\gamma^-,\gamma^+})$ are zero-divisor-free (or entire) semirings.*

Proof. Proceeding in the same way as Proposition 5.2.18, we have that, since $\forall A, B \in \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}},\ A \sqcap B = inf_{\sqsubseteq_{\gamma^-,\gamma^+}}(A, B)$, it follows that:
$A \sqcap B = (0; -\infty) = \varnothing \Rightarrow A = (0; -\infty) = \varnothing$ or $B = (0; -\infty; 0) = \varnothing$.
Similarly, as $\forall A, B \in \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}},\ A \sqcup B = sup_{\sqsubseteq_{\gamma^-,\gamma^+}}(A, B)$, it follows that:
$A \sqcup B = (0; +\infty) = \mathbb{R} \Rightarrow A = (0; +\infty) = \mathbb{R}$ or $B = (0; +\infty; 0) = \mathbb{R}$. □

Proposition 5.2.20. *$(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \oplus, \lessapprox_{\gamma^-,\gamma^+})$ and $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge, \oplus, \lessapprox_{\gamma^-,\gamma^+})$ are zero-divisor-free (or entire) semirings.*

Proof. The proof is immediate since, for definition, we have
$A \oplus B = (\widehat{a} + \widehat{b}; \widetilde{a} + \widetilde{b})$, $\forall A, B \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$, it follows that:
$A \oplus B = (-\infty; 0) = -\infty \Rightarrow A = (-\infty; 0) = -\infty$ or $B = (-\infty; 0) = -\infty$.
Similarly, $A \oplus B = (+\infty; 0) = +\infty \Rightarrow A = (+\infty; 0) = +\infty$ or $B = (+\infty; 0) = +\infty$, $\forall A, B \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$. □

Proposition 5.2.21. *$(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup, \oplus, \sqsubseteq_{\gamma^-,\gamma^+})$ and $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcap, \oplus, \sqsubseteq_{\gamma^-,\gamma^+})$ are zero-divisor-free (or entire) semirings.*

Proof. Proceeding in the same way as proposition (5.2.20), we have that:
$A \oplus B = (0; -\infty) = \varnothing \Rightarrow A = (0; -\infty) = \varnothing$ or $B = (0; -\infty) = \varnothing$
as well as $A \oplus B = (0; +\infty) = \mathbb{R} \Rightarrow A = (0; +\infty) = \mathbb{R}$ or $B = (0; +\infty) = \mathbb{R}$, $\forall A, B \in \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$. □

We also add that the semirings $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \wedge, \lessapprox_{\gamma^-,\gamma^+})$, $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge, \vee, \lessapprox_{\gamma^-,\gamma^+})$, $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup, \sqcap, \sqsubseteq_{\gamma^-,\gamma^+})$, $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcap, \sqcup, \sqsubseteq_{\gamma^-,\gamma^+})$, as well as $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \oplus, \lessapprox_{\gamma^-,\gamma^+})$, $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge, \oplus, \lessapprox_{\gamma^-,\gamma^+})$, $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup, \oplus, \sqsubseteq_{\gamma^-,\gamma^+})$ and $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcap, \oplus, \sqsubseteq_{\gamma^-,\gamma^+})$, being zero-sum-free and zero-divisor-free, are Information Algebras (see [47]).

Moreover, $\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$ and $\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$ are also additively-idempotent, specifically we have that:

- $\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$ is $\vee$-idempotent, as $\forall A \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$, it is $A \vee A = A$;
- $\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$ is $\wedge$-idempotent, as $\forall A \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$, it is $A \wedge A = A$;
- $\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$ is $\sqcup$-idempotent, as $\forall A \in \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$, it is $A \sqcup A = A$;
- $\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$ is $\sqcap$-idempotent, as $\forall A \in \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$, it is $A \sqcap A = A$.

On the other hand, some of the semiring described above (specifically $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \oplus, \lessapprox_{\gamma^-,\gamma^+})$, $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge, \oplus, \lessapprox_{\gamma^-,\gamma^+})$, $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup, \oplus, \sqsubseteq_{\gamma^-,\gamma^+})$, $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcap, \oplus, \sqsubseteq_{\gamma^-,\gamma^+})$) are not multiplicatively-idempotent (or $\oplus$-idempotent) as only 0 is $\oplus$-idempotent: $0 \oplus 0 = 0$ and $A \oplus A = A \Leftrightarrow A = 0$.

Furthermore, since an idempotent semiring $(S, +, \cdot)$ is complete if it is closed for infinitive sums and if the product distributes over infinite sums too (see Subsection 5.1.2), we have that all the semirings just considered are complete; therefore, the following proposition holds.

Proposition 5.2.22. *The structures $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \wedge, \lessapprox_{\gamma^-,\gamma^+})$, $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge, \vee, \lessapprox_{\gamma^-,\gamma^+})$, $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup, \sqcap, \sqsubseteq_{\gamma^-,\gamma^+})$, $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcap, \sqcup, \sqsubseteq_{\gamma^-,\gamma^+})$ as well as $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \oplus, \lessapprox_{\gamma^-,\gamma^+})$, $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge, \oplus, \lessapprox_{\gamma^-,\gamma^+})$, $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup, \oplus, \sqsubseteq_{\gamma^-,\gamma^+})$ and $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcap, \oplus, \sqsubseteq_{\gamma^-,\gamma^+})$ are complete semirings.*

Again, the structures examined in this Subsection can be summarized as follows.

1) $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \wedge; -\infty, +\infty, \lessapprox_{\gamma^-,\gamma^+})$ is a commutative, idempotent, zero-sum-free, zero-divisor free and complete semiring, as:

1.1 $\vee$ is associative;

1.2 $\vee$ is commutative;

1.3 $\vee$ has the neutral element: $-\infty$ (zero of the semiring);

1.4 $\vee$ is idempotent, as: $\forall A \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$, $A \vee A = A$;

[so $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee)$ is a commutative, idempotent monoid]

1.5 $\wedge$ is associative;

1.6 $\wedge$ is commutative;

1.7 $\wedge$ has the neutral element: $+\infty$ (unity of the semiring);

1.8 $\wedge$ is idempotent, as: $\forall A \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$, $A \wedge A = A$;

[so $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge)$ is a commutative, idempotent monoid]

1.9 $\wedge$ is distributive with respect to $\vee$;

1.10 $-\infty$ is the absorbing element for $\wedge$

[so $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \wedge)$ is a commutative, idempotent semiring]

1.11 $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \wedge)$ is zero-sum-free: $A \vee B = -\infty \Leftrightarrow A = B = -\infty$;

1.12 $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \wedge)$ is zero-divisor-free: $A \wedge B \neq -\infty \Leftrightarrow A \neq -\infty \neq B$;

1.13 $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \wedge)$ is complete: $\vee$ distributes over infinite $\wedge$;

[so $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \wedge)$ is a zero-sum-free, zero-divisor-free and complete semiring]

2) $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge, \vee; +\infty, -\infty, \lessapprox_{\gamma^-,\gamma^+})$ is a commutative, idempotent, zero-sum-free, zero-divisor free and complete semiring, as:

2.1 $\wedge$ is associative;

2.2 $\wedge$ is commutative;

2.3 $\wedge$ has the neutral element: $(-\infty)^*$ (zero of the semiring);

2.4 $\wedge$ is idempotent, as: $\forall A \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$, $A \wedge A = A$;

[so $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge)$ is a commutative, idempotent monoid]

2.5 $\vee$ is associative;

2.6 $\vee$ is commutative;

2.7 $\vee$ has the neutral element: $-\infty$ (unity of the semiring);

2.8 $\vee$ is idempotent, as: $\forall A \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$, $A \vee A = A$;

[so $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee)$ is a commutative, idempotent monoid]

2.9 $\vee$ is distributive with respect to $\wedge$;

2.10 $(-\infty)^*$ is the absorbing element for $\vee$

[so $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge, \vee)$ is a commutative, idempotent semiring]

2.11 $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge, \vee)$ is zero-sum-free: $A \wedge B = +\infty \Leftrightarrow A = B = +\infty$;

2.12 $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge, \vee)$ is zero-divisor-free: $A \vee B \neq +\infty \Leftrightarrow A \neq +\infty \neq B$;

2.13 $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge, \vee)$ is complete: $\wedge$ distributes over infinite $\vee$;

[so $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge, \vee)$ is a zero-sum-free, zero-divisor-free and complete semiring]

3) $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup, \sqcap; \varnothing, \mathbb{R}, \sqsubseteq_{\gamma^-,\gamma^+})$ is a commutative, idempotent, zero-sum-free, zero-divisor-free and complete semiring, as:

3.1 $\sqcup$ is associative;

3.2 $\sqcup$ is commutative;

3.3 $\sqcup$ has the neutral element: $\varnothing$ (zero of the semiring);

3.4 $\sqcup$ is idempotent, as: $\forall A \in \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$, $A \sqcup A = A$;

[so $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup)$ is a commutative, idempotent monoid]

3.5 $\sqcap$ is associative;

3.6 $\sqcap$ is commutative;

3.7 $\sqcap$ has the neutral element: $\mathbb{R}$ (unity of the semiring);

3.8 $\sqcap$ is idempotent, as: $\forall A \in \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$, $A \sqcap A = A$;

[so $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcap)$ is a commutative, idempotent monoid]

3.9 $\sqcap$ is distributive with respect to $\sqcup$;

3.10 $\varnothing$ is the absorbing element for $\sqcap$

[so $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup, \sqcap)$ is a commutative, idempotent semiring]

3.11 $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup, \sqcap)$ is zero-sum-free: $A \sqcup B = \varnothing \Leftrightarrow A = B = \varnothing$;

3.12 $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup, \sqcap)$ is zero-divisor-free: $A \sqcap B \neq \varnothing \Leftrightarrow A \neq \varnothing \neq B$;

3.13 $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup, \sqcap)$ is complete: $\sqcup$ distributes over infinite $\sqcap$;

[so $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup, \sqcap)$ is a zero-sum-free, zero-divisor-free and complete semiring]

3.14 $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup, \sqcap)$ is a De Morgan algebra:
$(A^*)^* = A$ and $A \sqsubseteq_{\gamma^-,\gamma^+} B \Longrightarrow B^* \sqsubseteq_{\gamma^-,\gamma^+} A^*$, $\forall A, B \in \mathcal{K}_{C}^{\pm\varnothing\mathbb{R}}$.

4) $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcap, \sqcup; \mathbb{R}, \varnothing, \sqsubseteq_{\gamma^-,\gamma^+})$ is a commutative, idempotent, zero-sum-free, zero-divisor free and complete semiring, as:

4.1 $\sqcap$ is associative;

4.2 $\sqcap$ is commutative;

4.3 $\sqcap$ has the neutral element: $\mathbb{R}$ (zero of the semiring);

4.4 $\sqcap$ is idempotent, as: $\forall A \in \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$, $A \sqcap A = A$;

[so $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcap)$ is a commutative, idempotent monoid]

4.5 $\sqcup$ is associative;

4.6 $\sqcup$ is commutative;

4.7 $\sqcup$ has the neutral element: $\varnothing$ (unity of the semiring);

4.8 $\sqcup$ is idempotent, as: $\forall A \in \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$, $A \sqcup A = A$;

[so $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup)$ is a commutative, idempotent monoid]

4.9 $\sqcup$ is distributive with respect to $\sqcap$;

4.10 $\mathbb{R}$ is the absorbing element for $\sqcup$;

[so $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcap, \sqcup)$ is a commutative, idempotent semiring]

4.11 $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcap, \sqcup)$ is zero-sum-free: $A \sqcap B = \mathbb{R} \Leftrightarrow A = B = \mathbb{R}$;

4.12 $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcap, \sqcup)$ is zero-divisor-free: $A \sqcup B \neq \mathbb{R} \Leftrightarrow A \neq \mathbb{R} \neq B$;

4.13 $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcap, \sqcup)$ is complete: $\sqcap$ distributes over infinite $\sqcup$;

[so $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcap, \sqcup)$ is a zero-sum-free, zero-divisor-free and complete semiring]

4.14 $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcap, \sqcup)$ is a De Morgan algebra:
$(A^*)^* = A$ and $A \sqsubseteq_{\gamma^-,\gamma^+} B \Longrightarrow B^* \sqsubseteq_{\gamma^-,\gamma^+} A^*$, $\forall A, B \in \mathcal{K}_{C}^{\pm\varnothing\mathbb{R}}$.

5) $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \oplus; -\infty, 0, \lessapprox_{\gamma^-,\gamma^+})$ is a commutative, zero-sum-free, zero-divisor-free and complete semiring, as:

5.1 $\vee$ is associative;

5.2 $\vee$ is commutative;

5.3 $\vee$ has the neutral element: $-\infty$ (zero of the semiring);

5.4 $\vee$ is idempotent, as: $\forall A \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$, $A \vee A = A$;

[so $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee)$ is a commutative, idempotent monoid]

5.5 $\oplus$ is associative;

5.6 $\oplus$ is commutative;

5.7 $\oplus$ has the neutral element: 0 (unity of the semiring);

[so $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \oplus)$ is a commutative monoid]

5.8 $\oplus$ is distributive with respect to $\vee$;

5.9 $-\infty$ is the absorbing element for $\oplus$;

[so $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \oplus)$ is a commutative semiring]

5.10 $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \oplus)$ is zero-sum-free: $A \vee B = -\infty \Leftrightarrow A = B = -\infty$;

5.11 $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \oplus)$ is zero-divisor-free: $A \oplus B \neq -\infty \Leftrightarrow A \neq -\infty \neq B$;

5.12 $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \oplus)$ is complete: it is closed for infinite sums and $\vee$ distributes over infinite sums;

[so $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \oplus)$ is a zero-sum-free, zero-divisor-free and complete semiring]

6) $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge, \oplus; +\infty, 0, \lessapprox_{\gamma^-,\gamma^+})$ is a commutative, zero-sum-free, zero-divisor-free and complete semiring, as:

6.1 $\wedge$ is associative;

6.2 $\wedge$ is commutative;

6.3 $\wedge$ has the neutral element: $+\infty$ (zero of the semiring);

6.4 $\wedge$ is idempotent, as: $\forall A \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$, $A \wedge A = A$;

[so $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge)$ is a commutative, idempotent monoid]

6.5 $\oplus$ is associative;

6.6 $\oplus$ is commutative;

6.7 $\oplus$ has the neutral element: 0 (unity of the semiring);

[so $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$ is a commutative monoid]

6.8 $\oplus$ is distributive with respect to $\wedge$;

6.9 $+\infty$ is the absorbing element for $\oplus$;

[so $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge, \oplus)$ is a commutative semiring]

6.10 $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge, \oplus)$ is zero-sum-free: $A \wedge B = +\infty \Leftrightarrow A = B = +\infty$;

6.11 $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge, \oplus)$ is zero-divisor-free: $A \oplus B \neq +\infty \Leftrightarrow A \neq +\infty \neq B$;

6.12 $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge, \oplus)$ is complete: it is closed for infinite sums and $\wedge$ distributes over infinite sums;

[so $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge, \oplus)$ is a zero-sum-free, zero-divisor-free and complete semiring]

7) $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup, \oplus; \varnothing, 0, \sqsubseteq_{\gamma^-,\gamma^+})$ is a commutative, zero-sum-free, zero-divisor-free and complete semiring, as:

7.1 $\sqcup$ is associative;

7.2 $\sqcup$ is commutative;

7.3 $\sqcup$ has the neutral element: $\varnothing$ (zero of the semiring);

7.4 $\sqcup$ is idempotent, as: $\forall A \in \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$, $A \sqcup A = A$;

[so $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup)$ is a commutative, idempotent monoid]

7.5 $\oplus$ is associative;

7.6 $\oplus$ is commutative;

7.7 $\oplus$ has the neutral element: 0 (unity of the semiring);

[so $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \oplus)$ is a commutative monoid]

7.8 $\oplus$ is distributive with respect to $\sqcup$;

7.9 $\varnothing$ is the absorbing element for $\oplus$;

[so $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup, \oplus)$ is a commutative semiring]

7.10 $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup, \oplus)$ is zero-sum-free: $A \sqcup B = \varnothing \Leftrightarrow A = B = \varnothing$;

7.11 $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup, \oplus)$ is zero-divisor-free: $A \oplus B \neq \varnothing \Leftrightarrow A \neq \varnothing \neq B$;

7.12 $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup, \oplus)$ is complete: it is closed for infinite sums and $\sqcup$ distributes over infinite sums;

[so $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup, \oplus)$ is a zero-sum-free, zero-divisor-free and complete semiring]

8) $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcap, \oplus; \mathbb{R}, 0, \sqsubseteq_{\gamma^-,\gamma^+})$ is a commutative, zero-sum-free, zero-divisor-free and complete semiring, as:

8.1 $\sqcap$ is associative;

8.2 $\sqcap$ is commutative;

8.3 $\sqcap$ has the neutral element: $\mathbb{R}$ (zero of the semiring);

8.4 $\sqcap$ is idempotent, as: $\forall A \in \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$, $A \sqcap A = A$;

[so $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcap)$ is a commutative, idempotent monoid]

8.5 $\oplus$ is associative;

8.6 $\oplus$ is commutative;

8.7 $\oplus$ has the neutral element: 0 (unity of the semiring);

[so $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \oplus)$ is a commutative monoid]

8.8 $\oplus$ is distributive with respect to $\sqcap$;

8.9 $\mathbb{R}$ is the absorbing element for $\oplus$;

[so $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcap, \oplus)$ is a commutative semiring]

8.10 $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup, \oplus)$ is zero-sum-free: $A \sqcup B = \mathbb{R} \Leftrightarrow A = B = \mathbb{R}$;

8.11 $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcap, \oplus)$ is zero-divisor-free: $A \oplus B \neq \mathbb{R} \Leftrightarrow A \neq \mathbb{R} \neq B$;

8.12 $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcap, \oplus)$ is complete: it is closed for infinite sums and $\sqcap$ distributes over infinite sums;

[so $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcap, \oplus)$ is a zero-sum-free, zero-divisor-free and complete semiring]

Table 5.4 summarizes the different types of interval semirings we have defined in this subsection with the properties associated with.

Semiring	**0 – element**	**1 – element**	**Properties**
$(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \wedge, \lesssim_{\gamma^-,\gamma^+})$	$-\infty = (-\infty; 0)$	$+\infty = (+\infty; 0)$	C, ZS, ZD, I, E
$(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge, \vee, \lesssim_{\gamma^-,\gamma^+})$	$+\infty = (+\infty; 0)$	$-\infty = (-\infty; 0)$	C, ZS, ZD, I, E
$(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup, \sqcap, \sqsubseteq_{\gamma^-,\gamma^+})$	$\varnothing = (0; -\infty)$	$\mathbb{R} = (0; +\infty)$	C, ZS, ZD, I, E, DM
$(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcap, \sqcup, \sqsubseteq_{\gamma^-,\gamma^+})$	$\mathbb{R} = (0; +\infty)$	$\varnothing = (0; -\infty)$	C, ZS, ZD, I, E, DM
$(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \oplus, \lesssim_{\gamma^-,\gamma^+})$	$-\infty = (-\infty; 0)$	$0 = (0; 0)$	C, ZS, ZD, E
$(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge, \oplus, \lesssim_{\gamma^-,\gamma^+})$	$+\infty = (+\infty; 0)$	$0 = (0; 0)$	C, ZS, ZD, E
$(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup, \oplus, \sqsubseteq_{\gamma^-,\gamma^+})$	$\varnothing = (0; -\infty)$	$0 = (0; 0)$	C, ZS, ZD, E
$(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcap, \oplus, \sqsubseteq_{\gamma^-,\gamma^+})$	$\mathbb{R} = (0; +\infty)$	$0 = (0; 0)$	C, ZS, ZD, E

Table 5.4: Classification of interval "polar" semirings. C= commutative, ZS= zero-sum-free (or antinegative), ZD= zero-divisor-free (or entire), I= idempotent, E= complete, DE= De Morgan algebra.

5.2.3 The barycentric approach (intervals of intervals)

Besides the one examined in Subsection 5.2.2, it is also possible to consider a different approach to interval structures, such that the validity of the distributive property is still ensured.

This second modality foresees the use of the concept of interval of intervals introduced in Subsections 4.1.1 and 4.1.6.

Therefore, according to Definitions 4.1.19 and 4.1.21, for each $A, B \in \mathcal{K}_{\mathcal{C}}$ such that $A \cap B \neq \varnothing$ with respect to inclusion order $\subseteqq_{\gamma^-,\gamma^+}$ defined in (4.11), we can consider the set:

$$[[A, B]]_{\precsim_{\gamma^-,\gamma^+}} = \left\{X \in \mathcal{K}_{\mathcal{C}} \mid A \wedge B \precsim_{\gamma^-,\gamma^+} X \precsim_{\gamma^-,\gamma^+} A \vee B\right\} \tag{5.12}$$

or, respectively,

$$[[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}} = \left\{X \in \mathcal{K}_{\mathcal{C}} \mid A \cap B \subseteqq_{\gamma^-,\gamma^+} X \subseteqq_{\gamma^-,\gamma^+} A \cup B\right\}. \tag{5.13}$$

Moreover, according to (4.24), we also know that

$$[[A, B]]_{\precsim_{\gamma^-,\gamma^+}} = [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}$$

which, in case $A \precsim_{\gamma^-,\gamma^+} B$, from (4.25), it can also be written as

$$[[A, B]]_{\precsim_{\gamma^-,\gamma^+}} = [[A \cap B, A \cup B]]_{\subseteqq_{\gamma^-,\gamma^+}}$$

or, if $A \subseteqq_{\gamma^-,\gamma^+} B$, from (4.26), as:

$$[[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}} = [[A \wedge B, A \vee B]]_{\precsim_{\gamma^-,\gamma^+}}.$$

Therefore, since the two different notations can be represented by the same interval of intervals (as well described by Figure 4.19), from now on, in case no misunderstandings arise, we could also choose to simply write

$$[[A, B]]_{\gamma^-,\gamma^+}$$

instead of $[[A, B]]_{\precsim_{\gamma^-,\gamma^+}}$ or $[[A, B]]_{\subseteq_{\gamma^-,\gamma^+}}$.

It is interesting to note that in the case where $A \cap B \neq \varnothing$, being the elements of the set $[[A, B]]_{\gamma^-,\gamma^+}$ two-by-two non-disjoint intervals, a great advantage is obtained in validating the distributive property.

Furthermore, as the classic Minkowski addition is not an internal operation to the interval $[[A, B]]_{\gamma^-,\gamma^+}$, instead of such addition, it is more suitable to introduce the "barycentric function" of $X = (\widehat{x}; \widetilde{x})$ and $Y = (\widehat{y}; \widetilde{y})$ in $\mathcal{K}_{\mathcal{C}}$, defined as

$$X \boxplus Y \overset{def}{=} \left(\frac{\widehat{x} + \widehat{y}}{2}; \frac{\widetilde{x} + \widetilde{y}}{2} \right) \tag{5.14}$$

and extend it to the set $[[A, B]]_{\gamma^-,\gamma^+}$ as follows:

$$\boxplus : [[A, B]]_{\gamma^-,\gamma^+} \times [[A, B]]_{\gamma^-,\gamma^+} \longrightarrow [[A, B]]_{\gamma^-,\gamma^+}$$

$$\text{such that: } (X, Y) \longrightarrow X \boxplus Y = \left(\frac{\widehat{x} + \widehat{y}}{2}; \frac{\widetilde{x} + \widetilde{y}}{2} \right).$$

Hence, as a consequence of the fact that all the elements of $[[A, B]]_{\gamma^-,\gamma^+}$ are two-by-two non-disjoint intervals (i.e., for all $X, Y \in [[A, B]]_{\gamma^-,\gamma^+}$ it is $X \cap Y \neq \varnothing$), we have that the left and right distributive property of $\boxplus$ with respect to $\cup$ and to $\cap$ holds for all $X, Y, Z \in [[A, B]]_{\gamma^-,\gamma^+}$:

1.a) $X \boxplus (Y \cup Z) = (X \boxplus Y) \cup (X \boxplus Z)$ and $(X \cup Y) \boxplus Z = (X \boxplus Z) \cup (Y \boxplus Z)$;

1.b) $X \boxplus (Y \cap Z) = (X \boxplus Y) \cap (X \boxplus Z)$ and $(X \cap Y) \boxplus Z = (X \boxplus Z) \cap (Y \boxplus Z)$.

Moreover, it goes without saying that the same property is valid if the set $[[A, B]]_{\gamma^-,\gamma^+}$ is associated with operation $\vee$ as well as with $\wedge$; indeed for all $X, Y, Z \in [[A, B]]_{\gamma^-,\gamma^+}$ we have:

2.a) $X \boxplus (Y \vee Z) = (X \boxplus Y) \vee (X \boxplus Z)$ and $(X \vee Y) \boxplus Z = (X \boxplus Z) \vee (Y \boxplus Z)$;

2.b) $X \boxplus (Y \wedge Z) = (X \boxplus Y) \wedge (X \boxplus Z)$ and $(X \wedge Y) \boxplus Z = (X \boxplus Z) \wedge (Y \boxplus Z)$.

It is also trivial to check the validity of the following properties:

3) $X \boxplus Y = Y \boxplus X$, $\forall X, Y \in [[A, B]]_{\gamma^-,\gamma^+}$ (commutativity);

4) $X \boxplus (Y \boxplus Z) = (X \boxplus Y) \boxplus X$, $\forall X, Y, Z \in [[A, B]]_{\gamma^-,\gamma^+}$ (associativity);

5) $X \boxplus X = X$, $\forall X \in [[A, B]]_{\gamma^-,\gamma^+}$ (idempotency).

Consequently, according to Definition 4.1.7, the following propositions hold.

Proposition 5.2.23. *Let $A, B \in \mathcal{K}_{\mathcal{C}}$ such that $A \cap B \neq \varnothing$. The structure $([[A, B]]_{\gamma^-,\gamma^+}, \boxplus)$ is a semilattice.*

Proof. $([[A, B]]_{\gamma^-,\gamma^+}, \boxplus)$ is a semilattice, as:

(1) $\boxplus$ is associative;

(2) $\boxplus$ is commutative;

(3) $\boxplus$ is idempotent.

□

Proposition 5.2.24. *Let $A, B \in \mathcal{K}_{\mathcal{C}}$ such that $A \cap B \neq \varnothing$.*

i) If $A \precapprox_{\gamma^-,\gamma^+} B$, then we have that the structures $([[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee)$ and $([[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge)$ are bounded semilattices.

ii) If $A \subseteqq_{\gamma^-,\gamma^+} B$, then we have that the structures $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup)$ and $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap)$ are bounded semilattices.

Proof. According to Definition 4.1.7, we have that if $A \precapprox_{\gamma^-,\gamma^+} B$, then $([[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee)$ is a bounded semilattice, as:

(1a) $\vee$ is associative;

(2a) $\vee$ is commutative;

(3a) $\vee$ is idempotent;

(4a) $\vee$ has a neutral element $i_\vee = A : A \vee X = X \vee A = X, \ \forall X \in [[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}$.

Likewise, $([[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge)$ is a bounded semilattice, as:

(1b) $\wedge$ is associative;

(2b) $\wedge$ is commutative;

(3b) $\wedge$ is idempotent;

(4b) $\wedge$ has a neutral element $i_\wedge = B : B \wedge X = X \wedge B = X, \ \forall X \in [[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}$.

Similarly, in case $A \subseteqq_{\gamma^-,\gamma^+} B$, then $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup)$ is a bounded semilattice, as:

(1c) $\cup$ is associative;

(2c) $\cup$ is commutative;

(3c) $\cup$ is idempotent;

(4c) $\cup$ has a neutral element $i_\cup = A:\ A \cup X = X \cup A = X,\ \forall X \in\ [[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}$.

Finally, $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap)$ is a bounded semilattice too, as:

(1d) $\cap$ is associative;

(2d) $\cap$ is commutative;

(3d) $\cap$ is idempotent;

(4d) $\cap$ has a neutral element $i_\cap = B:\ B \cap X = X \cap B = X,\ \forall X \in\ [[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}$.

□

According to Definition 5.1.2, other important results concerning the structures associated with intervals of intervals are the following.

Proposition 5.2.25. *Let $A, B \in \mathcal{K}_\mathcal{C}$ such that $A \cap B \neq \varnothing$.*

i) If $A \precapprox_{\gamma^-,\gamma^+} B$, then the two structures $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \boxplus)$ and $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \boxplus)$, are commutative, idempotent pre-semirings with zero.

ii) If $A \subseteqq_{\gamma^-,\gamma^+} B$, then the two structures $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \boxplus)$ and $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap, \boxplus)$ are commutative, idempotent pre-semirings with zero.

Proof. If $A \precapprox_{\gamma^-,\gamma^+} B$, then $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \boxplus)$ is a commutative, idempotent pre-semiring with zero, as:

1a) $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee)$ is a commutative idempotent monoid:

(i) $\vee$ is associative;

(ii) $\vee$ is commutative;

(iii) $\vee$ has the neutral element $i_\vee = A \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}:\ A \vee X = X \vee A = X,\ \forall X \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}$ (so, $i_\vee = A$ is the 0-element of the pre-semiring);

(iv) $\vee$ is idempotent.

2a) $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \boxplus)$ is a commutative, idempotent semigroup:

(i) $\boxplus$ is associative;

(ii) $\boxplus$ is commutative;

(iii) $\boxplus$ is idempotent.

3a) $\boxplus$ is left and right distributive over $\vee$:

(i) $X \boxplus (Y \vee Z) = (X \boxplus Y) \vee (X \boxplus Z), \forall X, Y, Z \in [[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}$;

(ii) $(X \vee Y) \boxplus Z = (X \boxplus Z) \vee (Y \boxplus Z), \forall X, Y, Z \in [[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}$.

Similarly, $([[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \boxplus)$ is a commutative, idempotent pre-semiring with zero, as:

1b) $([[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge)$ is a commutative, idempotent monoid:

(i) $\wedge$ is associative;

(ii) $\wedge$ is commutative;

(iii) $\wedge$ has the neutral element $i_\wedge = B \in [[A, B]]_{\precapprox_{\gamma^-,\gamma^+}} : B \wedge X = X \wedge B = X, \forall X \in [[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}$ (so, $i_\wedge = B$ is the 0-element of the pre-semiring);

(iv) $\wedge$ is idempotent.

2b) $([[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}, \boxplus)$ is a commutative, idempotent semigroup:

(i) $\boxplus$ is associative;

(ii) $\boxplus$ is commutative;

(iii) $\boxplus$ is idempotent.

3b) $\boxplus$ is left and right distributive over $\wedge$:

(i) $X \boxplus (Y \wedge Z) = (X \boxplus Y) \wedge (X \boxplus Z), \forall X, Y, Z \in [[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}$;

(ii) $(X \wedge Y) \boxplus Z = (X \boxplus Z) \wedge (Y \boxplus Z), \forall X, Y, Z \in [[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}$.

In a similar way, if $A \subseteqq_{\gamma^-,\gamma^+} B$, we have that $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \boxplus)$ is a commutative, idempotent pre-semiring with zero, as:

1c) $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup)$ is a commutative, idempotent monoid:

(i) $\cup$ is associative;

(ii) $\cup$ is commutative;

(iii) $\cup$ has the neutral element $i_\cup = A \in [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}} : A \cup X = X \cup A = X, \forall X \in [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}$ (so, $i_\cup = A$ is the 0-element of the pre-semiring);

(iv) $\cup$ is idempotent.

2c) $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \boxplus)$ is a commutative, idempotent semigroup:

(i) $\boxplus$ is associative;

(ii) $\boxplus$ is commutative;

(iii) $\boxplus$ is idempotent.

3c) $\boxplus$ is left and right distributive over $\cup$:

(i) $X \boxplus (Y \cup Z) = (X \boxplus Y) \cup (X \boxplus Z), \forall X, Y, Z \in [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}$;

(ii) $(X \cup Y) \boxplus Z = (X \boxplus Z) \cup (Y \boxplus Z), \forall X, Y, Z \in [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}$.

Likewise, we have that $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap, \boxplus)$ is a commutative, idempotent pre-semiring with zero, as:

1d) $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap)$ is a commutative, idempotent monoid:

(i) $\cap$ is associative;

(ii) $\cap$ is commutative;

(iii) $\cap$ has the neutral element $i_\cap = B \in [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}} : B \cap X = X \cap B = X, \forall X \in [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}$ (so, $i_\cap = B$ is the 0-element of the pre-semiring);

(iv) $\cup$ is idempotent.

2d) $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \boxplus)$ is a commutative, idempotent semigroup:

(i) $\boxplus$ is associative;

(ii) $\boxplus$ is commutative;

(iii) $\boxplus$ is idempotent.

3d) $\boxplus$ is left and right distributive over $\cap$:

(i) $X \boxplus (Y \cap Z) = (X \boxplus Y) \cap (X \boxplus Z), \forall X, Y, Z \in [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}$;

(ii) $(X \cup Y) \boxplus Z = (X \boxplus Z) \cap (Y \boxplus Z), \forall X, Y, Z \in [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}$.

□

Remark 5.2.6. *Note that in Proposition 5.2.25 we considered the classical set union $\cup$ instead of the convex union $\uplus$ that we had used in Subsection 5.1.2 because, as all the elements of $[[A, B]]_{\gamma^-,\gamma^+}$ are intervals not disjoint from each other, it follows that the use of such an operation is no longer justified.*

Proposition 5.2.26. *Let $A, B \in \mathcal{K}_\mathcal{C}$ such that $A \cap B \neq \varnothing$.*

i) If $A \precsim\approx_{\gamma^-,\gamma^+} B$, then the two structures $([[A, B]]_{\precsim\approx_{\gamma^-,\gamma^+}}, \vee, \wedge)$ and $([[A, B]]_{\precsim\approx_{\gamma^-,\gamma^+}}, \wedge, \vee)$ are commutative, idempotent semirings.

ii) If $A \subseteqq_{\gamma^-,\gamma^+} B$, then the two structures $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \cap)$ and $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap, \cup)$ are commutative, idempotent semirings.

Proof. If $A \precapprox_{\gamma^-,\gamma^+} B$, we have that $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \wedge)$ is a commutative, idempotent semiring, as:

1a) $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee)$ is a commutative, idempotent monoid with neutral element $i_\vee = A$:

  (i) $\vee$ is associative;

  (ii) $\vee$ is commutative;

  (iii) $\vee$ has the neutral element $i_\vee = A \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}} : X \vee A = A \vee X = X$, $\forall X \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}$ (so, $i_\vee = A$ is the 0-element of the semiring);

  (iv) $\vee$ is idempotent.

2a) $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge)$ is a commutative, idempotent monoid with neutral element $i_\wedge = B$:

  (i) $\wedge$ is associative;

  (ii) $\wedge$ is commutative;

  (iii) $\wedge$ has the neutral element $i_\wedge = B \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}} : X \wedge B = B \wedge X = X$, $\forall X \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}$ (so, $i_\wedge = B$ is the 1-element of the semiring);

  (iv) $\wedge$ is idempotent.

3a) $\wedge$ is left and right distributive over $\vee$:

  (i) $X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Z), \forall X, Y, Z \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}$;

  (ii) $(X \vee Y) \wedge Z = (X \wedge Z) \vee (Y \wedge Z), \forall X, Y, Z \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}$.

4a) $i_\vee = A$ is the absorbing element for $\wedge$:

$$A \wedge X = X \wedge A = A, \forall X \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}.$$

Likewise, $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \vee)$ is a commutative, idempotent semiring, as:

1b) $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge)$ is a commutative, idempotent monoid with neutral element $i_\wedge = B$:

  (i) $\wedge$ is associative;

  (ii) $\wedge$ is commutative;

  (iii) $\wedge$ has the neutral element $i_\wedge = B \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}} : X \wedge B = B \wedge X = X$, $\forall X \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}$ (so, $i_\wedge = B$ is the 0-element of the semiring);

  (iv) $\wedge$ is idempotent.

2b) $([[A,B]]_{\precsim_{\gamma^-,\gamma^+}}, \vee)$ is a commutative, idempotent monoid with neutral element $i_\vee = A$:

(i) $\vee$ is associative;

(ii) $\vee$ is commutative;

(iii) $\vee$ has the neutral element $i_\vee = A \in [[A,B]]_{\precsim_{\gamma^-,\gamma^+}} : X \vee A = A \vee X = X, \forall X \in [[A,B]]_{\precsim_{\gamma^-,\gamma^+}}$ (so, $i_\vee = A$ is the 1-element of the semiring);

(iv) $\vee$ is idempotent.

3b) $\vee$ is left and right distributive over $\wedge$:

(i) $X \vee (Y \wedge Z) = (X \vee Y) \wedge (X \vee Z), \forall X, Y, Z \in [[A,B]]_{\precsim_{\gamma^-,\gamma^+}}$;

(ii) $(X \wedge Y) \vee Z = (X \vee Z) \wedge (Y \vee Z), \forall X, Y, Z \in [[A,B]]_{\precsim_{\gamma^-,\gamma^+}}$.

4b) $i_\wedge = B$ is the absorbing element for $\vee$:

$$B \vee X = X \vee B = B, \forall X \in [[A,B]]_{\precsim_{\gamma^-,\gamma^+}}.$$

Similarly, if $A \subseteqq_{\gamma^-,\gamma^+} B$, then $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \cap)$ is a commutative, idempotent semiring, as:

1c) $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup)$ is a commutative, idempotent monoid with neutral element $i_\cup = A$:

(i) $\cup$ is associative;

(ii) $\cup$ is commutative;

(iii) $\cup$ has the neutral element $i_\cup = A \in [[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}} : X \cup A = A \cup X = X, \forall X \in [[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}$ (so, $i_\cup = A$ is the 0-element of the semiring);

(iv) $\cup$ is idempotent.

2c) $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap)$ is a commutative, idempotent monoid with neutral element $i_\cap = B$:

(i) $\cap$ is associative;

(ii) $\cap$ is commutative;

(iii) $\cap$ has the neutral element $i_\cap = B \in [[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}} : X \cap B = B \cap X = X, \forall X \in [[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}$ (so, $i_\cap = B$ is the 1-element of the semiring);

(iv) $\cap$ is idempotent.

3c) $\cap$ is left and right distributive over $\cup$:

(i) $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z), \forall X, Y, Z \in [[A, B]]_{\subseteq_{\gamma^-,\gamma^+}}$;

(ii) $(X \cup Y) \cap Z = (X \cap Z) \cup (Y \wedge Z), \forall X, Y, Z \in [[A, B]]_{\subseteq_{\gamma^-,\gamma^+}}$.

4c) $i_\cup = A$ is the absorbing element for $\cap$:

$$A \cap X = X \cap A = A, \forall X \in [[A, B]]_{\subseteq_{\gamma^-,\gamma^+}}.$$

Likewise, $([[A, B]]_{\subseteq_{\gamma^-,\gamma^+}}, \cap, \cup)$ is a commutative, idempotent semiring, as:

1d) $([[A, B]]_{\subseteq_{\gamma^-,\gamma^+}}, \cap)$ is a commutative, idempotent monoid with neutral element $i_\cap = B$:

(i) $\cap$ is associative;

(ii) $\cap$ is commutative;

(iii) $\cap$ has the neutral element $i_\cap = B \in [[A, B]]_{\subseteq_{\gamma^-,\gamma^+}} : X \cap B = B \cap X = X$, $\forall X \in [[A, B]]_{\subseteq_{\gamma^-,\gamma^+}}$ (so, $i_\cap = B$ is the 0-element of the semiring);

(iv) $\cap$ is idempotent.

2d) $([[A, B]]_{\subseteq_{\gamma^-,\gamma^+}}, \cup)$ is a commutative, idempotent monoid with neutral element $i_\cup = A$:

(i) $\cup$ is associative;

(ii) $\cup$ is commutative;

(iii) $\cup$ has the neutral element $i_\cup = A \in [[A, B]]_{\subseteq_{\gamma^-,\gamma^+}} : X \cup A = A \cup X = X$, $\forall X \in [[A, B]]_{\subseteq_{\gamma^-,\gamma^+}}$ (so, $i_\cup = A$ is the 1-element of the semiring);

(iv) $\cup$ is idempotent.

3d) $\cup$ is left and right distributive over $\cap$:

(i) $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z), \forall X, Y, Z \in [[A, B]]_{\subseteq_{\gamma^-,\gamma^+}}$;

(ii) $(X \cap Y) \cup Z = (X \cup Z) \cap (Y \cup Z), \forall X, Y, Z \in [[A, B]]_{\subseteq_{\gamma^-,\gamma^+}}$.

4d) $i_\cap = B$ is the absorbing element for $\cup$:

$$B \cup X = X \cup B = B, \forall X \in [[A, B]]_{\subseteq_{\gamma^-,\gamma^+}}.$$

□

Furthermore, according to Definitions 5.1.12 and 5.1.13 , we have that the following properties hold.

Proposition 5.2.27. *Let $A, B \in \mathcal{K}_\mathcal{C}$ such that $A \cap B \neq \varnothing$, then the structures $([[A, B]]_{\precsim_{\gamma^-,\gamma^+}}, \vee, \wedge)$ and $([[A, B]]_{\precsim_{\gamma^-,\gamma^+}}, \wedge, \vee)$ are zero-sum-free semirings.*

Proof. Considering the case in which $A \precapprox_{\gamma^-,\gamma^+} B$ (the other cases are analogous), the proof is immediate since, for definition, we have
$\forall X, Y \in [[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}$, $X \vee Y = sup_{\precapprox_{\gamma^-,\gamma^+}}(X, Y)$, it follows that:
$X \vee Y = A \Rightarrow X = Y = A$.
Similarly, as $\forall X, Y \in [[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}$, $X \wedge Y = inf_{\precapprox_{\gamma^-,\gamma^+}}(X, Y)$, it follows that: $X \wedge B = B \Rightarrow X = Y = B$. □

Proposition 5.2.28. *Let $A, B \in \mathcal{K_C}$ such that $A \cap B \neq \varnothing$, then the structures $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \cap)$ and $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap, \cup)$ are zero-sum-free semirings.*

Proof. Considering the case in which $A \subseteqq_{\gamma^-,\gamma^+} B$ (the other cases are analogous) and proceeding in the same way as Proposition 5.2.27, we have that, since $\forall X, Y \in [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}$, $X \cup Y = sup_{\subseteqq_{\gamma^-,\gamma^+}}(X, Y)$, it follows that:
$X \cup Y = A \Rightarrow X = Y = A$.
Similarly, as $\forall X, Y \in [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}$, $X \cap Y = inf_{\subseteqq_{\gamma^-,\gamma^+}}(X, Y)$, it follows that: $X \cap Y = B \Rightarrow X = Y = B$. □

As a consequence of Propositions 5.2.27 and 5.2.28, we also have the next one.

Proposition 5.2.29. *Let $A, B \in \mathcal{K_C}$ such that $A \cap B \neq \varnothing$.*

i) The two structures $([[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \boxplus)$ and $([[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \boxplus)$ are zero-sum-free pre-semirings.

ii) The two structures $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \boxplus)$ and $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap, \boxplus)$ are zero-sum-free pre-semirings.

Proposition 5.2.30. *Let $A, B \in \mathcal{K_C}$ such that $A \cap B \neq \varnothing$, then the structures $([[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \wedge)$ and $([[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \vee)$ are zero-divisor-free semirings.*

Proof. Considering the case in which $A \precapprox_{\gamma^-,\gamma^+} B$ (the other cases are analogous), the proof is immediate since, for definition, we have
$\forall X, Y \in [[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}$, $X \wedge Y = inf_{\precapprox_{\gamma^-,\gamma^+}}(X, Y)$, it follows that:
$X \wedge Y = A \Rightarrow X = A$ or $Y = A$.
Similarly, as $\forall X, Y \in [[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}$, $X \vee Y = sup_{\precapprox_{\gamma^-,\gamma^+}}(X, Y)$, it follows that: $X \vee Y = B \Rightarrow X = B$ or $Y = B$. □

Proposition 5.2.31. *Let $A, B \in \mathcal{K_C}$, then the structures $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \cap)$ and $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap, \cup)$ are zero-divisor-free semirings.*

Proof. Considering the case in which $A \subseteqq_{\gamma^-,\gamma^+} B$ (the other cases are analogous) and proceeding in the same way as Proposition 5.2.30, we have that, since $\forall X, Y \in [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}$, $X \cap Y = inf_{\subseteqq_{\gamma^-,\gamma^+}}(X, Y)$, it follows that:
$X \cap Y = A \Rightarrow X = A$ or $Y = A$.
Similarly, as $\forall X, Y \in [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}$, $X \cup Y = sup_{\subseteqq_{\gamma^-,\gamma^+}}(X, Y)$, it follows that: $X \cup Y = B \Rightarrow X = B$ or $Y = B$. □

Now, on the basis of what has been analysed so far and taking into account the polarity highlighted between the two orders $\precapprox_{\gamma^-,\gamma^+}$ and $\subseteqq_{\gamma^-,\gamma^+}$ (see Section 4.1), it may be interesting to try placing them in the same type of structure. What we get are the following statements.

Proposition 5.2.32. *Let $A, B \in \mathcal{K}_\mathcal{C}$ such that $A \cap B \neq \varnothing$.*

i) The structures $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \cup)$ and $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \cup)$ as well as $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \cap)$ and $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \cap)$ are commutative, idempotent pre-semirings with zero and unity.

ii) The structures $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \vee)$ and $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap, \vee)$ as well as $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \wedge)$ and $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap, \wedge)$ are commutative, idempotent pre-semirings with zero and unity.

Proof. In the demonstration of part *i)* we only consider the case in which $A \precapprox_{\gamma^-,\gamma^+} B$ (the other cases are analogous). Therefore, we have that $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \cup)$ is a commutative, idempotent pre-semiring as:

1a) $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee)$ is a commutative, idempotent monoid with neutral element $i_\vee = A$:

- (i) $\vee$ is associative;
- (ii) $\vee$ is commutative;
- (iii) $\vee$ has the neutral element $i_\vee = A \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}$:
  $X \vee A = A \vee X = X$, $\forall X \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}$ (so, $i_\vee = A$ is the 0-element of the pre-semiring);
- (iv) $\vee$ is idempotent.

2a) $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \cup)$ is a commutative, idempotent monoid with neutral element $i_\cup = A \cap B$:

- (i) $\cup$ is associative;
- (ii) $\cup$ is commutative;
- (iii) $\cup$ has the neutral element $i_\cup = A \cap B \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}$:
  $X \cup (A \cap B) = (A \cap B) \cup X = X$, $\forall X \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}$ (so, $i_\cup = A \cap B$ is the 1-element of the pre-semiring);
- (iv) $\cup$ is idempotent.

3a) $\cup$ is left and right distributive over $\vee$:

- (i) $X \cup (Y \vee Z) = (X \cup Y) \vee (X \cup Z), \forall X, Y, Z \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}$;
- (ii) $(X \vee Y) \cup Z = (X \cup Z) \vee (Y \cup Z), \forall X, Y, Z \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}$.

Likewise, $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \cup)$ is a commutative, idempotent pre-semiring as:

1b) $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge)$ is a commutative, idempotent monoid with neutral element $i_\wedge = B$:

   (i) $\wedge$ is associative;

   (ii) $\wedge$ is commutative;

   (iii) $\wedge$ has the neutral element $i_\wedge = B \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}$:
   $X \wedge B = B \wedge X = X,\ \forall X \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}$ (so, $i_\wedge = B$ is the 0-element of the pre-semiring);

   (iv) $\wedge$ is idempotent.

2b) $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \cup)$ is a commutative, idempotent monoid with neutral element $i_\cup = A \cap B$:

   (i) $\cup$ is associative;

   (ii) $\cup$ is commutative;

   (iii) $\cup$ has the neutral element $i_\cup = A \cap B \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}$:
   $X \cup (A \cap B) = (A \cap B) \cup X = X,\ \forall X \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}$ (so, $i_\cup = (A \cap B)$ is the 1-element of the pre-semiring);

   (iv) $\cup$ is idempotent.

3b) $\cup$ is left and right distributive over $\wedge$:

   (i) $X \cup (Y \wedge Z) = (X \cup Y) \wedge (X \cup Z), \forall X,Y,Z \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}$;

   (ii) $(X \wedge Y) \cup Z = (X \cup Z) \wedge (Y \cup Z), \forall X,Y,Z \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}$.

Similarly $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \cap)$ is a commutative, idempotent pre-semiring as:

1c) $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee)$ is a commutative, idempotent monoid with neutral element $i_\vee = A$ as the 0-element of the pre-semiring (verified in 1a).

2c) $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \cap)$ is a commutative, idempotent monoid with neutral element $i_\cap = A \cup B$:

   (i) $\cap$ is associative;

   (ii) $\cap$ is commutative;

   (iii) $\cap$ has the neutral element $i_\cap = A \cup B \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}$:
   $X \cap (A \cup B) = (A \cup B) \cap X = X,\ \forall X \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}$ (so, $i_\cap = A \cup B$ is the 1-element of the pre-semiring);

   (iv) $\cap$ is idempotent.

3c) $\cap$ is left and right distributive over $\vee$:

(i) $X \cap (Y \vee Z) = (X \cap Y) \vee (X \cap Z), \forall X, Y, Z \in [[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}$;

(ii) $(X \vee Y) \cap Z = (X \cap Z) \vee (Y \cap Z), \forall X, Y, Z \in [[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}$.

Likewise, $([[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \cap)$ is a commutative, idempotent pre-semiring as:

1d) $([[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge)$ is a commutative, idempotent monoid with neutral element $i_\wedge = B$ as the 0-element of the pre-semiring (verified in 1b).

2d) $([[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}, \cap)$ is a commutative, idempotent monoid with neutral element $i_\cap = A \cup B$ as the 1-element of the pre-semiring (verified in 2c).

3d) $\cap$ is left and right distributive over $\wedge$:

(i) $X \cap (Y \wedge Z) = (X \cap Y) \wedge (X \cap Z), \forall X, Y, Z \in [[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}$;

(ii) $(X \wedge Y) \cap Z = (X \cap Z) \wedge (Y \cap Z), \forall X, Y, Z \in [[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}$.

In a similar way, in the demonstration of part *ii*) we only consider the case in which $A \subseteqq_{\gamma^-,\gamma^+} B$ (the other cases are analogous). Therefore, we have that $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \vee)$ is a commutative, idempotent semiring as:

1e) $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup)$ is a commutative, idempotent monoid with neutral element $i_\cup = A$:

(i) $\cup$ is associative;

(ii) $\cup$ is commutative;

(iii) $\cup$ has the neutral element $i_\cup = A \in [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}} : X \cup A = A \cup X = X, \forall X \in [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}$ (so, $i_\cup = A$ is the 0-element of the pre-semiring);

(iv) $\cup$ is idempotent.

2e) $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \vee)$ is a commutative, idempotent monoid with neutral element $i_\vee = A \wedge B$:

(i) $\vee$ is associative;

(ii) $\vee$ is commutative;

(iii) $\vee$ has the neutral element $i_\vee = A \wedge B \in [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}} : X \vee (A \wedge B) = (A \wedge B) \vee X = X, \forall X \in [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}$ (so, $i_\vee = A \wedge B$ is the 1-element of the semiring);

(iv) $\vee$ is idempotent.

3e) $\vee$ is left and right distributive over $\cup$:

(i) $X \vee (Y \cup Z) = (X \vee Y) \cup (X \vee Z), \forall X, Y, Z \in [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}$;

(ii) $(X \cup Y) \vee Z = (X \vee Z) \cup (Y \vee Z), \forall X, Y, Z \in [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}$.

Likewise, $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap, \vee)$ is a commutative, idempotent pre-semiring as:

1f) $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap)$ is a commutative, idempotent monoid with neutral element $i_\cap = B$:

(i) $\cap$ is associative;

(ii) $\cap$ is commutative;

(iii) $\cap$ has the neutral element $i_\cap = B \in [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}} : X \cap B = B \cap X = X$, $\forall X \in [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}$ (so, $i_\cap = B$ is the 0-element of the pre-semiring);

(iv) $\cap$ is idempotent.

2f) $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \vee)$ is a commutative, idempotent monoid with neutral element $i_\vee = A \wedge B$ as the 1-element of the pre-semiring (verified in 2e).

3f) $\vee$ is left and right distributive over $\cap$:

(i) $X \vee (Y \cap Z) = (X \vee Y) \cap (X \vee Z), \forall X, Y, Z \in [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}$;

(ii) $(X \cap Y) \vee Z = (X \vee Z) \cap (Y \vee Z), \forall X, Y, Z \in [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}$.

Similarly, $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \wedge)$ is a commutative, idempotent semiring as:

1g) $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup)$ is a commutative, idempotent monoid with neutral element $i_\cup = A$ as the 0-element of the pre-semiring (verified in 1e).

2g) $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \wedge)$ is a commutative, idempotent monoid with neutral element $i_\wedge = A \vee B$:

(i) $\wedge$ is associative;

(ii) $\wedge$ is commutative;

(iii) $\wedge$ has the neutral element $i_\wedge = A \vee B \in [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}} : X \wedge (A \vee B) = (A \vee B) \wedge X = X$, $\forall X \in [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}$ (so, $i_\wedge = A \vee B$ is the 1-element of the pre-semiring);

(iv) $\wedge$ is idempotent.

3g) $\wedge$ is left and right distributive over $\cup$:

(i) $X \wedge (Y \cup Z) = (X \wedge Y) \cup (X \wedge Z), \forall X, Y, Z \in [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}$;

(ii) $(X \cup Y) \wedge Z = (X \wedge Z) \cup (Y \wedge Z), \forall X, Y, Z \in [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}$.

Likewise, $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap, \wedge)$ is a commutative, idempotent pre-semiring as:

1h) $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}},\cap)$ is a commutative, idempotent monoid with neutral element $i_\cap = B$ as the 0-element of the pre-semiring (verified in 1f).

2h) $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}},\wedge)$ is a commutative, idempotent monoid with neutral element $i_\wedge = A \vee B$ as the 1-element of the pre-semiring (verified in 2g).

3h) $\wedge$ is left and right distributive over $\cap$:

(i) $X \wedge (Y \cap Z) = (X \wedge Y) \cap (X \wedge Z), \forall X, Y, Z \in [[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}$;

(ii) $(X \cap Y) \wedge Z = (X \wedge Z) \cap (Y \wedge Z), \forall X, Y, Z \in [[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}$.

□

Also this time, according to Definitions 5.1.12 and 5.1.13, we have the following statements.

Proposition 5.2.33. *Let $A, B \in \mathcal{K}_\mathcal{C}$ such that $A \cap B \neq \varnothing$, then the structures $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}},\vee,\cup)$ and $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}},\wedge,\cup)$, as well as $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}},\vee,\cap)$ and $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}},\wedge,\cap)$, are zero-sum-free pre-semirings.*

Proof. See proof of Proposition 5.2.27. □

Proposition 5.2.34. *Let $A, B \in \mathcal{K}_\mathcal{C}$ such that $A \cap B \neq \varnothing$, then the structures $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}},\cup,\vee)$ and $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}},\cap,\vee)$, as well as $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}},\cup,\wedge)$ and $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}},\cap,\wedge)$, are zero-sum-free pre-semirings.*

Proof. See proof of Proposition 5.2.28. □

Proposition 5.2.35. *Let $A, B \in \mathcal{K}_\mathcal{C}$ such that $A \cap B \neq \varnothing$, then we have that $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}},\vee,\cup)$ and $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}},\wedge,\cup)$, as well as $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}},\vee,\cap)$ and $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}},\wedge,\cap)$, are zero-divisor-free pre-semirings.*

Proof. Considering the case in which $A \precapprox_{\gamma^-,\gamma^+} B$ (the other cases are analogous), the proof is immediate since, for definition, we have
$\forall X, Y \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}$, $X \cup Y = sup_{\subseteqq_{\gamma^-,\gamma^+}}(X,Y)$, it follows that:
$X \cup Y = A \Rightarrow X = A$ or $Y = A$. Similarly, $X \cup Y = B \Rightarrow X = B$ or $Y = B$.
Likewise way we have that,
since $\forall X, Y \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}$, $X \cap Y = inf_{\subseteqq_{\gamma^-,\gamma^+}}(X,Y)$, it follows that:
$X \cap Y = A \Rightarrow X = A$ or $Y = A$.
Similarly, $X \cap Y = B \Rightarrow X = B$ or $Y = B$. □

Proposition 5.2.36. *Let $A, B \in \mathcal{K}_\mathcal{C}$ such that $A \cap B \neq \varnothing$, then we have that the structures $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}},\cup,\vee)$ and $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}},\cap,\vee)$, as well as $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}},\cup,\wedge)$ and $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}},\cap,\wedge)$, are zero-divisor-free pre-semirings.*

Proof. Considering the case in which $A \subseteq_{\gamma^-,\gamma^+} B$ (the other cases are analogous) and proceeding in the same way as Proposition 5.2.35, we have that, since $\forall X, Y \in [[A,B]]_{\subseteq_{\gamma^-,\gamma^+}}$, $X \vee Y = sup_{\precapprox_{\gamma^-,\gamma^+}}(X,Y)$, it follows that: $X \vee Y = A \Rightarrow X = A$ or $Y = A$. Similarly, $X \vee Y = B \Rightarrow X = B$ or $Y = B$. Likewise way we have that,
since $\forall X, Y \in [[A,B]]_{\subseteq_{\gamma^-,\gamma^+}}$, $X \wedge Y = inf_{\precapprox_{\gamma^-,\gamma^+}}(X,Y)$, it follows that:
$X \wedge Y = A \Rightarrow X = A$ or $Y = A$. Similarly, $X \wedge Y = B \Rightarrow X = B$ or $Y = B$.
□

Finally, according to Definition 5.1.14, it is easy to verify the following property.

Proposition 5.2.37. *Let $A, B \in \mathcal{K}_\mathcal{C}$ such that $A \cap B \neq \varnothing$.*

i) The two structures $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \wedge)$ and $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \vee)$ are complete semirings while $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \boxplus)$ and $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \boxplus)$ as well as the structures $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \cup)$, $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \cup)$, $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \cap)$ and $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \cap)$ are complete pre-semirings.

ii) The two structures $([[A,B]]_{\subseteq_{\gamma^-,\gamma^+}}, \cup, \cap)$ and $([[A,B]]_{\subseteq_{\gamma^-,\gamma^+}}, \cap, \cup)$ are complete semirings while $([[A,B]]_{\subseteq_{\gamma^-,\gamma^+}}, \cup, \boxplus)$ and $([[A,B]]_{\subseteq_{\gamma^-,\gamma^+}}, \cap, \boxplus)$ as well as the structures $([[A,B]]_{\subseteq_{\gamma^-,\gamma^+}}, \cup, \vee)$, $([[A,B]]_{\subseteq_{\gamma^-,\gamma^+}}, \cap, \vee)$, $([[A,B]]_{\subseteq_{\gamma^-,\gamma^+}}, \cup, \wedge)$ and $([[A,B]]_{\subseteq_{\gamma^-,\gamma^+}}, \cap, \wedge)$ are complete pre-semirings.

Similarly to what was done at the end of Subsections 5.1.2 and 5.2.2, all the structures examined in this paragraph can be summarized as follows.

1) $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \wedge; A \wedge B, A \vee B, \precapprox_{\gamma^-,\gamma^+})$ is a commutative, idempotent, zero-sum-free, zero-divisor free and complete semiring, as:

    1.1 $\vee$ is associative;

    1.2 $\vee$ is commutative;

    1.3 $\vee$ has the neutral element: $A \wedge B$ (zero of the semiring);

    1.4 $\vee$ is idempotent, as: $\forall X \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}$, $X \vee X = X$;

    [so $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee)$ is a commutative, idempotent monoid]

    1.5 $\wedge$ is associative;

    1.6 $\wedge$ is commutative;

    1.7 $\wedge$ has the neutral element: $A \vee B$ (unity of the semiring);

    1.8 $\wedge$ is idempotent, as: $\forall X \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}$, $X \wedge X = X$;

    [so $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge)$ is a commutative, idempotent monoid]

    1.9 $\wedge$ is distributive with respect to $\vee$;

1.10 $A \wedge B$ is the absorbing element for $\wedge$

[so $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \wedge)$ is a commutative, idempotent semiring]

1.11 $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \wedge)$ is zero-sum-free: $X \vee Y = A \wedge B \Leftrightarrow X = Y = A \wedge B$;

1.12 $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \wedge)$ is zero-divisor-free: $X \wedge Y \neq A \wedge B \Leftrightarrow X \neq A \wedge B \neq Y$;

1.13 $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \wedge)$ is complete: $\vee$ distributes over infinite $\wedge$;

[so $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \wedge)$ is a zero-sum-free, zero-divisor-free and complete semiring]

2) $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \vee; A \vee B, A \wedge B, \precapprox_{\gamma^-,\gamma^+})$ is a commutative, idempotent, zero-sum-free, zero-divisor free and complete semiring, as:

2.1 $\wedge$ is associative;

2.2 $\wedge$ is commutative;

2.3 $\wedge$ has the neutral element: $A \vee B$ (zero of the semiring);

2.4 $\wedge$ is idempotent;

[so $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge)$ is a commutative, idempotent monoid]

2.5 $\vee$ is associative;

2.6 $\vee$ is commutative;

2.7 $\vee$ has the neutral element: $A \wedge B$ (unity of the semiring);

2.8 $\vee$ is idempotent;

[so $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee)$ is a commutative, idempotent monoid]

2.9 $\vee$ is distributive with respect to $\wedge$;

2.10 $A \vee B$ is the absorbing element for $\vee$

[so $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \vee)$ is a commutative, idempotent semiring]

2.11 $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \vee)$ is zero-sum-free: $X \wedge Y = A \vee B \Leftrightarrow X = Y = A \vee B$;

2.12 $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \vee)$ is zero-divisor-free: $X \vee Y \neq A \vee B \Leftrightarrow X \neq A \vee B \neq Y$;

2.13 $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \vee)$ is complete: $\wedge$ distributes over infinite $\vee$;

[so $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \vee)$ is a zero-sum-free, zero-divisor-free and complete semiring]

3) $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \cap; A \cap B, A \cup B, \subseteqq_{\gamma^-,\gamma^+})$ is a commutative, idempotent, zero-sum-free, zero-divisor-free and complete semiring, as:

3.1 $\cup$ is associative;

3.2 $\cup$ is commutative;

3.3 $\cup$ has the neutral element: $A \cap B$ (zero of the semiring);

3.4 $\cup$ is idempotent;

[so $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup)$ is a commutative, idempotent monoid]

3.5 $\cap$ is associative;

3.6 $\cap$ is commutative;

3.7 $\cap$ has the neutral element: $A \cup B$ (unity of the semiring);

3.8 $\cap$ is idempotent;

[so $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap)$ is a commutative, idempotent monoid]

3.9 $\cap$ is distributive with respect to $\cup$;

3.10 $A \cap B$ is the absorbing element for $\cap$

[so $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \cap)$ is a commutative, idempotent semiring]

3.11 $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \cap)$ is zero-sum-free: $X \cup Y = A \cap B \Leftrightarrow X = Y = A \cap B$;

3.12 $([[A, B]]_{\precsim_{\gamma^-,\gamma^+}}, \cup, \cap)$ is zero-divisor-free: $X \cap Y \neq A \cap B \Leftrightarrow X \neq A \cap B \neq Y$;

3.13 $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \cap)$ is complete: $\cup$ distributes over infinite $\cap$;

[so $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \cap)$ is a zero-sum-free, zero-divisor-free and complete semiring]

4) $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap, \cup; A \cup B, A \cap B, \subseteqq_{\gamma^-,\gamma^+})$ is a commutative, idempotent, zero-sum-free, zero-divisor free and complete semiring, as:

4.1 $\cap$ is associative;

4.2 $\cap$ is commutative;

4.3 $\cap$ has the neutral element: $A \cup B$ (zero of the semiring);

4.4 $\cap$ is idempotent;

[so $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap)$ is a commutative, idempotent monoid]

4.5 $\cup$ is associative;

4.6 $\cup$ is commutative;

4.7 $\cup$ has the neutral element: $A \cap B$ (unity of the semiring);

4.8 $\cup$ is idempotent;

[so $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup)$ is a commutative, idempotent monoid]

4.9 $\cup$ is distributive with respect to $\cap$;

4.10 $A \cup B$ is the absorbing element for $\cup$;

[so $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap, \cup)$ is a commutative, idempotent semiring]

4.11 $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap, \cup)$ is zero-sum-free: $X \cap Y = A \cup B \Leftrightarrow X = Y = A \cup B$;

4.12 $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap, \cup)$ is zero-divisor-free: $X \cup Y \neq A \cup B \Leftrightarrow X \neq A \cup B \neq Y$;

4.13 $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap, \cup)$ is complete: $\cap$ distributes over infinite $\cup$;

[so $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap, \cup)$ is a zero-sum-free, zero-divisor-free and complete semiring]

5) $(([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \boxplus; A \wedge B, \precapprox_{\gamma^-,\gamma^+})$ is a commutative, idempotent, zero-sum-free and complete pre-semiring with zero, as:

5.1 $\vee$ is associative;

5.2 $\vee$ is commutative;

5.3 $\vee$ has the neutral element: $A \wedge B$ (zero of the pre-semiring);

5.4 $\vee$ is idempotent;

[so $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee)$ is a commutative, idempotent monoid]

5.5 $\boxplus$ is associative;

5.6 $\boxplus$ is commutative;

5.7 $\boxplus$ is idempotent;

[so $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \boxplus)$ is a commutative, idempotent semigroup]

5.8 $\boxplus$ is distributive with respect to $\vee$;

[so $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \boxplus)$ is a commutative pre-semiring with zero element A]

5.9 $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \boxplus)$ is zero-sum-free: $X \vee Y = A \wedge B \Leftrightarrow X = Y = A \wedge B$;

5.10 $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \boxplus)$ is complete: it is closed for infinite sums and $\vee$ distributes over infinite sums;

[so $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \boxplus)$ is a zero-sum-free and complete pre-semiring]

6) $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \boxplus; A \vee B, \precapprox_{\gamma^-,\gamma^+})$ is a commutative, idempotent, zero-sum-free and complete pre-semiring with zero, as:

6.1 $\wedge$ is associative;

6.2 $\wedge$ is commutative;

6.3 $\wedge$ has the neutral element: $A \vee B$ (zero of the pre-semiring);

6.4 $\wedge$ is idempotent;

[so $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge)$ is a commutative, idempotent monoid]

6.5 $\boxplus$ is associative;

6.6 $\boxplus$ is commutative;

6.7 $\boxplus$ is idempotent;

[so $([[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}, \boxplus)$ is a commutative, idempotent semigroup]

6.8 $\boxplus$ is distributive with respect to $\wedge$;

[so $([[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \boxplus)$ is a commutative pre-semiring with zero element B]

6.9 $([[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \boxplus)$ is zero-sum-free: $X \wedge Y = A \vee B \Leftrightarrow X = Y = A \vee B$;

6.10 $([[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \boxplus)$ is complete: it is closed for infinite sums and $\wedge$ distributes over infinite sums;

[so $([[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \oplus)$ is a zero-sum-free and complete pre-semiring]

7) $(([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \boxplus; A \cap B, \subseteqq_{\gamma^-,\gamma^+})$ is a commutative, idempotent, zero-sum-free and complete pre-semiring with zero, as:

7.1 $\cup$ is associative;

7.2 $\cup$ is commutative;

7.3 $\cup$ has the neutral element: $A \cap B$ (zero of the pre-semiring);

7.4 $\cup$ is idempotent;

[so $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \vee)$ is a commutative, idempotent monoid]

7.5 $\boxplus$ is associative;

7.6 $\boxplus$ is commutative;

7.7 $\boxplus$ is idempotent;

[so $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \boxplus)$ is a commutative, idempotent semigroup]

7.8 $\boxplus$ is distributive with respect to $\cup$;

[so $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \boxplus)$ is a commutative pre-semiring with zero element A]

7.9 $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \boxplus)$ is zero-sum-free: $X \cup Y = A \cap B \Leftrightarrow X = Y = A \cap B$;

7.10 $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \boxplus)$ is complete: it is closed for infinite sums and $\cup$ distributes over infinite sums;

[so $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \boxplus)$ is a zero-sum-free and complete pre-semiring]

8) $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap, \boxplus; A \cup B, \subseteqq_{\gamma^-,\gamma^+})$ is a commutative, idempotent, zero-sum-free and complete pre-semiring with zero, as:

8.1 $\cap$ is associative;

8.2 $\cap$ is commutative;

8.3 $\cap$ has the neutral element: $A \cup B$ (zero of the pre-semiring);

8.4 $\cap$ is idempotent;

[so $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap)$ is a commutative, idempotent monoid]

8.5 $\boxplus$ is associative;

8.6 $\boxplus$ is commutative;

8.7 $\boxplus$ is idempotent;

[so $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \boxplus)$ is a commutative, idempotent semigroup]

8.8 $\boxplus$ is distributive with respect to $\cap$;

[so $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap, \boxplus)$ is a commutative pre-semiring with zero element B]

8.9 $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap, \boxplus)$ is zero-sum-free: $X \cap Y = A \cup B \Leftrightarrow X = Y = A \cup B$;

8.10 $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap, \boxplus)$ is complete: it is closed for infinite sums and $\cap$ distributes over infinite sums;

[so $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap, \oplus)$ is a zero-sum-free and complete pre-semiring]

9) $([[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \cup; A \wedge B, A \cap B, \precapprox_{\gamma^-,\gamma^+})$ is a commutative, idempotent, zero-sum-free, zero-divisor free and complete pre-semiring with zero and unity, as:

9.1 $\vee$ is associative;

9.2 $\vee$ is commutative;

9.3 $\vee$ has the neutral element: $A \wedge B$ (zero of the pre-semiring);

9.4 $\vee$ is idempotent, as: $\forall X \in [[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}$, $X \vee X = X$;

[so $([[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee)$ is a commutative, idempotent monoid]

9.5 $\cup$ is associative;

9.6 $\cup$ is commutative;

9.7 $\cup$ has the neutral element: $A \cap B$ (unity of the pre-semiring);

9.8 $\cup$ is idempotent, as: $\forall X \in [[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}$, $X \cup X = X$;

[so $([[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge)$ is a commutative, idempotent monoid]

9.9 $\cup$ is distributive with respect to $\vee$;

[so $([[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \wedge)$ is a commutative, idempotent pre-semiring with zero and unity]

9.10 $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \cup)$ is zero-sum-free: $X \vee Y = A \wedge B \Leftrightarrow X = Y = A \wedge B$;

9.11 $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \cup)$ is zero-divisor-free: $X \cup Y \neq A \wedge B \Leftrightarrow X \neq A \wedge B \neq Y$;

9.12 $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \wedge)$ is complete: $\vee$ distributes over infinite $\cup$;

[so $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \wedge)$ is a zero-sum-free, zero-divisor-free and complete pre-semiring]

10) $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \cup; A \vee B, A \cap B, \precapprox_{\gamma^-,\gamma^+})$ is a commutative, idempotent, zero-sum-free, zero-divisor free and complete pre-semiring with zero and unity, as:

10.1 $\wedge$ is associative;

10.2 $\wedge$ is commutative;

10.3 $\wedge$ has the neutral element: $A \vee B$ (zero of the pre-semiring);

10.4 $\wedge$ is idempotent;

[so $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge)$ is a commutative, idempotent monoid]

10.5 $\cup$ is associative;

10.6 $\cup$ is commutative;

2.7 $\cup$ has the neutral element: $A \cap B$ (unity of the pre-semiring);

10.8 $\cup$ is idempotent;

[so $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \cup)$ is a commutative, idempotent monoid]

10.9 $\cup$ is distributive with respect to $\wedge$;

[so $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \cup)$ is a commutative, idempotent pre-semiring with zero and unity]

10.10 $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \cup)$ is zero-sum-free: $X \wedge Y = A \vee B \Leftrightarrow X = Y = A \vee B$;

10.11 $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \cup)$ is zero-divisor-free: $X \cup Y \neq A \vee B \Leftrightarrow X \neq A \vee B \neq Y$;

10.12 $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \cup)$ is complete: $\wedge$ distributes over infinite $\cup$;

[so $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \cup)$ is a zero-sum-free, zero-divisor-free and complete pre-semiring]

11) $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \cap; A \wedge B, A \cup B, \precapprox_{\gamma^-,\gamma^+})$ is a commutative, idempotent, zero-sum-free and complete pre-semiring with zero and unity, as:

11.1 $\vee$ is associative;

11.2 $\vee$ is commutative;

11.3 $\vee$ has the neutral element: $A \wedge B$ (zero of the pre-semiring);

11.4 $\vee$ is idempotent;

[so $([[A,B]]_{\precsim_{\gamma^-,\gamma^+}}, \vee)$ is a commutative, idempotent monoid]

11.5 $\cap$ is associative;

11.6 $\cap$ is commutative;

11.7 $\cap$ has the neutral element: $A \cup B$ (unity of the pre-semiring);

11.8 $\cap$ is idempotent;

[so $([[A,B]]_{\precsim_{\gamma^-,\gamma^+}}, \cap)$ is a commutative, idempotent monoid]

11.9 $\cap$ is distributive with respect to $\vee$;

[so $([[A,B]]_{\precsim_{\gamma^-,\gamma^+}}, \cup, \cap)$ is a commutative, idempotent pre-semiring with zero and unity]

11.10 $([[A,B]]_{\precsim_{\gamma^-,\gamma^+}}, \vee, \cap)$ is zero-sum-free: $X \vee Y = A \wedge B \Leftrightarrow X = Y = A \wedge B$;

11.11 $([[A,B]]_{\precsim_{\gamma^-,\gamma^+}}, \cup, \cap)$ is zero-divisor-free:
$X \cap Y \neq A \wedge B \Leftrightarrow X \neq A \wedge B \neq Y$;

11.12 $([[A,B]]_{\precsim_{\gamma^-,\gamma^+}}, \vee, \cap)$ is complete: $\vee$ distributes over infinite $\cap$;

[so $([[A,B]]_{\precsim_{\gamma^-,\gamma^+}}, \cup, \cap)$ is a zero-sum-free, zero-divisor-free and complete pre-semiring]

12) $([[A,B]]_{\precsim_{\gamma^-,\gamma^+}}, \wedge, \cap; A \vee B, A \cup B, \precsim_{\gamma^-,\gamma^+})$ is a commutative, idempotent, zero-sum-free, zero-divisor free and complete pre-semiring with zero and unity, as:

12.1 $\wedge$ is associative;

12.2 $\wedge$ is commutative;

12.3 $\wedge$ has the neutral element: $A \vee B$ (zero of the pre-semiring);

12.4 $\wedge$ is idempotent;

[so $([[A,B]]_{\precsim_{\gamma^-,\gamma^+}}, \wedge)$ is a commutative, idempotent monoid]

12.5 $\cap$ is associative;

12.6 $\cap$ is commutative;

12.7 $\cap$ has the neutral element: $A \cup B$ (unity of the pre-semiring);

12.8 $\cap$ is idempotent;

[so $([[A,B]]_{\precsim_{\gamma^-,\gamma^+}}, \cap)$ is a commutative, idempotent monoid]

12.9 $\cap$ is distributive with respect to $\wedge$;

[so $([[A,B]]_{\precsim_{\gamma^-,\gamma^+}}, \wedge, \cap)$ is a commutative, idempotent pre-semiring with zero and unity]

12.10 $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \cap)$ is zero-sum-free: $X \wedge Y = A \vee B \Leftrightarrow X = Y = A \vee B$;

12.11 $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \cap)$ is zero-divisor-free: $X \cap Y \neq A \vee B \Leftrightarrow X \neq A \vee B \neq Y$;

12.12 $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \cap, \cup)$ is complete: $\cap$ distributes over infinite $\cup$;

[so $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \cap)$ is a zero-sum-free, zero-divisor-free and complete pre-semiring]

13) $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \vee; A \cap B, A \wedge B, \subseteqq_{\gamma^-,\gamma^+})$ is a commutative, idempotent, zero-sum-free, zero-divisor-free and complete pre-semiring with zero and unity, as:

13.1 $\cup$ is associative;

13.2 $\cup$ is commutative;

13.3 $\cup$ has the neutral element: $A \cap B$ (zero of the pre-semiring);

13.4 $\cup$ is idempotent;

[so $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup)$ is a commutative, idempotent monoid]

13.5 $\vee$ is associative;

13.6 $\vee$ is commutative;

13.7 $\vee$ has the neutral element: $A \wedge B$ (unity of the pre-semiring);

13.8 $\vee$ is idempotent;

[so $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \vee)$ is a commutative, idempotent monoid]

13.9 $\vee$ is distributive with respect to $\cup$;

[so $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \vee)$ is a commutative, idempotent pre-semiring with zero and unity]

13.10 $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \vee)$ is zero-sum-free: $X \cup Y = A \cap B \Leftrightarrow X = Y = A \cap B$;

13.11 $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \cup, \vee)$ is zero-divisor-free: $X \vee Y \neq A \cap B \Leftrightarrow X \neq A \cap B \neq Y$;

13.12 $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \vee)$ is complete: $\cup$ distributes over infinite $\vee$;

[so $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \vee)$ is a zero-sum-free, zero-divisor-free and complete pre-semiring]

14) $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap, \vee; A \cup B, A \wedge B, \subseteqq_{\gamma^-,\gamma^+})$ is a commutative, idempotent, zero-sum-free, zero-divisor free and complete pre-semiring with zero and unity, as:

14.1 $\cap$ is associative;

14.2 $\cap$ is commutative;

14.3 $\cap$ has the neutral element: $A \cup B$ (zero of the pre-semiring);

14.4 $\cap$ is idempotent;

[so $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap)$ is a commutative, idempotent monoid]

14.5 $\vee$ is associative;

14.6 $\vee$ is commutative;

14.7 $\vee$ has the neutral element: $A \wedge B$ (unity of the pre-semiring);

14.8 $\vee$ is idempotent;

[so $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \vee)$ is a commutative, idempotent monoid]

14.9 $\vee$ is distributive with respect to $\cap$;

[so $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap, \vee)$ is a commutative, idempotent pre-semiring with zero and unity]

14.10 $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap, \vee)$ is zero-sum-free: $X \cap Y = A \cup B \Leftrightarrow X = Y = A \cup B$;

14.11 $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap, \vee)$ is zero-divisor-free: $X \vee Y \neq A \cup B \Leftrightarrow X \neq A \cup B \neq Y$;

14.12 $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap, \vee)$ is complete: $\cap$ distributes over infinite $\vee$;

[so $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap, \vee)$ is a zero-sum-free, zero-divisor-free and complete pre-semiring]

15) $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \wedge; A \cap B, A \vee B, \subseteqq_{\gamma^-,\gamma^+})$ is a commutative, idempotent, zero-sum-free, zero-divisor-free and complete pre-semiring with zero and unity, as:

15.1 $\cup$ is associative;

15.2 $\cup$ is commutative;

15.3 $\cup$ has the neutral element: $A \cap B$ (zero of the pre-semiring);

15.4 $\cup$ is idempotent;

[so $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup)$ is a commutative, idempotent monoid]

15.5 $\wedge$ is associative;

15.6 $\wedge$ is commutative;

15.7 $\wedge$ has the neutral element: $A \vee B$ (unity of the pre-semiring);

15.8 $\wedge$ is idempotent;

[so $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \wedge)$ is a commutative, idempotent monoid]

15.9 $\wedge$ is distributive with respect to $\cup$;

[so $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \wedge)$ is a commutative, idempotent pre-semiring with zero and unity]

15.10 $([[A,B]]_{\underset{\equiv}{\subset}_{\gamma^-,\gamma^+}}, \cup, \wedge)$ is zero-sum-free: $X \cup Y = A \cap B \Leftrightarrow X = Y = A \cap B$;

15.11 $([[A,B]]_{\underset{\approx}{\prec}_{\gamma^-,\gamma^+}}, \cup, \wedge)$ is zero-divisor-free:
$X \wedge Y \neq A \cap B \Leftrightarrow X \neq A \cap B \neq Y$;

15.12 $([[A,B]]_{\underset{\equiv}{\subset}_{\gamma^-,\gamma^+}}, \cup, \vee)$ is complete: $\cup$ distributes over infinite $\wedge$;

[so $([[A,B]]_{\underset{\equiv}{\subset}_{\gamma^-,\gamma^+}}, \cup, \wedge)$ is a zero-sum-free, zero-divisor-free and complete pre-semiring]

16) $([[A,B]]_{\underset{\equiv}{\subset}_{\gamma^-,\gamma^+}}, \cap, \wedge; A \cup B, A \vee B, \underset{\equiv}{\subset}_{\gamma^-,\gamma^+})$ is a commutative, idempotent, zero-sum-free, zero-divisor free and complete pre-semiring with zero and unity, as:

16.1 $\cap$ is associative;

16.2 $\cap$ is commutative;

16.3 $\cap$ has the neutral element: $A \cup B$ (zero of the pre-semiring);

16.4 $\cap$ is idempotent;

[so $([[A,B]]_{\underset{\equiv}{\subset}_{\gamma^-,\gamma^+}}, \cap)$ is a commutative, idempotent monoid]

16.5 $\wedge$ is associative;

16.6 $\wedge$ is commutative;

16.7 $\wedge$ has the neutral element: $A \vee B$ (unity of the pre-semiring);

16.8 $\wedge$ is idempotent;

[so $([[A,B]]_{\underset{\equiv}{\subset}_{\gamma^-,\gamma^+}}, \wedge)$ is a commutative, idempotent monoid]

16.9 $\wedge$ is distributive with respect to $\cap$;

[so $([[A,B]]_{\underset{\equiv}{\subset}_{\gamma^-,\gamma^+}}, \cap, \wedge)$ is a commutative, idempotent pre-semiring with zero and unity]

16.10 $([[A,B]]_{\underset{\equiv}{\subset}_{\gamma^-,\gamma^+}}, \cap, \wedge)$ is zero-sum-free: $X \cap Y = A \cup B \Leftrightarrow X = Y = A \cup B$;

16.11 $([[A,B]]_{\underset{\equiv}{\subset}_{\gamma^-,\gamma^+}}, \cap, \wedge)$ is zero-divisor-free:
$X \wedge Y \neq A \cup B \Leftrightarrow X \neq A \cup B \neq Y$;

16.12 $([[A,B]]_{\underset{\equiv}{\subset}_{\gamma^-,\gamma^+}}, \cap, \wedge)$ is complete: $\cap$ distributes over infinite $\wedge$;

[so $([[A,B]]_{\underset{\equiv}{\subset}_{\gamma^-,\gamma^+}}, \cap, \wedge)$ is a zero-sum-free, zero-divisor-free and complete pre-semiring]

Table 5.5 summarizes the different types of interval structures we have defined in this subsection and the properties associated with.

Type	**Structure**	**0**	**1**	**Properties**
Semiring	$([[A,B]]_{\gamma^-,\gamma^+},\vee,\wedge)$	$A\wedge B$	$A\vee B$	C, ZS, ZD, I, E
Semiring	$([[A,B]]_{\gamma^-,\gamma^+},\wedge,\vee)$	$A\vee B$	$A\wedge B$	C, ZS, ZD, I, E
Semiring	$([[A,B]]_{\gamma^-,\gamma^+},\cup,\cap)$	$A\cap B$	$A\cup B$	C, ZS, ZD, I, E
Semiring	$([[A,B]]_{\gamma^-,\gamma^+},\cap,\cup)$	$A\cup B$	$A\cap B$	C, ZS, ZD, I, E
Pre-semiring with 0	$([[A,B]]_{\gamma^-,\gamma^+},\vee,\boxplus)$	$A\wedge B$	/	C, ZS, I, E
Pre-semiring with 0	$([[A,B]]_{\gamma^-,\gamma^+},\wedge,\boxplus)$	$A\vee B$	/	C, ZS, I, E
Pre-semiring with 0	$([[A,B]]_{\gamma^-,\gamma^+},\cup,\boxplus)$	$A\cap B$	/	C, ZS, I, E
Pre-semiring with 0	$([[A,B]]_{\gamma^-,\gamma^+},\cap,\boxplus)$	$A\cup B$	/	C, ZS, I, E
Pre-semiring with 0 and 1	$([[A,B]]_{\gamma^-,\gamma^+},\vee,\cup)$	$A\wedge B$	$A\cap B$	C, ZS, ZD, I, E
Pre-semiring with 0 and 1	$([[A,B]]_{\gamma^-,\gamma^+},\wedge,\cup)$	$A\vee B$	$A\cap B$	C, ZS, ZD, I, E
Pre-semiring with 0 and 1	$([[A,B]]_{\gamma^-,\gamma^+},\vee,\cap)$	$A\wedge B$	$A\cup B$	C, ZS, ZD, I, E
Pre-semiring with 0 and 1	$([[A,B]]_{\gamma^-,\gamma^+},\wedge,\cap)$	$A\vee B$	$A\cup B$	C, ZS, ZD, I, E
Pre-semiring with 0 and 1	$([[A,B]]_{\gamma^-,\gamma^+},\cup,\vee)$	$A\cap B$	$A\wedge B$	C, ZS, ZD, I, E
Pre-semiring with 0 and 1	$([[A,B]]_{\gamma^-,\gamma^+},\cap,\vee)$	$A\cup B$	$A\wedge B$	C, ZS, ZD, I, E
Pre-semiring with 0 and 1	$([[A,B]]_{\gamma^-,\gamma^+},\cup,\wedge)$	$A\cap B$	$A\vee B$	C, ZS, ZD, I, E
Pre-semiring with 0 and 1	$([[A,B]]_{\gamma^-,\gamma^+},\cap,\wedge)$	$A\cup B$	$A\vee B$	C, ZS, ZD, I, E

Table 5.5: General classification of interval structures on set $[[A,B]]_{\gamma^-,\gamma^+}$ such that $A\cap_{\subseteq_{\gamma^-,\gamma^+}} B\neq\varnothing$. C= commutative, ZS= zero-sum-free (or antinegative), ZD= zero-divisor-free (or entire), I= idempotent, E= complete.

Interval translation

As repeatedly highlighted, in the structures examined in Subsection 5.2.3, problems may arise when $A \cap B = \varnothing$ (with respect to inclusion order $\subseteqq_{\gamma^-,\gamma^+}$) since in that case the distributive property is not ensured; that is why it was necessary to introduce the condition: $A \cap B \neq \varnothing$.

However, it is possible to overcome this drawback also without imposing any restriction in the definition of the set $[[A, B]]_{\gamma^-,\gamma^+}$ but through the use of a kind of interval translation.

For this purpose, considering, e.g., the LU-case, as shown in Figure 5.11, for each $A, B \in \mathcal{K_C}$ with $A \precsim_{\gamma^-,\gamma^+} B$, where $A = (\widehat{a}; \widetilde{a}) = [a^-, a^+]$ and $B = (\widehat{b}; \widetilde{b}) = [b^-, b^+]$, we can define

$$\Delta(A, B) = \begin{cases} (0; 0) & \text{if } A \cap B \neq \varnothing; \\ \left(0; \dfrac{b^- - a^+}{2}\right) & \text{if } A \cap B = \varnothing. \end{cases}$$

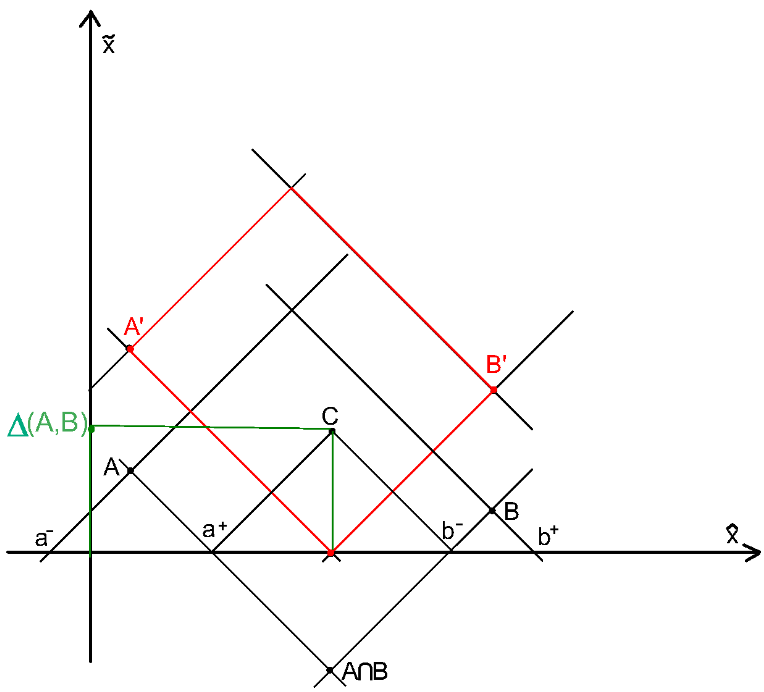


Figure 5.11: Translation of the set $[[A, B]]_{\gamma^-,\gamma^+}$ in such a way that the intersection point between intervals A and B belongs to the horizontal axis of the half-plane.

Therefore, since $\mathcal{K_C}$ is located by definition in the positive half-plane $(\widehat{z}; \widetilde{z})$, then, in case $A \cap B = \varnothing$, the representation of the intersection point is not possible as it would fall out of the half-plane; however, we can overcome this inconvenience by translating the entire interval of intervals $[[A, B]]_{\gamma^-,\gamma^+}$ in such a way as to make the intersection "reenter" in the positive half-plane $\mathcal{K}_\mathcal{C}$. This means that intersection can be chosen to be, e.g.,

$$\left(\frac{b^- + a^+}{2}; 0\right).$$

In order to achieve this, it is necessary to carry out the following translation

$$\begin{cases} \widehat{x}' = \widehat{x} \\ \widetilde{x}' = \widetilde{x} + \Delta(A, B) \end{cases}, \quad \text{i.e.,} \quad \begin{cases} A \to A' \\ B \to B' \end{cases}$$

such that we get

$$A' \cap B' = \left(\frac{a^+ + b^-}{2}; 0\right).$$

As for the intervals of intervals we have

$$[[A', B']]_{\gamma^-,\gamma^+} = [[A, B]]_{\gamma^-,\gamma^+} + \Delta(A, B),$$

so, it is indifferent to consider one interval or the other in $\mathcal{K}_\mathcal{C}$, since they maintain the same characteristics unaltered.

Therefore, henceforth it is not restrictive to consider only intervals of intervals entirely contained in $\mathcal{K}_\mathcal{C}$, namely such that $[[A, B]]_{\gamma^-,\gamma^+} \subseteq \mathcal{K}_\mathcal{C}$.

Finally, note that, in case $A \cap B = \varnothing$, i.e., the point representing the intersection does not belong to the half-plane $\mathcal{K}_\mathcal{C}$, if we consider the interval

$$C = (\widehat{c}; \widetilde{c}) = \left(\frac{a^+ + b^-}{2}; \frac{b^- - a^+}{2}\right)$$

as the symmetric point of $A \cap B$ in the half-plane $\mathcal{K}_\mathcal{C}$, we obtain that convexification of union is exactly

$$A \uplus B = conv(A \cup B) = A \cup B \cup C$$

which represents the smallest interval with this property.

5.3 Interval combined structures

In abstract algebra, in addition to the common algebraic structures, there are also particular ones, which from now on we will simply call combined structures, characterized by the fact that three or more binary operations are defined on the underlying set; the coexistence of these operations is ensured by the fact that they satisfy a certain number of properties which link them to each other.

In this section we will deal with some of them also giving an interval interpretation as we believe that theories associated with these structures, such as, e.g., ordered lattice monoids (see [10]), offer a conceptually elegant and compact way to express structurally rich and articulated situations.

5.3.1 Combined structures in classic algebra

Before continuing, let us review some useful definitions of combined algebraic structures (see also [10], [40], [53], [54] and [90]).

Definition 5.3.1. *We define (commutative) lattice-ordered semigroup/monoid, l-semigroup/monoid for short, a structure* $(L, \vee, \wedge, \star)$ *such that:*

1) $(L, \vee, \wedge)$ *is a lattice;*

2) $(L, \star)$ *is a (commutative) semigroup/monoid;*

3) the right and left distributive laws of $\star$ *over* $\vee$ *and over* $\wedge$ *both hold:*
$x \star (y \vee z) = (x \star y) \vee (x \star z), \quad (y \vee z) \star x = (y \star x) \vee (z \star x),$
$x \star (y \wedge z) = (x \star y) \wedge (x \star z), \quad (y \wedge z) \star x = (y \star x) \wedge (z \star x),$
$\forall x, y, z \in L.$

Note that a lattice-ordered group (*l-group* for short) is defined similarly.

Moreover, suppose now that L is also a semigroup/monoid under a dual operation $\star'$ that distributes over $\vee$ and $\wedge$ too. This means that L has four binary operations.

Definition 5.3.2. *We define (commutative) lattice-ordered double monoid a structure* $(L, \vee, \wedge, \star, \star', \leq)$ *such that:*

1) $(L, \vee, \wedge)$ *is a lattice whose associated partial order is* $\leq$*;*

2) $(L, \star)$ *is a (commutative) monoid whose operation* $\star$ *distributes over* $\vee$*:*

$x \star (y \vee z) = (x \star y) \vee (x \star z), \quad (y \vee z) \star x = (y \star x) \vee (z \star x),$
$\forall x, y, z \in L;$

3) $(L, \star')$ *is a (commutative) monoid whose operation* $\star'$ *distributes over* $\wedge$*:*

$x \star (y \wedge z) = (x \star y) \wedge (x \star z), \quad (y \wedge z) \star x = (y \star x) \wedge (z \star x),$
$\forall x, y, z \in L.$

To the above definitions we add the word *complete* if L is a complete lattice and the distributivities involved are infinite.
In the case of a complete lattice-ordered double monoid, the structure is simply called *clodum*.

5.3.2 Combined structures in interval algebra

Let consider two intervals $A, B \in \mathcal{K}_{\mathcal{C}}$ such that $A \cap B \neq \varnothing$ (with respect to inclusion order $\subseteqq_{\gamma^-,\gamma^+}$). According to (4.24), if there is no risk of misunderstanding we could simply denote

$$[[A,B]]_{\gamma^-,\gamma^+} = [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}} = [[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}$$

where, according to (5.12) and (5.13), it is

$$[[A,B]]_{\precapprox_{\gamma^-,\gamma^+}} = \left\{X \in \mathcal{K}_{\mathcal{C}} \mid A \wedge B \precapprox_{\gamma^-,\gamma^+} X \precapprox_{\gamma^-,\gamma^+} A \vee B\right\}$$

and

$$[[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}} = \left\{X \in \mathcal{K}_{\mathcal{C}} \mid A \cap B \subseteqq_{\gamma^-,\gamma^+} X \subseteqq_{\gamma^-,\gamma^+} A \cup B\right\}),$$

as shown in Figure 5.12, where the particular case $A \precapprox_{\gamma^-,\gamma^+} B$ is analysed.

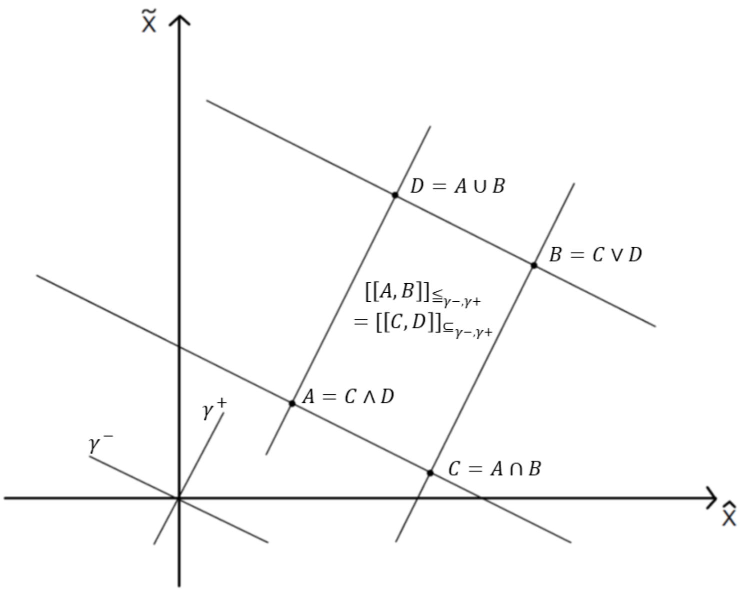


Figure 5.12: The interval of intervals $[[A,B]]_{\precapprox_{\gamma^-,\gamma^+}} = [[C,D]]_{\subseteqq_{\gamma^-,\gamma^+}}$ with $C = A \cap B$, $D = A \cup B$ and $A = C \wedge D$, $B = C \vee D$.

In this regard it is possible to consider the well known different kind of operations associated with the two orders $\precapprox_{\gamma^-,\gamma^+}$ and $\subseteqq_{\gamma^-,\gamma^+}$ to apply them to the set $[[A,B]]_{\gamma^-,\gamma^+}$ (of which they are internal) as shown below:

$\vee : [[A,B]]_{\gamma^-,\gamma^+} \times [[A,B]]_{\gamma^-,\gamma^+} \to [[A,B]]_{\gamma^-,\gamma^+}$ such that: $(X,Y) \to X \vee Y$;

$\wedge : [[A,B]]_{\gamma^-,\gamma^+} \times [[A,B]]_{\gamma^-,\gamma^+} \to [[A,B]]_{\gamma^-,\gamma^+}$ such that: $(X,Y) \to X \wedge Y$;

$\cup : [[A,B]]_{\gamma^-,\gamma^+} \times [[A,B]]_{\gamma^-,\gamma^+} \to [[A,B]]_{\gamma^-,\gamma^+}$ such that: $(X,Y) \to X \cup Y$;

$\cap : [[A,B]]_{\gamma^-,\gamma^+} \times [[A,B]]_{\gamma^-,\gamma^+} \to [[A,B]]_{\gamma^-,\gamma^+}$ such that: $(X,Y) \to X \cap Y$;

$\boxplus : [[A,B]]_{\gamma^-,\gamma^+} \times [[A,B]]_{\gamma^-,\gamma^+} \to [[A,B]]_{\gamma^-,\gamma^+}$ such that: $(X,Y) \to X \boxplus Y$.

What we get are the following interesting properties.

Proposition 5.3.1. *Let be $A,B \in \mathcal{K_C}$ such that $A \cap B \neq \varnothing$, then the structure $([[A,B]]_{\precsim_{\gamma^-,\gamma^+}}, \vee, \wedge, \boxplus)$ is a commutative complete lattice-ordered semigroup.*

Proof. $([[A,B]]_{\precsim_{\gamma^-,\gamma^+}}, \vee, \wedge, \boxplus)$ is a complete and commutative l-semigroup as:

1) $([[A,B]]_{\precsim_{\gamma^-,\gamma^+}}, \vee, \wedge)$ is a complete lattice, according to Proposition 5.2.37; indeed all subsets have both a supremum (join) and an infimum (meet) which graphically correspond to the extreme left and right points of the subset.

2) $([[A,B]]_{\precsim_{\gamma^-,\gamma^+}}, \boxplus)$ is a commutative semigroup:

   (i) $\boxplus$ is associative;

   (ii) $\boxplus$ is commutative.

3) $\boxplus$ is left and right distributive over $\vee$ and $\wedge$:

   (i) $X \boxplus (Y \vee Z) = (X \boxplus Y) \vee (X \boxplus Z), \forall X,Y,Z \in [[A,B]]_{\precsim_{\gamma^-,\gamma^+}}$;

   (ii) $(X \vee Y) \boxplus Z = (X \boxplus Z) \vee (Y \boxplus Z), \forall X,Y,Z \in [[A,B]]_{\precsim_{\gamma^-,\gamma^+}}$;

   (iii) $X \boxplus (Y \wedge Z) = (X \boxplus Y) \wedge (X \boxplus Z), \forall X,Y,Z \in [[A,B]]_{\precsim_{\gamma^-,\gamma^+}}$;

   (iv) $(X \wedge Y) \boxplus Z = (X \boxplus Z) \wedge (Y \boxplus Z), \forall X,Y,Z \in [[A,B]]_{\precsim_{\gamma^-,\gamma^+}}$.

□

Proposition 5.3.2. *Let be $C,D \in \mathcal{K_C}$, such that $C \cap D \neq \varnothing$, then the structure $([[C,D]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \cap, \boxplus)$ is a commutative complete lattice-ordered semigroup .*

Proof. $([[C,D]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \cap, \boxplus)$ is a complete and commutative l-semigroup as:

1) $([[C,D]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \cap)$ is a complete lattice, according to Proposition 5.2.37.

2) $([[C,D]]_{\subseteqq_{\gamma^-,\gamma^+}}, \boxplus)$ is a commutative semigroup:

(i) $\boxplus$ is associative;

(ii) $\boxplus$ is commutative.

3) $\boxplus$ is left and right distributive over $\cup$ and $\cap$:

(i) $X \boxplus (Y \cup Z) = (X \boxplus Y) \cup (X \boxplus Z), \forall X, Y, Z \in [[C, D]]_{\subseteqq_{\gamma^-,\gamma^+}}$;

(ii) $(X \cup Y) \boxplus Z = (X \boxplus Z) \cup (Y \boxplus Z), \forall X, Y, Z \in [[C, D]]_{\subseteqq_{\gamma^-,\gamma^+}}$;

(iii) $X \boxplus (Y \cap Z) = (X \boxplus Y) \cap (X \boxplus Z), \forall X, Y, Z \in [[C, D]]_{\subseteqq_{\gamma^-,\gamma^+}}$;

(iv) $(X \cap Y) \boxplus Z = (X \boxplus Z) \cap (Y \boxplus Z), \forall X, Y, Z \in [[C, D]]_{\subseteqq_{\gamma^-,\gamma^+}}$.

□

The following statements also hold.

Proposition 5.3.3. *Let be $A, B \in \mathcal{K}_{\mathcal{C}}$, such that $A \cap B \neq \varnothing$, then the structures $([[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \wedge, \cup)$ and $([[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \wedge, \cap)$ are commutative complete lattice-ordered monoids.*

Proof. Considering the case in which $A \precapprox_{\gamma^-,\gamma^+} B$ (the other cases are analogous), we have that $([[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \wedge, \cup)$ is a complete and commutative l-monoid as:

1) $([[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \wedge)$ is a complete lattice, according to Proposition 5.2.37.

2) $([[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}, \cup)$ is a commutative monoid with neutral element $i_\cup = A \cap B$:

(i) $\cup$ is associative;

(ii) $\cup$ has the neutral element $i_\cup = A \cap B \in [[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}$:
$X \cup (A \cap B) = (A \cap B) \cup X = X, \ \forall X \in [[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}$;

(iii) $\cup$ is commutative.

3) $\cup$ is left and right distributive over $\vee$ and $\wedge$:

(i) $X \cup (Y \vee Z) = (X \cup Y) \vee (X \cup Z), \forall X, Y, Z \in [[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}$;

(ii) $(X \vee Y) \cup Z = (X \cup Z) \vee (Y \cup Z), \forall X, Y, Z \in [[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}$;

(iii) $X \cup (Y \wedge Z) = (X \cup Y) \wedge (X \cup Z), \forall X, Y, Z \in [[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}$;

(iv) $(X \wedge Y) \cup Z = (X \cup Z) \wedge (Y \cup Z), \forall X, Y, Z \in [[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}$.

Likewise, $([[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \wedge, \cap)$ is a complete and commutative l-monoid as:

1) $([[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \wedge)$ is a complete lattice (see Proposition 5.2.37).

2) $([[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}, \cap)$ is a commutative monoid with neutral element $i_\cap = A \cup B$:

(i) $\cap$ is associative;

(ii) $\cap$ has the neutral element $i_\cap = A \cup B \in [[A,B]]_{\precsim_{\approx\gamma^-,\gamma^+}}$:
$X \cap (A \cup B) = (A \cup B) \cap X = X$, $\forall X \in [[A,B]]_{\precsim_{\approx\gamma^-,\gamma^+}}$;

(iii) $\cap$ is commutative.

3) $\cap$ is left and right distributive over $\vee$ and $\wedge$:

(i) $X \cap (Y \vee Z) = (X \cap Y) \vee (X \cap Z), \forall X,Y,Z \in [[A,B]]_{\precsim_{\approx\gamma^-,\gamma^+}}$;

(ii) $(X \vee Y) \cap Z = (X \cap Z) \vee (Y \cap Z), \forall X,Y,Z \in [[A,B]]_{\precsim_{\approx\gamma^-,\gamma^+}}$;

(iii) $X \cap (Y \wedge Z) = (X \cap Y) \wedge (X \cap Z), \forall X,Y,Z \in [[A,B]]_{\precsim_{\approx\gamma^-,\gamma^+}}$;

(iv) $(X \wedge Y) \cap Z = (X \cap Z) \wedge (Y \cap Z), \forall X,Y,Z \in [[A,B]]_{\precsim_{\approx\gamma^-,\gamma^+}}$.

□

Proposition 5.3.4. *Let be $C, D \in \mathcal{K}_\mathcal{C}$, such that $C \cap D \neq \varnothing$, then the structures $([[C,D]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \cap, \vee)$ and $([[C,D]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \cap, \wedge)$ are commutative complete lattice-ordered monoids.*

Proof. Considering the case in which $C \subseteqq_{\gamma^-,\gamma^+} D$ (the other cases are analogous), we have that $([[C,D]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \cap, \vee)$ is a complete and commutative l-monoid as:

1) $([[C,D]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \cap)$ is a complete lattice, according to Proposition 5.2.37.

2) $([[C,D]]_{\subseteqq_{\gamma^-,\gamma^+}}, \vee)$ is a commutative monoid with neutral element $i_\vee = C \wedge D$:

(i) $\vee$ is associative;

(ii) $\vee$ has the neutral element $i_\vee = C \wedge D \in [[C,D]]_{\subseteqq_{\gamma^-,\gamma^+}}$:
$X \vee (C \wedge D) = (C \wedge D) \vee X = X$, $\forall X \in [[C,D]]_{\subseteqq_{\gamma^-,\gamma^+}}$;

(iii) $\vee$ is commutative.

3) $\vee$ is left and right distributive over $\cup$ and $\cap$:

(i) $X \vee (Y \cup Z) = (X \vee Y) \cup (X \vee Z), \forall X,Y,Z \in [[C,D]]_{\subseteqq_{\gamma^-,\gamma^+}}$;

(ii) $(X \cup Y) \vee Z = (X \vee Z) \cup (Y \vee Z), \forall X,Y,Z \in [[C,D]]_{\subseteqq_{\gamma^-,\gamma^+}}$;

(iii) $X \vee (Y \cap Z) = (X \vee Y) \cap (X \vee Z), \forall X,Y,Z \in [[C,D]]_{\subseteqq_{\gamma^-,\gamma^+}}$;

(iv) $(X \cap Y) \vee Z = (X \vee Z) \cap (Y \vee Z), \forall X,Y,Z \in [[C,D]]_{\subseteqq_{\gamma^-,\gamma^+}}$.

Likewise, $([[C,D]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \cap, \wedge)$ is a complete and commutative l-monoid as:

1) $([[C,D]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \cap)$ is a complete lattice (see Proposition 5.2.37).

2) $([[C,D]]_{\subseteqq_{\gamma^-,\gamma^+}},\wedge)$ is a commutative monoid with neutral element $i_\wedge = C \vee D$:

   (i) $\wedge$ is associative;

   (ii) $\wedge$ has the neutral element $i_\wedge = C \vee D \in [[C,D]]_{\subseteqq_{\gamma^-,\gamma^+}}$:
   $X \wedge (C \vee D) = (C \vee D) \wedge X = X,\ \forall X \in [[C,D]]_{\subseteqq_{\gamma^-,\gamma^+}}$;

   (iii) $\wedge$ is commutative.

3) $\wedge$ is left and right distributive over $\cup$ and $\cap$:

   (i) $X \wedge (Y \cup Z) = (X \wedge Y) \cup (X \wedge Z), \forall X,Y,Z \in [[C,D]]_{\subseteqq_{\gamma^-,\gamma^+}}$;

   (ii) $(X \cup Y) \wedge Z = (X \wedge Z) \cup (Y \wedge Z), \forall X,Y,Z \in [[C,D]]_{\subseteqq_{\gamma^-,\gamma^+}}$;

   (iii) $X \wedge (Y \cap Z) = (X \wedge Y) \cap (X \wedge Z), \forall X,Y,Z \in [[C,D]]_{\subseteqq_{\gamma^-,\gamma^+}}$;

   (iv) $(X \cap Y) \wedge Z = (X \wedge Z) \cap (Y \wedge Z), \forall X,Y,Z \in [[C,D]]_{\subseteqq_{\gamma^-,\gamma^+}}$.

□

Proposition 5.3.5. *Let be $A,B \in \mathcal{K_C}$, such that $A \cap B \neq \varnothing$, then the structure $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}},\vee,\wedge,\cup,\cap,\precapprox_{\gamma^-,\gamma^+})$ is a commutative clodum (or complete lattice-ordered double monoid) .*

Proof. Considering the case in which $A \precapprox_{\gamma^-,\gamma^+} B$ such that $A \cap B \neq \varnothing$ (the other cases are analogous), we have that $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}},\vee,\wedge,\cup,\cap,\precapprox_{\gamma^-,\gamma^+})$ is a commutative clodum, as:

1a) $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}},\vee,\wedge)$ is a complete lattice whose associated partial order is $\precapprox_{\gamma^-,\gamma^+}$ (see Proposition 5.2.37).

2a) $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}},\cup)$ is a commutative monoid with neutral element $i_\cup = A \cap B$:

   (i) $\cup$ is associative;

   (ii) $\cup$ has the neutral element $i_\cup = A \cap B \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}$:
   $X \cup (A \cap B) = (A \cap B) \cup X = X,\ \forall X \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}$;

   (iii) $\cup$ is commutative.

2a') $\cup$ is left and right distributive over $\vee$:

   (i) $X \cup (Y \vee Z) = (X \cup Y) \vee (X \cup Z), \forall X,Y,Z \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}$;

   (ii) $(X \vee Y) \cup Z = (X \cup Z) \vee (Y \cup Z), \forall X,Y,Z \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}$.

3a) $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}},\cap)$ is a commutative monoid with neutral element $i_\cap = A \cup B$:

(i) $\cap$ is associative;

(ii) $\cap$ has the neutral element $i_\cap = A \cup B \in [[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}$:
$X \cap (A \cup B) = (A \cup B) \cap X = X$, $\forall X \in [[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}$;

(iii) $\cap$ is commutative.

3a') $\cap$ is left and right distributive over $\wedge$:

(i) $X \cap (Y \wedge Z) = (X \cap Y) \wedge (X \cap Z), \forall X, Y, Z \in [[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}$;

(ii) $(X \wedge Y) \cap Z = (X \cap Z) \wedge (Y \cap Z), \forall X, Y, Z \in [[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}$.

□

Proposition 5.3.6. *Let be $C, D \in \mathcal{K}_\mathcal{C}$, such that $C \cap D \neq \varnothing$, then the structure $([[C, D]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \cap, \vee, \wedge, \subseteqq_{\gamma^-,\gamma^+})$ is a commutative clodum (or complete lattice-ordered double monoid) .*

Proof. Considering the case in which $C \subseteqq_{\gamma^-,\gamma^+} D$ (the other cases are analogous), we have that $([[C, D]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \cap, \vee, \wedge, \subseteqq_{\gamma^-,\gamma^+})$ is a commutative clodum, as:

1b) $([[C, D]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \cap)$ is a complete lattice whose associated partial order is $\subseteqq_{\gamma^-,\gamma^+}$ (see Proposition 5.2.37).

2b) $([[C, D]]_{\subseteqq_{\gamma^-,\gamma^+}}, \vee)$ is a commutative monoid with neutral element $i_\vee = C \wedge D$:

(i) $\vee$ is associative;

(ii) $\vee$ has the neutral element $i_\vee = C \wedge D \in [[C, D]]_{\subseteqq_{\gamma^-,\gamma^+}}$:
$X \vee (C \wedge D) = (C \wedge D) \vee X = X$, $\forall X \in [[C, D]]_{\subseteqq_{\gamma^-,\gamma^+}}$;

(iii) $\vee$ is commutative.

2b') $\vee$ is left and right distributive over $\cup$:

(i) $X \vee (Y \cup Z) = (X \vee Y) \cup (X \vee Z), \forall X, Y, Z \in [[C, D]]_{\subseteqq_{\gamma^-,\gamma^+}}$;

(ii) $(X \cup Y) \vee Z = (X \vee Z) \cup (Y \vee Z), \forall X, Y, Z \in [[C, D]]_{\subseteqq_{\gamma^-,\gamma^+}}$.

3b) $([[C, D]]_{\subseteqq_{\gamma^-,\gamma^+}}, \wedge)$ is a commutative monoid with neutral element $i_\wedge = C \vee D$:

(i) $\wedge$ is associative;

(ii) $\wedge$ has the neutral element $i_\wedge = C \vee D \in [[C, D]]_{\subseteqq_{\gamma^-,\gamma^+}}$:
$X \wedge (C \vee D) = (C \vee D) \wedge X = X$, $\forall X \in [[C, D]]_{\subseteqq_{\gamma^-,\gamma^+}}$;

(iii) $\wedge$ is commutative.

3b') $\wedge$ is left and right distributive over $\cap$:

Type	**Structure**	**Properties**
l-semigroup	$([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \wedge, \boxplus)$	C, E
l-semigroup	$([[C,D]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \cap, \boxplus)$	C, E
l-monoid	$([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \wedge, \cup)$	C, E
l-monoid	$([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \wedge, \cap)$	C, E
l-monoid	$([[C,D]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \cap, \vee)$	C, E
l-monoid	$([[C,D]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \cap, \wedge)$	C, E
Clodum	$([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \vee, \wedge, \cup, \cap)$	C
Clodum	$([[C,D]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \cap, \vee, \wedge)$	C

Table 5.6: Classification of interval lattice-ordered structures. C = commutative, E = complete.

(i) $X \wedge (Y \cap Z) = (X \wedge Y) \cap (X \wedge Z), \forall X, Y, Z \in [[C,D]]_{\subseteqq_{\gamma^-,\gamma^+}}$;

(ii) $(X \cap Y) \wedge Z = (X \wedge Z) \cap (Y \wedge Z), \forall X, Y, Z \in [[C,D]]_{\subseteqq_{\gamma^-,\gamma^+}}$.

□

Table 5.6 summarizes the different types of interval structures we have defined in this subsection and the properties associated with.

Chapter 6

An additional interpretation of interval structures

In this Chapter we continue the work undertaken in Chapter 5, proposing further interpretations of interval structures. We will first focus on structures of the Boolean type, expanding what has already been briefly anticipated, in order to define the most common Boolean algebraic structures from an interval point of view .

More specifically, following the study of complementation properties, through the use of an innovative model and the related graphical representation, we will be able to configure also interval-type Boolean structures (such as interval Boolean lattices, interval Boolean algebras and interval Boolean rings).

After that, thanks to an ingenious definition of the equivalence relation between intervals, we will shift the attention to the concept of quotient set; indeed, the construction of an interval quotient set will be proposed, thanks to which it will be possible to determine even more solid structures, up to providing an example of an interval quotient pseudoring.

6.1 Interval Boolean structures

We recall that three types of structures are defined whose name refers to that of George Boole: we have Boolean lattices, which are by definition distributive and complemented lattices, Boolean rings, i.e., rings whose elements are all idempotents, and finally Boolean algebras, particular algebraic structures that we have already introduced in Definition 4.1.9 but we will better redefine later on. What links these objects together is the fact that, defining a structure of one of these three types (Boolean ring, Boolean lattice, Boolean algebra) is equivalent to defining one of each of the other two types; in this way the study of Boolean rings, that of Boolean algebras and that of Boolean lattices is completely equivalent.

Using an innovative model and its graphical representation, what we intend to do is to carry out an accurate study of these structures from an interval point of view.

6.1.1 Complementation

In Section 5.1 relevant properties have been analized referring to various cases of interval semirings but nothing has been said about complementation. We remember the following definition.

Definition 6.1.1. *In a Semiring* $(S, +, \cdot, 0, 1)$*, an element* $x \in S$ *is complemented if and only if* $\exists x^c \in S$ *such that* $x + x^c = x^c + x = 1$ *and* $x \cdot x^c = x^c \cdot x = 0$.

Analizing the cases considered in Subsection 5.1.2 (summarized in Table 5.2), we have that:

1.1 $X \in \overline{\mathcal{K}_\mathcal{C}}$ is complemented in $(\overline{\mathcal{K}_\mathcal{C}}, \vee, \wedge; -\infty, +\infty, \precsim_{\gamma^-,\gamma^+})$ iff $\exists X^c \in \overline{\mathcal{K}_\mathcal{C}}$ such that $X \vee X^c = +\infty \equiv (+\infty; 0)$ and $X \wedge X^c = -\infty \equiv (-\infty; 0)$; as $-\infty \vee +\infty = +\infty$ and $-\infty \wedge +\infty = -\infty$, it results that only $-\infty$ and $+\infty$ are complemented, with complement $+\infty$ and $-\infty$, respectively.

1.2 $X \in \overline{\mathcal{K}_\mathcal{C}}$ is complemented in $(\overline{\mathcal{K}_\mathcal{C}}, \wedge, \vee; +\infty, -\infty, \precsim_{\gamma^-,\gamma^+})$ iff $\exists X^c \in \overline{\mathcal{K}_\mathcal{C}}$ such that $X \wedge X^c = -\infty \equiv (-\infty; 0)$ and $X \vee X^c = +\infty \equiv (+\infty; 0)$; as $+\infty \wedge -\infty = -\infty$ and $+\infty \wedge -\infty = +\infty$, it results that only $+\infty$ and $-\infty$ are complemented, with complement $-\infty$ and $+\infty$, respectively.

1.3 $X \in \mathcal{K}_\mathcal{C}^{\varnothing\mathbb{R}}$ is complemented in $(\mathcal{K}_\mathcal{C}^{\varnothing\mathbb{R}}, \uplus, \cap; \varnothing, \mathbb{R}, \subseteqq_{\gamma^-,\gamma^+})$ iff $\exists X^c \in \mathcal{K}_\mathcal{C}^{\varnothing\mathbb{R}}$ such that $X \uplus X^c = \mathbb{R} \equiv (0; +\infty)$ and $X \cap X^c = \varnothing \equiv (0; -\infty)$; as $\varnothing \uplus \mathbb{R} = \mathbb{R}$ and $\varnothing \cap \mathbb{R} = \varnothing$, it results that only $\varnothing$ and $\mathbb{R}$ are complemented, with complement $\mathbb{R}$ and $\varnothing$, respectively.

1.4 $X \in \mathcal{K}_\mathcal{C}^{\varnothing\mathbb{R}}$ is complemented in $(\mathcal{K}_\mathcal{C}^{\varnothing\mathbb{R}}, \cap, \uplus; \mathbb{R}, \varnothing, \subseteqq_{\gamma^-,\gamma^+})$ iff $\exists X^c \in \mathcal{K}_\mathcal{C}^{\varnothing\mathbb{R}}$ such that $X \cap X^c = \varnothing \equiv (0; -\infty)$ and $X \uplus X^c = \mathbb{R} \equiv (0; +\infty)$;

as $\mathbb{R} \cap \varnothing = \varnothing$ and $\mathbb{R} \uplus \varnothing = \mathbb{R}$, it results that only $\mathbb{R}$ and $\varnothing$ are complemented, with complement $\varnothing$ and $\mathbb{R}$, respectively.

1.5 $X \in \mathcal{K}_{\mathcal{C}}^{-\infty}$ is complemented in $(\mathcal{K}_{\mathcal{C}}^{-\infty}, \vee, \oplus; -\infty, 0, \precapprox_{\gamma^-,\gamma^+})$ iff $\exists X^c \in \mathcal{K}_{\mathcal{C}}^{-\infty}$ such that $X \vee X^c = 0 \equiv (0;0)$ and $X \oplus X^c = -\infty \equiv (-\infty;0)$; as $-\infty \vee 0 = 0$ and $-\infty \oplus 0 = -\infty$, it results that only $-\infty$ and 0 are complemented, with complement 0 and $-\infty$, respectively.

1.6 $X \in \mathcal{K}_{\mathcal{C}}^{+\infty}$ is complemented in $(\mathcal{K}_{\mathcal{C}}^{+\infty}, \wedge, \oplus; +\infty, 0, \precapprox_{\gamma^-,\gamma^+})$ iff $\exists X^c \in \mathcal{K}_{\mathcal{C}}^{+\infty}$ such that $X \wedge X^c = 0 \equiv (0;0)$ and $X \oplus X^c = +\infty \equiv (+\infty;0)$; as $+\infty \wedge 0 = 0$ and $+\infty \oplus 0 = +\infty$, it results that only $+\infty$ and 0 are complemented, with complement 0 and $+\infty$, respectively.

1.7 $X \in \mathcal{K}_{\mathcal{C}}^{\varnothing}$ is complemented in $(\mathcal{K}_{\mathcal{C}}^{\varnothing}, \uplus, \oplus; \varnothing, 0, \subseteqq_{\gamma^-,\gamma^+})$ iff $\exists X^c \in \mathcal{K}_{\mathcal{C}}^{\varnothing}$ such that $X \uplus X^c = 0 \equiv (0;0)$ and $X \oplus X^c = \varnothing \equiv (0;-\infty)$; as $\varnothing \uplus 0 = 0$ and $\varnothing \oplus 0 = \varnothing$, it results that only $\varnothing$ and 0 are complemented, with complement 0 and $\varnothing$, respectively.

Likewise, considering the structures analysed in Subsection 5.2.2 (summarized in Table 5.4), we have:

2.1 $X \in \overline{\mathcal{K}_{\mathcal{C}}}$ is complemented in $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \wedge; -\infty, +\infty, \precapprox_{\gamma^-,\gamma^+})$ iff $\exists X^c \in \overline{\mathcal{K}_{\mathcal{C}}}$ such that $X \vee X^c = +\infty \equiv (+\infty;0)$ and $X \wedge X^c = -\infty \equiv (-\infty;0)$; as $-\infty \vee +\infty = +\infty$ and $-\infty \wedge +\infty = -\infty$, it results that only $-\infty$ and $+\infty$ are complemented, with complement $+\infty$ and $-\infty$, respectively.

2.2 $X \in \overline{\mathcal{K}_{\mathcal{C}}}$ is complemented in $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge, \vee; +\infty, -\infty, \precapprox_{\gamma^-,\gamma^+})$ iff $\exists X^c \in \overline{\mathcal{K}_{\mathcal{C}}}$ such that $X \wedge X^c = -\infty \equiv (-\infty;0)$ and $X \vee X^c = +\infty \equiv (+\infty;0)$; as $+\infty \wedge -\infty = -\infty$ and $+\infty \wedge -\infty = +\infty$, it results that only $+\infty$ and $-\infty$ are complemented, with complement $-\infty$ and $+\infty$, respectively.

2.3 $X \in \mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}$ is complemented in $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup, \sqcap; \varnothing, \mathbb{R}, \sqsubseteq_{\gamma^-,\gamma^+})$ iff $\exists X^c \in \mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}$ such that $X \sqcup X^c = \mathbb{R} \equiv (0;+\infty)$ and $X \sqcap X^c = \varnothing \equiv (0;-\infty)$; as $\varnothing \sqcup \mathbb{R} = \mathbb{R}$ and $\varnothing \sqcap \mathbb{R} = \varnothing$, it results that only $\varnothing$ and $\mathbb{R}$ are complemented, with complement $\mathbb{R}$ and $\varnothing$, respectively.

2.4 $X \in \mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}$ is complemented in $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcap, \sqcup; \mathbb{R}, \varnothing, \sqsubseteq_{\gamma^-,\gamma^+})$ iff $\exists X^c \in \mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}$ such that $X \sqcap X^c = \varnothing \equiv (0;-\infty)$ and $X \sqcup X^c = \mathbb{R} \equiv (0;+\infty)$; as $\mathbb{R} \sqcap \varnothing = \varnothing$ and $\mathbb{R} \sqcup \varnothing = \mathbb{R}$, it results that only $\mathbb{R}$ and $\varnothing$ are complemented, with complement $\varnothing$ and $\mathbb{R}$, respectively.

2.5 $X \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$ is complemented in $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \oplus; -\infty, 0, \precapprox_{\gamma^-,\gamma^+})$ iff $\exists X^c \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$ such that $X \vee X^c = 0 \equiv (0;0)$ and $X \oplus X^c = -\infty \equiv (-\infty;0)$; as $-\infty \vee 0 = 0$ and $-\infty \oplus 0 = -\infty$, it results that only $-\infty$ and 0 are complemented, with complement 0 and $-\infty$, respectively.

2.6 $X \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$ is complemented in $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge, \oplus; +\infty, 0, \precapprox_{\gamma^-,\gamma^+})$ iff $\exists X^c \in \overline{\mathcal{K}_{\mathcal{C}}^{\pm}}$ such that $X \wedge X^c = 0 \equiv (0;0)$ and $X \oplus X^c = +\infty \equiv (+\infty;0)$;
as $+\infty \wedge 0 = 0$ and $+\infty \oplus 0 = +\infty$, it results that only $+\infty$ and 0 are complemented, with complement 0 and $+\infty$, respectively.

2.7 $X \in \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$ is complemented in $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup, \oplus; \varnothing, 0, \sqsubseteqq_{\gamma^-,\gamma^+})$ iff $\exists X^c \in \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$ such that $X \sqcup X^c = 0 \equiv (0;0)$ and $X \oplus X^c = \varnothing \equiv (0;-\infty)$;
as $\varnothing \sqcup 0 = 0$ and $\varnothing \oplus 0 = \varnothing$, it results that only $\varnothing$ and 0 are complemented, with complement 0 and $\varnothing$, respectively.

2.8 $X \in \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$ is complemented in $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcap, \oplus; \mathbb{R}, 0, \sqsubseteqq_{\gamma^-,\gamma^+})$ iff $\exists X^c \in \mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}$ such that $X \sqcap X^c = 0 \equiv (0;0)$ and $X \oplus X^c = \mathbb{R} \equiv (0;+\infty)$;
as $\mathbb{R} \sqcap 0 = 0$ and $\mathbb{R} \oplus 0 = \mathbb{R}$, it results that only $\mathbb{R}$ and 0 are complemented, with complement 0 and $\mathbb{R}$, respectively.

In the end, with reference to the structures studied in the Subsection 5.2.3 (summarized in Table 5.5), the main cases considered are reported below (the other cases are completely analogous):

3.1 $X \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}$, with $A \precapprox_{\gamma^-,\gamma^+} B$ and $A \cap B \neq \varnothing$, is complemented in $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \wedge; A, B, \precapprox_{\gamma^-,\gamma^+})$ iff $\exists X^c \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}$ such that $X \vee X^c = B$ and $X \wedge X^c = A$;
as $A \vee B = B$ and $A \wedge B = A$, it results that only A and B are complemented, with complement B and A, respectively.

3.2 $X \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}$, with $A \precapprox_{\gamma^-,\gamma^+} B$ and $A \cap B \neq \varnothing$, is complemented in $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \vee; B, A, \precapprox_{\gamma^-,\gamma^+})$ iff $\exists X^c \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}$ such that $X \wedge X^c = A$ and $X \vee X^c = B$;
as $B \wedge A = A$ and $B \vee A = B$, it results that only B and A are complemented, with complement A and B, respectively.

3.3 $X \in [[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}$, with $A \subseteqq_{\gamma^-,\gamma^+} B$, is complemented in $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \cap; A, B, \subseteqq_{\gamma^-,\gamma^+})$ iff $\exists X^c \in [[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}$ such that $X \cup X^c = B$ and $X \cap X^c = A$;
as $A \cup B = B$ and $A \cap B = A$, it results that only A and B are complemented, with complement B and A, respectively.

3.4 $X \in [[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}$, with $A \subseteqq_{\gamma^-,\gamma^+} B$, is complemented in $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap, \cup; B, A, \subseteqq_{\gamma^-,\gamma^+})$ iff $\exists X^c \in [[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}$ such that $X \cap X^c = A$ and $X \cup X^c = B$;
as $B \cap A = A$ and $B \cup A = B$, it results that only B and A are complemented, with complement A and B, respectively.

3.5 $X \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}$, with $A \precapprox_{\gamma^-,\gamma^+} B$ and $A \cap B \neq \varnothing$, is complemented in $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \cup; A, A \cap B, \precapprox_{\gamma^-,\gamma^+})$ iff $\exists X^c \in [[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}$ such

that $X \vee X^c = A \cap B$ and $X \cup X^c = A$;
as $A \vee (A \cap B) = A \cap B$ and $A \cup (A \cap B) = A$, it results that only A and $A \cap B$ are complemented, with complement $A \cap B$ and A, respectively.

3.6 $X \in [[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}$, with $A \precapprox_{\gamma^-,\gamma^+} B$ and $A \cap B \neq \varnothing$, is complemented in $([[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \cup; B, A \cap B, \precapprox_{\gamma^-,\gamma^+})$ iff $\exists X^c \in [[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}$ such that $X \wedge X^c = A \cap B$ and $X \cup X^c = B$;
as $B \wedge (A \cap B) = A \cap B$ and $B \cup (A \cap B) = B$, it results that only B and $A \cap B$ are complemented, with complement $A \cap B$ and B, respectively.

3.7 $X \in [[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}$, with $A \precapprox_{\gamma^-,\gamma^+} B$ and $A \cap B \neq \varnothing$, is complemented in $([[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \cap; A, A \cup B, \precapprox_{\gamma^-,\gamma^+})$ iff $\exists X^c \in [[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}$ such that $X \vee X^c = A \cup B$ and $X \cap X^c = A$;
as $A \vee (A \cup B) = A \cup B$ and $A \cap (A \cup B) = A$, it results that only A and $A \cup B$ are complemented, with complement $A \cup B$ and A, respectively.

3.8 $X \in [[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}$, with $A \precapprox_{\gamma^-,\gamma^+} B$ and $A \cap B \neq \varnothing$, is complemented in $([[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \cap; B, A \cup B, \precapprox_{\gamma^-,\gamma^+})$ iff $\exists X^c \in [[A, B]]_{\precapprox_{\gamma^-,\gamma^+}}$ such that $X \wedge X^c = A \cup B$ and $X \cap X^c = B$;
as $B \wedge (A \cup B) = A \cup B$ and $B \cap (A \cup B) = B$, it results that only B and $A \cup B$ are complemented, with complement $A \cup B$ and B, respectively.

3.9 $X \in [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}$, with $A \subseteqq_{\gamma^-,\gamma^+} B$, is complemented in $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \vee; A, A \wedge B, \subseteqq_{\gamma^-,\gamma^+})$ iff $\exists X^c \in [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}$ such that $X \cup X^c = A \wedge B$ and $X \vee X^c = A$;
as $A \cup (A \wedge B) = A \wedge B$ and $A \vee (A \wedge B) = A$, it results that only A and $A \wedge B$ are complemented, with complement $A \wedge B$ and A, respectively.

3.10 $X \in [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}$, with $A \subseteqq_{\gamma^-,\gamma^+} B$, is complemented in $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap, \vee; B, A \wedge B, \subseteqq_{\gamma^-,\gamma^+})$ iff $\exists X^c \in [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}$ such that $X \cap X^c = A \wedge B$ and $X \vee X^c = B$;
as $B \cap (A \wedge B) = A \wedge B$ and $B \vee (A \wedge B) = B$, it results that only B and $A \wedge B$ are complemented, with complement $A \wedge B$ and B, respectively.

3.11 $X \in [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}$, with $A \subseteqq_{\gamma^-,\gamma^+} B$, is complemented in $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \wedge; A, A \vee B, \subseteqq_{\gamma^-,\gamma^+})$ iff $\exists X^c \in [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}$ such that $X \cup X^c = A \vee B$ and $X \wedge X^c = A$;
as $A \cup (A \vee B) = A \vee B$ and $A \wedge (A \vee B) = A$, it results that only A and $A \vee B$ are complemented, with complement $A \vee B$ and A, respectively.

3.12 $X \in [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}$, with $A \subseteqq_{\gamma^-,\gamma^+} B$, is complemented in $([[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap, \wedge; B, A \vee B, \subseteqq_{\gamma^-,\gamma^+})$ iff $\exists X^c \in [[A, B]]_{\subseteqq_{\gamma^-,\gamma^+}}$ such that $X \cap X^c = A \vee B$ and $X \wedge X^c = B$;
as $B \cap (A \vee B) = A \vee B$ and $B \wedge (A \vee B) = B$, it results that only B and $A \vee B$ are complemented, with complement $A \vee B$ and B, respectively.

We now consider the general concept of complementation with the structure of (algebraic) lattices (see [11] and [27]).

Let first recall the following statements.

Definition 6.1.2. *If* $(L, \vee, \wedge, 0, 1)$ *is a bounded lattice, we say that* $x^c \in L$ *is a complement of* $x \in L$ *if and only if*

$$x \wedge x^c = 0\ (= minL) \ \text{ and } \ x \vee x^c = \ 1\ (= maxL).$$

In this case x *is said to be a complemented element of* L *and* x^c *is its complement.*

It should be clear that, with the notations of Definition 6.1.2, saying that x^c is a complement of x is equivalent to saying that x is a complement of x^c. Clearly, every complement of a complemented element is itself complemented. Equally obvious is that $minL$ and $maxL$ are complementary to each other (indeed, $minL$ is the only complement of $maxL$ in L and, dually, $maxL$ is the only complement of $minL$ in L.)

Definition 6.1.3. *A lattice* L *is said to be complemented if and only if each of its elements has at least one complement in* L.

It is evident that, in order to be complemented, a lattice must be bounded, otherwise in it there can be no elements with complements.

Another algebraic property referred to lattices is distributivity.

Definition 6.1.4. *A lattice* $(L, \leq, \vee, \wedge)$ *is said to be distributive if and only if each of the two lattice operations is distributive with respect to the other. In more explicit terms,* $(L, \leq, \vee, \wedge)$ *is distributive if and only if,* $\forall x, y, z \in L$ *we have:*

(d1) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$;

(d2) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

Actually it is possible to prove that if in a lattice L at least one of the two conditions (d1) and (d2) is true for each triplet (x, y, z) of elements of L, then the other is also true and therefore L is distributive.

Instead, according to [24], it is a well known fact that neither the diamond lattices nor the pentagonal lattices are distributive. This fact follows from next result, as it is easy to verify that, in these two kind of lattices (diamond and pentagonal), there are elements with more complements.

Proposition 6.1.1. *Let* $(L, \leq, \vee, \wedge)$ *be a distributive lattice. Then each element of* L *has at most one complement in* L.

Proof. Suppose that in lattice L there are two element, x^{c_1} and x^{c_2}, which are both complements of $x \in L$. To prove the statement it is necessary (and that's enough) to check that $x^{c_2} = x^{c_1}$.
Since it has complements, the Lattice L must be bounded; therefore, indicating with 1 and 0, in order, the maximum and minimum of L, i.e., the neutral elements for $\wedge$ and $\vee$ respectively, we have that $x \wedge x^{c_1} = x \wedge x^{c_2} = 0$ and $x \vee x^{c_1} = x \vee x^{c_2} = 1$. Using the distributive property we have that:
$x^{c_1} = x^{c_1} \wedge 1 = x^{c_1} \wedge (x \vee x^{c_2}) = (x^{c_1} \wedge x) \vee (x^{c_1} \wedge x^{c_2}) = 0 \vee (x^{c_1} \wedge x^{c_2}) = x^{c_1} \wedge x^{c_2}$.
Similarly, by swapping the roles between x^{c_1} and x^{c_2}, we can get : $x^{c_2} = x^{c_1} \wedge x^{c_2}$. Thus $x^{c_1} = x^{c_2}$. □

It should be evident from Definition 6.1.4 that every sublattice of a distributive lattice is itself distributive. Consequently, a distributive lattice cannot have sublattices that are isomorphic to the diamond or pentagonal lattice. A remarkable result of lattice theory (see [11]) shows that the converse is also true: the absence of such sublattices is sufficient to prove that a lattice is distributive.

Theorem 6.1.1 (Birkhoff distribution criterion)**.** *Let L be a lattice. L is distributive if and only if it has no isomorphic sublattices to one between the diamond lattice and the pentagonal lattice.*

Another important consequence of Proposition 6.1.1 is the following result.

Proposition 6.1.2. *In a bounded distributive lattice, if the elements x and y have complements x^c and y^c, then De Morgan's identities are true:*

$$(x \vee y)^c = x^c \wedge y^c;$$

$$(x \wedge y)^c = x^c \vee y^c.$$

In general the (uniquely) complemented distributive lattices represent the heart of lattice theory with relevant applications.

In Section 4.1 we have verified that the four structures
$(\overline{\mathcal{K}_{\mathcal{C}}}, \vee, \wedge, -\infty, +\infty, \precapprox_{\gamma^-,\gamma^+})$, $(\overline{\mathcal{K}_{\mathcal{C}}}, \wedge, \vee, +\infty, -\infty, \precapprox_{\gamma^-,\gamma^+})$,
$(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}, \uplus, \cap, \varnothing, \mathbb{R}, \subseteqq_{\gamma^-,\gamma^+})$ and $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}, \cap, \uplus, \mathbb{R}, \varnothing, \subseteqq_{\gamma^-,\gamma^+})$
are distributive algebraic lattices and they are all bounded since, as shown in Subsection 5.1.2, they have a zero and a unit; so, according to Proposition 6.1.1, all the complements that exist are unique.

In a similar way it is possible to affirm that even the following structures (see Subsection 5.1.2) are bounded distributive lattices:

- $(\mathcal{K}_{\mathcal{C}}^{-\infty}, \vee, \oplus; -\infty, 0, \precapprox_{\gamma^-,\gamma^+})$,
- $(\mathcal{K}_{\mathcal{C}}^{+\infty}, \wedge, \oplus; +\infty, 0, \precapprox_{\gamma^-,\gamma^+})$,

- $(\mathcal{K}_{\mathcal{C}}^{\varnothing}, \uplus, \oplus; \varnothing, 0, \subseteqq_{\gamma^-,\gamma^+})$,

as well as (see Subsection 5.2.2)

- $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \wedge; -\infty, +\infty, \lessapprox_{\gamma^-,\gamma^+})$,
- $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge, \vee; +\infty, -\infty, \lessapprox_{\gamma^-,\gamma^+})$,
- $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup, \sqcap; \varnothing, \mathbb{R}, \sqsubseteq_{\gamma^-,\gamma^+})$,
- $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcap, \sqcup; \mathbb{R}, \varnothing, \sqsubseteq_{\gamma^-,\gamma^+})$,
- $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \vee, \oplus; -\infty, 0, \lessapprox_{\gamma^-,\gamma^+})$,
- $(\overline{\mathcal{K}_{\mathcal{C}}^{\pm}}, \wedge, \oplus; +\infty, 0, \lessapprox_{\gamma^-,\gamma^+})$,
- $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcup, \oplus; \varnothing, 0, \sqsubseteq_{\gamma^-,\gamma^+})$,
- $(\mathcal{K}_{\mathcal{C}}^{\pm\varnothing\mathbb{R}}, \sqcap, \oplus; \mathbb{R}, 0, \sqsubseteq_{\gamma^-,\gamma^+})$,

and also (see Subsection 5.2.3)

- $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \wedge; A \wedge B, A \vee B, \precapprox_{\gamma^-,\gamma^+})$,
- $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \vee; A \vee B, A \wedge B, \precapprox_{\gamma^-,\gamma^+})$,
- $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \cap; A \cap B, A \cup B, \subseteqq_{\gamma^-,\gamma^+})$,
- $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap, \cup; A \cup B, A \cap B, \subseteqq_{\gamma^-,\gamma^+})$

as well as

- $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \cup; A \wedge B, A \cap B, \precapprox_{\gamma^-,\gamma^+})$,
- $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \cup; A \vee B, A \cap B, \precapprox_{\gamma^-,\gamma^+})$,
- $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \vee, \cap; A \wedge B, A \cup B, \precapprox_{\gamma^-,\gamma^+})$,
- $([[A,B]]_{\precapprox_{\gamma^-,\gamma^+}}, \wedge, \cap; A \vee B, A \cup B, \precapprox_{\gamma^-,\gamma^+})$,
- $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \vee; A \cap B, A \wedge B, \subseteqq_{\gamma^-,\gamma^+})$,
- $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap, \vee; A \cup B, A \wedge B, \subseteqq_{\gamma^-,\gamma^+})$,
- $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cup, \wedge; A \cap B, A \vee B, \subseteqq_{\gamma^-,\gamma^+})$,
- $([[A,B]]_{\subseteqq_{\gamma^-,\gamma^+}}, \cap, \wedge; A \cup B, A \vee B, \subseteqq_{\gamma^-,\gamma^+})$.

Therefore also in these cases the possible complements are unique, as we have already anticipated at the beginning of this subsection, showing how only the minimum and maximum elements of each structures are complementary to each other.

6.1.2 Extension of $\mathcal{K}_{\mathcal{C}}$

Taking into account what we have seen in Subsection 6.1.1, it emerges that, in order to determine a set whose elements are all equipped with complement, a further extension of $\mathcal{K}_{\mathcal{C}}$ is necessary.

We can denote the set:

$$\cup\mathcal{K}_{\mathcal{C}} = \{A | A = \bigcup_j A_j, A_j \in I\}$$

with $I = \{[a,b],]a,b[, [a,b[,]a,b],]-\infty,b],]-\infty,b[,]a,+\infty[, [a,+\infty[\mid a,b \in \mathbb{R}\}$

which correspond to the Power-set of the interval of real numbers $\mathcal{P}(\mathbb{R})$.

Therefore, we have

$$\cup\mathcal{K}_{\mathcal{C}} = \mathcal{P}(\mathbb{R}).$$

It follows that each interval $A = [a^-, a^+] = (\widehat{a}; \widetilde{a}) \in \cup\mathcal{K}_{\mathcal{C}}$.

Moreover, $\forall A = [a^-, a^+] = (\widehat{a}; \widetilde{a}) \in \cup\mathcal{K}_{\mathcal{C}}$ we can also define the correspondent interval

$$A^c =]-\infty, a^-[\cup]a^+, +\infty[\stackrel{def}{=} (\widehat{a}; -\widetilde{a}) \in \cup\mathcal{K}_{\mathcal{C}} \tag{6.1}$$

as shown in Figure 6.1.

We also denote $\varnothing = (0; -\infty)$ and $\mathbb{R} = (0; +\infty)$ as element of $\cup\mathcal{K}_{\mathcal{C}}$ such that

$$\varnothing = (0; -\infty) \stackrel{def}{=} inf \cup \mathcal{K}_C,$$

$$\mathbb{R} = (0; +\infty) \stackrel{def}{=} sup \cup \mathcal{K}_C.$$

Obviously we have:

$$\varnothing = \mathbb{R}^c \text{ and } \mathbb{R} = \varnothing^c.$$

Remark 6.1.1. *Note that, considering the graphical representation of $\cup\mathcal{K}_{\mathcal{C}}$, similarly to the case of the dual approach and the polar approach of $\mathcal{K}_{\mathcal{C}}^{\pm}$, analyzed respectively in Subsection 5.2.1 and in Subsection 5.2.2, the whole plane is occupied (even the negative half-plane) but, unlike the other cases, here the order is not taken into account.*

Moreover, we have that:

- *if $\widetilde{a} \geqslant 0$ then we represent $A = (\widehat{a}; \widetilde{a})$ as usual, in the upper half-plane (it correspond to the closed interval $[a^-, a^+]$);*

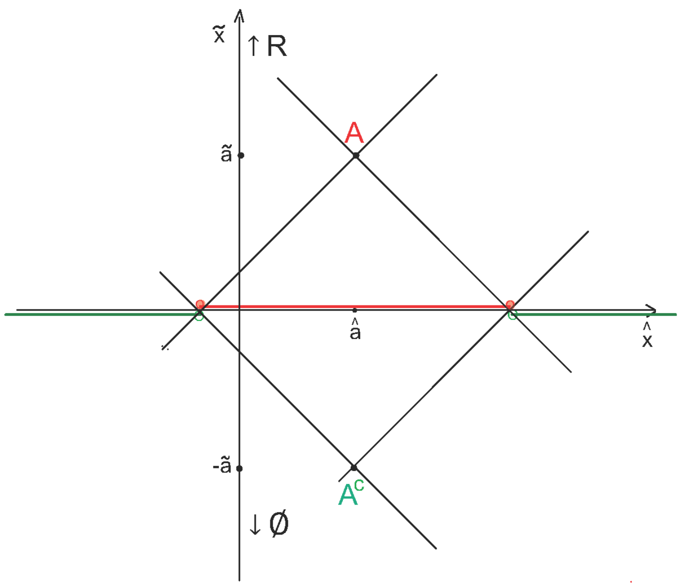


Figure 6.1: Representation of interval $A = [a^-, a^+] = (\widehat{a}; \widetilde{a}) \in \cup\mathcal{K}_\mathcal{C}$ and its complement $A^c =]-\infty, a^-[\cup]a^+, +\infty[= (\widehat{a}; -\widetilde{a}) \in \cup\mathcal{K}_\mathcal{C}$ in the midpoint plane $(\widehat{x}; \widetilde{x})$.

- *if* $\widetilde{a} \leqslant 0$, *we represent* $A = (\widehat{a}; \widetilde{a})$ *in the lower half-plane and decide to make it match to the open interval* $]-\infty, a^-[\cup]a^+, +\infty[$.

It is immediate to verify that:

- $A \cup A^C = [a^-, a^+] \cup (]-\infty, a^-[\cup]a^+, +\infty[) =]-\infty, +\infty[= \mathbb{R}$;
- $A \cap A^C = [a^-, a^+] \cap (]-\infty, a^-[\cup]a^+, +\infty[) = \varnothing$.

Furthermore, we can consider the two operations:

- $A \cup B$ as the usual union between intervals A and B;
- $A \cap B$ as the usual intersection between intervals A and B.

Both being binary operations, they can be applied to a pair of elements $X, Y \in \cup\mathcal{K}_\mathcal{C}$ to yeld again an element of $\cup\mathcal{K}_\mathcal{C}$. So we can define the internal operations:

$$\cap : \ \cup\mathcal{K}_\mathcal{C} \times \cup\mathcal{K}_\mathcal{C} \to \cup\mathcal{K}_\mathcal{C} \text{ such that: } (X, Y) \to X \cap Y;$$

$$\cup : \cup\mathcal{K}_\mathcal{C} \times \cup\mathcal{K}_\mathcal{C} \to \cup\mathcal{K}_\mathcal{C} \text{ such that: } (X, Y) \to X \cup Y.$$

It is possible to visualize the two operations in the various cases as shown in Figures 6.2 and 6.3.

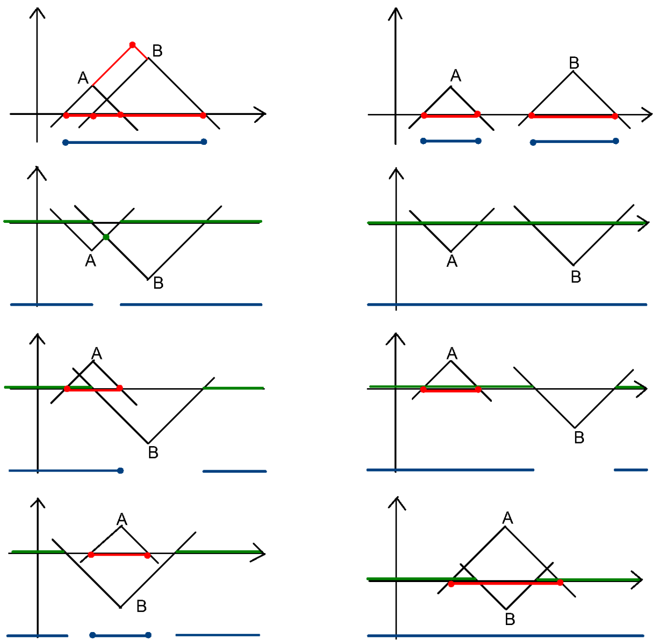


Figure 6.2: Union operation in the various cases in which the intervals A and B can occur in $\cup\mathcal{K}_{\mathcal{C}}$; the corresponding set operation is shown in blue below.

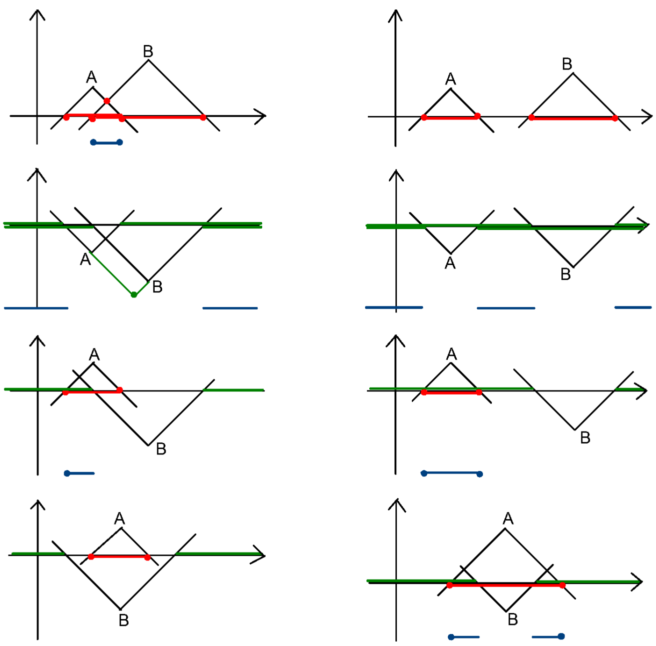


Figure 6.3: Intersection operation in the various cases in which the intervals A and B can occur in $\cup\mathcal{K}_{\mathcal{C}}$; the corresponding set operation is shown in blue below.

We can also define a relation of inclusion $\subseteq$ (see Figure 6.4) in a similar way to what was done in Section 4.1, referring to the $(\subseteqq_{-1,1})$-case. Therefore, it can be stated that:

i) Interval A dominates interval B (with respect to $\subseteq$) if and only if $A \subseteqq_{-1,1} B$;

ii) Interval A is dominated by interval C (with respect to $\subseteq$) if and only if $C \subseteqq_{-1,1} A$.

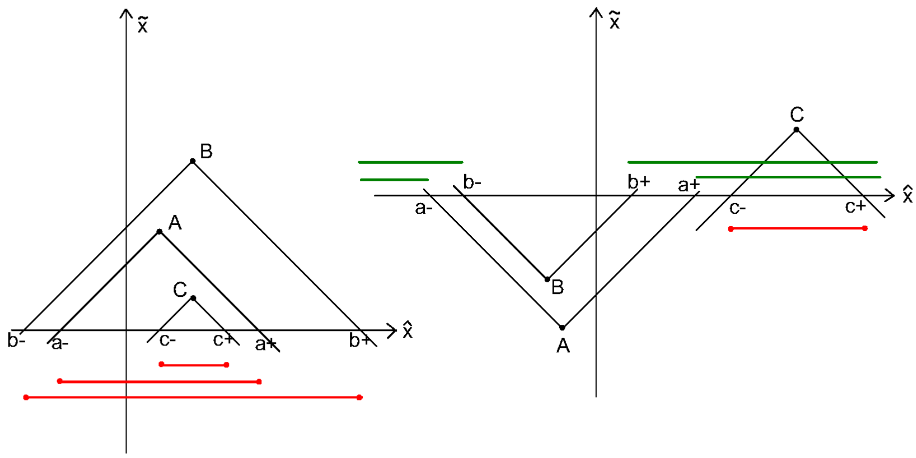


Figure 6.4: interval A dominates interval B ($A \subseteq B$) and is dominated by interval C ($C \subseteq A$) in both part of the picture.

It is easy now to verify the following properties:

1. $\subseteq$ is a partial order, as $\forall A, B, C \in \cup\mathcal{K}_\mathcal{C}$:

   - $\subseteq$ is reflexive: $A \subseteq A$;
   - $\subseteq$ is antisymmetric: $A \subseteq B$ and $B \subseteq A \Rightarrow A = B$;
   - $\subseteq$ is transitive: $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$;

   so the structure $(\cup\mathcal{K}_\mathcal{C}, \subseteq)$ is a poset.

2. $\forall A, B \in \cup\mathcal{K}_\mathcal{C}$ are defined:

   - $A \cap B \overset{def}{=} inf_{\subseteq}\{A, B\}$,
   - $A \cup B \overset{def}{=} sup_{\subseteq}\{A, B\}$,

   with $\varnothing = (0; -\infty) \overset{def}{=} inf_{\subseteq}\cup\mathcal{K}_C$ and $\mathbb{R} = (0; +\infty) \overset{def}{=} sup_{\subseteq}\cup\mathcal{K}_C$;

   so $(\cup\mathcal{K}_\mathcal{C}, \subseteq)$ is a complete lattice (exactly just like $(\mathcal{P}(\mathbb{R}), \subseteq)$ is too).

3. $\forall A, B, C \in \cup\mathcal{K}_{\mathcal{C}}$ we have that:

   - $\cap$ is left and right distributive over $\cup$:

     i $A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \forall A, B, C \in \cup\mathcal{K}_C$;

     ii $(A \cup B) \cap C = (A \cap C) \cup (B \cap C), \forall A, B, C \in \cup\mathcal{K}_C$;

   - $\cup$ is left and right distributive over $\cap$:

     i $A \cup (B \cap C) = (A \cup B) \cap (A \cup C), \forall A, B, C \in \cup\mathcal{K}_C$;

     ii $(A \cap B) \cup C = (A \cup C) \cap (B \cup C), \forall A, B, C \in \cup\mathcal{K}_C$.

As a result we have the following preposition.

Proposition 6.1.3. *The structure* $(\cup\mathcal{K}_{\mathcal{C}}, \cap, \cup, \subseteq)$ *is a complete, distributive lattice.*

Proposition 6.1.4. *If* $A = (\widehat{a}; \widetilde{a}) \in \cup\mathcal{K}_{\mathcal{C}}$*, then the interval* $A^c = (\widehat{a}; -\widetilde{a})$ *is the unique complement of* A *in* $\cup\mathcal{K}_{\mathcal{C}}$*.*

Proof. It is easy to verify that $A \cup A^c = \mathbb{R}$ and $A \cap A^c = \varnothing$. In addition, as $\cup\mathcal{K}_{\mathcal{C}}$ is a distributive lattice, from Proposition 6.1.1 the complement is unique. □

Remark 6.1.2. *In case* $\widetilde{a} \geqslant 0$ *we consider the interval* $A = (\widehat{a}; \widetilde{a})$ *from above, so if* $\widetilde{a} = 0$*, then the interval is reduced to a point as in the classical interval arithmetic. On the other hand, in case* $\widetilde{a} \leqslant 0$*, things are different because the interval is seen from below and this means that if* $\widetilde{a} = 0$*, the interval can be considered just as a single "hole" in the interval* $\mathbb{R}$ *of real numbers. So, it is possible to differentiate the two points by writing:*

> $A = (a; 0^+) = [a, a]$ *to consider the interval from above (as a point);*
>
> $A = (a; 0^-) =]-\infty, a[\cup]a, +\infty[$ *to indicate the same interval seen from below (as a hole).*

6.1.3 From Boolean lattices to interval Boolean lattices

Let recall the following definition (see [27]).

Definition 6.1.5. *A lattice is said to be Boolean if and only if it is distributive and complemented.*

This means that a Boolean lattice $(L, \vee, \wedge, 0, 1, \leq)$ is a distributive lattice in which, for each element $x \in L$ there exists a complement x^c such that:

- $x \wedge x^c = 0$;

- $x \vee x^c = 1$;

- $(x^c)^c = x$;
- $(x \wedge y)^c = x^c \vee y^c, \;\; \forall x, y \in L$;
- $(x \vee y)^c = x^c \wedge y^c, \;\; \forall x, y \in L$.

As a consequence of Definition 6.1.5 and Proposition 6.1.1, we have:

Proposition 6.1.5. *If L is a Boolean lattice, each element of L has one and only one complement in L.*

Now we try to apply these concepts to the interval case.

Proposition 6.1.6. *The structure $(\cup\mathcal{K}_{\mathcal{C}}, \cap, \cup, \subseteq)$ is a Boolean lattice.*

Proof. according to Definition 6.1.5, a lattice is Boolean if it is complemented and distributive. In our case we have that from Proposition 6.1.3, $\cup\mathcal{K}_{\mathcal{C}}$ is a complete distributive lattice; so being complete means that it also bounded. In addition, according to Proposition 6.1.4, in $\cup\mathcal{K}_{\mathcal{C}}$ every element has a complement.
Hence $\cup\mathcal{K}_{\mathcal{C}}$ is a bounded lattice in which every element has a complement, and this means exactly that $\cup\mathcal{K}_{\mathcal{C}}$ is a complemented lattice. This conclude the proof. □

Until now we have looked at lattices as special type of poset $(L, \leq)$ with $\vee$ and $\wedge$ defined by the relation $\leq$; however, according to Subsection 4.1.2, lattices can also be characterized as algebraic structures satisfying certain axiomatic identities. Since the two definitions are equivalent, lattice theory draws on both order theory and universal algebra.

Hence, now we are interested in considering the alternative method of studying lattices, which consist in starting with a set equipped with the $\vee$ and $\wedge$ operators and define the relation $\leq$ based on these operations.
In short, instead of the structure $(L, \vee, \wedge, \leq)$, we can consider the new one $(L, \vee, \wedge, 0, 1, (.)^c)$, called algebraic lattice.
In our case, instead of the structure $(\cup\mathcal{K}_{\mathcal{C}}, \cap, \cup, \subseteq)$, we consider

$$(\cup\mathcal{K}_{\mathcal{C}}, \cap, \cup, \varnothing, \mathbb{R}, (.)^c).$$

Therefore, according to Definition 4.1.6, we have the following result.

Proposition 6.1.7. *The structure $(\cup\mathcal{K}_{\mathcal{C}}, \cap, \cup, \varnothing, \mathbb{R}, (.)^c)$ is an algebraic lattice.*

Indeed, if we consider the operations defined above:

$\cap: \; \cup\mathcal{K}_{\mathcal{C}} \times \cup\mathcal{K}_{\mathcal{C}} \to \cup\mathcal{K}_{\mathcal{C}}$ such that: $(X, Y) \to X \cap Y$;

$\cup: \cup\mathcal{K}_{\mathcal{C}} \times \cup\mathcal{K}_{\mathcal{C}} \to \cup\mathcal{K}_{\mathcal{C}}$ such that: $(X, Y) \to X \cup Y$, it is trivial to verify that:

1) $\cup$ and $\cap$ are commutative:

   i $X \cup Y = Y \cup X$, $\forall X, Y \in \cup\mathcal{K}_{\mathcal{C}}$,
   ii $X \cap Y = Y \cap X$, $\forall X, Y \in \cup\mathcal{K}_{\mathcal{C}}$;

2) $\cup$ and $\cap$ are associative:

   i $X \cup (Y \cup Z) = (X \cup Y) \cup Z$, $\forall X, Y, Z \in \cup\mathcal{K}_{\mathcal{C}}$,
   ii $X \cap (Y \cap Z) = (X \cap Y) \cap Z, \forall X, Y, Z \in \cup\mathcal{K}_{\mathcal{C}}$;

3) the absortion laws hold:

   i $X \cup (X \cap Y) = X$,$\forall X, Y \in \cup\mathcal{K}_{\mathcal{C}}$,
   ii $X \cap (X \cup Y) = X$,$\forall X, Y \in \cup\mathcal{K}_{\mathcal{C}}$;

4) the idempotent laws hold:

   i $X \cup X = X$, $\forall X \in \cup\mathcal{K}_{\mathcal{C}}$,
   ii $X \cap X = X$, $\forall X \in \cup\mathcal{K}_{\mathcal{C}}$.

6.1.4 From Boolean algebras to interval Boolean algebras

As already mentioned in Section 4.1, the algebraic interpretation of lattices plays an essential role in universal algebra.

We can extend the Definition 6.1.5 (see [11]) and then rephrase Definition 4.1.9 as follows.

Definition 6.1.6. *A Boolean lattice is defined as any lattice L that is complemented and distributive. The set L, equipped with the two binary operations of supremum and infimum and with the unary operation of the complement, becomes an algebra. When so considered, a Boolean lattice is called a Boolean algebra.*

As we said, the notion of lattice can be given in purely algebraic terms, that is, exclusively in terms of operations, without referring to order relations: we are talking about lattices 'as algebraic structures'.

So, let $(L, \leq, \wedge, \vee, 0, 1, (.)^c)$ be an algebraic lattice structure; we want to analyse what conditions on operations must be imposed for lattice L to be Boolean.

First of all it turns out that, other than the commutative and associative properties and the laws of absorption, the distributive properties (of $\wedge$ with respect to $\vee$ and of $\vee$ with respect to $\wedge$) must also hold, which cause that the lattice L is distributive.

We then know from Lemma 4.1.1 that, as L is bounded, neutral elements for $\vee$ and $\wedge$ exist.
Finally, as we have seen in Subsection 6.1.1, in a Boolean lattice each element has a unique complement; then we can consider the application

$$(.)^c : L \longrightarrow L$$

which associates each $x \in L$ with its complement $x^c \in L$.
These considerations suggest the following definition:

Definition 6.1.7. *A Boolean algebra is an algebraic structure* $(L; \vee; \wedge; 0; 1;\ (.)^c)$, *where* $\vee$ *and* $\wedge$ *are binary operations, 0 and 1 are nullary operations and* $(.)^c$ *is unary operation, such that:*

(1) $(L; \vee; 0)$ *and* $(L; \wedge; 1)$ *are commutative monoids;*

(2) the laws of absorption are valid:
$\forall x, y \in L$, $x \vee (x \wedge y) = x = x \wedge (x \vee y)$;

(3) the distributive law applies to both operations:
$\vee$ *is distributive with respect to* $\wedge$,
$\wedge$ *and distributive with respect to* $\vee$;

(4) for each $x \in L$, $\exists x^c \in L$ *such that:*
$x \vee x^c = 1$,
$x \wedge x^c = 0$,

where we indicate with x^c *the image of* x *with respect to* $(.)^c$.

As stated above, each Boolean lattice gives rise to a Boolean algebra and, vice-versa, a Boolean algebra can always be regarded as a Boolean lattice. Indeed, we have that:
(1) and (2) express exactly the fact that $(L, \vee, \wedge, \leq)$ is a bounded lattice, with minimum 0 and maximum 1, as follows from Proposition 4.1.2 and Lemma 4.1.1;
(3) says that this lattice is distributive;
(4) guarantees that every element a of L has as its complement the element a^c of L.

So we can say that the notion of Boolean algebra is the 'purely algebraic' version of the notion of Boolean lattice.

Therefore, as for all types of algebraic structures, we have a notion of isomorphism among Boolean algebras. In fact, let us recall the following definition.

Definition 6.1.8. *An isomorphism from a Boolean algebra* $(L_1, \vee_1, \wedge_1, 0_1, 1_1, (.)^{c_1})$ *to a Boolean algebra* $(L_2, \vee_2, \wedge_2, 0_2, 1_2, (.)^{c_2}))$ *is a bijective application*

$$f : L_1 \longrightarrow L_2$$

that 'keeps the operations', such that, for each $x, y \in L_1$ *we have:*

(i) $f(x \vee_1 y) = f(x) \vee_2 f(y)$ *and* $f(x \wedge_1 y) = f(x) \wedge_2 f(y)$*;*

(ii) $f(0_1) = 0_2$ *and* $f(1_1) = f(1_2)$*;*

(iii) $f(x^{c_1}) = (f(x))^{c_2}$*.*

Specifically, we have that property (i) (i.e., f preserves lattice operations) is equivalent to the fact that the bijection f is an isomorphism of lattices. If this property is verified, then also (ii) and (iii) apply. Indeed, if f is an isomorphism of lattices from L_1 to L_2, then f must send the minimum 0_1 of L_1 into the minimum 0_2 of L_2 and, similarly, 1_1, which is the maximum of L_1, into the maximum 1_2 of L_2 . So is (ii).

Furthermore, for every $x \in L_1$, since x^{c_1} is a complement of x in L_1, its image $f(x^{c_1})$ must be a complement of $f(x)$ in L_2. But, since L_2 is Boolean, we have that $(f(x))^{c_2}$ is the only one complement of $f(x)$ in L_2, hence $f(x^{c_1}) = (f(x))^{c_2}$.

In conclusion, what we have verified is that the isomorphisms of Boolean algebras from L_1 to L_2 are all and only isomorphisms of lattices from L_1 to L_2. In particular, two Boolean algebras are isomorphic (like Boolean algebras) if and only if they are isomorphic as lattices.

At this point we can really conclude that the study of Boolean algebras is equivalent to the study of Boolean lattices.

The next sentence lists some identities that hold in Boolean algebras. The third is expressed by saying that the complement operation is involuntary, i.e., it coincides with the inverse application of itself (and, in particular, it is bijective); the last two properties are known as de Morgan's laws for Boolean algebras.

Proposition 6.1.8. *Let* $(L, \vee, \wedge, 0, 1, (.)^c)$ *be a Boolean algebra. Then, for each elements* $x, y \in L$ *we have:*

- $1 \vee x = 1$ *and* $0 \wedge x = 0$*;*
- $1^c = 0$ *and* $0^c = 1$*;*
- $(x^c)^c = x$*;*
- $(x \wedge y)^c = x^c \vee y^c$ *and* $(x \vee y)^c = x^c \wedge y^c$*.*

Finally, we can recall the principal properties of complementation in a Boolean algebra.

Theorem 6.1.2. *([11]) If L is a Boolean algebra then the following facts hold:*

(i) $\forall x, y \in L$, $(x \wedge y)^c = x^c \vee y^c$, $(x \vee y)^c = x^c \wedge y^c$ *(de Morgan's Laws);*

(ii) $\forall x, y \in L$, $x \leqslant y \Leftrightarrow x^c \geqslant y^c$*;*

(iii) $\forall x, y, z \in L,\ x \wedge y \leqslant z \Leftrightarrow x \leqslant z \vee y^c$;

(iv) $\forall x, y, z \in L,\ x \vee y \geqslant z \Leftrightarrow x \geqslant x \wedge y^c$.

Extending the theory to the interval case, which is what we set out to do, the following proposition holds.

Proposition 6.1.9. *The structure* $(\cup\mathcal{K}_{\mathcal{C}}, \cap, \cup, \varnothing, \mathbb{R}, (.)^c)$ *is a Boolean algebra.*

Indeed we have that:

1.a $(\cup\mathcal{K}_{\mathcal{C}}, \cup)$ is a commutative monoid as:

- $\cup$ is associative and commutative;
- $\cup$ has the neutral element $i_{\cup} = \varnothing = (0; -\infty)$ (called the $\cup$- identity): $X \cup \varnothing = X$, $\forall X \in \cup\mathcal{K}_{\mathcal{C}}$.

1.b $(\cup\mathcal{K}_{\mathcal{C}}, \cap)$ is a commutative monoid as:

- $\cap$ is associative and commutative;
- $\cap$ has the neutral element $i_{\cap} = \mathbb{R} = (0; +\infty)$ (called the $\cap$- identity): $X \cap \mathbb{R} = X$, $\forall X \in \cup\mathcal{K}_{\mathcal{C}}$.

2 Absortion laws hold, as $\forall X, Y \in \cup\mathcal{K}_{\mathcal{C}}$

- $X \cup (X \cap Y) = X$;
- $X \cap (X \cup Y) = X$.

3 Distributive laws hold, as:

- $\cap$ is left and right distributive over $\cup$;
- $\cup$ is left and right distributive over $\cap$.

4 For each element X of $\cup\mathcal{K}_{\mathcal{C}}$, exists a unique complement $X^c \in \cup\mathcal{K}_{\mathcal{C}}$ such that:

- $X \cup X^c = \mathbb{R}$;
- $X \cap X^c = \varnothing$.

6.1.5 From Boolean rings to interval Boolean rings

By definition (see [11]), a Boolean ring is a ring in which each element is idempotent under multiplication, i.e., $x^2 = x$ for all element x of the Boolean ring.

Proposition 6.1.10. *Let R be a Boolean ring. Then R is commutative and, if $|R| > 1$, then R has characteristic 2.*

Indeed, $\forall x, y \in R$ we have:
$x + y = (x + y)^2 = x^2 + xy + yx + y^2 = x + xy + yx + y$, which implies that $xy + yx = 0$. If we set $x = y = 1$, then we have: $1 + 1 = 1 + 1 + 1 + 1$ which implies $1 + 1 = 0$. Thus, the characteristic of R is at most 2. Since R is not trivial, we have $1 \neq 0$; thus we obtain char(R) = 2.

Now, char(R) = 2 implies that $xy + xy = 0$; this, with the fact that $xy + yx = 0$, implies that $xy = yx$, i.e., R is commutative.

The following fact is well known.

Theorem 6.1.3 (Stone)**.** *Let R be a Boolean ring. Then:*

(i) there exists a set S such that R is isomorphic to a subring of $(\mathcal{P}(S), \triangle, \cap)$;

(ii) if R is finite, there exists a set S such that R is isomorphic to $(\mathcal{P}(S), \triangle, \cap)$;

where $\mathcal{P}(S)$ is the power set of S.

It should be noted, with regard to point (i), that all subrings of $(\mathcal{P}(S), \triangle, \cap)$ are Boolean rings.

At this point, before continuing, we want to recall the fact that the notions of a Boolean ring and a Boolean lattice (i.e., Boolean algebra) are essentially interchangeable, in the sense that a Boolean lattice structure can be constructed on each Boolean ring and, vice-versa, a Boolean ring structure on each Boolean lattice, so that these two constructions are inverse of each other.

In particular, starting from a Boolean ring $(R, +, \cdot)$ we want to define a Boolean lattice structure on R and the example of the ring of the power set can suggest how to proceed.

Fixed a set S, in fact, $(\mathcal{P}(S), \triangle, \cap)$ is a Boolean ring while $(\mathcal{P}(S), \cup, \cap)$ is a Boolean lattice, with lattice operations $\cup$ and $\cap$. The second lattice operation is precisely the multiplication operation in the ring. Even the first lattice operation can be expressed in terms of the operations of the ring as for each $A, B \in \mathcal{P}(S)$ we have:

$$A \cup B = (A \triangle B) \cup (A \cap B) = (A \triangle B) \triangle (A \cap B).$$

Furthermore, the minimum and maximum of the lattice are $\varnothing$ and S, i.e., the zero and the unity of the ring, and each A of $\mathcal{P}(S)$ has, as complement in the lattice $(\mathcal{P}(S), \subseteq)$, the set

$$S \backslash A = S \triangle A = 1_{\mathcal{P}(S)} \triangle A$$

where $1_{\mathcal{P}(S)}$ stands for the unit S.

Turning now to an arbitrary Boolean ring $(R, +, \cdot, 0_R, 1_R)$, where 0_R and 1_R are the zero and the unit of the ring, the example of $\mathcal{P}(S)$ suggests defining the binary operation in R by setting, for each $a, b \in R$:

$$a \vee b = a + b + ab$$

and application

$$(.)^c : R \longrightarrow R \text{ such that } (.)^c : a \longrightarrow a^c = 1_R + a,$$

to be used as complementary unary operation.

Therefore, the following fact applies.

Proposition 6.1.11. *With the notations we have just stated, the structure* $(R, \vee, \cdot, 0_R, 1_R, (.)^c)$*, is a Boolean algebra.*

We now describe the inverse construction: a Boolean ring starting from a Boolean algebra.

Also in this case we let ourselves be guided by the example of the algebra of the power set of S: $(\mathcal{P}(S), \cup, \cap, \varnothing, S, (.)^c)$.

Considering the two binary operations of the (Boolean) ring $(\mathcal{P}(S), \triangle, \cap)$, the multiplication $\cap$ is already among the operations of Boolean algebra, while in order to express the addition (symmetric difference) using the operations of Boolean algebra, it is useful to observe that if A and B are parts of S, then

$$A \setminus B = A \cap (S \setminus B) = A \cap B^c.$$

Therefore, $A \triangle B$ can be written as

$$(A \setminus B) \cup (B \setminus A) = (A \cap B^c) \cup (B \cap A^c)$$

or also as

$$(A \cup B) \setminus (A \cap B) = (A \cup B) \cap (A \cap B)^c.$$

This example suggests two possible ways to define, in an arbitrary Boolean algebra $(L, \vee, \wedge, 0, 1, (.)^c)$, a binary operation of addition $+$ analogous to the symmetric difference in $\mathcal{P}(S)$.

The next lemma shows that these two possibilities lead to the same result.

Lemma 6.1.1. *Let* $(L, \vee, \wedge, 0, 1, (.)^c)$ *be a Boolean algebra. Then, for each* $a, b \in L$ *we have:* $(a \wedge b^c) \vee (a^c \wedge b) = (a \vee b) \wedge (a \wedge b)^c$.

Hence the following (well known) proposition follows.

Proposition 6.1.12. *Let* $(L, \vee, \wedge, 0, 1, (.)^c)$ *be a Boolean algebra. If* $+$ *is the binary operation definite in* L *by setting, for each* $a, b \in L$,

$$a + b = (a \wedge b^c) \vee (a^c \wedge b),$$

then $(L, +, \wedge)$ *is a Boolean ring, with zero* 0 *and unity* 1.

Thus we can say that every Boolean ring $(R, +, \cdot)$ determines a Boolean algebra structure on its own support: $(R, \vee, \cdot\ , 0_R, 1_R, (.)^c)$, defined as in

Proposition 6.1.11. By Proposition 6.1.12, this defines, in turn, a Boolean ring, that we indicate as $(R, \triangle, \cdot)$, with additive operation definite by

$$a \triangle b = (a \cdot b^c) \vee (a^c \cdot b) \in R, \ \forall a, b \in R.$$

Now, choose $a, b \in R$, since, according to Proposition 6.1.11 , for every $u, v \in R$, we have

$$u^c = 1_R + u, \quad u \cdot u^c = 0_R \ \text{ and } \ u \vee v = u + v + u \cdot v,$$

it follows that

$$(a \cdot b^c) \vee (a^c \cdot b) = (a \cdot b^c) + (a^c \cdot b) + (a \cdot b^c) \cdot (a^c \cdot b) = (a \cdot b^c) + (a^c \cdot b) + a \cdot a^c \cdot b \cdot b^c$$

$$= a \cdot (1_R + b) + (1_R + a) \cdot b + 0_R \cdot 0_R = a + a \cdot b + b + a \cdot b = a + b,$$

remembering that, being R Boolean, therefore of characteristic 2, we have $a \cdot b + a \cdot b = 0_R$.

Hence, the additive operation $\triangle$ of the Boolean ring constructed from $(R, \vee, \cdot, 0_R, 1_R, (.)^c)$ corresponds to the original addition in $(R, +, \cdot)$. This means that, given a Boolean ring R, if we construct a Boolean algebra on R as indicated in Proposition 6.1.11 and then, starting from the latter, we construct a Boolean ring as indicated in Proposition 6.1.12, this ring is precisely the ring R from which we started.

The same is true if we do the reverse. Indeed, if starting from a Boolean algebra $(L, \vee, \wedge, 0, 1, (.)^{c_1})$, we define the Boolean ring $(L, +, \wedge)$ as in Proposition 6.1.12 and then use Proposition 6.1.11 to construct a Boolean algebra $(L, \curlyvee, \wedge, 0, 1, (.)^{c_2})$ starting from this ring, then the algebra obtained is the original one. To prove it, just check that the operation $\curlyvee$ coincides with $\vee$ since, once this is established, we have the two Boolean lattice structures on L, the original $(L, \vee, \wedge)$ and the "new" $(L, \curlyvee, \wedge)$ coincide, so the same is true for the corresponding Boolean algebras. The fact that $\curlyvee$ coincides with $\vee$ can be easily proved using propositions 6.1.11 and 6.1.12 and the properties we have analized above.

In conclusion we can summarize what we have just shown in the following theorem.

Theorem 6.1.4. *Let L be a set. Let $\mathcal{A}$ be the set of ordered pairs $(\vee, \wedge)$ of binary operations in L that give L the structure of Boolean algebra, and let $\mathcal{B}$ be the set of ordered pairs $(\vee, \wedge)$ of binary operations in L which give L the structure of a Boolean ring. Then it is possible to define two applications, from $\mathcal{B}$ to $\mathcal{A}$ and from $\mathcal{A}$ to $\mathcal{B}$, which are inverse of each other, hence bijective.*

This correspondence between Boolean algebras and Boolean rings preserves the notion of isomorphism.

Proposition 6.1.13. *Let* $(L_1, \vee_1, \wedge_1, 0_1, 1_1, (.)^{c_1})$ *and* $(L_2, \vee_2, \wedge_2, 0_2, 1_2, (.)^{c_2})$ *be Boolean algebras and* $(L_1, +_1, \wedge_1)$ *and* $(L_2, +_2, \wedge_2)$ *the corresponding (in the sense of Theorem 6.1.4) Boolean rings. Let also*

$$f : L_1 \longrightarrow L_2$$

be a bijective application. Then f *is an isomorphism of Boolean algebras if and only if it is an isomorphism of Boolean rings.*

At this point we can conclude that the study of Boolean rings is equivalent to that of Boolean algebras and, therefore, to that of the Boolean lattices.

We also see that Boolean subalgebras correspond precisely to Boolean subrings.

Proposition 6.1.14. *Let* $(L, \vee, \wedge, 0, 1, (.)^c)$ *be a Boolean algebra and let* $(L, +, \wedge)$ *be the corresponding Boolean ring (in sense of Theorem 6.1.4). Let* $K \subseteq L$. *Then* K *is a Boolean subalgebra of* $(L, \vee, \wedge, 0, 1, (.)^c)$ *if and only if it is a subring of* $(L, +, \wedge)$.

From these last results and from Stone's theorem for Boolean rings (see Theorem 6.1.3), Stone's theorems for Boolean algebras and for Boolean lattices immediately follow.

Theorem 6.1.5 (Stone's theorem for Boolean algebras)**.** *Let* L *be a Boolean algebra. Then:*

(i) there exists a set S *such that* L *is isomorphic to a Boolean subalgebra of the algebra* $(\mathcal{P}(S), \cup, \cap, \varnothing, S, (.)^c)$ *of the power set of* S*;*

(ii) if L *is finite, there exists a set* S *such that* L *is isomorphic to the algebra* $(\mathcal{P}(S), \cup, \cap, \varnothing, S, (.)^c)$ *of the power set of* S.

Theorem 6.1.6 (Stone's theorem for Boolean lattices)**.** *Let* L *be a Boolean lattice. Then:*

(i) there exists a set S *such that* L *is isomorphic to a sublattice of the lattice* $(\mathcal{P}(S), \subseteq)$ *of the power set of* S*;*

(ii) if L *is finite, there exists a set* S *such that* L *is isomorphic to the lattice* $(\mathcal{P}(S), \subseteq)$ *of the power set of* S.

Now, returning to the interval case, we can define the binary operation in $\cup\mathcal{K}_{\mathcal{C}}$:

$$\triangle : \cup\mathcal{K}_{\mathcal{C}} \times \cup\mathcal{K}_{\mathcal{C}} \longrightarrow \cup\mathcal{K}_{\mathcal{C}}$$

such that

$$\triangle : (X, Y) \longrightarrow X \triangle Y \overset{def}{=} (X \cap Y^c) \cup (X^c \cap Y)$$

where $X = (\widehat{x}, \widetilde{x})$, $Y = (\widehat{y}, \widetilde{y})$, $X^c = (\widehat{x}, -\widetilde{x})$ and $Y^c = (\widehat{y}, -\widetilde{y})$ belong to $\cup\mathcal{K}_{\mathcal{C}}$.

According to proposition 6.1.12 and remembering that a Boolean ring is a ring in which each element is idempotent, the following fact holds.

Proposition 6.1.15. *The structure* $(\cup\mathcal{K}_{\mathcal{C}}, \triangle, \cap)$ *is a Boolean ring.*

Proof. We have that $(\cup\mathcal{K}_{\mathcal{C}}, \triangle, \cap)$ is a commutative ring as:

1) $(\cup\mathcal{K}_{\mathcal{C}}, \triangle)$ is an abelian group:

   1a $\triangle$ is associative and commutative;

   1b $\triangle$ has the neutral element $i_{\triangle} = \varnothing = (0; -\infty)$ (called the $\triangle$-identity) : $X \triangle \varnothing = X$, for each $X \in \cup\mathcal{K}_{\mathcal{C}}$,
   which represents the zero of the Boolean ring

   (so, $(\cup\mathcal{K}_{\mathcal{C}}, \triangle)$ is a commutative monoid);

   1c every element $X \in \cup\mathcal{K}_{\mathcal{C}}$ is the additive inverse (or symmetrical element) of itself: $X \triangle X = \varnothing$;

   (so, $(\cup\mathcal{K}_{\mathcal{C}}, \triangle)$ is an abelian group);

2) $(\cup\mathcal{K}_{\mathcal{C}}, \cap)$ is a commutative monoid:

   2a $\cap$ is associative;

   2b $\cap$ has the neutral element $i_{\cap} = \mathbb{R} = (0; +\infty)$ (called the $\cap$-identity) : $X \cap \mathbb{R} = X$, for each $X \in \cup\mathcal{K}_{\mathcal{C}}$,
   which represents the unity of the Boolean ring

   (so, $(\cup\mathcal{K}_{\mathcal{C}}, \cap)$ is a commutative monoid);

3) $\cap$ is left and right distributive over $\triangle$;

4) $\varnothing$ is the absorbing element for $\cap$: $X \cap \varnothing = X$, for each $X \in \cup\mathcal{K}_{\mathcal{C}}$

   (so, $(\cup\mathcal{K}_{\mathcal{C}}, \triangle, \cap)$ is a ring);

5) $\cap$ is commutative

   (so, $(\cup\mathcal{K}_{\mathcal{C}}, \triangle, \cap)$ is a commutative ring);

Furthermore, every element X of $\cup\mathcal{K}_{\mathcal{C}}$ is idempotent with respect to $\cap$: $X \cap X = X$

(this means that $(\cup\mathcal{K}_{\mathcal{C}}, \triangle, \cap)$ is a Boolean ring). □

6.2 Interval quotient set

According to [52] the passage from a set S to the quotient set $S/\sim$, being $\sim$ an equivalence relation, schematizes and specifies the process of formation of concepts starting from objects and, more generally, the ordinary process of abstraction, consisting in the identification of different elements but all with a common property.

In fact, it is a question of considering classes of elements (of S) as elements of a new set $S/\sim$; more specifically, the elements of S can be considered as given (objects), while the elements of $S/\sim$ can be considered as "conceptual abstractions" (classes of objects thought of as a single object). We can also say that the passage to the quotient set is the mathematization of the thought process that leads us to identify elements that can be replaced with one another in a given context.

Therefore, the mathematical concept of equivalence expands and clarifies the concept of "equality" of common language, highlighting its relative character, where absolute equality becomes a particular case of equivalence: identity.

So, what interests us is to construct a quotient set of the type $\mathcal{K_C}/\sim$, where $\sim$ is an equivalence relationship between intervals; then, the search for such a relationship is the first step to take.

6.2.1 The γ-equivalence relation

We recall that an equivalence relation in a set S is a relation that verifies the three formal properties: reflexive, symmetric and transitive. In order to find such a relation in $\mathcal{K_C}$, the best candidate seems to be the one defined below.

Definition 6.2.1. *Let A and B be two intervals in $\mathcal{K_C}$ such that $\widehat{a} \neq \widehat{b}$ and let γ be a real number. We say that A and B are in γ-relation and write*

$$A \sim_\gamma B \text{ (or, equivalentely } (\widehat{a};\widetilde{a}) \sim_\gamma (\widehat{b};\widetilde{b})) \text{ iff } \gamma = \frac{\widetilde{a}-\widetilde{b}}{\widehat{a}-\widehat{b}}.$$

This means that for all $\gamma \in \mathbb{R}$:

$$A \sim_\gamma B \Leftrightarrow \gamma = \frac{\widetilde{a}-\widetilde{b}}{\widehat{a}-\widehat{b}} \Leftrightarrow \widetilde{a}-\widetilde{b} = \gamma(\widehat{a}-\widehat{b}) \Leftrightarrow \widetilde{a}-\gamma\widehat{a} = \widetilde{b}-\gamma\widehat{b}$$

which is equivalent to say that γ represents the slope of the line through A and B.

In particular, considering the simplest case where $\gamma = \pm 1$ (the well known LU-case), it follows that two relations are possible:

1) $A \sim_1 B \Leftrightarrow (\widehat{a};\widetilde{a}) \sim_1 (\widehat{b};\widetilde{b}) \Leftrightarrow 1 = \dfrac{\widetilde{a}-\widetilde{b}}{\widehat{a}-\widehat{b}}$ (as represented in Figure 6.5);

2) $A \sim_{-1} B \Leftrightarrow (\widehat{a};\widetilde{a}) \sim_{-1} (\widehat{b};\widetilde{b}) \Leftrightarrow -1 = \dfrac{\widetilde{a}-\widetilde{b}}{\widehat{a}-\widehat{b}}$.

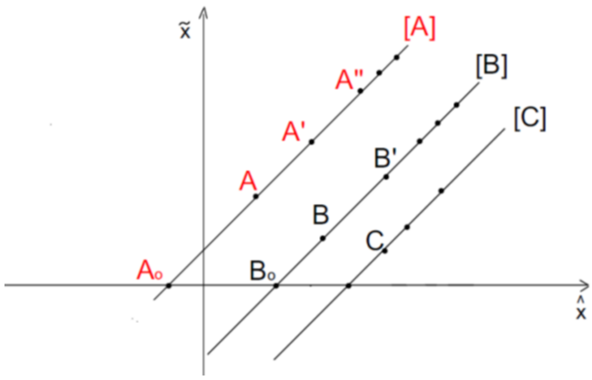


Figure 6.5: Intervals in γ-relation to each other belong to the same line which thus indicates an equivalence class (here is LU-case: $\gamma = 1$).

It is easy to proof the following propety.

Proposition 6.2.1. *The relation $\sim_\gamma$ of Definiton 6.2.1 is an equivalence relation in $\mathcal{K_C}$.*

Proof. The following properties are verified:

1) $A \sim_\gamma A$ (reflexive), as $\widetilde{a} - \gamma\widehat{a} = \widetilde{a} - \gamma\widehat{a}$, $\forall A \in \mathcal{K_C}$;

2) $A \sim_\gamma B \Leftrightarrow B \sim_\gamma A$ (symmetric), as $A \sim_\gamma B \Leftrightarrow \widetilde{a} - \gamma\widehat{a} = \widetilde{b} - \gamma\widehat{b} \Leftrightarrow \widetilde{b} - \gamma\widehat{b} = \widetilde{a} - \gamma\widehat{a} \Leftrightarrow B \sim_\gamma A$, $\forall A, B \in \mathcal{K_C}$;

3) if $A \sim_\gamma B$ and $B \sim_\gamma C$, then $A \sim_\gamma C$ (transitive), indeed if $\widetilde{a} - \gamma\widehat{a} = \widetilde{b} - \gamma\widehat{b}$ and $\widetilde{b} - \gamma\widehat{b} = \widetilde{c} - \gamma\widehat{c}$, then $\widetilde{a} - \gamma\widehat{a} = \widetilde{c} - \gamma\widehat{c}$, $\forall A, B, C \in \mathcal{K_C}$.

□

As a consequence we obtain that the set of all the equivalence classes $[A]_{\sim_\gamma}$ associated with the equivalence relation $\sim_\gamma$ is exactly the quotient set

$$\mathcal{K_C}/\sim_\gamma = \{[A]_{\sim_\gamma} |\ A \in \mathcal{K_C}\} = \{[(\widehat{a};\widetilde{a})]_{\sim_\gamma} |\ (\widehat{a};\widetilde{a}) \in \mathcal{K_C}.\} \tag{6.2}$$

Furthermore, we can decide to consider, as representative element of a class $[A]_{\sim_\gamma}$, the intersection point of the straight line with the horizontal axis, i.e., the interval denoted by $A_0 = \left(\widehat{a} - \dfrac{\widetilde{a}}{\gamma^+}; 0\right)$. So we have:

$$[A]_{\sim_\gamma} = [A_0]_{\sim_\gamma} = \left[\left(\widehat{a} - \frac{\widetilde{a}}{\gamma^+}; 0\right)\right]_{\sim_\gamma} \tag{6.3}$$

which, in the LU-case ($\gamma = 1$), it simply corresponds to

$$[A_0]_{\sim_1} = \left[\left(a^-; 0\right)\right]_{\sim_1}.$$

6.2.2 Interval quotient semirings

It is possible to extend the operations $\wedge$ and $\vee$ of $\mathcal{K}_{\mathcal{C}}$, introduced in Subsection 4.1.3, to all equivalence classes of the quotient-set $(\mathcal{K}_{\mathcal{C}}/\sim_\gamma)$; therefore, for all $[X]_{\sim_\gamma}, [Y]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}/\sim_\gamma$ we define:

$$\wedge_{\sim_\gamma} : (\mathcal{K}_{\mathcal{C}}/\sim_\gamma) \times (\mathcal{K}_{\mathcal{C}}/\sim_\gamma) \to (\mathcal{K}_{\mathcal{C}}/\sim_\gamma)$$

such that: $([X]_{\sim_\gamma}, [Y]_{\sim_\gamma}) \to [X]_{\sim_\gamma} \wedge_{\sim_\gamma} [Y]_{\sim_\gamma} \stackrel{def}{=} [X \wedge Y]_{\sim_\gamma}$,

$$\vee_{\sim_\gamma} : (\mathcal{K}_{\mathcal{C}}/\sim_\gamma) \times (\mathcal{K}_{\mathcal{C}}/\sim_\gamma) \to (\mathcal{K}_{\mathcal{C}}/\sim_\gamma)$$

such that: $([X]_{\sim_\gamma}, [Y]_{\sim_\gamma}) \to [X]_{\sim_\gamma} \vee_{\sim_\gamma} [Y]_{\sim_\gamma} \stackrel{def}{=} [X \vee Y]_{\sim_\gamma}$,

with operation $\vee$ and $\wedge$ defined as in (4.8) and (4.9) and where

$$[X]_{\sim_\gamma} \wedge_{\sim_\gamma} [Y]_{\sim_\gamma} \stackrel{def}{=} inf_{\sim_\gamma}\{[X]_{\sim_\gamma}, [Y]_{\sim_\gamma}\} \qquad (6.4)$$

and

$$[X]_{\sim_\gamma} \vee_{\sim_\gamma} [Y]_{\sim_\gamma} \stackrel{def}{=} sup_{\sim_\gamma}\{[X]_{\sim_\gamma}, [Y]_{\sim_\gamma}\}. \qquad (6.5)$$

In particular, when we consider the LU-relation ($\gamma = 1$), the situation obtained is well described in Figure 6.6.

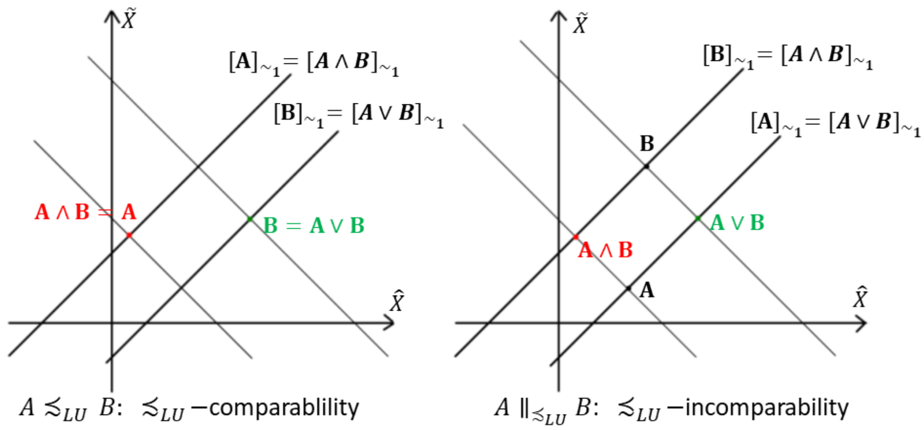


Figure 6.6: Operations $\wedge_{\sim_\gamma}$ and $\vee_{\sim_\gamma}$ in LU-case ($\gamma = 1$).

It will also be useful to consider, as usual, $\overline{\mathcal{K}_C} = \mathcal{K}_{\mathcal{C}} \cup \{-\infty, +\infty\}$, where

$$-\infty = (-\infty; 0) = inf_{\sim_\gamma}\overline{\mathcal{K}_C} \quad \text{and} \quad +\infty = (+\infty; 0) = sup_{\sim_\gamma}\overline{\mathcal{K}_C}$$

so that

$$\overline{\mathcal{K}_C}/\sim_\gamma = \mathcal{K}_\mathcal{C}/\sim_\gamma \cup \left\{[-\infty]_{\sim_\gamma}, [+\infty]_{\sim_\gamma}\right\}$$

with

$$[-\infty]_{\sim_\gamma} = [(-\infty;0)]_{\sim_\gamma} \stackrel{def}{=} inf_{\sim_\gamma}(\overline{\mathcal{K}_C}/\sim_\gamma)$$

and

$$[+\infty]_{\sim_\gamma} = [(+\infty;0)]_{\sim_\gamma} \stackrel{def}{=} sup_{\sim_\gamma}(\overline{\mathcal{K}_C}/\sim_\gamma).$$

Moreover, it is also possible to define $\mathcal{K}_\mathcal{C}/\sim_\gamma$ (or, depending on the case, $\overline{\mathcal{K}_\mathcal{C}}/\sim_\gamma$) as an algebraic structure with the following properties.

(1) $\vee_{\sim_\gamma}$ and $\wedge_{\sim_\gamma}$ are commutative, as $\forall[X]_{\sim_\gamma}, [Y]_{\sim_\gamma} \in \mathcal{K}_\mathcal{C}/\sim_\gamma$ we have:

(1.a) $[X]_{\sim_\gamma} \vee_{\sim_\gamma} [Y]_{\sim_\gamma} = [X \vee Y]_{\sim_\gamma} = [Y \vee X]_{\sim_\gamma} = [Y]_{\sim_\gamma} \vee_{\sim_\gamma} [X]_{\sim_\gamma}$;

(1.b) $[X]_{\sim_\gamma} \wedge_{\sim_\gamma} [Y]_{\sim_\gamma} = [X \wedge Y]_{\sim_\gamma} =_{\sim_\gamma} [Y \wedge X] = [Y]_{\sim_\gamma} \wedge_{\sim_\gamma} [X]_{\sim_\gamma}$.

(2) $\vee_{\sim_\gamma}$ and $\wedge_{\sim_\gamma}$ are associative, as $\forall[X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in \mathcal{K}_\mathcal{C}/\sim_\gamma$ we have:

(2.a) $([X]_{\sim_\gamma} \vee_{\sim_\gamma} [Y]_{\sim_\gamma}) \vee_{\sim_\gamma} [Z]_{\sim_\gamma} = ... = [(X \vee Y) \vee X]_{\sim_\gamma} = [X \vee (Y \vee Z)]_{\sim_\gamma} = ... = [X]_{\sim_\gamma} \vee_{\sim_\gamma} ([Y]_{\sim_\gamma} \vee_{\sim_\gamma} [Z]_{\sim_\gamma})$;

(2.b) $([X]_{\sim_\gamma} \wedge_{\sim_\gamma} [Y]_{\sim_\gamma}) \wedge_{\sim_\gamma} [Z]_{\sim_\gamma} = ... = [(X \wedge Y) \wedge X]_{\sim_\gamma} = [X \wedge (Y \wedge Z)]_{\sim_\gamma} = ... = [X]_{\sim_\gamma} \wedge_{\sim_\gamma} ([Y]_{\sim_\gamma} \wedge_{\sim_\gamma} [Z]_{\sim_\gamma})$.

(3) The absorption identities are satisfied for both $\vee_{\sim_\gamma}$ and $\wedge_{\sim_\gamma}$, as $\forall[X]_{\sim_\gamma}, [Y]_{\sim_\gamma} \in \mathcal{K}_\mathcal{C}/\sim_\gamma$, we have:

(3.a) $[X]_{\sim_\gamma} \vee \sim_\gamma([X]_{\sim_\gamma} \wedge \sim_\gamma[Y]_{\sim_\gamma}) = [X \vee (X \wedge Y)]_{\sim_\gamma} = [X]_{\sim_\gamma}$;

(3.b) $[X]_{\sim_\gamma} \wedge \sim_\gamma([X]_{\sim_\gamma} \vee \sim_\gamma[Y]_{\sim_\gamma}) = [X \wedge (X \vee Y)]_{\sim_\gamma} = [X]_{\sim_\gamma}$.

(4) The idempotency is satisfied for both $\vee_{\sim_\gamma}$ and $\wedge_{\sim_\gamma}$, as $\forall[X]_{\sim_\gamma} \in \mathcal{K}_\mathcal{C}/\sim_\gamma$ we have:

(4.a) $[X]_{\sim_\gamma} \vee \sim_\gamma[X]_{\sim_\gamma} = [X \vee X]_{\sim_\gamma} = [X]_{\sim_\gamma}$;

(4.b) $[X]_{\sim_\gamma} \wedge \sim_\gamma[X]_{\sim_\gamma} = [X \wedge X]_{\sim_\gamma} = [X]_{\sim_\gamma}$.

It is also easy to verify that:

(5.a) $\wedge_{\sim_\gamma}$ is left and right distributive over $\vee_{\sim_\gamma}$:

(5.a.1) $[X]_{\sim_\gamma} \wedge_{\sim_\gamma} ([Y]_{\sim_\gamma} \vee_{\sim_\gamma} [Z]_{\sim_\gamma}) = ([X]_{\sim_\gamma} \wedge_{\sim_\gamma} [Y]_{\sim_\gamma}) \vee_{\sim_\gamma} ([X]_{\sim_\gamma} \wedge_{\sim_\gamma} [Z]_{\sim_\gamma}), \forall[X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in \mathcal{K}_\mathcal{C}/\sim_\gamma$,

(5.a.2) $([X]_{\sim_\gamma} \vee_{\sim_\gamma} [Y]_{\sim_\gamma}) \wedge_{\sim_\gamma} [Z]_{\sim_\gamma} = ([X]_{\sim_\gamma} \wedge_{\sim_\gamma} [Z]_{\sim_\gamma}) \vee_{\sim_\gamma} ([Y]_{\sim_\gamma} \wedge_{\sim_\gamma} [Z]_{\sim_\gamma}), \forall[X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in \mathcal{K}_\mathcal{C}/\sim_\gamma$;

(5b) $\vee_{\sim_\gamma}$ is left and right distributive over $\wedge_{\sim_\gamma}$:

(5.b.1) $[X]_{\sim_\gamma} \vee_{\sim_\gamma} ([Y]_{\sim_\gamma} \wedge_{\sim_\gamma} [Z]_{\sim_\gamma}) = ([X]_{\sim_\gamma} \vee_{\sim_\gamma} [Y]_{\sim_\gamma}) \wedge_{\sim_\gamma} ([X]_{\sim_\gamma} \vee_{\sim_\gamma} [Z]_{\sim_\gamma}), \forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in \mathcal{K_C}/\sim_\gamma$,

(5.b.2) $([X]_{\sim_\gamma} \wedge_{\sim_\gamma} [Y]_{\sim_\gamma}) \vee_{\sim_\gamma} [Z]_{\sim_\gamma} = ([X]_{\sim_\gamma} \vee_{\sim_\gamma} [Z]_{\sim_\gamma}) \wedge_{\sim_\gamma} ([Y]_{\sim_\gamma} \vee_{\sim_\gamma} [Z]_{\sim_\gamma}), \forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in \mathcal{K_C}/\sim_\gamma$.

Other relevant properties are the following.

(6.a) $\vee_{\sim_\gamma}$ has a neutral element in $\overline{\mathcal{K_C}}/\sim_\gamma$ which is $i_{\vee_\sim} = [-\infty]_{\sim_\gamma} = [(-\infty;0)]_{\sim_\gamma}$; indeed, $[X]_{\sim_\gamma} \vee_{\sim_\gamma} [(-\infty;0)]_{\sim_\gamma} = [X \vee (-\infty;0)]_{\sim_\gamma} = [X]_{\sim_\gamma}, \forall [X]_{\sim_\gamma} \in \overline{\mathcal{K_C}}/\sim_\gamma$.

(6.b) $\wedge_{\sim_\gamma}$ has a neutral element in $\overline{\mathcal{K_C}}/\sim_\gamma$ which is $i_{\wedge_\sim} = [+\infty]_{\sim_\gamma} = [(+\infty;0)]_{\sim_\gamma}$; indeed, $[X]_{\sim_\gamma} \wedge_{\sim_\gamma} [(+\infty;0)]_{\sim_\gamma} = [X \wedge (+\infty;0)]_{\sim_\gamma} = [X]_{\sim_\gamma}, \forall [X]_{\sim_\gamma} \in \overline{\mathcal{K_C}}/\sim_\gamma$.

(7.a) $i_{\wedge_\sim} = [(+\infty;0)]_{\sim_\gamma}$ is the absorbing element for $\vee_{\sim_\gamma}$; indeed, $[X]_{\sim_\gamma} \vee_{\sim_\gamma} [(+\infty;0)]_{\sim_\gamma} = [X \vee (+\infty;0)]_{\sim_\gamma} = [(+\infty;0)]_{\sim_\gamma}, \forall [X]_{\sim_\gamma} \in \overline{\mathcal{K_C}}/\sim_\gamma$.

(7.b) $i_{\vee_\sim} = [(-\infty;0)]_{\sim_\gamma}$ is the absorbing element for $\wedge_{\sim_\gamma}$; indeed, $[X]_{\sim_\gamma} \wedge_{\sim_\gamma} [(-\infty;0)]_{\sim_\gamma} = [X \wedge (-\infty;0)]_{\sim_\gamma} = [(-\infty;0)]_{\sim_\gamma}, \forall [X]_{\sim_\gamma} \in \overline{\mathcal{K_C}}/\sim_\gamma$.

(8.a) $[X]_{\sim_\gamma} \vee_{\sim_\gamma} [Y]_{\sim_\gamma} = [-\infty]_{\sim_\gamma} \Rightarrow [X]_{\sim_\gamma} = [X]_{\sim_\gamma} = [-\infty]_{\sim_\gamma}, \forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma} \in \overline{\mathcal{K_C}}/\sim_\gamma$.

(8.b) $[X]_{\sim_\gamma} \wedge_{\sim_\gamma} [Y]_{\sim_\gamma} = [+\infty]_{\sim_\gamma} \Rightarrow [X]_{\sim_\gamma} = [Y]_{\sim_\gamma} = [+\infty]_{\sim_\gamma}, \forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma} \in \overline{\mathcal{K_C}}/\sim_\gamma$.

(9.a) $[X]_{\sim_\gamma} \wedge_{\sim_\gamma} [Y]_{\sim_\gamma} = [-\infty]_{\sim_\gamma} \Rightarrow [X]_{\sim_\gamma} = [-\infty]_{\sim_\gamma}$ or $[Y]_{\sim_\gamma} = [-\infty]_{\sim_\gamma}$, $\forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma} \in \overline{\mathcal{K_C}}/\sim_\gamma$.

(9.b) $[X]_{\sim_\gamma} \vee_{\sim_\gamma} [Y]_{\sim_\gamma} = [+\infty]_{\sim_\gamma} \Rightarrow [X]_{\sim_\gamma} = [+\infty]_{\sim_\gamma}$ or $[X]_{\sim_\gamma} = [+\infty]_{\sim_\gamma}$, $\forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma} \in \overline{\mathcal{K_C}}/\sim_\gamma$.

On the other hand, in a similar way, it is also possible to extend the operations of intersection and union between intervals in $\mathcal{K_C}$ to all equivalence classes of the quotient-set $(\mathcal{K_C}/\sim_\gamma)$; therefore, for all $[X]_{\sim_\gamma}, [Y]_{\sim_\gamma} \in \mathcal{K_C}/\sim_\gamma$ we define:

$\cap_{\sim_\gamma} : (\mathcal{K_C}/\sim_\gamma) \times (\mathcal{K_C}/\sim_\gamma) \to (\mathcal{K_C}/\sim_\gamma)$

such that: $([X]_{\sim_\gamma}, [Y]_{\sim_\gamma}) \to [X]_{\sim_\gamma} \cap_{\sim_\gamma} [Y]_{\sim_\gamma} \overset{def}{=} [X \cap Y]_{\sim_\gamma}$,

$\cup_{\sim_\gamma} : (\mathcal{K_C}/\sim_\gamma) \times (\mathcal{K_C}/\sim_\gamma) \to (\mathcal{K_C}/\sim_\gamma)$

such that: $([X]_{\sim_\gamma}, [Y]_{\sim_\gamma}) \to [X]_{\sim_\gamma} \cup_{\sim_\gamma} [Y]_{\sim_\gamma} \overset{def}{=} [X \cup Y]_{\sim_\gamma}$,

with operation $\cap$ and $\cup$ defined as in (4.19) and (4.18) and where

$$[X]_{\sim_\gamma} \cap_{\sim_\gamma} [Y]_{\sim_\gamma} \overset{def}{=} inf_{\sim_\gamma}\{[X]_{\sim_\gamma}, [Y]_{\sim_\gamma}\} \tag{6.6}$$

and

$$[X]_{\sim_\gamma} \cup_{\sim_\gamma} [Y]_{\sim_\gamma} \stackrel{def}{=} sup_{\sim_\gamma}\{[X]_{\sim_\gamma}, [Y]_{\sim_\gamma}\}. \tag{6.7}$$

In particular, when we consider the LU-relation ($\gamma = 1$), the situation that we obtained is well described in Figure 6.7.

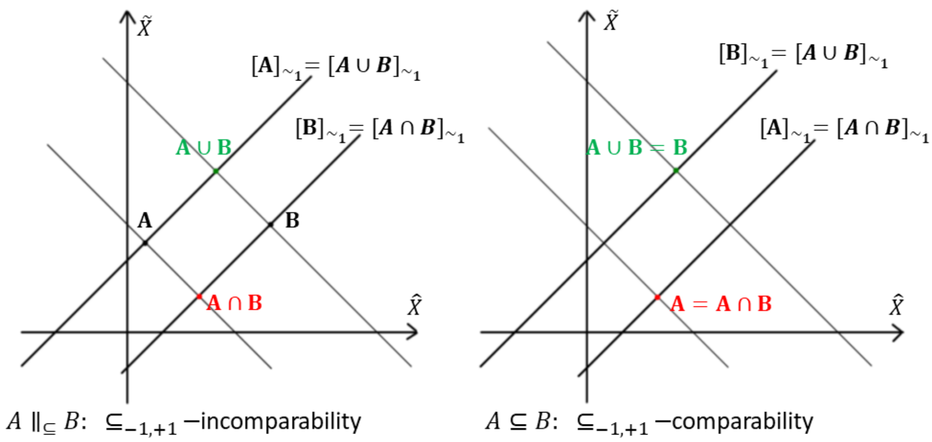


Figure 6.7: Operations $\cup_{\sim_\gamma}$ and $\cap_{\sim_\gamma}$ in LU-case ($\gamma = 1$).

It will also be useful to consider, as usual, $\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}} = \mathcal{K}_{\mathcal{C}} \cup \{\varnothing, \mathbb{R}\}$, where

$$\varnothing = (0; -\infty) = inf_{\gamma^-,\gamma^+}\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}} \text{ and } \mathbb{R} = (0; +\infty) = sup_{\gamma^-,\gamma^+}\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}$$

so that

$$\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma = \mathcal{K}_{\mathcal{C}}/\sim_\gamma \cup \left\{[\varnothing]_{\sim_\gamma}, [\mathbb{R}]_{\sim_\gamma}\right\}$$

with

$$[\varnothing]_{\sim_\gamma} = [(0 : -\infty)]_{\sim_\gamma} \stackrel{def}{=} inf_{\sim_\gamma}(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma)$$

and

$$[\mathbb{R}]_{\sim_\gamma} = [(0; +\infty)]_{\sim_\gamma} \stackrel{def}{=} sup_{\sim_\gamma}(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma).$$

Remark 6.2.1. *It is easy to verify, also with the help of Figures 6.6 and 6.7, that class* $[\varnothing]_{\sim_\gamma} = [(0; -\infty)]_{\sim_\gamma}$ *coincides with* $[+\infty]_{\sim_\gamma} = [(+\infty; 0)]_{\sim_\gamma}$ *as well as class* $[\mathbb{R}]_{\sim_\gamma} = [(0; +\infty)]_{\sim_\gamma}$ *coincides with* $[-\infty]_{\sim_\gamma} = [(-\infty; 0)]_{\sim_\gamma}$.

Also in this case we can define $\mathcal{K}_{\mathcal{C}}/\sim_\gamma$ (or, depending on the case, $\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma$) as an algebraic structure with the following properties.

(1)' $\cup_{\sim_\gamma}$ and $\cap_{\sim_\gamma}$ are commutative, as $\forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}/\sim_\gamma$ we have:

(1.a)' $[X]_{\sim_\gamma} \cup_{\sim_\gamma} [Y]_{\sim_\gamma} = [X \cup Y]_{\sim_\gamma} = [Y \cup X]_{\sim_\gamma} = [Y]_{\sim_\gamma} \cup_{\sim_\gamma} [X]_{\sim_\gamma}$;

(1.b)' $[X]_{\sim_\gamma} \cap_{\sim_\gamma} [Y]_{\sim_\gamma} = [X \cap Y]_{\sim_\gamma} =_{\sim_\gamma} [Y \cap X] = [Y]_{\sim_\gamma} \cap_{\sim_\gamma} [X]_{\sim_\gamma}$.

(2)' $\cup_{\sim_\gamma}$ and $\cap_{\sim_\gamma}$ are associative, as $\forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in \mathcal{K}_\mathcal{C}/\sim_\gamma$ we have:

(2.a)' $([X]_{\sim_\gamma} \cup_{\sim_\gamma} [Y]_{\sim_\gamma}) \cup_{\sim_\gamma} [Z]_{\sim_\gamma} = ... = [(X \cup Y) \cup Z]_{\sim_\gamma} = [X \cup (Y \cup Z)]_{\sim_\gamma} = ... = [X]_{\sim_\gamma} \cup_{\sim_\gamma} ([Y]_{\sim_\gamma} \cup_{\sim_\gamma} [Z]_{\sim_\gamma})$;

(2.b)' $([X]_{\sim_\gamma} \cap_{\sim_\gamma} [Y]_{\sim_\gamma}) \cap_{\sim_\gamma} [Z]_{\sim_\gamma} = ... = [(X \cap Y) \cap X]_{\sim_\gamma} = [X \cap (Y \cap Z)]_{\sim_\gamma} = ... = [X]_{\sim_\gamma} \cap_{\sim_\gamma} ([Y]_{\sim_\gamma} \cap_{\sim_\gamma} [Z]_{\sim_\gamma})$.

(3)' The absorpion identities are satified for both $\cup_{\sim_\gamma}$ and $\cap_{\sim_\gamma}$, as $\forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma} \in \mathcal{K}_\mathcal{C}/\sim_\gamma$, we have:

(3.a)' $[X]_{\sim_\gamma} \cup \sim_\gamma ([X]_{\sim_\gamma} \cap \sim_\gamma [Y]_{\sim_\gamma}) = [X \cup (X \cap Y)]_{\sim_\gamma} = [X]_{\sim_\gamma}$;

(3.b)' $[X]_{\sim_\gamma} \cap \sim_\gamma ([X]_{\sim_\gamma} \cup \sim_\gamma [Y]_{\sim_\gamma}) = [X \cap (X \cup Y)]_{\sim_\gamma} = [X]_{\sim_\gamma}$.

(4)' The idempotency is satisfied for both $\cup_{\sim_\gamma}$ and $\cap_{\sim_\gamma}$, as $\forall [X]_{\sim_\gamma} \in \mathcal{K}_\mathcal{C}/\sim_\gamma$, we have:

(4.a)' $[X]_{\sim_\gamma} \cup \sim_\gamma [X]_{\sim_\gamma} = [X \cup X]_{\sim_\gamma} = [X]_{\sim_\gamma}$;

(4.b)' $[X]_{\sim_\gamma} \cap \sim_\gamma [X]_{\sim_\gamma} = [X \cap X]_{\sim_\gamma} = [X]_{\sim_\gamma}$.

It is also easy to verify that:

(5.a)' $\cap_{\sim_\gamma}$ is left and right distributive over $\cup_{\sim_\gamma}$:

(5.a.1)' $[X]_{\sim_\gamma} \cap_{\sim_\gamma} ([Y]_{\sim_\gamma} \cup_{\sim_\gamma} [Z]_{\sim_\gamma}) = ([X]_{\sim_\gamma} \cap_{\sim_\gamma} [Y]_{\sim_\gamma}) \cup_{\sim_\gamma} ([X]_{\sim_\gamma} \cap_{\sim_\gamma} [Z]_{\sim_\gamma}), \forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in \mathcal{K}_\mathcal{C}/\sim_\gamma$,

(5.a.2)' $([X]_{\sim_\gamma} \cup_{\sim_\gamma} [Y]_{\sim_\gamma}) \cap_{\sim_\gamma} [Z]_{\sim_\gamma} = ([X]_{\sim_\gamma} \cap_{\sim_\gamma} [Z]_{\sim_\gamma}) \cup_{\sim_\gamma} ([Y]_{\sim_\gamma} \cap_{\sim_\gamma} [Z]_{\sim_\gamma}), \forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in \mathcal{K}_\mathcal{C}/\sim_\gamma$;

(5.b)' $\cup_{\sim_\gamma}$ is left and right distributive over $\cap_{\sim_\gamma}$:

(5.b.1)' $[X]_{\sim_\gamma} \cup_{\sim_\gamma} ([Y]_{\sim_\gamma} \cap_{\sim_\gamma} [Z]_{\sim_\gamma}) = ([X]_{\sim_\gamma} \cup_{\sim_\gamma} [Y]_{\sim_\gamma}) \cap_{\sim_\gamma} ([X]_{\sim_\gamma} \cup_{\sim_\gamma} [Z]_{\sim_\gamma}), \forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in \mathcal{K}_\mathcal{C}/\sim_\gamma$,

(5.b.2)' $([X]_{\sim_\gamma} \cap_{\sim_\gamma} [Y]_{\sim_\gamma}) \cup_{\sim_\gamma} [Z]_{\sim_\gamma} = ([X]_{\sim_\gamma} \cup_{\sim_\gamma} [Z]_{\sim_\gamma}) \cap_{\sim_\gamma} ([Y]_{\sim_\gamma} \cup_{\sim_\gamma} [Z]_{\sim_\gamma}), \forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in \mathcal{K}_\mathcal{C}/\sim_\gamma$.

Other relevant properties are the following.

(6.a)' $\cup_{\sim_\gamma}$ has a neutral element in $\mathcal{K}_\mathcal{C}^{\varnothing\mathbb{R}}/\sim_\gamma$ which is $i_{\cup_\sim} = [\varnothing]_{\sim_\gamma} = [(0; -\infty)]_{\sim_\gamma}$; indeed, $[X]_{\sim_\gamma} \cup_{\sim_\gamma} [(0; -\infty)]_{\sim_\gamma} = [X \cup (0; -\infty)]_{\sim_\gamma} = [X]_{\sim_\gamma}, \forall [X]_{\sim_\gamma} \in \mathcal{K}_\mathcal{C}^{\varnothing\mathbb{R}}/\sim_\gamma$.

(6.b)' $\cap_{\sim_\gamma}$ has a neutral element in $\mathcal{K}_\mathcal{C}^{\varnothing\mathbb{R}}/\sim_\gamma$ which is $i_{\cap_\sim} = [\mathbb{R}]_{\sim_\gamma} = [(0; +\infty)]_{\sim_\gamma}$; indeed, $[X]_{\sim_\gamma} \cap_{\sim_\gamma} [(0; +\infty)]_{\sim_\gamma} = [X \cap (0; +\infty)]_{\sim_\gamma} = [X]_{\sim_\gamma}, \forall [X]_{\sim_\gamma} \in \mathcal{K}_\mathcal{C}^{\varnothing\mathbb{R}}/\sim_\gamma$.

(7.a)' $i_{\cap_\sim} = [\mathbb{R}]_{\sim_\gamma} = [(0;+\infty)]_{\sim_\gamma}$ is the absorbing element for $\cup_{\sim_\gamma}$; indeed, $[X]_{\sim_\gamma} \cup_{\sim_\gamma} [(0;+\infty)]_{\sim_\gamma} = [X \cup (0;+\infty)]_{\sim_\gamma} = [(0;+\infty)]_{\sim_\gamma}, \forall [X]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma$.

(7.b)' $i_{\cup_\sim} = [\varnothing]_{\sim_\gamma} = [(0;-\infty)]_{\sim_\gamma}$ is the absorbing element for $\cap_{\sim_\gamma}$; indeed, $[X]_{\sim_\gamma} \cap_{\sim_\gamma} [(0;-\infty;0)]_{\sim_\gamma} = [X \cap (0;-\infty)]_{\sim_\gamma} = [(0;-\infty)]_{\sim_\gamma}, \forall [X]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma$.

(8.a)' $[X]_{\sim_\gamma} \cup_{\sim_\gamma} [Y]_{\sim_\gamma} = [\varnothing]_{\sim_\gamma} \Rightarrow [X]_{\sim_\gamma} = [X]_{\sim_\gamma} = [\varnothing]_{\sim_\gamma}, \forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma$.

(8.b)' $[X]_{\sim_\gamma} \cap_{\sim_\gamma} [Y]_{\sim_\gamma} = [\mathbb{R}]_{\sim_\gamma} \Rightarrow [X]_{\sim_\gamma} = [Y]_{\sim_\gamma} = [\mathbb{R}]_{\sim_\gamma}, \forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma$.

(9.a)' $[X]_{\sim_\gamma} \cap_{\sim_\gamma} [Y]_{\sim_\gamma} = [\varnothing]_{\sim_\gamma} \Rightarrow [X]_{\sim_\gamma} = [\varnothing]_{\sim_\gamma}$ or $[Y]_{\sim_\gamma} = [\varnothing]_{\sim_\gamma}$, $\forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma$.

(9.b)' $[X]_{\sim_\gamma} \cup_{\sim_\gamma} [Y]_{\sim_\gamma} = [\mathbb{R}]_{\sim_\gamma} \Rightarrow [X]_{\sim_\gamma} = [\mathbb{R}]_{\sim_\gamma}$ or $[X]_{\sim_\gamma} = [\mathbb{R}]_{\sim_\gamma}$, $\forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma$.

As a direct consequence of the properties seen, we have that the following propositions hold.

Proposition 6.2.2. *The structures:* $(\overline{\mathcal{K}_{\mathcal{C}}}/\sim_\gamma, \vee_{\sim_\gamma}, \wedge_{\sim_\gamma})$, $(\overline{\mathcal{K}_{\mathcal{C}}}/\sim_\gamma, \wedge_{\sim_\gamma}, \vee_{\sim_\gamma})$, $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma, \cup_{\sim_\gamma}, \cap_{\sim_\gamma})$ *and* $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma, \cap_{\sim_\gamma}, \cup_{\sim_\gamma})$ *are commutative, idempotent semirings.*

Proof. $(\overline{\mathcal{K}_{\mathcal{C}}}/\sim_\gamma, \vee_{\sim_\gamma}, \wedge_{\sim_\gamma})$ is a commutative, idempotent semiring, as:

1a) $(\overline{\mathcal{K}_{\mathcal{C}}}/\sim_\gamma, \vee_{\sim_\gamma})$ is a commutative, idempotent monoid with neutral element $i_{\vee_\sim} = [-\infty]_{\sim_\gamma}$:

   (i) $\vee_{\sim_\gamma}$ is associative;
   (ii) $\vee_{\sim_\gamma}$ has the neutral element $i_{\vee_\sim} = [-\infty]_{\sim_\gamma} = [(-\infty;0)]_{\sim_\gamma} \in \overline{\mathcal{K}_{\mathcal{C}}}/\sim_\gamma : [X]_{\sim_\gamma} \vee_{\sim_\gamma} [(-\infty;0)]_{\sim_\gamma} = [X \vee (-\infty;0)]_{\sim_\gamma} = [X]_{\sim_\gamma}, \forall [X]_{\sim_\gamma} \in \overline{\mathcal{K}_{\mathcal{C}}}/\sim_\gamma$
   (so, $i_{\vee_\sim} = [-\infty]_{\sim_\gamma} = [(-\infty;0)]$ is the 0-element of the semiring);
   (iii) $\vee_{\sim_\gamma}$ is commutative;
   (iv) $\vee_{\sim_\gamma}$ is idempotent.

2a) $(\overline{\mathcal{K}_{\mathcal{C}}}/\sim_\gamma, \wedge_{\sim_\gamma})$ is a commutative, idempotent monoid with neutral element $i_{\wedge_\sim} = [+\infty]_{\sim_\gamma}$:

   (i) $\wedge_{\sim_\gamma}$ is associative;
   (ii) $\wedge_{\sim_\gamma}$ has the neutral element $i_{\wedge_\sim} = [+\infty]_{\sim_\gamma} = [(+\infty;0)]_{\sim_\gamma} \in \overline{\mathcal{K}_{\mathcal{C}}}/\sim_\gamma : [X]_{\sim_\gamma} \wedge_{\sim_\gamma} [(+\infty;0)]_{\sim_\gamma} = [X \wedge (+\infty;0)]_{\sim_\gamma} = [X]_{\sim_\gamma}, \forall [X]_{\sim_\gamma} \in \overline{\mathcal{K}_{\mathcal{C}}}/\sim_\gamma$
   (so, $i_{\vee_\sim} = [+\infty]_{\sim_\gamma} = [(+\infty;0)]$ is the 1-element of the semiring);

(iii) $\wedge_{\sim_\gamma}$ is commutative;

(iv) $\wedge_{\sim_\gamma}$ is idempotent.

3a) $\wedge_{\sim_\gamma}$ is left and right distributive over $\vee_{\sim_\gamma}$

(i) $[X]_{\sim_\gamma} \wedge_{\sim_\gamma} ([Y]_{\sim_\gamma} \vee_{\sim_\gamma} [Z]_{\sim_\gamma}) = ([X]_{\sim_\gamma} \wedge_{\sim_\gamma} [Y]_{\sim_\gamma}) \vee_{\sim_\gamma} ([X]_{\sim_\gamma} \wedge_{\sim_\gamma} [Z]_{\sim_\gamma}), \forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in \overline{\mathcal{K}_\mathcal{C}}/\sim_\gamma$;

(ii) $([X]_{\sim_\gamma} \vee_{\sim_\gamma} [Y]_{\sim_\gamma}) \wedge_{\sim_\gamma} [Z]_{\sim_\gamma} = ([X]_{\sim_\gamma} \wedge_{\sim_\gamma} [Z]_{\sim_\gamma}) \vee_{\sim_\gamma} ([Y]_{\sim_\gamma} \wedge_{\sim_\gamma} [Z]_{\sim_\gamma}), \forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in \overline{\mathcal{K}_\mathcal{C}}/\sim_\gamma$.

4a) $i_{\vee_\sim} = [-\infty]_{\sim_\gamma} = [(-\infty;0)]_{\sim_\gamma}$ is the absorbing element for $\wedge_{\sim_\gamma}$:

$[(-\infty;0)]_{\sim_\gamma} \wedge_{\sim_\gamma} [X]_{\sim_\gamma} = [X]_{\sim_\gamma} \wedge_{\sim_\gamma} [(-\infty;0)]_{\sim_\gamma} = [(-\infty;0)]_{\sim_\gamma}, \forall [X]_{\sim_\gamma} \in \overline{\mathcal{K}_\mathcal{C}}/\sim_\gamma$.

Analogously, also the structure $(\overline{\mathcal{K}_\mathcal{C}}/\sim_\gamma, \wedge_{\sim_\gamma}, \vee_{\sim_\gamma})$ is a commutative, idempotent semiring, as:

1b) $(\overline{\mathcal{K}_\mathcal{C}}/\sim_\gamma, \wedge_{\sim_\gamma})$ is a commutative, idempotent monoid with neutral element $i_{\wedge_\sim} = [+\infty]_{\sim_\gamma}$:

(i) $\wedge_{\sim_\gamma}$ is associative;

(ii) $\wedge_{\sim_\gamma}$ has the neutral element $i_{\wedge_\sim} = [+\infty]_{\sim_\gamma} = [(+\infty;0)]_{\sim_\gamma} \in \overline{\mathcal{K}_\mathcal{C}}/\sim_\gamma : [X]_{\sim_\gamma} \wedge_{\sim_\gamma} [(+\infty;0)]_{\sim_\gamma} = [X \wedge (+\infty;0)]_{\sim_\gamma} = [X]_{\sim_\gamma}, \forall [X]_{\sim_\gamma} \in \overline{\mathcal{K}_\mathcal{C}}/\sim_\gamma$
(so, $i_{\vee_\sim} = [+\infty]_{\sim_\gamma} = [(+\infty;0)]$ is the 0-element of the semiring);

(iii) $\wedge_{\sim_\gamma}$ is commutative;

(iv) $\wedge_{\sim_\gamma}$ is idempotent.

2b) $(\overline{\mathcal{K}_\mathcal{C}}/\sim_\gamma, \vee_{\sim_\gamma})$ is a commutative, idempotent monoid with neutral element $i_{\vee_\sim} = [-\infty]_{\sim_\gamma}$:

(i) $\vee_{\sim_\gamma}$ is associative;

(ii) $\vee_{\sim_\gamma}$ has the neutral element $i_{\vee_\sim} = [-\infty]_{\sim_\gamma} = [(-\infty;0)]_{\sim_\gamma} \in \overline{\mathcal{K}_\mathcal{C}}/\sim_\gamma : [X]_{\sim_\gamma} \vee_{\sim_\gamma} [(-\infty;0)]_{\sim_\gamma} = [X \vee (-\infty;0)]_{\sim_\gamma} = [X]_{\sim_\gamma}, \forall [X]_{\sim_\gamma} \in \overline{\mathcal{K}_\mathcal{C}}/\sim_\gamma$
(so, $i_{\vee_\sim} = [-\infty]_{\sim_\gamma} = [(-\infty;0)]$ is the 1-element of the semiring);

(iii) $\vee_{\sim_\gamma}$ is commutative;

(iv) $\vee_{\sim_\gamma}$ is idempotent.

3b) $\vee_{\sim_\gamma}$ is left and right distributive over $\wedge_{\sim_\gamma}$

(i) $[X]_{\sim_\gamma} \vee_{\sim_\gamma} ([Y]_{\sim_\gamma} \wedge_{\sim_\gamma} [Z]_{\sim_\gamma}) = ([X]_{\sim_\gamma} \vee_{\sim_\gamma} [Y]_{\sim_\gamma}) \wedge_{\sim_\gamma} ([X]_{\sim_\gamma} \vee_{\sim_\gamma} [Z]_{\sim_\gamma}), \forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in \overline{\mathcal{K}_\mathcal{C}}/\sim_\gamma$;

(ii) $([X]_{\sim_\gamma} \wedge_{\sim_\gamma} [Y]_{\sim_\gamma}) \vee_{\sim_\gamma} [Z]_{\sim_\gamma} = ([X]_{\sim_\gamma} \vee_{\sim_\gamma} [Z]_{\sim_\gamma}) \wedge_{\sim_\gamma} ([Y]_{\sim_\gamma} \vee_{\sim_\gamma} [Z]_{\sim_\gamma}), \forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in \overline{\mathcal{K}_\mathcal{C}}/\sim_\gamma$.

4b) $i_{\wedge\sim} = [+\infty]_{\sim_\gamma} = [(+\infty;0)]_{\sim_\gamma}$ is the absorbing element for $\vee_{\sim_\gamma}$:

$[(+\infty;0)]_{\sim_\gamma} \vee_{\sim_\gamma} [X]_{\sim_\gamma} = [X]_{\sim_\gamma} \vee_{\sim_\gamma} [(+\infty;0)]_{\sim_\gamma} = [(+\infty;0)]_{\sim_\gamma}, \forall [X]_{\sim_\gamma} \in \overline{\mathcal{K}_\mathcal{C}} / \sim_\gamma$.

Similarly, $(\mathcal{K}_\mathcal{C}^{\varnothing\mathbb{R}} / \sim_\gamma, \cup_{\sim_\gamma}, \cap_{\sim_\gamma})$ is a commutative, idempotent semiring, as:

1c) $(\mathcal{K}_\mathcal{C}^{\varnothing\mathbb{R}} / \sim_\gamma, \cup_{\sim_\gamma})$ is a commutative, idempotent monoid with neutral element $i_{\cup\sim} = [\varnothing]_{\sim_\gamma}$:

(i) $\cup_{\sim_\gamma}$ is associative;

(ii) $\cup_{\sim_\gamma}$ has the neutral element $i_{\cup\sim} = [\varnothing]_{\sim_\gamma} = [(0;-\infty)]_{\sim_\gamma} \in \mathcal{K}_\mathcal{C}^{\varnothing\mathbb{R}} / \sim_\gamma : [X]_{\sim_\gamma} \cup_{\sim_\gamma} [(0;-\infty)]_{\sim_\gamma} = [X \cup (0;-\infty)]_{\sim_\gamma} = [X]_{\sim_\gamma}, \forall [X]_{\sim_\gamma} \in \mathcal{K}_\mathcal{C}^{\varnothing\mathbb{R}} / \sim_\gamma$
(so, $i_{\cup\sim} = [\varnothing]_{\sim_\gamma} = [(0;-\infty)]$ is the 0-element of the semiring);

(iii) $\cup_{\sim_\gamma}$ is commutative;

(iv) $\cup_{\sim_\gamma}$ is idempotent.

2c) $(\mathcal{K}_\mathcal{C}^{\varnothing\mathbb{R}} / \sim_\gamma, \cap_{\sim_\gamma})$ is a commutative, idempotent monoid with neutral element $i_{\cap\sim} = [\mathbb{R}]_{\sim_\gamma}$:

(i) $\cap_{\sim_\gamma}$ is associative;

(ii) $\cap_{\sim_\gamma}$ has the neutral element $i_{\cap\sim} = [\mathbb{R}]_{\sim_\gamma} = [(0;+\infty)]_{\sim_\gamma} \in \mathcal{K}_\mathcal{C}^{\varnothing\mathbb{R}} / \sim_\gamma : [X]_{\sim_\gamma} \cap_{\sim_\gamma} [(0;+\infty)]_{\sim_\gamma} = [X \cap (0;+\infty)]_{\sim_\gamma} = [X]_{\sim_\gamma}, \forall [X]_{\sim_\gamma} \in \mathcal{K}_\mathcal{C}^{\varnothing\mathbb{R}} / \sim_\gamma$
(so, $i_{\cap\sim} = [\mathbb{R}]_{\sim_\gamma} = [(0;+\infty)]$ is the 1-element of the semiring);

(iii) $\cap_{\sim_\gamma}$ is commutative;

(iv) $\cap_{\sim_\gamma}$ is idempotent.

3c) $\cap_{\sim_\gamma}$ is left and right distributive over $\cup_{\sim_\gamma}$

(i) $[X]_{\sim_\gamma} \cap_{\sim_\gamma} ([Y]_{\sim_\gamma} \cup_{\sim_\gamma} [Z]_{\sim_\gamma}) = ([X]_{\sim_\gamma} \cap_{\sim_\gamma} [Y]_{\sim_\gamma}) \cup_{\sim_\gamma} ([X]_{\sim_\gamma} \cap_{\sim_\gamma} [Z]_{\sim_\gamma}), \forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in K_C^{\varnothing\mathbb{R}} / \sim_\gamma$;

(ii) $([X]_{\sim_\gamma} \cup_{\sim_\gamma} [Y]_{\sim_\gamma}) \cap_{\sim_\gamma} [Z]_{\sim_\gamma} = ([X]_{\sim_\gamma} \cap_{\sim_\gamma} [Z]_{\sim_\gamma}) \cup_{\sim_\gamma} ([Y]_{\sim_\gamma} \cap_{\sim_\gamma} [Z]_{\sim_\gamma}), \forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in K_C^{\varnothing\mathbb{R}} / \sim_\gamma$.

4c) $i_{\cup\sim} = [\varnothing]_{\sim_\gamma} = [(0;-\infty)]_{\sim_\gamma}$ is the absorbing element for $\cap_{\sim_\gamma}$:

$[(0;-\infty)]_{\sim_\gamma} \cap_{\sim_\gamma} [X]_{\sim_\gamma} = [X]_{\sim_\gamma} \cap_{\sim_\gamma} [(0;-\infty)]_{\sim_\gamma} = [(0;-\infty)]_{\sim_\gamma}, \forall [X]_{\sim_\gamma} \in K_C^{\varnothing\mathbb{R}} / \sim_\gamma$.

Likewise, $(\mathcal{K}_\mathcal{C}^{\varnothing\mathbb{R}} / \sim_\gamma, \cap_{\sim_\gamma}, \cup_{\sim_\gamma})$ is a commutative, idempotent semiring, as:

1d) $(\mathcal{K}_\mathcal{C}^{\varnothing\mathbb{R}} / \sim_\gamma, \cap_{\sim_\gamma})$ is a commutative, idempotent monoid with neutral element $i_{\cap\sim} = [\mathbb{R}]_{\sim_\gamma}$:

(i) $\cap_{\sim_\gamma}$ is associative;

(ii) $\cap_{\sim_\gamma}$ has the neutral element $i_{\cap_\sim} = [\mathbb{R}]_{\sim_\gamma} = [(0;+\infty)]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma : [X]_{\sim_\gamma}\cap_{\sim_\gamma}[(0;+\infty)]_{\sim_\gamma} = [X\cap(0;+\infty)]_{\sim_\gamma} = [X]_{\sim_\gamma}, \forall [X]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma$
(so, $i_{\cap_\sim} = [\mathbb{R}]_{\sim_\gamma} = [(0;+\infty)]$ is the 0-element of the semiring);

(iii) $\cap_{\sim_\gamma}$ is commutative;

(iv) $\cap_{\sim_\gamma}$ is idempotent.

2d) $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma, \cup_{\sim_\gamma})$ is a commutative, idempotent monoid with neutral element $i_{\cup_\sim} = [\varnothing]_{\sim_\gamma}$:

(i) $\cup_{\sim_\gamma}$ is associative;

(ii) $\cup_{\sim_\gamma}$ has the neutral element $i_{\cup_\sim} = [\varnothing]_{\sim_\gamma} = [(0;-\infty)]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma : [X]_{\sim_\gamma}\cup_{\sim_\gamma}[(0;-\infty)]_{\sim_\gamma} = [X\cup(0;-\infty)]_{\sim_\gamma} = [X]_{\sim_\gamma}, \forall [X]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma$
(so, $i_{\cup_\sim} = [\varnothing]_{\sim_\gamma} = [(0;-\infty)]$ is the 1-element of the semiring);

(iii) $\cup_{\sim_\gamma}$ is commutative;

(iv) $\cup_{\sim_\gamma}$ is idempotent.

3d) $\cup_{\sim_\gamma}$ is left and right distributive over $\cap_{\sim_\gamma}$

(i) $[X]_{\sim_\gamma} \cup_{\sim_\gamma} ([Y]_{\sim_\gamma} \cap_{\sim_\gamma} [Z]_{\sim_\gamma}) = ([X]_{\sim_\gamma} \cup_{\sim_\gamma} [Y]_{\sim_\gamma}) \cap_{\sim_\gamma} ([X]_{\sim_\gamma} \cup_{\sim_\gamma} [Z]_{\sim_\gamma}), \forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in K_C^{\varnothing\mathbb{R}}/\sim_\gamma$;

(ii) $([X]_{\sim_\gamma} \cap_{\sim_\gamma} [Y]_{\sim_\gamma}) \cup_{\sim_\gamma} [Z]_{\sim_\gamma} = ([X]_{\sim_\gamma} \cup_{\sim_\gamma} [Z]_{\sim_\gamma}) \cap_{\sim_\gamma} ([Y]_{\sim_\gamma} \cup_{\sim_\gamma} [Z]_{\sim_\gamma}), \forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in K_C^{\varnothing\mathbb{R}}/\sim_\gamma$.

4d) $i_{\cap_\sim} = [\mathbb{R}]_{\sim_\gamma} = [(0;+\infty)]_{\sim_\gamma}$ is the absorbing element for $\cup_{\sim_\gamma}$:

$[(0;+\infty)]_{\sim_\gamma}\cup_{\sim_\gamma}[X]_{\sim_\gamma} = [X]_{\sim_\gamma}\cup_{\sim_\gamma}[(0;+\infty)]_{\sim_\gamma} = [(0;+\infty)]_{\sim_\gamma}, \forall [X]_{\sim_\gamma} \in K_C^{\varnothing\mathbb{R}}/\sim_\gamma$.

□

Proposition 6.2.3. *The structures* $(\overline{\mathcal{K}_{\mathcal{C}}}/\sim_\gamma, \vee_{\sim_\gamma}, \wedge_{\sim_\gamma})$, $(\overline{\mathcal{K}_{\mathcal{C}}}/\sim_\gamma, \wedge_{\sim_\gamma}, \vee_{\sim_\gamma})$, $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma, \cup_{\sim_\gamma}, \cap_{\sim_\gamma})$ *and* $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma, \cap_{\sim_\gamma}, \cup_{\sim_\gamma})$ *are zero-sum-free semirings.*

Proof. The proof is immediate since, for definition, we have
$\forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma} \in \overline{\mathcal{K}_{\mathcal{C}}}/\sim_\gamma$, $[X]_{\sim_\gamma} \vee_{\sim_\gamma} [Y]_{\sim_\gamma} = sup_{\sim_\gamma}([X]_{\sim_\gamma}, [Y]_{\sim_\gamma})$ and
$[X]_{\sim_\gamma} \wedge_{\sim_\gamma} [Y]_{\sim_\gamma} = inf_{\sim_\gamma}([X]_{\sim_\gamma}, [Y]_{\sim_\gamma})$, it follows that:
$[X]_{\sim_\gamma} \vee_{\sim_\gamma} [Y]_{\sim_\gamma} = [-\infty]_{\sim_\gamma} \Rightarrow [X]_{\sim_\gamma} = [Y]_{\sim_\gamma} = [-\infty]_{\sim_\gamma}$ and
$[X]_{\sim_\gamma} \wedge_{\sim_\gamma} [Y]_{\sim_\gamma} = [+\infty]_{\sim_\gamma} \Rightarrow [X]_{\sim_\gamma} = [Y]_{\sim_\gamma} = [+\infty]_{\sim_\gamma}$ respectively.
So $(\overline{\mathcal{K}_{\mathcal{C}}}/\sim_\gamma, \vee_{\sim_\gamma}, \wedge_{\sim_\gamma})$ and $(\overline{\mathcal{K}_{\mathcal{C}}}/\sim_\gamma, \wedge_{\sim_\gamma}, \vee_{\sim_\gamma})$ are zero-sum-free semirings.
The other two cases are analogous since, for definition, we have

$\forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}$, $[X]_{\sim_\gamma} \cup_{\sim_\gamma} [Y]_{\sim_\gamma} = sup_{\sim_\gamma}([X]_{\sim_\gamma}, [Y]_{\sim_\gamma})$ and $[X]_{\sim_\gamma} \cap_{\sim_\gamma} [Y]_{\sim_\gamma} = inf_{\sim_\gamma}([X]_{\sim_\gamma}, [Y]_{\sim_\gamma})$, it follows that:
$[X]_{\sim_\gamma} \cup_{\sim_\gamma} [Y]_{\sim_\gamma} = [\varnothing]_{\sim_\gamma} \Rightarrow [X]_{\sim_\gamma} = [Y]_{\sim_\gamma} = [\varnothing]_{\sim_\gamma}$ and
$[X]_{\sim_\gamma} \cap_{\sim_\gamma} [Y]_{\sim_\gamma} = [\mathbb{R}]_{\sim_\gamma} \Rightarrow [X]_{\sim_\gamma} = [Y]_{\sim_\gamma} = [\mathbb{R}]_{\sim_\gamma}$ respectively.
Therefore, $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma, \cup_{\sim_\gamma}, \cap_{\sim_\gamma})$ and $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma, \cap_{\sim_\gamma}, \cup_{\sim_\gamma})$ are zero-sum-free semirings. □

Proposition 6.2.4. *The structures* $(\overline{\mathcal{K}_{\mathcal{C}}}/\sim_\gamma, \vee_{\sim_\gamma}, \wedge_{\sim_\gamma})$, $(\overline{\mathcal{K}_{\mathcal{C}}}/\sim_\gamma, \wedge_{\sim_\gamma}, \vee_{\sim_\gamma})$, $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma, \cup_{\sim_\gamma}, \cap_{\sim_\gamma})$ *and* $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma, \cap_{\sim_\gamma}, \cup_{\sim_\gamma})$ *are zero-divisor-free semirings.*

Proof. The proof is similar to that of Proposition 6.2.3. Indeed, since, for definition, we have $\forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma} \in \overline{\mathcal{K}_{\mathcal{C}}}/\sim_\gamma$,
$[X]_{\sim_\gamma} \wedge_{\sim_\gamma} [Y]_{\sim_\gamma} = inf_{\sim_\gamma}([X]_{\sim_\gamma}, [Y]_{\sim_\gamma})$ and
$[X]_{\sim_\gamma} \vee_{\sim_\gamma} [Y]_{\sim_\gamma} = sup_{\sim_\gamma}([X]_{\sim_\gamma}, [Y]_{\sim_\gamma})$, it follows that:
$[X]_{\sim_\gamma} \wedge_{\sim_\gamma} [Y]_{\sim_\gamma} = [-\infty]_{\sim_\gamma} \Rightarrow [X]_{\sim_\gamma} = [-\infty]_{\sim_\gamma}$ or $[Y]_{\sim_\gamma} = [-\infty]_{\sim_\gamma}$ and
$[X]_{\sim_\gamma} \vee_{\sim_\gamma} [Y]_{\sim_\gamma} = [+\infty]_{\sim_\gamma} \Rightarrow [X]_{\sim_\gamma} = [+\infty]_{\sim_\gamma}$ or $[Y]_{\sim_\gamma} = [+\infty]_{\sim_\gamma}$ respectively. Therefore, $(\overline{\mathcal{K}_{\mathcal{C}}}/\sim_\gamma, \vee_{\sim_\gamma}, \wedge_{\sim_\gamma})$ and $(\overline{\mathcal{K}_{\mathcal{C}}}/\sim_\gamma, \wedge_{\sim_\gamma}, \vee_{\sim_\gamma})$ are zero-divisor-free semirings.
The other two cases are analogous since, for definition, we have
$\forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}$, $[X]_{\sim_\gamma} \cap_{\sim_\gamma} [Y]_{\sim_\gamma} = inf_{\sim_\gamma}([X]_{\sim_\gamma}, [Y]_{\sim_\gamma})$ and
$[X]_{\sim_\gamma} \cup_{\sim_\gamma} [Y]_{\sim_\gamma} = sup_{\sim_\gamma}([X]_{\sim_\gamma}, [Y]_{\sim_\gamma})$, it follows that:
$[X]_{\sim_\gamma} \cap_{\sim_\gamma} [Y]_{\sim_\gamma} = [\varnothing]_{\sim_\gamma} \Rightarrow [X]_{\sim_\gamma} = [\varnothing]_{\sim_\gamma}$ or $[Y]_{\sim_\gamma} = [\varnothing]_{\sim_\gamma}$ and
$[X]_{\sim_\gamma} \cup_{\sim_\gamma} [Y]_{\sim_\gamma} = [\mathbb{R}]_{\sim_\gamma} \Rightarrow [X]_{\sim_\gamma} = [\mathbb{R}]_{\sim_\gamma}$ or $[Y]_{\sim_\gamma} = [\mathbb{R}]_{\sim_\gamma}$ respectively.
Therefore, $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma, \cup_{\sim_\gamma}, \cap_{\sim_\gamma})$ and $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma, \cap_{\sim_\gamma}, \cup_{\sim_\gamma})$ are zero-divisor-free semirings. □

The four semirings of Propositions 6.2.3 and 6.2.4, being zero-sum-free and zero-divisor-free, are information algebras.

Moreover, since we know that any commutative, idempotent semiring under join and meet is a bounded, distributive lattice and, having defined, from (6.4) and (6.5), the operations:

$$[X]_{\sim_\gamma} \wedge_{\sim_\gamma} [Y]_{\sim_\gamma} = inf\{[X]_{\sim_\gamma}, [Y]_{\sim_\gamma}\}$$

and

$$[X]_{\sim_\gamma} \vee_{\sim_\gamma} [Y]_{\sim_\gamma} = sup\{[X]_{\sim_\gamma}, [Y]_{\sim_\gamma}\},$$

it is possible to consider the quotient structures $(\overline{\mathcal{K}_{\mathcal{C}}}/\sim_\gamma, \vee_{\sim_\gamma}, \wedge_{\sim_\gamma})$ and $(\overline{\mathcal{K}_{\mathcal{C}}}/\sim_\gamma, \wedge_{\sim_\gamma}, \vee_{\sim_\gamma})$, as bounded, distributive, algebraic lattices, denoted by

$$(\overline{\mathcal{K}_{\mathcal{C}}}/\sim_\gamma, \vee_{\sim_\gamma}, \wedge_{\sim_\gamma}, [-\infty]_{\sim_\gamma}, [+\infty]_{\sim_\gamma})$$

and

$$(\overline{\mathcal{K}_{\mathcal{C}}}/\sim_\gamma, \wedge_{\sim_\gamma}, \vee_{\sim_\gamma}, [+\infty]_{\sim_\gamma}, [-\infty]_{\sim_\gamma}).$$

Indeed, according to Definition 4.1.6, the following properties hold.

(1) $\vee_{\sim_\gamma}$ and $\wedge_{\sim_\gamma}$ are commutative, as $\forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma} \in \overline{\mathcal{K}_{\mathcal{C}}} / \sim_\gamma$ we have:

(1.a) $[X]_{\sim_\gamma} \vee_{\sim_\gamma} [Y]_{\sim_\gamma} = [X \vee Y]_{\sim_\gamma} = [Y \vee X]_{\sim_\gamma} = [Y]_{\sim_\gamma} \vee_{\sim_\gamma} [X]_{\sim_\gamma}$;

(1.b) $[X]_{\sim_\gamma} \wedge_{\sim_\gamma} [Y]_{\sim_\gamma} = [X \wedge Y]_{\sim_\gamma} = [Y \wedge X] = [Y]_{\sim_\gamma} \wedge_{\sim_\gamma} [X]_{\sim_\gamma}$.

(2) $\vee_{\sim_\gamma}$ and $\wedge_{\sim_\gamma}$ are associative, as $\forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in \overline{\mathcal{K}_{\mathcal{C}}} / \sim_\gamma$ we have:

(2.a) $([X]_{\sim_\gamma} \vee_{\sim_\gamma} [Y]_{\sim_\gamma}) \vee_{\sim_\gamma} [Z]_{\sim_\gamma} = ... = [(X \vee Y) \vee X]_{\sim_\gamma} = [X \vee (Y \vee Z)]_{\sim_\gamma} = ... = [X]_{\sim_\gamma} \vee_{\sim_\gamma} ([Y]_{\sim_\gamma} \vee_{\sim_\gamma} [Z]_{\sim_\gamma})$;

(2.b) $([X]_{\sim_\gamma} \wedge_{\sim_\gamma} [Y]_{\sim_\gamma}) \wedge_{\sim_\gamma} [Z]_{\sim_\gamma} = ... = [(X \wedge Y) \wedge X]_{\sim_\gamma} = [X \wedge (Y \wedge Z)]_{\sim_\gamma} = ... = [X]_{\sim_\gamma} \wedge_{\sim_\gamma} ([Y]_{\sim_\gamma} \wedge_{\sim_\gamma} [Z]_{\sim_\gamma})$.

(3) The absorption laws hold, as $\forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma} \in \overline{\mathcal{K}_{\mathcal{C}}} / \sim_\gamma$, we have:

(3.a) $[X]_{\sim_\gamma} \vee \sim_\gamma ([X]_{\sim_\gamma} \wedge \sim_\gamma [Y]_{\sim_\gamma}) = [X \vee (X \wedge Y)]_{\sim_\gamma} = [X]_{\sim_\gamma}$;

(3.b) $[X]_{\sim_\gamma} \wedge \sim_\gamma ([X]_{\sim_\gamma} \vee \sim_\gamma [Y]_{\sim_\gamma}) = [X \wedge (X \vee Y)]_{\sim_\gamma} = [X]_{\sim_\gamma}$.

(4) The idempotent laws hold, as $\forall [X]_{\sim_\gamma} \in \overline{\mathcal{K}_{\mathcal{C}}} / \sim_\gamma$ we have:

(4.a) $[X]_{\sim_\gamma} \vee \sim_\gamma [X]_{\sim_\gamma} = [X \vee X]_{\sim_\gamma} = [X]_{\sim_\gamma}$;

(4.b) $[X]_{\sim_\gamma} \wedge \sim_\gamma [X]_{\sim_\gamma} = [X \wedge X]_{\sim_\gamma} = [X]_{\sim_\gamma}$.

In the same way, since we had defined, from (6.6) and (6.7),

$$[X]_{\sim_\gamma} \cap_{\sim_\gamma} [Y]_{\sim_\gamma} = inf\{[X]_{\sim_\gamma}, [Y]_{\sim_\gamma}\}$$

and

$$[X]_{\sim_\gamma} \cup_{\sim_\gamma} [Y]_{\sim_\gamma} = sup\{[X]_{\sim_\gamma}, [Y]_{\sim_\gamma}\},$$

then is also possible to consider the structures $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}} / \sim_\gamma, \cup_{\sim_\gamma}, \cap_{\sim_\gamma})$and $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}} / \sim_\gamma, \cap_{\sim_\gamma}, \cup_{\sim_\gamma})$, as bounded, distributive, algebraic lattices, denoted by

$$(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}} / \sim_\gamma, \cup_{\sim_\gamma}, \cap_{\sim_\gamma}, [\varnothing]_{\sim_\gamma}, [\mathbb{R}]_{\sim_\gamma})$$

and

$$(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}} / \sim_\gamma, \cap_{\sim_\gamma}, \cup_{\sim_\gamma}, [\mathbb{R}]_{\sim_\gamma}, [\varnothing]_{\sim_\gamma}).$$

Indeed, according to Definition 4.1.6, the following properties hold.

(1) $\cup_{\sim_\gamma}$ and $\cap_{\sim_\gamma}$ are commutative, as $\forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}} / \sim_\gamma$ we have:

(1.a) $[X]_{\sim_\gamma} \cup_{\sim_\gamma} [Y]_{\sim_\gamma} = [X \cup Y]_{\sim_\gamma} = [Y \cup X]_{\sim_\gamma} = [Y]_{\sim_\gamma} \cup_{\sim_\gamma} [X]_{\sim_\gamma}$;

(1.b) $[X]_{\sim_\gamma} \cap_{\sim_\gamma} [Y]_{\sim_\gamma} = [X \cap Y]_{\sim_\gamma} = [Y \cap X]_{\sim_\gamma} = [Y]_{\sim_\gamma} \cap_{\sim_\gamma} [X]_{\sim_\gamma}$.

(2) $\cup_{\sim_\gamma}$ and $\cap_{\sim_\gamma}$ are associative, as $\forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}} / \sim_\gamma$ we have:

(2.a) $([X]_{\sim_\gamma} \cup_{\sim_\gamma} [Y]_{\sim_\gamma}) \cup_{\sim_\gamma} [Z]_{\sim_\gamma} = ... = [(X \cup Y) \cup X]_{\sim_\gamma} = [X \cup (Y \cup Z)]_{\sim_\gamma} = ... = [X]_{\sim_\gamma} \cup_{\sim_\gamma} ([Y]_{\sim_\gamma} \cup_{\sim_\gamma} [Z]_{\sim_\gamma})$;

(2.b) $([X]_{\sim_\gamma} \cap_{\sim_\gamma} [Y]_{\sim_\gamma}) \cap_{\sim_\gamma} [Z]_{\sim_\gamma} = ... = [(X \cap Y) \cap X]_{\sim_\gamma} = [X \cap (Y \cap Z)]_{\sim_\gamma} = ... = [X]_{\sim_\gamma} \cap_{\sim_\gamma} ([Y]_{\sim_\gamma} \cap_{\sim_\gamma} [Z]_{\sim_\gamma})$.

(3) The absorption laws hold, as $\forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma$, we have:

(3.a) $[X]_{\sim_\gamma} \cup \sim_\gamma ([X]_{\sim_\gamma} \cap \sim_\gamma [Y]_{\sim_\gamma}) = [X \cup (X \cap Y)]_{\sim_\gamma} = [X]_{\sim_\gamma}$;

(3.b) $[X]_{\sim_\gamma} \cap \sim_\gamma ([X]_{\sim_\gamma} \cup \sim_\gamma [Y]_{\sim_\gamma}) = [X \cap (X \cup Y)]_{\sim_\gamma} = [X]_{\sim_\gamma}$.

(4) The idempotent laws hold, as $\forall [X]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma$ we have:

(4.a) $[X]_{\sim_\gamma} \cup \sim_\gamma [X]_{\sim_\gamma} = [X \cup X]_{\sim_\gamma} = [X]_{\sim_\gamma}$;

(4.b) $[X]_{\sim_\gamma} \cap \sim_\gamma [X]_{\sim_\gamma} = [X \cap X]_{\sim_\gamma} = [X]_{\sim_\gamma}$.

Comparing now γ-relation in case $\gamma = 1$ and $\gamma = -1$, we obtain the following proposition.

Proposition 6.2.5. *Let A and B two intervals in $\mathcal{K}_{\mathcal{C}} \cup \{-\infty, +\infty, \varnothing, \mathbb{R}\}$ and considering the two equivalence relations associated to the well known LU-case: $\gamma = 1$ and $\gamma = -1$, we have that*

$$[A]_{\sim_1} \cup_{\sim_1} [B]_{\sim_1} = [A]_{\sim_1} \wedge_{\sim_1} [B]_{\sim_1}, \tag{6.8}$$

$$[A]_{\sim_1} \cap_{\sim_1} [B]_{\sim_1} = [A]_{\sim_1} \vee_{\sim_1} [B]_{\sim_1}, \tag{6.9}$$

$$[A]_{\sim_{-1}} \cup_{\sim_{-1}} [B]_{\sim_{-1}} = [A]_{\sim_{-1}} \vee_{\sim_{-1}} [B]_{\sim_{-1}}, \tag{6.10}$$

$$[A]_{\sim_{-1}} \cap_{\sim_{-1}} [B]_{\sim_{-}} = [A]_{\sim_{-1}} \wedge_{\sim_{-1}} [B]_{\sim_{-1}}. \tag{6.11}$$

Proof. As shown in Figure 6.8, is easy to verify that
$[A]_{\sim_1} \cup_{\sim_1} [B]_{\sim_1} = [A \cup B]_{\sim_1} = [A \wedge B]_{\sim_1} = [A]_{\sim_\gamma} \wedge_{\sim_1} [B]_{\sim_1}$ and
$[A]_{\sim_1} \cap_{\sim_1} [B]_{\sim_1} = [A \cap B]_{\sim_1} = [A \vee B]_{\sim_1} = [A]_{\sim_1} \vee_{\sim_1} [B]_{\sim_1}$.

In a similar way we have that
$[A]_{\sim_{-1}} \cup_{\sim_{-1}} [B]_{\sim_{-1}} = [A \cup B]_{\sim_{-1}} = [A \vee B]_{\sim_{-1}} = [A]_{\sim_{-1}} \vee_{\sim_{-1}} [B]_{\sim_{-1}}$ and
$[A]_{\sim_{-1}} \cap_{\sim_{-1}} [B]_{\sim_{-}} = [A \cap B]_{\sim_{-1}} = [A \wedge B]_{\sim_{-1}} = [A]_{\sim_{-1}} \wedge_{\sim_{-1}} [B]_{\sim_{-1}}$. □

Similarly to what was done in Subsection 5.1.2, also in this case is possible to extend the Minkowski operation $\oplus$ to all equivalence classes of the quotient-set $(\mathcal{K}_{\mathcal{C}}/\sim_\gamma)$; therefore, for all $[X]_{\sim_\gamma}, [Y]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}/\sim_\gamma$ we define:

$$\oplus_{\sim_\gamma} : (\mathcal{K}_{\mathcal{C}}/\sim_\gamma) \times (\mathcal{K}_{\mathcal{C}}/\sim_\gamma) \to (\mathcal{K}_{\mathcal{C}}/\sim_\gamma)$$

such that: $([X]_{\sim_\gamma}, [Y]_{\sim_\gamma}) \to [X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Y]_{\sim_\gamma} \stackrel{def}{=} [X_0 \oplus Y_0]_{\sim_\gamma}$

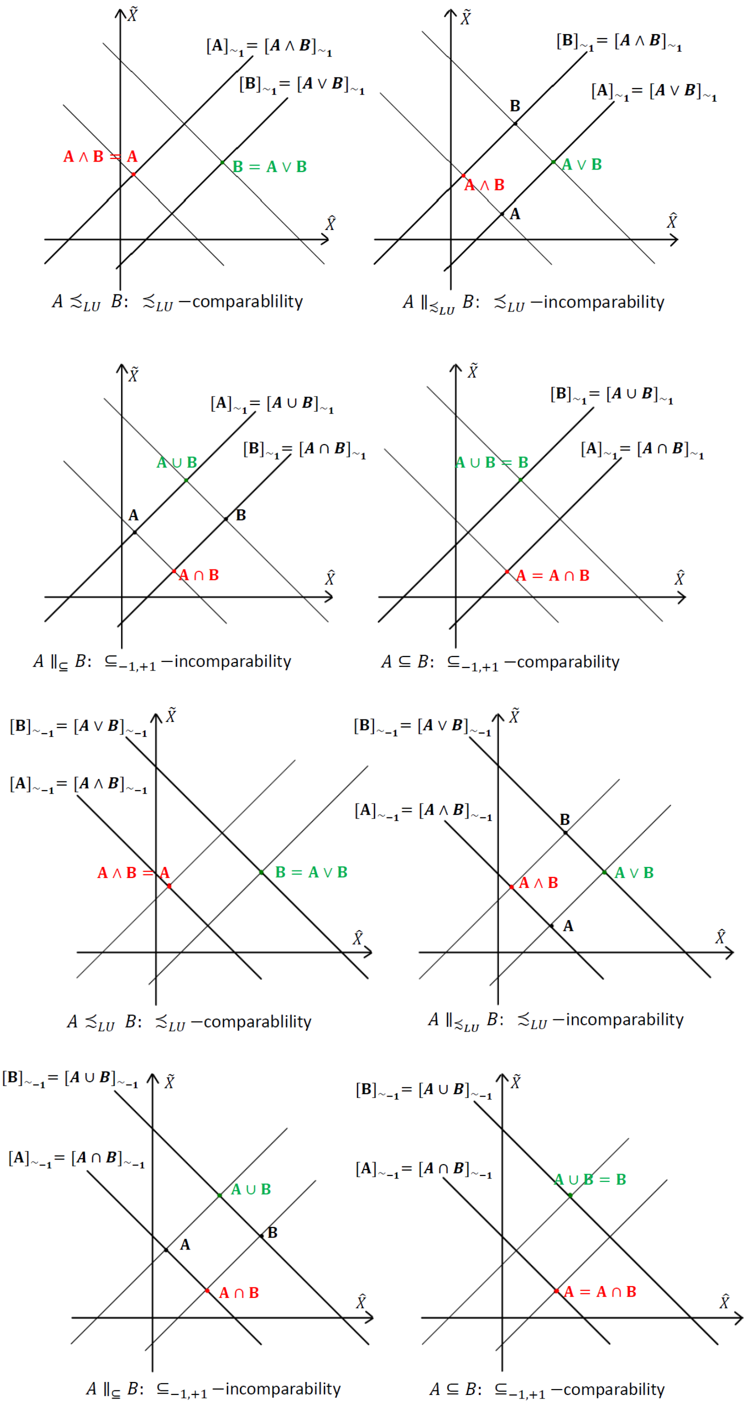


Figure 6.8: Representation of the properties related to : $\cup_{\sim_\gamma}$, $\cap_{\sim_\gamma}$, $\vee_{\sim_\gamma}$ and $\wedge_{\sim_\gamma}$ in case $\gamma = \pm 1$.

where $\forall X = (\widehat{x}; \widetilde{x}) \in \mathcal{K}_\mathcal{C}$, $X_0 = (\widehat{x}_0; 0)$ stands for the representative element of the class $[X]_{\sim_\gamma} \in \mathcal{K}_\mathcal{C}/\sim_\gamma$ and the addition $\oplus$ is the classical Minkowski operation in $\mathcal{K}_\mathcal{C}$:

$$X_0 \oplus Y_0 = (\widehat{x}_0; 0) \oplus (\widehat{y}_0; 0) = (\widehat{x}_0 + \widehat{y}_0; 0).$$

It follows that, considering the LU-case ($\gamma = 1$), for all $[X]_{\sim_1}, [Y]_{\sim_1} \in \mathcal{K}_\mathcal{C}/\sim_1$, we have

$$[X]_{\sim_1} \oplus_{\sim_1} [Y]_{\sim_1} = [X_0 \oplus Y_0]_{\sim_1} = [(\widehat{x}_0; 0) \oplus (\widehat{y}_0; 0)]_{\sim_1} = [(x^-; 0) \oplus (y^-; 0)]_{\sim_1}$$
$$= [(x^- + y^-; 0)]_{\sim_1}$$

(see also Figure 6.9) while, in the more general case, according to (6.3), it is

$$[X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Y]_{\sim_\gamma} = [X_0 \oplus Y_0]_{\sim_\gamma} = [(\widehat{x}_0; 0) \oplus (\widehat{y}_0; 0)]_{\sim_\gamma}$$
$$= \left[\left(\widehat{x} - \frac{\widetilde{x}}{\gamma^+}; 0\right) \oplus \left(\widehat{y} - \frac{\widetilde{y}}{\gamma^+}; 0\right)\right]_{\sim_\gamma} = \left[\left(\widehat{x} - \frac{\widetilde{x}}{\gamma^+} + \widehat{y} - \frac{\widetilde{y}}{\gamma^+}; 0\right)\right]_{\sim_\gamma}.$$

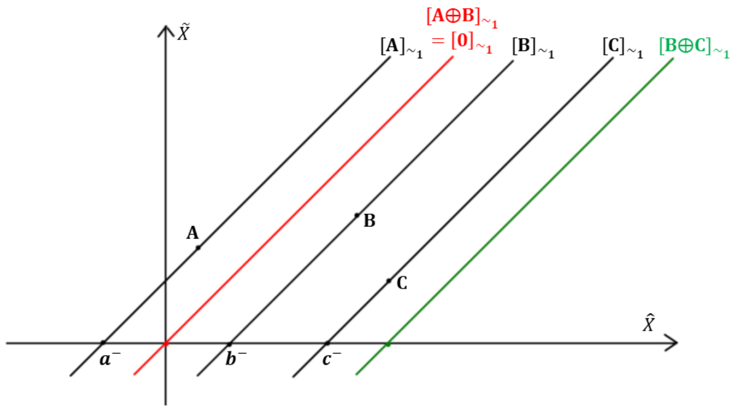


Figure 6.9: Representation of the operation $\oplus_{\sim_\gamma}$ in case $\gamma = 1$; in particular in the example we have $a^- = -b^-$.

It is not difficult to prove that the following properties hold.

(1) $\oplus_{\sim_\gamma}$ is commutative; indeed, $\forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma} \in \mathcal{K}_\mathcal{C}/\sim_\gamma$ we have: $[X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Y]_{\sim_\gamma} = [X_0 \oplus Y_0]_{\sim_\gamma} = [Y_0 \oplus X_0]_{\sim_\gamma} = [Y]_{\sim_\gamma} \oplus_{\sim_\gamma} [X]_{\sim_\gamma}$.

(2) $\oplus_{\sim_\gamma}$ is associative; indeed, $\forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in \mathcal{K}_\mathcal{C}/\sim_\gamma$ we have: $([X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Y]_{\sim_\gamma}) \oplus_{\sim_\gamma} [Z]_{\sim_\gamma} = ... = [(X_0 \oplus Y_0) \oplus Z_0]_{\sim_\gamma} = [X_0 \oplus (Y_0 \oplus Z_0)]_{\sim_\gamma} = ... = [X]_{\sim_\gamma} \oplus_{\sim_\gamma} ([Y]_{\sim_\gamma} \oplus_{\sim_\gamma} [Z]_{\sim_\gamma})$.

(3.a) $\oplus_{\sim_\gamma}$ is left and right distributive over $\vee_{\sim_\gamma}$ and over $\wedge_{\sim_\gamma}$; indeed, $\forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in \mathcal{K}_\mathcal{C}/\sim_\gamma$, we have:

(3.a.1) $[X]_{\sim_\gamma} \oplus_{\sim_\gamma} ([Y]_{\sim_\gamma} \vee_{\sim_\gamma} [Z]_{\sim_\gamma}) = ([X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Y]_{\sim_\gamma}) \vee_{\sim_\gamma} ([X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Z]_{\sim_\gamma})$;

(3.a.2) $([X]_{\sim_\gamma} \vee_{\sim_\gamma} [Y]_{\sim_\gamma}) \oplus_{\sim_\gamma} [Z]_{\sim_\gamma} = ([X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Z]_{\sim_\gamma}) \vee_{\sim_\gamma} ([Y]_{\sim_\gamma} \oplus_{\sim_\gamma} [Z]_{\sim_\gamma})$;

(3.a.3) $[X]_{\sim_\gamma} \oplus_{\sim_\gamma} ([Y]_{\sim_\gamma} \wedge_{\sim_\gamma} [Z]_{\sim_\gamma}) = ([X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Y]_{\sim_\gamma}) \wedge_{\sim_\gamma} ([X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Z]_{\sim_\gamma})$;

(3.a.4) $([X]_{\sim_\gamma} \wedge_{\sim_\gamma} [Y]_{\sim_\gamma}) \oplus_{\sim_\gamma} [Z]_{\sim_\gamma} = ([X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Z]_{\sim_\gamma}) \wedge_{\sim_\gamma} ([Y]_{\sim_\gamma} \oplus_{\sim_\gamma} [Z]_{\sim_\gamma})$.

(3.b) $\oplus_{\sim_\gamma}$ is left and right distributive over $\cup_{\sim_\gamma}$ and over $\cap_{\sim_\gamma}$; indeed, $\forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in \mathcal{K}_\mathcal{C}/\sim_\gamma$, we have:

(3.b.1) $[X]_{\sim_\gamma} \oplus_{\sim_\gamma} ([Y]_{\sim_\gamma} \cup_{\sim_\gamma} [Z]_{\sim_\gamma}) = ([X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Y]_{\sim_\gamma}) \cup_{\sim_\gamma} ([X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Z]_{\sim_\gamma})$;

(3.b.2) $([X]_{\sim_\gamma} \cup_{\sim_\gamma} [Y]_{\sim_\gamma}) \oplus_{\sim_\gamma} [Z]_{\sim_\gamma} = ([X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Z]_{\sim_\gamma}) \cup_{\sim_\gamma} ([Y]_{\sim_\gamma} \oplus_{\sim_\gamma} [Z]_{\sim_\gamma})$;

(3.b.3) $[X]_{\sim_\gamma} \oplus_{\sim_\gamma} ([Y]_{\sim_\gamma} \cap_{\sim_\gamma} [Z]_{\sim_\gamma}) = ([X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Y]_{\sim_\gamma}) \cap_{\sim_\gamma} ([X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Z]_{\sim_\gamma})$;

(3.b.4) $([X]_{\sim_\gamma} \cap_{\sim_\gamma} [Y]_{\sim_\gamma}) \oplus_{\sim_\gamma} [Z]_{\sim_\gamma} = ([X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Z]_{\sim_\gamma}) \cap_{\sim_\gamma} ([Y]_{\sim_\gamma} \oplus_{\sim_\gamma} [Z]_{\sim_\gamma})$.

Other relevant properties are the following.

(4) $\oplus_{\sim_\gamma}$ has a neutral element in $\mathcal{K}_\mathcal{C}/\sim_\gamma$ which is $i_{\oplus_\sim} = [0]_{\sim_\gamma} = [(0;0)]_{\sim_\gamma}$; indeed, $[X]_{\sim_\gamma} \oplus_{\sim_\gamma} [0]_{\sim_\gamma} = [(\widehat{x}_0;0) \oplus (0;0)]_{\sim_\gamma} = [(\widehat{x}_0 + 0; 0+0)]_{\sim_\gamma} = [(\widehat{x}_0;0)]_{\sim_\gamma} = [X]_{\sim_\gamma}, \forall [X]_{\sim_\gamma} \in \mathcal{K}_\mathcal{C}/\sim_\gamma$.

(5) $\oplus_{\sim_\gamma}$ has different absorbing elements which are:

(5.a) $[-\infty]_{\sim_\gamma} = [(-\infty;0)]_{\sim_\gamma}$, as $\forall [X]_{\sim_\gamma} \in \mathcal{K}_\mathcal{C}^{-\infty}/\sim_\gamma$, we have that $[X]_{\sim_\gamma} \oplus_{\sim_\gamma} [(-\infty;0)]_{\sim_\gamma} = [X_0 \oplus (-\infty;0)]_{\sim_\gamma} = [(-\infty;0)]_{\sim_\gamma}$;

(5.b) $[+\infty]_{\sim_\gamma} = [(+\infty;0)]_{\sim_\gamma}$, as $\forall [X]_{\sim_\gamma} \in \mathcal{K}_\mathcal{C}^{+\infty}/\sim_\gamma$, we have that $[X]_{\sim_\gamma} \oplus_{\sim_\gamma} [(+\infty;0)]_{\sim_\gamma} = [X_0 \oplus (+\infty;0)]_{\sim_\gamma} = [(+\infty;0)]_{\sim_\gamma}$;
or, we can also consider:

(5.c) $[\varnothing]_{\sim_\gamma} = [(0;-\infty)]_{\sim_\gamma} \equiv [+\infty]_{\sim_\gamma} = [(+\infty;0)]_{\sim_\gamma}$, as $\forall [X]_{\sim_\gamma} \in \mathcal{K}_\mathcal{C}^{\varnothing}/\sim_\gamma$, we have that $[X]_{\sim_\gamma} \oplus_{\sim_\gamma} [\varnothing]_{\sim_\gamma} = [X_0 \oplus (+\infty;0)]_{\sim_\gamma} = [(+\infty;0)]_{\sim_\gamma} \equiv [\varnothing]_{\sim_\gamma}$;

(5.d) $[\mathbb{R}]_{\sim_\gamma} = [(0;+\infty)]_{\sim_\gamma} \equiv [-\infty]_{\sim_\gamma} = [(-\infty;0)]_{\sim_\gamma}$, as $\forall [X]_{\sim_\gamma} \in \mathcal{K}_\mathcal{C}^{\mathbb{R}}/\sim_\gamma$, we have that $[X]_{\sim_\gamma} \oplus_{\sim_\gamma} [\mathbb{R}]_{\sim_\gamma} = [X_0 \oplus (-\infty;0)]_{\sim_\gamma} = [(-\infty;0)]_{\sim_\gamma} \equiv [\mathbb{R}]_{\sim_\gamma}$.

(6) Existance of the additive inverse (opposite element with respect to $\oplus_{\sim_\gamma}$):

$\forall [X]_{\sim_\gamma} = [X_0]_{\sim_\gamma} = [(\widehat{x}_0;0)]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}/\sim_\gamma$, it exists the element $[X]'_{\sim_\gamma} = [X'_0]_{\sim_\gamma} = [-X_0]_{\sim_\gamma} \stackrel{def}{=} [(-\widehat{x}_0;0)]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}/\sim_\gamma$ such that: $[X]_{\sim_\gamma} \oplus_{\sim_\gamma} [X']_{\sim_\gamma} = [X_0 \oplus X'_0]_{\sim_\gamma} = [(\widehat{x}_0 - \widehat{x}_0;0)]_{\sim_\gamma} = [0]_{\sim_\gamma}$.

(7.a) $[X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Y]_{\sim_\gamma} = [+\infty]_{\sim_\gamma} \Rightarrow [X]_{\sim_\gamma} = [+\infty]_{\sim_\gamma}$ or $[Y]_{\sim_\gamma} = [+\infty]_{\sim_\gamma}$, $\forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{+\infty}/\sim_\gamma$.

(7.b) $[X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Y]_{\sim_\gamma} = [-\infty]_{\sim_\gamma} \Rightarrow [X]_{\sim_\gamma} = [-\infty]_{\sim_\gamma}$ or $[Y]_{\sim_\gamma} = [-\infty]_{\sim_\gamma}$, $\forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{-\infty}/\sim_\gamma$.

(7.c) $[X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Y]_{\sim_\gamma} = [\mathbb{R}]_{\sim_\gamma} \Rightarrow [X]_{\sim_\gamma} = [R]_{\sim_\gamma}$ or $[Y]_{\sim_\gamma} = [R]_{\sim_\gamma}$, $\forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{\mathbb{R}}/\sim_\gamma$.

(7.d) $[X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Y]_{\sim_\gamma} = [\varnothing]_{\sim_\gamma} \Rightarrow [X]_{\sim_\gamma} = [\varnothing]_{\sim_\gamma}$ or $[Y]_{\sim_\gamma} = [\varnothing]_{\sim_\gamma}$, $\forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{\varnothing}/\sim_\gamma$.

The properties we have just analized allow us to introduce the following propositions.

Proposition 6.2.6. *The structures* $(\mathcal{K}_{\mathcal{C}}^{-\infty}/\sim_\gamma, \vee_{\sim_\gamma}, \oplus_{\sim_\gamma})$, $(\mathcal{K}_{\mathcal{C}}^{+\infty}/\sim_\gamma, \wedge_{\sim_\gamma}, \oplus_{\sim_\gamma})$, $(\mathcal{K}_{\mathcal{C}}^{\varnothing}/\sim_\gamma, \cup_{\sim_\gamma}, \oplus_{\sim_\gamma})$ *and* $(\mathcal{K}_{\mathcal{C}}^{\mathbb{R}}/\sim_\gamma, \cap_{\sim_\gamma}, \oplus_{\sim_\gamma})$ *are commutative semirings.*

Proof. $(\mathcal{K}_{\mathcal{C}}^{-\infty}/\sim_\gamma, \vee_{\sim_\gamma}, \oplus_{\sim_\gamma})$ is a commutative semiring, as:

1a) $(\mathcal{K}_{\mathcal{C}}^{-\infty}/\sim_\gamma, \vee_{\sim_\gamma})$ is a commutative monoid with neutral element $i_{\vee_\sim} = [-\infty]_{\sim_\gamma}$:

- (i) $\vee_{\sim_\gamma}$ is associative;
- (ii) $\vee_{\sim_\gamma}$ has the neutral element $i_{\vee_\sim} = [-\infty]_{\sim_\gamma} = [(-\infty;0)]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{-\infty}/\sim_\gamma$: $[X]_{\sim_\gamma} \vee_{\sim_\gamma} [(-\infty;0)]_{\sim_\gamma} = [X \vee (-\infty;0)]_{\sim_\gamma} = [X]_{\sim_\gamma}, \forall [X]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{-\infty}/\sim_\gamma$
  (so, $i_{\vee_\sim} = [-\infty]_{\sim_\gamma} = [(-\infty;0)]$ is the 0-element of the semiring);
- (iii) $\vee_{\sim_\gamma}$ is commutative.

2a) $(\mathcal{K}_{\mathcal{C}}^{-\infty}/\sim_\gamma, \oplus_{\sim_\gamma})$ is a commutative monoid with neutral element $i_{\oplus_\sim} = [0]_{\sim_\gamma}$:

- (i) $\oplus_{\sim_\gamma}$ is associative;
- (ii) $\oplus_{\sim_\gamma}$ has the neutral element $i_{\oplus_\sim} = [0]_{\sim_\gamma} = [(0;0)]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{-\infty}/\sim_\gamma$: $[X]_{\sim_\gamma} \oplus_{\sim_\gamma} [(0,0)]_{\sim_\gamma} = [(0;0)]_{\sim_\gamma} \oplus_{\sim_\gamma} [X]_{\sim_\gamma} = [X]_{\sim_\gamma}$, $\forall [X]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{-\infty}/\sim_\gamma$
  (so, $i_{\oplus_\sim} = [0]_{\sim_\gamma} = [(0;0)]_{\sim_\gamma}$ is the 1-element of the semiring);
- (iii) $\oplus_{\sim_\gamma}$ is commutative.

3a) $\oplus_{\sim_\gamma}$ is left and right distributive over $\vee_{\sim_\gamma}$:

- (i) $[X]_{\sim_\gamma} \oplus_{\sim_\gamma} ([Y]_{\sim_\gamma} \vee_{\sim_\gamma} [Z]_{\sim_\gamma}) = ([X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Y]_{\sim_\gamma}) \vee_{\sim_\gamma} ([X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Z]_{\sim_\gamma}), \forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{-\infty}/\sim_\gamma$;

(ii) $([X]_{\sim_\gamma} \vee_{\sim_\gamma} [Y]_{\sim_\gamma}) \oplus_{\sim_\gamma} [Z]_{\sim_\gamma} = ([X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Z]_{\sim_\gamma}) \vee_{\sim_\gamma} ([Y]_{\sim_\gamma} \oplus_{\sim_\gamma} [Z]_{\sim_\gamma}), \forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{-\infty}/\sim_\gamma$.

4a) $i_{\vee_\sim} = [-\infty]_{\sim_\gamma} = [(-\infty;0)]_{\sim_\gamma}$ is the absorbing element for $\oplus_{\sim_\gamma}$:

$[(-\infty;0)]_{\sim_\gamma} \oplus_{\sim_\gamma} [X]_{\sim_\gamma} = [X]_{\sim_\gamma} \oplus_{\sim_\gamma} [(-\infty;0)]_{\sim_\gamma} = [(-\infty;0)]_{\sim_\gamma}, \forall [X]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{-\infty}/\sim_\gamma$.

Analogously, also the structure $(\mathcal{K}_{\mathcal{C}}^{+\infty}/\sim_\gamma, \wedge_{\sim_\gamma}, \oplus_{\sim_\gamma})$ is a commutative semiring, as:

1b) $(\mathcal{K}_{\mathcal{C}}^{+\infty}/\sim_\gamma, \wedge_{\sim_\gamma})$ is a commutative monoid with neutral element $i_{\wedge_\sim} = [+\infty]_{\sim_\gamma}$:

(i) $\wedge_{\sim_\gamma}$ is associative;

(ii) $\wedge_{\sim_\gamma}$ has the neutral element $i_{\wedge_\sim} = [+\infty]_{\sim_\gamma} = [(+\infty;0)]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{+\infty}/\sim_\gamma : [X]_{\sim_\gamma} \wedge_{\sim_\gamma} [(+\infty;0)]_{\sim_\gamma} = [X \wedge (+\infty;0)]_{\sim_\gamma} = [X]_{\sim_\gamma}, \forall [X]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{+\infty}/\sim_\gamma$
(so, $i_{\wedge_\sim} = [+\infty]_{\sim_\gamma} = [(+\infty;0)]$ is the 0-element of the semiring);

(iii) $\wedge_{\sim_\gamma}$ is commutative.

2b) $(\mathcal{K}_{\mathcal{C}}^{+\infty}/\sim_\gamma, \oplus_{\sim_\gamma})$ is a commutative monoid with neutral element $i_{\oplus_\sim} = [0]_{\sim_\gamma}$:

(i) $\oplus_{\sim_\gamma}$ is associative;

(ii) $\oplus_{\sim_\gamma}$ has the neutral element $i_{\oplus_\sim} = [0]_{\sim_\gamma} = [(0;0)]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{+\infty}/\sim_\gamma$: $[X]_{\sim_\gamma} \oplus_{\sim_\gamma} [(0,0)]_{\sim_\gamma} = [(0;0)]_{\sim_\gamma} \oplus_{\sim_\gamma} [X]_{\sim_\gamma} = [X]_{\sim_\gamma}$, $\forall [X]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{+\infty}/\sim_\gamma$
(so, $i_{\oplus_\sim} = [0]_{\sim_\gamma} = [(0;0)]_{\sim_\gamma}$ is the 1-element of the semiring);

(iii) $\oplus_{\sim_\gamma}$ is commutative.

3b) $\oplus_{\sim_\gamma}$ is left and right distributive over $\wedge_{\sim_\gamma}$:

(i) $[X]_{\sim_\gamma} \oplus_{\sim_\gamma} ([Y]_{\sim_\gamma} \wedge_{\sim_\gamma} [Z]_{\sim_\gamma}) = ([X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Y]_{\sim_\gamma}) \wedge_{\sim_\gamma} ([X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Z]_{\sim_\gamma}), \forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \mathcal{K}_{\mathcal{C}}^{+\infty}/\sim_\gamma$;

(ii) $([X]_{\sim_\gamma} \wedge_{\sim_\gamma} [Y]_{\sim_\gamma}) \oplus_{\sim_\gamma} [Z]_{\sim_\gamma} = ([X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Z]_{\sim_\gamma}) \wedge_{\sim_\gamma} ([Y]_{\sim_\gamma} \oplus_{\sim_\gamma} [Z]_{\sim_\gamma}), \forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{+\infty}/\sim_\gamma$.

4b) $i_{\wedge_\sim} = [+\infty]_{\sim_\gamma} = [(+\infty;0)]_{\sim_\gamma}$ is the absorbing element for $\oplus_{\sim_\gamma}$:

$[(+\infty;0)]_{\sim_\gamma} \oplus_{\sim_\gamma} [X]_{\sim_\gamma} = [X]_{\sim_\gamma} \oplus_{\sim_\gamma} [(+\infty;0)]_{\sim_\gamma} = [(+\infty;0)]_{\sim_\gamma}, \forall [X]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{+\infty}/\sim_\gamma$.

Similarly, $(\mathcal{K}_{\mathcal{C}}^{\varnothing}/\sim_\gamma, \cup_{\sim_\gamma}, \oplus_{\sim_\gamma})$ is a commutative semiring, as:

1c) $(\mathcal{K}_{\mathcal{C}}^{\varnothing}/\sim_\gamma, \cup_{\sim_\gamma})$ is a commutative monoid with neutral element $i_{\cup_\sim} = [\varnothing]_{\sim_\gamma}$:

(i) $\cup_{\sim_\gamma}$ is associative;

(ii) $\cup_{\sim_\gamma}$ has the neutral element $i_{\cup_\sim} = [\varnothing]_{\sim_\gamma} = [(0;-\infty)]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{\varnothing}/\sim_\gamma : [X]_{\sim_\gamma} \cup_{\sim_\gamma} [(0;-\infty)]_{\sim_\gamma} = [X \cup (0;-\infty)]_{\sim_\gamma} = [X]_{\sim_\gamma}, \forall [X]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{\varnothing}/\sim_\gamma$
(so, $i_{\cup_\sim} = [\varnothing]_{\sim_\gamma} = [(0;-\infty)]$ is the 0-element of the semiring);

(iii) $\cup_{\sim_\gamma}$ is commutative.

2c) $(\mathcal{K}_{\mathcal{C}}^{\varnothing}/\sim_\gamma, \oplus_{\sim_\gamma})$ is a commutative monoid with neutral element $i_{\oplus_\sim} = [0]_{\sim_\gamma}$:

(i) $\oplus_{\sim_\gamma}$ is associative;

(ii) $\oplus_{\sim_\gamma}$ has the neutral element $i_{\oplus_\sim} = [0]_{\sim_\gamma} = [(0;0)]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{\oplus}/\sim_\gamma$: $[X]_{\sim_\gamma} \oplus_{\sim_\gamma} [(0,0)]_{\sim_\gamma} = [(0;0)]_{\sim_\gamma} \oplus_{\sim_\gamma} [X]_{\sim_\gamma} = [X]_{\sim_\gamma}, \forall [X]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{\varnothing}/\sim_\gamma$
(so, $i_{\oplus_\sim} = [0]_{\sim_\gamma} = [(0;0)]_{\sim_\gamma}$ is the 1-element of the semiring);

(iii) $\oplus_{\sim_\gamma}$ is commutative.

3c) $\oplus_{\sim_\gamma}$ is left and right distributive over $\cup_{\sim_\gamma}$:

(i) $[X]_{\sim_\gamma} \oplus_{\sim_\gamma} ([Y]_{\sim_\gamma} \cup_{\sim_\gamma} [Z]_{\sim_\gamma}) = ([X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Y]_{\sim_\gamma}) \cup_{\sim_\gamma} ([X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Z]_{\sim_\gamma}), \forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in K_C^{\varnothing}/\sim_\gamma$;

(ii) $([X]_{\sim_\gamma} \cup_{\sim_\gamma} [Y]_{\sim_\gamma}) \oplus_{\sim_\gamma} [Z]_{\sim_\gamma} = ([X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Z]_{\sim_\gamma}) \cup_{\sim_\gamma} ([Y]_{\sim_\gamma} \oplus_{\sim_\gamma} [Z]_{\sim_\gamma}), \forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in K_C^{\varnothing}/\sim_\gamma$.

4c) $i_{\cup_\sim} = [\varnothing]_{\sim_\gamma} = [(0;-\infty)]_{\sim_\gamma} \equiv [+\infty]_{\sim_\gamma} = [(+\infty;0)]_{\sim_\gamma}$ is the absorbing element for $\oplus_{\sim_\gamma}$:

$$[\varnothing]_{\sim_\gamma} \oplus_{\sim_\gamma} [X]_{\sim_\gamma} = [X]_{\sim_\gamma} \oplus_{\sim_\gamma} [\varnothing]_{\sim_\gamma} = [\varnothing]_{\sim_\gamma}, \forall [X]_{\sim_\gamma} \in K_C^{\varnothing}/\sim_\gamma.$$

Likewise, $(\mathcal{K}_{\mathcal{C}}^{\mathbb{R}}/\sim_\gamma, \cap_{\sim_\gamma}, \oplus_{\sim_\gamma})$ is a commutative semiring, as:

1d) $(\mathcal{K}_{\mathcal{C}}^{\mathbb{R}}/\sim_\gamma, \cap_{\sim_\gamma})$ is a commutative monoid with neutral element $i_{\wedge_\sim} = [\mathbb{R}]_{\sim_\gamma}$:

(i) $\cap_{\sim_\gamma}$ is associative;

(ii) $\cap_{\sim_\gamma}$ has the neutral element $i_{\cap_\sim} = [\mathbb{R}]_{\sim_\gamma} = [(0;+\infty)]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{\mathbb{R}}/\sim_\gamma$: $[X]_{\sim_\gamma} \cap_{\sim_\gamma} [(0;+\infty)]_{\sim_\gamma} = [X \cap (0;+\infty)]_{\sim_\gamma} = [X]_{\sim_\gamma}, \forall [X]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{\mathbb{R}}/\sim_\gamma$ (so, $i_{\cap_\sim} = [\mathbb{R}]_{\sim_\gamma} = [(0;+\infty)]$ is the 0-element of the semiring);

(iii) $\cap_{\sim_\gamma}$ is commutative.

2d) $(\mathcal{K}_{\mathcal{C}}^{\mathbb{R}}/\sim_\gamma, \oplus_{\sim_\gamma})$ is a commutative monoid with neutral element $i_{\oplus_\sim} = [0]_{\sim_\gamma}$:

(i) $\oplus_{\sim_\gamma}$ is associative;

(ii) $\oplus_{\sim_\gamma}$ has the neutral element $i_{\oplus_\sim} = [0]_{\sim_\gamma} = [(0;0)]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{\mathbb{R}}/\sim_\gamma$: $[X]_{\sim_\gamma} \oplus_{\sim_\gamma} [(0,0)]_{\sim_\gamma} = [(0;0)]_{\sim_\gamma} \oplus_{\sim_\gamma} [X]_{\sim_\gamma} = [X]_{\sim_\gamma}, \forall [X]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{\mathbb{R}}/\sim_\gamma$
(so, $i_{\oplus_\sim} = [0]_{\sim_\gamma} = [(0;0)]_{\sim_\gamma}$ is the 1-element of the semiring);

(iii) $\oplus_{\sim_\gamma}$ is commutative.

3d) $\oplus_{\sim_\gamma}$ is left and right distributive over $\cap_{\sim_\gamma}$:

(i) $[X]_{\sim_\gamma} \oplus_{\sim_\gamma} ([Y]_{\sim_\gamma} \cap_{\sim_\gamma} [Z]_{\sim_\gamma}) = ([X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Y]_{\sim_\gamma}) \cap_{\sim_\gamma} ([X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Z]_{\sim_\gamma}), \forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{\mathbb{R}} / \sim_\gamma$;

(ii) $([X]_{\sim_\gamma} \cap_{\sim_\gamma} [Y]_{\sim_\gamma}) \oplus_{\sim_\gamma} [Z]_{\sim_\gamma} = ([X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Z]_{\sim_\gamma}) \cap_{\sim_\gamma} ([Y]_{\sim_\gamma} \oplus_{\sim_\gamma} [Z]_{\sim_\gamma}), \forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{\mathbb{R}} / \sim_\gamma$.

4d) $i_{\cap_\sim} = [\mathbb{R}]_{\sim_\gamma} = [(0; +\infty)]_{\sim_\gamma} \equiv [-\infty]_{\sim_\gamma} = [(-\infty; 0)]_{\sim_\gamma}$ is the absorbing element for $\oplus_{\sim_\gamma}$:

$$[\mathbb{R}]_{\sim_\gamma} \oplus_{\sim_\gamma} [X]_{\sim_\gamma} = [X]_{\sim_\gamma} \oplus_{\sim_\gamma} [\mathbb{R}]_{\sim_\gamma} = [\mathbb{R}]_{\sim_\gamma}\ , \forall [X]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}^{\mathbb{R}} / \sim_\gamma.$$

□

Proposition 6.2.7. *The structures* $(\mathcal{K}_{\mathcal{C}}^{-\infty} / \sim_\gamma, \vee_{\sim_\gamma}, \oplus_{\sim_\gamma})$, $(\mathcal{K}_{\mathcal{C}}^{+\infty} / \sim_\gamma, \wedge_{\sim_\gamma}, \oplus_{\sim_\gamma})$, $(\mathcal{K}_{\mathcal{C}}^{\varnothing} / \sim_\gamma, \cup_{\sim_\gamma}, \oplus_{\sim_\gamma})$ *and* $(\mathcal{K}_{\mathcal{C}}^{\mathbb{R}} / \sim_\gamma, \cap_{\sim_\gamma}, \oplus_{\sim_\gamma})$ *are zero-sum-free semirings.*

Proof. The proof is the same as that of Proposition 6.2.3. □

Proposition 6.2.8. *The structures* $(\mathcal{K}_{\mathcal{C}}^{-} / \sim_\gamma, \vee_{\sim_\gamma}, \oplus_{\sim_\gamma})$, $(\mathcal{K}_{\mathcal{C}}^{+} / \sim_\gamma, \wedge_{\sim_\gamma}, \oplus_{\sim_\gamma})$, $(\mathcal{K}_{\mathcal{C}}^{\varnothing} / \sim_\gamma, \cup_{\sim_\gamma}, \oplus_{\sim_\gamma})$ *and* $(\mathcal{K}_{\mathcal{C}}^{\mathbb{R}} / \sim_\gamma, \cap_{\sim_\gamma}, \oplus_{\sim_\gamma})$ *are zero-divisor-free semirings.*

Proof. The proof is immediate since, for definition, we have
$[X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Y]_{\sim_\gamma} = [(\widehat{x} + \widehat{y}; \widetilde{x} + \widetilde{y})]_{\sim_\gamma}$, it follows that, respectively:
$[X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Y]_{\sim_\gamma} = [-\infty]_{\sim_\gamma} \Rightarrow [X]_{\sim_\gamma} = [-\infty]_{\sim_\gamma}$ or $[Y]_{\sim_\gamma} = [-\infty]_{\sim_\gamma}$,
$[X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Y]_{\sim_\gamma} = [+\infty]_{\sim_\gamma} \Rightarrow [X]_{\sim_\gamma} = [+\infty]_{\sim_\gamma}$ or $[Y]_{\sim_\gamma} = [+\infty]_{\sim_\gamma}$,
$[X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Y]_{\sim_\gamma} = [\varnothing]_{\sim_\gamma} \Rightarrow [X]_{\sim_\gamma} = [\varnothing]_{\sim_\gamma}$ or $[Y]_{\sim_\gamma} = [\varnothing]_{\sim_\gamma}$ and
$[X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Y]_{\sim_\gamma} = [\mathbb{R}]_{\sim_\gamma} \Rightarrow [X]_{\sim_\gamma} = [\mathbb{R}]_{\sim_\gamma}$ or $[Y]_{\sim_\gamma} = [\mathbb{R}]_{\sim_\gamma}$. □

The four semirings of proposition 6.2.7 and 6.2.8 , being zero-sum-free and zero-divisor-free, are information algebras.

Concluding, similarly to what was done in the previous Sections, all the quotient structures examined in this paragraph, i.e., $(\overline{\mathcal{K}_{\mathcal{C}}} / \sim_\gamma, \vee_{\sim_\gamma}, \wedge_{\sim_\gamma})$, $(\overline{\mathcal{K}_{\mathcal{C}}} / \sim_\gamma, \wedge_{\sim_\gamma}, \vee_{\sim_\gamma})$, $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}} / \sim_\gamma, \cup_{\sim_\gamma}, \cap_{\sim_\gamma})$, $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}} / \sim_\gamma, \cap_{\sim_\gamma}, \cup_{\sim_\gamma})$, as well as $(\mathcal{K}_{\mathcal{C}}^{-\infty} / \sim_\gamma, \vee_{\sim_\gamma}, \oplus_{\sim_\gamma})$, $(\mathcal{K}_{\mathcal{C}}^{+\infty} / \sim_\gamma, \wedge_{\sim_\gamma}, \oplus_{\sim_\gamma})$, $(\mathcal{K}_{\mathcal{C}}^{\varnothing} / \sim_\gamma, \cup_{\sim_\gamma}, \oplus_{\sim_\gamma})$ and $(\mathcal{K}_{\mathcal{C}}^{\mathbb{R}} / \sim_\gamma, \cap_{\sim_\gamma}, \oplus_{\sim_\gamma})$, can be summarized as follows.

1) $(\overline{\mathcal{K}_{\mathcal{C}}} / \sim_\gamma, \vee_{\sim_\gamma}, \wedge_{\sim_\gamma}; [-\infty]_{\sim_\gamma}, [+\infty]_{\sim_\gamma})$ is a commutative, idempotent, zero-sum-free, zero-divisor free semiring, as:

1.1 $\vee_{\sim_\gamma}$ is associative;

1.2 $\vee_{\sim_\gamma}$ is commutative;

1.3 $\vee_{\sim_\gamma}$ has the neutral element: $[-\infty]_{\sim_\gamma}$ (zero of the semiring);

1.4 $\vee_{\sim_\gamma}$ is idempotent;

[so $(\overline{\mathcal{K}_\mathcal{C}}/\sim_\gamma, \vee_{\sim_\gamma})$ is a commutative, idempotent monoid]

1.5 $\wedge_{\sim_\gamma}$ is associative;

1.6 $\wedge_{\sim_\gamma}$ is commutative;

1.7 $\wedge_{\sim_\gamma}$ has the neutral element: $[+\infty]_{\sim_\gamma}$ (unity of the semiring);

1.8 $\wedge$ is idempotent;

[so $(\overline{\mathcal{K}_\mathcal{C}}/\sim_\gamma, \wedge_{\sim_\gamma})$ is a commutative, idempotent monoid]

1.9 $\wedge_{\sim_\gamma}$ is distributive with respect to $\vee_{\sim_\gamma}$;

1.10 $[-\infty]_{\sim_\gamma}$ is the absorbing element for $\wedge_{\sim_\gamma}$

[so $(\overline{\mathcal{K}_\mathcal{C}}/\sim_\gamma, \vee_{\sim_\gamma}, \wedge_{\sim_\gamma})$ is a commutative, idempotent semiring]

1.11 $(\overline{\mathcal{K}_\mathcal{C}}/\sim_\gamma, \vee_{\sim_\gamma}, \wedge_{\sim_\gamma})$ is zero-sum-free;

1.12 $(\overline{\mathcal{K}_\mathcal{C}}/\sim_\gamma, \vee_{\sim_\gamma}, \wedge_{\sim_\gamma})$ is zero-divisor-free;

[so $(\overline{\mathcal{K}_\mathcal{C}}/\sim_\gamma, \vee_{\sim_\gamma}, \wedge_{\sim_\gamma})$ is a zero-sum-free, zero-divisor-free semiring]

2) $(\overline{\mathcal{K}_\mathcal{C}}/\sim_\gamma, \wedge_{\sim_\gamma}, \vee_{\sim_\gamma}; [+\infty]_{\sim_\gamma}, [-\infty]_{\sim_\gamma})$ is a commutative, idempotent, zero-sum-free, zero-divisor free semiring, as:

2.1 $\wedge_{\sim_\gamma}$ is associative;

2.2 $\wedge_{\sim_\gamma}$ is commutative;

2.3 $\wedge_{\sim_\gamma}$ has the neutral element: $[+\infty]_{\sim_\gamma}$ (zero of the semiring);

2.4 $\wedge_{\sim_\gamma}$ is idempotent;

[so $(\overline{\mathcal{K}_\mathcal{C}}/\sim_\gamma, \wedge_{\sim_\gamma})$ is a commutative, idempotent monoid]

2.5 $\vee_{\sim_\gamma}$ is associative;

2.6 $\vee_{\sim_\gamma}$ is commutative;

2.7 $\vee_{\sim_\gamma}$ has the neutral element: $[-\infty]_{\sim_\gamma}$ (unity of the semiring);

2.8 $\vee_{\sim_\gamma}$ is idempotent;

[so $(\overline{\mathcal{K}_\mathcal{C}}/\sim_\gamma, \vee_{\sim_\gamma})$ is a commutative, idempotent monoid]

2.9 $\vee_{\sim_\gamma}$ is distributive with respect to $\wedge_{\sim_\gamma}$;

2.10 $[+\infty]_{\sim_\gamma}$ is the absorbing element for $\vee_{\sim_\gamma}$

[so $(\overline{\mathcal{K}_\mathcal{C}}/\sim_\gamma, \wedge_{\sim_\gamma}, \vee_{\sim_\gamma})$ is a commutative, idempotent semiring]

2.11 $(\overline{\mathcal{K}_\mathcal{C}}/\sim_\gamma, \wedge_{\sim_\gamma}, \vee_{\sim_\gamma})$ is zero-sum-free;

2.12 $(\overline{\mathcal{K}_\mathcal{C}}/\sim_\gamma, \wedge_{\sim_\gamma}, \vee_{\sim_\gamma})$ is zero-divisor-free;

[so $(\overline{\mathcal{K}_\mathcal{C}}/\sim_\gamma, \wedge_{\sim_\gamma}, \vee_{\sim_\gamma})$ is a zero-sum-free, zero-divisor-free semiring]

3) $(\mathcal{K}_\mathcal{C}^{\varnothing\mathbb{R}}/\sim_\gamma, \cup_{\sim_\gamma}, \cap_{\sim_\gamma}; [\varnothing]_{\sim_\gamma}, [\mathbb{R}]_{\sim_\gamma})$ is a commutative, idempotent, zero-sum-free semiring, as:

3.1 $\cup_{\sim_\gamma}$ is associative;

3.2 $\cup_{\sim_\gamma}$ is commutative;

3.3 $\cup_{\sim_\gamma}$ has the neutral element: $[\varnothing]_{\sim_\gamma}$ (zero of the semiring);

3.4 $\cup_{\sim_\gamma}$ is idempotent;

[so $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma, \cup_{\sim_\gamma})$ is a commutative, idempotent monoid]

3.5 $\cap_{\sim_\gamma}$ is associative;

3.6 $\cap_{\sim_\gamma}$ is commutative;

3.7 $\cap_{\sim_\gamma}$ has the neutral element: $[\mathbb{R}]_{\sim_\gamma}$ (unity of the semiring);

3.8 $\cap_{\sim_\gamma}$ is idempotent;

[so $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma, \cap_{\sim_\gamma})$ is a commutative, idempotent monoid]

3.9 $\cap_{\sim_\gamma}$ is distributive with respect to $\cup_{\sim_\gamma}$;

3.10 $[\varnothing]_{\sim_\gamma}$ is the absorbing element for $\cap_{\sim_\gamma}$

[so $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma, \cup_{\sim_\gamma}, \cap_{\sim_\gamma})$ is a commutative, idempotent semiring]

3.11 $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma, \cup_{\sim_\gamma}, \cap_{\sim_\gamma})$ is zero-sum-free;

3.12 $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma, \cup_{\sim_\gamma}, \cap_{\sim_\gamma})$ is zero-divisor-free;

[so $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma, \cup_{\sim_\gamma}, \cap_{\sim_\gamma})$ is a zero-sum-free, zero-divisor-free semiring]

4) $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma, \cap_{\sim_\gamma}, \cup_{\sim_\gamma}; [\mathbb{R}]_{\sim_\gamma}, [\varnothing]_{\sim_\gamma})$ is a commutative, idempotent, zero-sum-free, zero-divisor free semiring, as:

4.1 $\cap_{\sim_\gamma}$ is associative;

4.2 $\cap_{\sim_\gamma}$ is commutative;

4.3 $\cap_{\sim_\gamma}$ has the neutral element: $[\mathbb{R}]_{\sim_\gamma}$ (zero of the semiring);

4.4 $\cap_{\sim_\gamma}$ is idempotent;

[so $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma, \cap_{\sim_\gamma})$ is a commutative, idempotent monoid]

4.5 $\cup_{\sim_\gamma}$ is associative;

4.6 $\cup_{\sim_\gamma}$ is commutative;

4.7 $\cup_{\sim_\gamma}$ has the neutral element: $[\varnothing]_{\sim_\gamma}$ (unity of the semiring);

4.8 $\cup_{\sim_\gamma}$ is idempotent;

[so $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma, \cup_{\sim_\gamma})$ is a commutative, idempotent monoid]

4.9 $\cup_{\sim_\gamma}$ is distributive with respect to $\cap_{\sim_\gamma}$;

4.10 $[\mathbb{R}]_{\sim_\gamma}$ is the absorbing element for $\cup_{\sim_\gamma}$;

[so $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma, \cap_{\sim_\gamma}, \cup_{\sim_\gamma})$ is a commutative, idempotent semiring]

4.11 $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma, \cap_{\sim_\gamma}, \cup_{\sim_\gamma})$ is zero-sum-free;

4.12 $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma, \cap_{\sim_\gamma}, \cup_{\sim_\gamma})$ is zero-divisor-free;

[so $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma, \cap_{\sim_\gamma}, \cup_{\sim_\gamma})$ is a zero-sum-free, zero-divisor-free semiring]

5) $(\mathcal{K}_{\mathcal{C}}^{-\infty}/\sim_\gamma, \vee_{\sim_\gamma}, \oplus_{\sim_\gamma}; [-\infty]_{\sim_\gamma}, [0]_{\sim_\gamma})$ is a commutative, zero-sum-free, zero-divisor free semiring, as:

5.1 $\vee_{\sim_\gamma}$ is associative;

5.2 $\vee_{\sim_\gamma}$ is commutative;

5.3 $\vee_{\sim_\gamma}$ has the neutral element: $[-\infty]_{\sim_\gamma}$ (zero of the semiring);

5.4 $\vee_{\sim_\gamma}$ is idempotent;

[so $(\mathcal{K}_{\mathcal{C}}^{-\infty}/\sim_\gamma, \vee_{\sim_\gamma})$ is a commutative, idempotent monoid]

5.5 $\oplus_{\sim_\gamma}$ is associative;

5.6 $\oplus_{\sim_\gamma}$ is commutative;

5.7 $\oplus_{\sim_\gamma}$ has the neutral element: $[0]_{\sim_\gamma}$ (unity of the semiring);

[so $(\mathcal{K}_{\mathcal{C}}^{-\infty}/\sim_\gamma, \wedge_{\sim_\gamma})$ is a commutative monoid]

5.8 $\oplus_{\sim_\gamma}$ is distributive with respect to $\vee_{\sim_\gamma}$;

5.9 $[-\infty]_{\sim_\gamma}$ is the absorbing element for $\oplus_{\sim_\gamma}$

[so $(\mathcal{K}_{\mathcal{C}}^{-\infty}/\sim_\gamma, \vee_{\sim_\gamma}, \oplus_{\sim_\gamma})$ is a commutative semiring]

5.10 $(\mathcal{K}_{\mathcal{C}}^{-\infty}/\sim_\gamma, \vee_{\sim_\gamma}, \oplus_{\sim_\gamma})$ is zero-sum-free;

5.11 $(\mathcal{K}_{\mathcal{C}}^{-\infty}/\sim_\gamma, \vee_{\sim_\gamma}, \oplus_{\sim_\gamma})$ is zero-divisor-free;

[so $(\mathcal{K}_{\mathcal{C}}^{-\infty}/\sim_\gamma, \vee_{\sim_\gamma}, \oplus_{\sim_\gamma})$ is a zero-sum-free, zero-divisor-free semiring]

6) $(\mathcal{K}_{\mathcal{C}}^{+\infty}/\sim_\gamma, \wedge_{\sim_\gamma}, \oplus_{\sim_\gamma}; [+\infty]_{\sim_\gamma}, [0]_{\sim_\gamma})$ is a commutative, zero-sum-free, zero-divisor free semiring, as:

6.1 $\wedge_{\sim_\gamma}$ is associative;

6.2 $\wedge_{\sim_\gamma}$ is commutative;

6.3 $\wedge_{\sim_\gamma}$ has the neutral element: $[+\infty]_{\sim_\gamma}$ (zero of the semiring);

6.4 $\wedge_{\sim_\gamma}$ is idempotent;

[so $(\overline{\mathcal{K}_{\mathcal{C}}}/\sim_\gamma, \wedge_{\sim_\gamma})$ is a commutative, idempotent monoid]

6.5 $\oplus_{\sim_\gamma}$ is associative;

6.6 $\oplus_{\sim_\gamma}$ is commutative;

6.7 $\oplus_{\sim_\gamma}$ has the neutral element: $[0]_{\sim_\gamma}$ (unity of the semiring);

[so $(\mathcal{K}_{\mathcal{C}}^{+\infty}/\sim_\gamma, \vee_{\sim_\gamma})$ is a commutative monoid]

6.8 $\oplus_{\sim_\gamma}$ is distributive with respect to $\wedge_{\sim_\gamma}$;

6.9 $[+\infty]_{\sim_\gamma}$ is the absorbing element for $\oplus_{\sim_\gamma}$

[so $(\mathcal{K}_{\mathcal{C}}^{+\infty}/\sim_\gamma, \wedge_{\sim_\gamma}, \oplus_{\sim_\gamma})$ is a commutative semiring]

6.10 $(\mathcal{K}_{\mathcal{C}}^{+\infty}/\sim_\gamma, \wedge_{\sim_\gamma}, \oplus_{\sim_\gamma})$ is zero-sum-free;

6.11 $(\mathcal{K}_{\mathcal{C}}^{+\infty}/\sim_\gamma, \wedge_{\sim_\gamma}, \oplus_{\sim_\gamma})$ is zero-divisor-free;

[so $(\mathcal{K}_{\mathcal{C}}^{+\infty}/\sim_\gamma, \wedge_{\sim_\gamma}, \oplus_{\sim_\gamma})$ is a zero-sum-free, zero-divisor-free semiring]

7) $(\mathcal{K}_{\mathcal{C}}^{\varnothing}/\sim_\gamma, \cup_{\sim_\gamma}, \oplus_{\sim_\gamma}; [\varnothing]_{\sim_\gamma}, [0]_{\sim_\gamma})$ is a commutative, zero-sum-free, zero-divisor-free semiring, as:

7.1 $\cup_{\sim_\gamma}$ is associative;

7.2 $\cup_{\sim_\gamma}$ is commutative;

7.3 $\cup_{\sim_\gamma}$ has the neutral element: $[\varnothing]_{\sim_\gamma}$ (zero of the semiring);

7.4 $\cup_{\sim_\gamma}$ is idempotent;

[so $(\mathcal{K}_{\mathcal{C}}^{\varnothing}/\sim_\gamma, \cup_{\sim_\gamma})$ is a commutative, idempotent monoid]

7.5 $\oplus_{\sim_\gamma}$ is associative;

7.6 $\oplus_{\sim_\gamma}$ is commutative;

7.7 $\oplus_{\sim_\gamma}$ has the neutral element: $[0]_{\sim_\gamma}$ (unity of the semiring);

[so $(\mathcal{K}_{\mathcal{C}}^{\varnothing\mathbb{R}}/\sim_\gamma, \oplus_{\sim_\gamma})$ is a commutative monoid]

7.8 $\oplus_{\sim_\gamma}$ is distributive with respect to $\cup_{\sim_\gamma}$;

7.9 $[\varnothing]_{\sim_\gamma}$ is the absorbing element for $\oplus_{\sim_\gamma}$

[so $(\mathcal{K}_{\mathcal{C}}^{\varnothing}/\sim_\gamma, \cup_{\sim_\gamma}, \oplus_{\sim_\gamma})$ is a commutative semiring]

7.10 $(\mathcal{K}_{\mathcal{C}}^{\varnothing}/\sim_\gamma, \cup_{\sim_\gamma}, \oplus_{\sim_\gamma})$ is zero-sum-free;

7.11 $(\mathcal{K}_{\mathcal{C}}^{\varnothing}/\sim_\gamma, \cup_{\sim_\gamma}, \oplus_{\sim_\gamma})$ is zero-divisor-free;

[so $(\mathcal{K}_{\mathcal{C}}^{\varnothing}/\sim_\gamma, \cup_{\sim_\gamma}, \oplus_{\sim_\gamma})$ is a zero-sum-free, zero-divisor-free semiring]

8) $(\mathcal{K}_{\mathcal{C}}^{\mathbb{R}}/\sim_\gamma, \cap_{\sim_\gamma}, \oplus_{\sim_\gamma}; [\mathbb{R}]_{\sim_\gamma}, [0]_{\sim_\gamma})$ is a commutative, zero-sum-free, zero-divisor free semiring, as:

8.1 $\cap_{\sim_\gamma}$ is associative;

8.2 $\cap_{\sim_\gamma}$ is commutative;

8.3 $\cap_{\sim_\gamma}$ has the neutral element: $[\mathbb{R}]_{\sim_\gamma}$ (zero of the semiring);

8.4 $\cap_{\sim_\gamma}$ is idempotent;

[so $(\mathcal{K}_{\mathcal{C}}^{\mathbb{R}}/\sim_\gamma, \cap_{\sim_\gamma})$ is a commutative, idempotent monoid]

8.5 $\oplus_{\sim_\gamma}$ is associative;

8.6 $\oplus_{\sim_\gamma}$ is commutative;

8.7 $\oplus_{\sim_\gamma}$ has the neutral element: $[0]_{\sim_\gamma}$ (unity of the semiring);

[so $(\mathcal{K}_{\mathcal{C}}^{\mathbb{R}}/\sim_\gamma, \oplus_{\sim_\gamma})$ is a commutative monoid]

8.8 $\oplus_{\sim_\gamma}$ is distributive with respect to $\cap_{\sim_\gamma}$;

Semiring	0 – element	1 – element	Properties
$(\overline{\mathcal{K}_\mathcal{C}}/\sim_\gamma, \vee_{\sim_\gamma}, \wedge_{\sim_\gamma})$	$[-\infty]_{\sim_\gamma} = [(-\infty;0)]_{\sim_\gamma}$	$[+\infty]_{\sim_\gamma} = [(+\infty;0)]_{\sim_\gamma}$	C, ZS, ZD, I
$(\overline{\mathcal{K}_\mathcal{C}}/\sim_\gamma, \wedge_{\sim_\gamma}, \vee_{\sim_\gamma})$	$[+\infty]_{\sim_\gamma} = [(+\infty;0)]_{\sim_\gamma}$	$[-\infty]_{\sim_\gamma} = [(-\infty;0)]_{\sim_\gamma}$	C, ZS, ZD, I
$(\mathcal{K}_\mathcal{C}^{\varnothing\mathbb{R}}/\sim_\gamma, \cup_{\sim_\gamma}, \cap_{\sim_\gamma})$	$[\varnothing]_{\sim_\gamma} = [(0;-\infty)]_{\sim_\gamma}$	$[\mathbb{R}]_{\sim_\gamma} = [(0;+\infty)]_{\sim_\gamma}$	C, ZS, ZD, I
$(\mathcal{K}_\mathcal{C}^{\varnothing\mathbb{R}}/\sim_\gamma, \cap_{\sim_\gamma}, \cup_{\sim_\gamma})$	$[\mathbb{R}]_{\sim_\gamma} = [(0;+\infty)]_{\sim_\gamma}$	$[\varnothing]_{\sim_\gamma} = [(0;-\infty)]_{\sim_\gamma}$	C, ZS, ZD, I
$(\mathcal{K}_\mathcal{C}^{-\infty}/\sim_\gamma, \vee_{\sim_\gamma}, \oplus_{\sim_\gamma})$	$[-\infty]_{\sim_\gamma} = [(-\infty;0)]_{\sim_\gamma}$	$[0]_{\sim_\gamma} = [(0;0)]_{\sim_\gamma}$	C, ZS, ZD
$(\mathcal{K}_\mathcal{C}^{+\infty}/\sim_\gamma, \wedge_{\sim_\gamma}, \oplus_{\sim_\gamma})$	$[+\infty]_{\sim_\gamma} = [(+\infty;0)]_{\sim_\gamma}$	$[0]_{\sim_\gamma} = [(0;0)]_{\sim_\gamma}$	C, ZS, ZD
$(\mathcal{K}_\mathcal{C}^{\varnothing}/\sim_\gamma, \cup_{\sim_\gamma}, \oplus_{\sim_\gamma})$	$[\varnothing]_{\sim_\gamma} = [(0;-\infty)]_{\sim_\gamma}$	$[0]_{\sim_\gamma} = [(0;0)]_{\sim_\gamma}$	C, ZS, ZD
$(\mathcal{K}_\mathcal{C}^{\mathbb{R}}/\sim_\gamma, \cap_{\sim_\gamma}, \oplus_{\sim_\gamma})$	$[\mathbb{R}]_{\sim_\gamma} = [(0;+\infty)]_{\sim_\gamma}$	$[0]_{\sim_\gamma} = [(0;0)]_{\sim_\gamma}$	C, ZS, ZD

Table 6.1: Classification of interval quotient semirings. C= commutative, ZS= zero-sum-free (or antinegative), ZD= zero-divisor-free (or entire), I= idempotent.

8.9 $[\mathbb{R}]_{\sim_\gamma}$ is the absorbing element for $\oplus_{\sim_\gamma}$;
[so $(\mathcal{K}_\mathcal{C}^{\mathbb{R}}/\sim_\gamma, \cap_{\sim_\gamma}, \oplus_{\sim_\gamma})$ is a commutative semiring]

8.10 $(\mathcal{K}_\mathcal{C}^{\mathbb{R}}/\sim_\gamma, \cap_{\sim_\gamma}, \oplus_{\sim_\gamma})$ is zero-sum-free;

8.11 $(\mathcal{K}_\mathcal{C}^{\mathbb{R}}/\sim_\gamma, \cap_{\sim_\gamma}, \oplus_{\sim_\gamma})$ is zero-divisor-free;
[so $(\mathcal{K}_\mathcal{C}^{\mathbb{R}}/\sim_\gamma, \cap_{\sim_\gamma}, \oplus_{\sim_\gamma})$ is a zero-sum-free, zero-divisor-free semiring]

Table 6.1 summarizes the eight different kind of interval quotient semirings we have just defined and the various properties associated with them.

6.2.3 An examples of interval pseudoring

It is interesting at this point to check if it is possible to build, in addition to the semirings analyzed so far, also other types of structures strong enough, such as for example a pseudorings (see Definition 5.1.4) which is not always taken for granted in mathematics.

Using the sum $\oplus_{\sim_\gamma}$ defined in Subsection 6.2.2 as first operation, since a second one is required, we realized that the most suitable to use could be the quotient extension of a multiplication in $\mathcal{K}_C$, based on so-called *pan-operations* (see [55], [60]) and defined as follow:

$$\otimes_P : \mathcal{K}_\mathcal{C} \times \mathcal{K}_\mathcal{C} \to \mathcal{K}_\mathcal{C}$$

such that:

$$(X,Y) \to X \otimes_P Y \stackrel{def}{=} (\widehat{x} \cdot \widehat{y}; \widetilde{x} \cdot \widetilde{y}), \ \ \forall X, Y \in \mathcal{K}_C. \tag{6.12}$$

Considering the pan-multiplication just introduced, we have that the following properties are verified.

(1) $\otimes_P$ is commutative:
$X \otimes_P Y = (\widehat{x} \cdot \widehat{y}; \widetilde{x} \cdot \widetilde{y}) = (\widehat{y} \cdot \widehat{x}; \widetilde{y} \cdot \widetilde{x}) = Y \otimes_P X, \ \ \forall X, Y \in \mathcal{K}_C$.

(2) $\otimes_P$ is associative:
$X \otimes_P (Y \otimes_P Z) = (\widehat{x} \cdot (\widehat{y} \cdot \widehat{z}); \widetilde{x} \cdot (\widetilde{y} \cdot \widetilde{z})) = ((\widehat{x} \cdot \widehat{y}) \cdot \widehat{z}; (\widetilde{x} \cdot \widetilde{y}) \cdot \widetilde{z}) = (X \otimes_P Y) \otimes_P Z, \ \ \forall X, Y, Z \in \mathcal{K}_C$.

(3) $\otimes_P$ is left and right distributive over Minkowski addition $\oplus$:

(i) $X \otimes_P (Y \oplus Z) = (X \otimes_P Y) \oplus (X \otimes_P Z), \ \ \forall X, Y, Z \in \mathcal{K}_C$;
(ii) $(X \oplus Y) \otimes_P Z = (X \otimes_P Z) \oplus (Y \otimes_P Z), \ \ \forall X, Y, Z \in \mathcal{K}_C$.

(4) $\otimes_P$ has a neutral element in $\mathcal{K}_\mathcal{C}$ which is $i_{\otimes_P} = (1; 1)$:
$X \otimes_P i_{\otimes_P} = X \otimes_P (1; 1) = X, \ \ \forall X \in \mathcal{K}_C$.

(5) $\otimes_P$ has an absorbing element in $\mathcal{K}_\mathcal{C}$ which is $0 = (0; 0)$:
$X \otimes_P 0 = X \otimes_P (0; 0) = 0, \ \ \forall X \in \mathcal{K}_C$.

Therefore, the following proposition holds.

Proposition 6.2.9. *$(\mathcal{K}_C, \oplus, \otimes_P)$ is a commutative semiring.*

Proof. $(\mathcal{K}_C, \oplus, \otimes_P)$ is a commutative semiring, as:

1a) $(\mathcal{K}_C, \oplus)$ is a commutative monoid with neutral element $(0; 0)$:

(i) $\oplus$ is associative: $(X \oplus Y) \oplus Z = X \oplus (Y \oplus Z), \forall X, Y, Z \in \mathcal{K}_C$;
(ii) $\oplus$ has a neutral element $i_\oplus = 0 = (0; 0) \in \mathcal{K}_C$:
$X \oplus 0 = 0 \oplus X = X, \forall X \in \mathcal{K}_C$ (so, $(0; 0)$ is the 0-element of the semiring $(\mathcal{K}_C, \oplus, \otimes_P)$);
(iii) $\oplus$ is commutative: $X \oplus Y = Y \oplus X, \forall X, Y \in \mathcal{K}_C$.

2a) $(\mathcal{K}_C, \otimes_P)$ is a commutative monoid with neutral element $(1; 1)$:

(i) $\otimes_P$ is associative: $(X \otimes_P Y) \otimes_P Z = X \otimes_P (Y \otimes_P Z), \forall X, Y, Z \in \mathcal{K}_C$;
(ii) $\otimes_P$ has a neutral element $i_{\otimes_P} = 1 = (1; 1) \in \mathcal{K}_C$:
$X \otimes_P 1 = 1 \otimes_P X = X, \forall X \in \mathcal{K}_C$ (so, $(1; 1)$ is the 1-element of the semiring $(\mathcal{K}_C, \oplus, \otimes_P)$);
(iii) $\otimes_P$ is commutative: $X \otimes_P Y = Y \otimes_P X, \forall X, Y \in \mathcal{K}_C$.

3a) $\otimes_P$ is left and right distributive over $\oplus$:

(i) $X \otimes_P (Y \oplus Z) = (X \otimes_P Y) \oplus (X \otimes_P Z), \forall X, Y, Z \in \mathcal{K}_C$;

(ii) $(X \oplus Y) \otimes_P Z = (X \otimes_P Z) \oplus (Y \otimes_P Z), \forall X, Y, Z \in \mathcal{K}_C$.

4a) $i_\oplus = 0 = (0; 0)$ is the absorbing element for $\otimes_P$:

$$X \otimes_P 0 = 0 \otimes_P X = 0, \forall X \in \overline{\mathcal{K}_C}.$$

□

At this point, as we have done in Subsectio 6.2.2, is possible to extend the operation $\otimes_P$ to the quotient set $\mathcal{K}_\mathcal{C}/\sim_\gamma$ where $\sim_\gamma$ is the γ-equivalence relation introduced by Definition 6.2.1.

Therefore, for all $[X]_{\sim_\gamma}, [Y]_{\sim_\gamma} \in \mathcal{K}_\mathcal{C}/\sim_\gamma$ we define:

$$\otimes_{P_{\sim_\gamma}} : (\mathcal{K}_\mathcal{C}/\sim_\gamma) \times (\mathcal{K}_\mathcal{C}/\sim_\gamma) \to (\mathcal{K}_\mathcal{C}/\sim_\gamma)$$

such that: $([X]_{\sim_\gamma}, [Y]_{\sim_\gamma}) \to [X]_{\sim_\gamma} \otimes_{P_{\sim_\gamma}} [Y]_{\sim_\gamma} \overset{def}{=} [X_0 \otimes_P Y_0]_{\sim_\gamma}$

where $\forall X = (\widehat{x}; \widetilde{x}) \in \mathcal{K}_\mathcal{C}$, it is $X_0 = (\widehat{x}_0; 0) = \left(\widehat{x} - \dfrac{\widetilde{x}}{\gamma^+}; 0\right)$ which, in the LU-case ($\gamma = 1$), simply corresponds to $X_0 \overset{def}{=} (x^-; 0)$, and the multiplication $\otimes_P$ is the operation in $\mathcal{K}_\mathcal{C}$ defined in (6.12), i.e.,

$$X_0 \otimes_P Y_0 = (\widehat{x}_0; 0) \otimes_P (\widehat{y}_0; 0) = (\widehat{x}_0 \cdot \widehat{y}_0; 0).$$

Thus, for all $[X]_{\sim_\gamma}, [Y]_{\sim_\gamma} \in \mathcal{K}_\mathcal{C}/\sim_\gamma$, we have:

$$[X]_{\sim_\gamma} \otimes_{P_{\sim_\gamma}} [Y]_{\sim_\gamma} = [X_0 \otimes_P Y_0]_{\sim_\gamma} = [(\widehat{x}_0; 0) \otimes_P (\widehat{y}_0; 0)]_{\sim_\gamma} = [(\widehat{x}_0 \cdot \widehat{y}_0; 0)]_{\sim_\gamma}.$$

In particular, considering the LU-case, it is:

$$[X]_{\sim_1} \otimes_{P_{\sim_1}} [Y]_{\sim_1} = [(\widehat{x}_0; 0) \otimes_P (\widehat{y}_0; 0)]_{\sim_1} = [(x^-; 0) \otimes_P (y^-; 0)]_{\sim_1} = [(x^- \cdot y^-; 0)]_{\sim_1}.$$

It is not difficult to prove that the following properties hold.

(1) $\otimes_{P_{\sim_\gamma}}$ is commutative:
$[X]_{\sim_\gamma} \otimes_{P_{\sim_\gamma}} [Y]_{\sim_\gamma} = [X_0 \otimes_P Y_0]_{\sim_\gamma} = [Y_0 \otimes_P X_0]_{\sim_\gamma} = [Y]_{\sim_\gamma} \otimes_{P_{\sim_\gamma}} [X]_{\sim_\gamma}$, $\forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma} \in \mathcal{K}_\mathcal{C}/\sim_\gamma$.

(2) $\otimes_{P_{\sim_\gamma}}$ is associative:
$([X]_{\sim_\gamma} \otimes_{P_{\sim_\gamma}} [Y]_{\sim_\gamma}) \otimes_{P_{\sim_\gamma}} [Z]_{\sim_\gamma} = ... = [(X_0 \otimes_P Y_0) \oplus Z_0]_{\sim_\gamma} = [X_0 \otimes_P (Y_0 \otimes_P Z_0)]_{\sim_\gamma} = ... = [X]_{\sim_\gamma} \otimes_{P_{\sim_\gamma}} ([Y]_{\sim_\gamma} \otimes_{P_{\sim_\gamma}} [Z]_{\sim_\gamma})$, $\forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in \mathcal{K}_\mathcal{C}/\sim_\gamma$.

(3) $\otimes_{P_{\sim_\gamma}}$ is left and right distributive over $\oplus_{\sim_\gamma}$:

(i) $[X]_{\sim_\gamma} \otimes_{P_{\sim_\gamma}} ([Y]_{\sim_\gamma} \oplus_{\sim_\gamma} [Z]_{\sim_\gamma}) = ([X]_{\sim_\gamma} \otimes_{P_{\sim_\gamma}} [Y]_{\sim_\gamma}) \oplus_{\sim_\gamma} ([X]_{\sim_\gamma} \otimes_{P_{\sim_\gamma}} [Z]_{\sim_\gamma})$, $\forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in \mathcal{K}_\mathcal{C}/\sim_\gamma$;

(ii) $([X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Y]_{\sim_\gamma}) \otimes_{P_{\sim_\gamma}} [Z]_{\sim_\gamma} = ([X]_{\sim_\gamma} \otimes_{P_{\sim_\gamma}} [Z]_{\sim_\gamma}) \oplus_{\sim_\gamma} ([Y]_{\sim_\gamma} \otimes_{P_{\sim_\gamma}} [Z]_{\sim_\gamma})$, $\forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}/\sim_\gamma$.

(4) $\otimes_{P_{\sim_\gamma}}$ has an absorbing element in $\mathcal{K}_{\mathcal{C}}/\sim_\gamma$ which is $[0]_{\sim_\gamma} = [(0;0)]_{\sim_\gamma}$: $[X]_{\sim_\gamma} \otimes_{P_{\sim_\gamma}} [0]_{\sim_\gamma} = [X_0 \otimes_P 0]_{\sim_\gamma} = [(\widehat{x}_0; 0) \otimes_P (0;0)]_{\sim_\gamma} = [(\widehat{x}_0 \cdot 0; 0 \cdot 0)]_{\sim_\gamma} = [(0;0)]_{\sim_\gamma} = [0]_{\sim_\gamma}, \forall [X]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}/\sim_\gamma$.

(5) $[X]_{\sim_\gamma} \otimes_{P_{\sim_\gamma}} [Y]_{\sim_\gamma} = [0]_{\sim_\gamma} \Rightarrow [X]_{\sim_\gamma} = [0]_{\sim_\gamma}$ or $[X]_{\sim_\gamma} = [0]_{\sim_\gamma}, \forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}/\sim_\gamma$.

Therefore, according to Definition 5.1.4, the following result holds.

Proposition 6.2.10. *The structure* $(\mathcal{K}_{\mathcal{C}}/\sim_\gamma, \oplus_{\sim_\gamma}, \otimes_{P_{\sim_\gamma}})$ *is a commutative pseudoring.*

Proof. $(\mathcal{K}_{\mathcal{C}}/\sim_\gamma, \oplus_{\sim_\gamma}, \otimes_{P_{\sim_\gamma}})$ is a commutative pseudoring, as:

1) $(\mathcal{K}_{\mathcal{C}}/\sim_\gamma, \oplus_{\sim_\gamma})$ is an abelian group with neutral element $i_{\oplus_\sim} = [0]_{\sim_\gamma}$:

   (i) $\oplus_{\sim_\gamma}$ is associative;

   (ii) $\oplus_{\sim_\gamma}$ has the neutral element $i_{\oplus_\sim} = [0]_{\sim_\gamma} = [(0;0)]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}/\sim_\gamma$: $[X]_{\sim_\gamma} \oplus_{\sim_\gamma} [(0;0)]_{\sim_\gamma} = [X \oplus (0;0)]_{\sim_\gamma} = [X]_{\sim_\gamma}, \forall [X]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}/\sim_\gamma$ (so, $i_{\oplus_\sim} = [0]_{\sim_\gamma} = [(0;0)]$ is the 0-element of the pseudoring);

   (iii) $\oplus_{\sim_\gamma}$ is commutative;

   (iv) existence of the additive inverse (opposite element with respect to $\oplus_{\sim_\gamma}$):
   $\forall [X]_{\sim_\gamma} = [X_0]_{\sim_\gamma} = [(\widehat{x}_0; 0)]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}/\sim_\gamma$,
   $\exists [X]'_{\sim_\gamma} = [X'_0]_{\sim_\gamma} = [-X_0]_{\sim_\gamma} = [(-\widehat{x}_0; 0)]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}/\sim_\gamma$ such that:
   $[X]_{\sim_\gamma} \oplus_{\sim_\gamma} [X']_{\sim_\gamma} = [X_0 \oplus X'_0]_{\sim_\gamma} = [(\widehat{x}_0 - \widehat{x}_0; 0)]_{\sim_\gamma} = [0]_{\sim_\gamma}$.

2) $(\mathcal{K}_{\mathcal{C}}/\sim_\gamma, \otimes_{P_{\sim_\gamma}})$ is a commutative semigroup:

   (i) $\otimes_{P_{\sim_\gamma}}$ is associative;

   (ii) $\otimes_{P_{\sim_\gamma}}$ is commutative.

3) $\otimes_{P_{\sim_\gamma}}$ is left and right distributive over $\oplus_{\sim_\gamma}$:

   (i) $[X]_{\sim_\gamma} \otimes_{P_{\sim_\gamma}} ([Y]_{\sim_\gamma} \oplus_{\sim_\gamma} [Z]_{\sim_\gamma}) = ([X]_{\sim_\gamma} \otimes_{P_{\sim_\gamma}} [Y]_{\sim_\gamma}) \oplus_{\sim_\gamma} ([X]_{\sim_\gamma} \otimes_{P_{\sim_\gamma}} [Z]_{\sim_\gamma}), \forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}/\sim_\gamma$;

   (ii) $([X]_{\sim_\gamma} \oplus_{\sim_\gamma} [Y]_{\sim_\gamma}) \otimes_{P_{\sim_\gamma}} [Z]_{\sim_\gamma} = ([X]_{\sim_\gamma} \otimes_{P_{\sim_\gamma}} [Z]_{\sim_\gamma}) \oplus_{\sim_\gamma} ([Y]_{\sim_\gamma} \otimes_{P_{\sim_\gamma}} [Z]_{\sim_\gamma}), \forall [X]_{\sim_\gamma}, [Y]_{\sim_\gamma}, [Z]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}/\sim_\gamma$.

4) $i_{\oplus_\sim} = [0]_{\sim_\gamma} = [(0;0)]_{\sim_\gamma}$ is the absorbing element for $\otimes_{P_{\sim_\gamma}}$:

   $[(0;0)]_{\sim_\gamma} \otimes_{P_{\sim_\gamma}} [X]_{\sim_\gamma} = [X]_{\sim_\gamma} \otimes_{P_{\sim_\gamma}} [(0;0)]_{\sim_\gamma} = [(0;0)]_{\sim_\gamma}, \forall [X]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}/\sim_\gamma$.

□

Remark 6.2.2. *The pseudoring* $(\mathcal{K}_{\mathcal{C}}/\sim_\gamma, \oplus_{\sim_\gamma}, \otimes_{P_{\sim_\gamma}})$ *is also zero-divisor-free since for all* $[X]_{\sim_\gamma}, [Y]_{\sim_\gamma} \in \mathcal{K}_{\mathcal{C}}/\sim_\gamma$*, it is:*
$[X]_{\sim_\gamma} \otimes_{P_{\sim_\gamma}} [Y]_{\sim_\gamma} = [0]_{\sim_\gamma} \Rightarrow [X]_{\sim_\gamma} = [0]_{\sim_\gamma}$ *or* $[X]_{\sim_\gamma} = [0]_{\sim_\gamma}$.
On the contrary, it is not a zero-sum-free structure, as can be easily seen from Figure 6.9.

As was done in the previous sections, the structures examined in this paragraph can be summarized as follows.

1) $(\mathcal{K}_C, \oplus, \otimes_P)$ is a commutative semiring, as:

1.1 $\oplus$ is associative;

1.2 $\oplus$ is commutative;

1.3 $\oplus$ has the neutral element: $i_\oplus = 0 = (0;0)$ (zero of the semiring);

[so $(\mathcal{K}_C, \oplus)$ is a commutative monoid]

1.4 $\otimes_P$ is associative;

1.5 $\otimes_P$ is commutative;

1.6 $\otimes_P$ has the neutral element: $i_{\otimes_P} = 1 = (1;1)$ (unity of the semiring);

[so $(\mathcal{K}_C, \otimes_P)$ is a commutative monoid]

1.7 $\otimes_P$ is left and right distributive over $\oplus$;

1.8 $i_\oplus = 0 = (0;0)$ is the absorbing element for $\otimes_P$;

[so $(\mathcal{K}_C, \oplus, \otimes_P)$ is a commutative semiring]

2) $(\mathcal{K}_{\mathcal{C}}/\sim_\gamma, \oplus_{\sim_\gamma}, \otimes_{P_{\sim_\gamma}})$ is a commutative, zero-divisor free pseudoring, as:

2.1 $\oplus_{\sim_\gamma}$ is associative;

2.2 $\oplus_{\sim_\gamma}$ is commutative;

2.3 $\oplus_{\sim_\gamma}$ has the neutral element: $i_{\oplus_\sim} = [0]_{\sim_\gamma} = [(0;0)]_{\sim_\gamma}$ (zero of the pseudoring);

2.4 existence of the additive inverse element;

[so $(\mathcal{K}_{\mathcal{C}}/\sim_\gamma, \oplus_{\sim_\gamma})$ is an abelian group]

2.5 $\otimes_{P_{\sim_\gamma}}$ is associative;

2.6 $\otimes_{P_{\sim_\gamma}}$ is commutative;

[so $(\mathcal{K}_{\mathcal{C}}/\sim_\gamma, \otimes_{P_{\sim_\gamma}})$ is a commutative semigroup]

2.7 $\otimes_{P_{\sim_\gamma}}$ is left and right distributive over $\oplus_{\sim_\gamma}$;

2.8 $i_{\oplus_\sim} = [0]_{\sim_\gamma} = [(0;0)]_{\sim_\gamma}$ is the absorbing element for $\otimes_{P_{\sim_\gamma}}$;

[so $(\mathcal{K}_{\mathcal{C}}/\sim_\gamma, \oplus_{\sim_\gamma}, \otimes_{P_{\sim_\gamma}})$ is a commutative pseudoring]

Type	**Structure**	**0**	**1**	**Properties**
Semiring	$(\mathcal{K}_C, \oplus, \otimes_P)$	$(0;0)$	$(1;1)$	C
Pseudoring	$(\mathcal{K}_\mathcal{C}/\sim_\gamma, \oplus_{\sim_\gamma}, \otimes_{P_{\sim_\gamma}})$	$[(0;0)]_{\sim_\gamma}$	/	C, ZD

Table 6.2: Interval structures with pan-operation $\otimes_P$. C= commutative, ZD= zero-divisor-free.

2.9 $(\mathcal{K}_\mathcal{C}/\sim_\gamma, \oplus_{\sim_\gamma}, \otimes_{P_{\sim_\gamma}})$ is zero-divisor-free;
[so $(\mathcal{K}_\mathcal{C}/\sim_\gamma, \oplus_{\sim_\gamma}, \otimes_{P_{\sim_\gamma}})$ is a zero-divisor-free pseudoring]

Table 6.2 summarizes the interval structures we have defined in this subsection and the properties associated with them.

Conclusions and further developments

As amply presented in the introduction, this work has been carried out motivated by a dual purpose: first of all we wanted to offer an updated state of the art on the concepts, problems and techniques of interval analysis, with a specific focus on the theoretical aspects of calculus in the setting of interval-valued functions of a single real variable; secondly, we intended to make a direct contribution to the study of the aforementioned topics, in particular by deepening the investigation from the algebraic point of view, also through approaches that could go beyond the classic representations, to endow the theory with a renewed and powerful algebraic framework so far absent in literature.

Following this approach, the work has been developed with an extensive and detailed treatment in which each result achieved has been justified both through exhaustive demonstrations and through citations and references to well-known books and articles.

The entire first part of the work was aimed at responding to the first of the above-mentioned purposes.

We started with a general overview of the theory of interval analysis, introducing notations and basic facts following the so-called classical approach and the main steps that have characterized its historical evolution, from its origins to the present day, starting from the reasons that originated it up to outlining its future prospects. The theory was presented through what have historically been the two main approaches: axiomatic and set.

Beginning with the basic concepts, the main algebraic operations with their properties were presented up to analyzing more complex structures, such as vectors, matrices and the fundamental elements of complex interval calculus. Finally, after having introduced the concept of interval-valued functions with some important applications, we concluded with a mention of the numerous alternative theories related to the interval method.

After that, we moved on to the presentation of contents inspired by the results of a recent work consisting of two separate and interconnected articles ([84] and [85]), concerning interval analysis and the calculus for interval-valued functions of a single real variable. Here, starting from a recently proposed comparison index, an innovative general setting for the partial order in the space of compact real intervals has been developed whose corresponding concepts have also been applied for the analysis and computation of interval-valued functions.

Specifically, in introducing the basic properties of the space of real intervals it was highlighted how the non-invertibility of the standard Minkowski addition and multiplication represent a non-trivial problem within the whole interval analysis, thus justifying the introduction of the Hukuhara operations precisely to overcome this shortcoming. We have also widely adopted the midpoint-radius representation of intervals in the real half-plane, especially showing its usefulness in calculus. However, compared to the two articles mentioned above, the contents have been expanded and enriched with various new elements, which offer innovative and interesting interpretative ideas. More specifically, several partial orders for the intervals with their properties have been analyzed, in terms of the midpoint representation and in relation to lattice theory, focusing particularly on the role of gH-difference also through numerous references to graphical representations.

All the notions presented were then applied to the analysis and calculus of interval-valued functions, up to the introduction and detailed investigation of concepts relating to limits, continuity, gH-differentiability and monotonicity, as well as a discussion concerning extremal points, concavity and convexity of interval-valued functions; all accompanied by a complete illustrative example. The same goes for the concept of periodicity, introduced and visualized with the aid of some well-known plane curves.

Starting from these results, new possibilities of using interval-valued functions were then introduced. First, a new notation to represent complex intervals was proposed, whose peculiarities and advantages were fully shown through an unprecedented and original visual approach. In addition, an interesting application was suggested concerning a topic, the q-calculus, which nowadays holds great interest in the scientific community and, therefore, could be taken into consideration also for future research. Indeed, it is a sector highly appreciated not only by physicists (from statictical mechanics to theory of relativity, up to the concepts of q-heat and q-wave recently introduced) but also by mathematicians because of its recent applications in different areas such as orthogonal polynomials, basic hypergeometric functions, combinatorics and calculus of variations.

On the other hand, as regards the second objective that was intended to be pursued, we can state that all the remaining part of the work was dedicated to this. In fact, although over the years many authors have ventured into the study of the algebraic properties of the intervals, looking for algebraic systems within which to configure them, however, even today such kind of structures have not been completely axiomatized; therefore, it was precisely to fill this gap that we introduced some innovative approaches which proved to be very useful for the determination of interval algebraic systems.

First of all, an attempt has been made to broaden the concept of order analyzed in the first part of the thesis, introducing a new one capable of giving a unitary interpretation, with the aim of obtaining a sort of polarity between the two types of orders; this also allowed us to determine a very important completion of the $\mathcal{K}_\mathcal{C}$ lattice. Indeed, since lattices are more interesting structures than general posets, we thought it might be useful to address the question of how to embed any poset into a complete lattice, thus arriving at the notion of interval lattice completion.

Clearly, all of this has undoubtedly been of vital importance in redefining the set of algebraic structures which underlie the interval theory. In fact, these concepts have been applied in an attempt to enrich the theory itself, thus overcoming the limitation that up to now has been found in the literature whenever there has been an attempt to define non-trivial interval algebraic structures. Trying to maintain the validity of important properties, several types of approaches to the problem have been proposed, from which as many algebraic structures have arisen.

In particular, by introducing different types of extensions of the set of intervals $\mathcal{K}_\mathcal{C}$, each equipped with a specific associated order, it has been possible to outline different types of structures capable of maintaining the validity of important properties or even of verifying further ones. In this way, hitherto unexplored algebraic structures were developed, some rather well-known, such as Kleene algebras, semirings or pre-semirings, others more unusual, ranging from l-semigroups to the so-called clodum, i.e., combined structures, characterized by the fact that on the underlying set three or more binary operations are defined whose coexistence is ensured by satisfyng a certain number of properties that link them together. Indeed, it seemed interesting to deal with some of them by giving an interval interpretation, since the theories associated with these structures, such as, for example, lattice-ordered monoids, offer a conceptually elegant and compact way to express rich patterns with multiple application possibilities.

Finally, from a study on the complementation properties, through the use of innovative models, we ventured into the configuration of interval-type Boolean structures, aware of the fact that defining a single structure of this kind (Boolean ring, Boolean lattice, Boolean algebra) is equivalent to defining the others as well; thus the study of Boolean rings, Boolean algebras and Boolean lattices is completely equivalent. After that, thanks to an ingenious definition of the equivalence relation between intervals, the construction of an interval quotient set has also been proposed, thanks to which it has been possible to determine further new solid structures, up to providing an example of interval quotient pseudoring.

We thought it was interesting to dwell on the passage to the quotient set as it schematizes and specifies the process of formation of concepts starting from objects and, more generally, the ordinary process of abstraction, important in mathematics and beyond, consisting in identifying different elements but all with a common property. Indeed, if the elements of a set can be considered as "data" (objects), then the elements of the corresponding quotient set can be understood as "conceptual abstractions" (classes of objects thought of as a single object); therefore, the transition from one form to another embodies the mathematization of a subtle thought process that leads us to identify elements that can be replaced in a given context. And thus, the logical-mathematical concept of equivalence expands and specifies the concept of "equality" of common language, highlighting its relative character and enhancing its application power in the various fields.

Clearly, the whole work fits into a broader research context and can be considered a first step as regards the mathematical analysis for interval-valued functions of a single variable as well as for subjects of a more strictly algebraic nature, offering ideas for research, study and application totally innovative with respect to the literature so far known. In addition, with regard to interval-valued functions it would also be interesting to consider appropriate extensions in order to investigate the case of multivariable interval functions, which could be the object of future research too.

It should also be remembered that here the setting has been the standard interval analysis, based on Minkowski-type operations, with the supplement of gH-difference and gH-addition; however, several additional properties and applications could be possibly established, in particular with reference to specific problems and questions such as the solution of interval differential equations (IDE), including the extension of ordinary and partial differential equations to the interval-valued case using gH-derivatives (examples in this sense are described in [16]).

On the other hand, as regards the algebraic aspect, it should be emphasized that, in addition to providing a solid algebraic structure to the theory, the goal here is also to launch new interpretative challenges against an algebraic backgoung to be completely rebuilt and reinvented. The adaptability and flexibility of the multiple structures built, sometimes even in a whimsical way, lend themselves to new interpretations and multiple uses, especially in the computer-logic field, thus favoring that process of mutual growth and progress between different subjects and frameworks, fundamental step to pave the way towards new scenarios of the entire scientific panorama.

Finally, it should definitely be highlighted that two papers are currently in preparation, to be submitted for publication, obtained from the contents of Chapters 4 and 5, specifically:

a) Introduction and analysis of polar orders and lattices for real intervals via midpoint representation (from Sections 4.1 and 4.2);

b) On the combination of algebraic structures in the space of real intervals, based on different partial orders (from Sections 5.1 and 5.3 and Subsection 5.2.3).

Further scientific publications (journal articles or conference papers) will refer to the following portions of the thesis:

- Chapter 3 - Subsection 3.2.2 (Interval-valued q-calculus interpreted according to some recent developments in interval analysis);
- Chapter 5 - Subsections 5.2.1 and 5.2.2 (Alternative approaches to interval semirings: how to extend the set of real intervals to keep crucial properties valid or verify additional ones);
- Chapter 6 - Section 6.1 (An innovative model to carry out an accurate study of Boolean structures from an interval point of view);
- Chapter 6 - Section 6.2 (Use of an interval quotient set, defined through a novel equivalence relation between intervals, in order to build some types of strong algebraic structures).

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