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A probabilistic modal logic for context-aware trust based on evidence

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A R T I C L E   I N F O

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A B S T R A C T

Trust is an extremely helpful construct when reasoning under uncertainty. Thus, being able to logically formalize the concept in a suitable language is important. However, doing so is problematic for three reasons. First, in order to keep track of the contextual nature of trust, situation trackers are required inside the language. Second, in order to produce trust estimations, agents rely on evidence personally gathered or reported by other agents; this requires elements in the language that can track which agents are used as referrals and how much weight is placed on their opinions. Finally, trust is subjective in nature, thus, personal thresholds are needed to track the trust-propensity of different evaluators. In this paper we propose an interpretation of a probabilistic modal language à la Hennessy-Milner in order to capture a context-aware quantitative notion of trust based on evidence. We also provide an axiomatization for the language and prove soundness, completeness, and decidability results.

1. Introduction

In environments where information is scarce and where data might be inaccurate or biased, trust plays a crucial role. Whenever explicit and direct knowledge is not available and attaining precise data is difficult, trust helps agents to make decisions. Trust enables those decisions through a combination of (i) risk acceptance [32,40], (ii) personal experiences by the agent making the decision [22], and (iii) recommendations by other agents [49]. By trusting, agents accept that the experiences they personally had and the ones reported by other parties provide a good enough ground for their decision-making procedures, and allow them to embrace the risk of being wrong and, consequently, being disappointed.

The role that trust plays as a facilitator for decision-making under uncertainty should highlight the importance of studying this concept thoroughly. Disciplines as diverse as sociology [25], economy [12], political science [24], evolutionary biology [47], and computer science [38,43,4], all dedicated some of their attention to trust, obviously prioritizing their specific needs. Among those research endeavors, a prolific subject has been that of logical representations of trust [30,27,8,35,44,42]. By representing trust formally, not only it is possible to understand the concept better, but it also becomes easier to implement trust models in online environments, which are prime candidates of environments where clear and reliable information is hard to come by.[1]

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[1] According to a recent report from 2021 of the Central Statistic Office, almost 62% of individuals believe that information found online is unreliable or false.

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However, obtaining satisfactory logical models of trust is difficult, especially when elements (i)-(iii) (i.e., risk acceptance, personal experiences and recommendations) have to be modeled in order to obtain trust values to apply in decision making. The difficulties are produced by two important limitations: (a) logical languages often employ qualitative evaluations of the concept of trust rather than quantitative ones; (b) personal experiences and recommendations produce further elements of uncertainty that have to be dealt with [5].

Problem (a) is tied to the risk-acceptance component of trust. Different agents have different risk-propensities, which could further change in intensity depending on the context (in some contexts an agent might be risk averse, while in others s/he might be risk prone) or the information to be trusted (information that is extremely impactful in a situation might require stricter analysis and better safeguards, compared to information that is less important).

Problem (b) is tied to the personal experiences and recommendations components, the relevance of which depends on the reliability of the source of the evidence (either direct or indirect), and the significance of such evidence to the specific decision context in which trust should be applied.

In order to provide a solution to those problems it is necessary to implement context-aware components and subjective expertise parameters to the logical languages that are built to model trust. The context-aware components (such as different trust thresholds that depend on the situation) should help in dealing with risk-acceptance, since specifying how difficult it is to trust in different circumstances can help to provide adequate estimations based on the situation. On the other hand, expertise parameters that can be set from a subjective standpoint can indicate which experts are consulted in order to gauge their opinions, and, moreover, how much weight those opinions have in the final considerations.

The aim of this paper is to show that those components can be implemented through the use of classical ingredients of probabilistic modal languages. In particular, we define a logical framework that can be used to analyze context-aware trust relations based on possessed and referred evidence. The resulting language, called *Trust Evidence Logic* (**TEL**), includes modalities, inspired by logics à la Hennessy-Milner [26], used to express estimations about trust towards logical formulas.

In order to achieve our goals, the paper is structured as follows. In Section 2, motives that inspired our approach are provided, hinting at alternative solutions and connections between them. In Section 3, the syntax and semantics of **TEL** are introduced and model theoretic results are given, by interpreting the theoretical machinery for the study of a context-aware computational notion of trust based on evidence. In Section 4, **TEL** is used to model and check a case study taken from the setting of mobile and sensor networks. In Section 5, soundness, completeness, and decidability results for **TEL** are proved, and in Section 6, conclusions and future works follow.

This paper expands previous work by the authors [3], both from a modeling and a technical perspective. On one hand, we detail the relation between the theory behind our language and the properties of the notions of trust we formalize (Sections 3 and 5), by illustrating the expressiveness of our framework in a real use case (Section 4). On the other hand, from a technical perspective, we expanded our completeness results and added decidability results (Section 5).

2. Modeling trust in modal logics

The formal modeling of trust received a lot of attention in the modal logics community in the last 20 years [14,37,39,1,36,2,35,45,42,46]. Since there is no general agreement on a precise notion of trust, all these approaches emphasize different aspects and properties of such a highly subjective concept, without unifying in a unique framework all the characteristics emphasized in Section 1. As a starting point for our proposed approach, a prominent analysis of the concept of trust that we rely on is that of Gambetta [19]. According to his definition of trust, when Alice (the trustor) trusts Bob (the trustee), Alice subjectively attributes a sufficiently high probability to the possibility that Bob will act in a beneficial way towards her. In this sense, what seems relevant for the presence of trust is that Alice is able to have enough information to form a subjective evaluation on the probability that Bob will act in a certain way. Moreover, the kind of information that Alice is seeking is context-dependent and might change in different scenarios and/or time frames, i.e., when and in which circumstances Alice is evaluating Bob's actions.

One of the main approaches to implement such form of trust in online environments is based on reputation models [31,30]. The advantage of these models is that they are well-structured to deal with indirect evidence of behavior and they provide a solid base for trust evaluations. In particular, agents provide their feedback after each interaction with a certain trustee on the basis of such a personal experience, and then, the reputation model estimates a unified value from the collection of such information; finally, the values are shared with all the agents, which use them when behaving as trustors to form their subjective evaluation about the trustee under consideration. Given that reputation models only need to manipulate data provided directly by agents, they are easy to implement in online environments and thus are widely employed as evidence-bases for trust evaluations.

Assuming a simple reputation model where agents can only provide Boolean evaluations for certain behaviors, e.g., the behavior is present or is absent, it would be easy to represent such a model through the use of graded modal logics (GML) [21,18,17]. In GML, the modal operator $\Diamond_n \phi$ specifies that in strictly more than n accessible states of the system ϕ holds. Hence, by interpreting states as agents and ϕ as the evaluated behavior, the modal operator can be adapted to decide whether a given number of evaluations are present and from there, provide an estimation of trust.[2] Taking the intuition behind GML as a starting point, two important

[2] An alternative approach is to employ *majority logics* [41]. Such logics are well suited to deal with dynamic scenarios where the number of evaluations is not fixed. The advantage of employing majority logics instead of GML is given by the fact that majority logics only specify that the (strict) majority of evaluations must be positive, without specifying a given number of those. On the contrary, in GML, this number must always be specified.

elements can be added in order to improve the expressiveness of the model. The first element is a parameter indicating the context of evaluation. Hence, an evaluation is not a general assessment, but strictly depends on a specific scenario that indicates in which circumstances the evaluation is made. The second element is a numeric parameter indicating the expertise of the evaluator, i.e., it establishes how much weight the trustor places on the evaluations of other agents. In this sense, not all evaluations are judged equally, but some will be more relevant than others. Those two additions produce a language (**TEL**) that encompasses some features from both the probabilistic version of Hennessy-Milner logic (HML), see for instance [34], and the Probabilistic Computation Tree Logic (PCTL) [23]. In particular, **TEL** introduces modal formulas that re-interpret the modality $\langle a\rangle_p^{\bowtie}\phi$, in which a represents the context of evaluation and p denotes the evaluation threshold used to govern trust-based decisions.

Following both [34] and [23], the semantics of **TEL** is defined on top of probabilistic labeled state-transition systems, which generalize Kripke semantics and allow for a straightforward logical characterization of bisimulation based equivalence. Moreover, we extend soundness, completeness, and decidability results of modal languages with graded modalities to our setting.

The upside of our approach is that, compared to the other modal logic formalizations of trust mentioned at the beginning of this section, we combine in a unique model all the aspects surveyed above that are useful to determine quantitatively trust evaluations. In particular, to the best of our knowledge, our formalization of trust for multi-agent systems represents the first effort of including in modal logics context sensitivity, agents' trust propensities, and weighted evaluations of (personal and reported) experiences.

3. Syntax and semantics of TEL

We start by introducing the language of *Trust Evidence Logic* (**TEL**). Formulas of **TEL** handle numbers from a fixed set called *base*.

Definition 1 *(Base)*. A recursively enumerable set $\{0,1\} \subseteq \mathbb{B} \subseteq [0,1]$ is called a *base* if it satisfies the following closure conditions:

– *Quasi-closure under addition*:

$$\forall r, r' \in \mathbb{B}. (r + r' \leq 1 \implies r + r' \in \mathbb{B})$$

– *Closure under complements*:

$$\forall r \in [0,1]. (r \in \mathbb{B} \implies 1 - r \in \mathbb{B})$$

– *Density*: if $\mathbb{B}$ is infinite then

$$\forall r, r' \in \mathbb{B}. (r < r' \implies \exists r^* \in \mathbb{B}. (r < r^* \wedge r^* < r'))$$

The above definition generalizes the notion of base introduced in [17], which is restricted to finite sets. Notice also that, in order to work with (semi)decidable logics, we require bases to be recursively enumerable. In what follows, we shall fix a base $\mathbb{B}$ and keep it throughout this section. As an example, the set of rational numbers $\mathbb{Q}_{[0,1]}$ could be a candidate base.

Definition 2 *(Language of* **TEL***)*. Let At be a countable set of *propositional atoms* ranging over $\alpha, \beta, \gamma, \ldots$, and let A be a countable set of *labels* ranging over $a, b, c, \ldots$. The language $\mathcal{L}_{\mathbf{TEL}}$ is generated by the following grammar:

$$\phi ::= \top \mid \alpha \mid \neg\phi \mid \phi \vee \phi \mid wT_p^a(\phi) \tag{1}$$

where $\alpha \in At$, $a \in A$, and $p \in \mathbb{B}$.

The elements of $\mathcal{L}_{\mathbf{TEL}}$ are called *formulas*, and range over ϕ, ψ. As usual, we define $\bot \triangleq \neg\top$, $\phi \wedge \psi \triangleq \neg(\neg\phi \vee \neg\psi)$ and $\phi \to \psi \triangleq \neg\phi \vee \psi$. The *weak trust* modality $wT_p^a(\phi)$ stands for *"it is weakly trusted that* ϕ". As we will see, in principle the semantics of $wT_p^a(\phi)$ corresponds to that of the probabilistic diamond modality $\langle a\rangle_p^{\geq}\phi$ of [34] and resembles an analogous operator of PCTL [23]; however, it is based on a different interpretation of the parameters p and a.

For the sake of brevity, we define the following derived modal operators for all $a \in A$ and all $p \in \mathbb{B}$:

– *trust* $T_p^a(\phi)$ to stand for $\neg wT_{1-p}^a(\neg\phi)$, thus corresponding to the probabilistic diamond modality $\langle a\rangle_p^{>}\phi$;
– *exact trust* $eT_p^a(\phi)$ to stand for $wT_p^a(\phi) \wedge \neg T_p^a(\phi)$, thus corresponding to the probabilistic diamond modality $\langle a\rangle_p^{=}\phi$;
– *weak distrust* $wD_p^a(\phi)$ to stand for $wT_p^a(\neg\phi)$;
– *distrust* $D_p^a(\phi)$ to stand for $T_p^a(\neg\phi)$;
– *exact distrust* $eD_p^a(\phi)$ to stand for $wD_p^a(\phi) \wedge \neg D_p^a(\phi)$.

To define the semantics for **TEL** we first introduce the underlying model, which follows the same lines of the semantic models of [34] and [23].

Definition 3 *(PLSTS)*. A *Probabilistic Labeled State-Transition System* is a tuple $\mathfrak{M} = (S, At, A, \{\mathscr{D}_a\}_{a\in A}, v)$, where S is a non-empty countable set of states, At is a countable set of state labels, A is a countable set of transition labels, v is a *valuation function*

$v : S \to \wp(At)$, and $\{\mathscr{D}_a\}_{a\in A}$ is a family of probabilistic transition functions of the form $\mathscr{D}_a : S \times S \to \mathbb{B}$ satisfying the following condition:

$$\forall s \in S : \quad \sum_{t\in S} \mathscr{D}_a(s,t) = 1. \tag{2}$$

If $X \subseteq S$, we let $\mathscr{D}_a(s, X)$ denote $\sum_{t\in X} \mathscr{D}_a(s,t)$.

PLSTS-like models defined in the setting of probabilistic HML and PCTL are typically employed to investigate the properties of probabilistic systems (see, e.g., [6,33]). In those cases, each state in S represents a system configuration, characterized by a set of atomic predicates in At expressing the statements that are true in the state. Then, each transition is enriched with a label in A describing the action that is executed through the transition and with probabilistic information expressing quantitatively the behavior associated to the action.

In our setting, we provide an interpretation of PLSTSs for reasoning about context-aware evidence-based trust. In particular, such an interpretation is intended to formalize a notion of *trust towards formulas*, which derives from a combination of personal opinions and recommendations by third parties, analogously as done in classical reputation systems. To this aim, the states of a PLSTSs represent the agents of a (social) network. The set of atomic propositions $v(s) \subseteq At$ labeling the state s associated to an agent represents the agent's *opinions* (*beliefs*), i.e., an agent is labeled with an atomic proposition if the agent believes that such proposition is true. Under this interpretation, notice that two distinct states might satisfy the same atomic propositions. Moreover, the transitions represent relations between agents that allow agents to exchange opinions concerning certain topics of interest. The probabilistic information associated to the transition represents the strength of the relation. More precisely, a transition connecting an agent s to an agent s' indicates that agent s considers the opinions of agent s' when evaluating her/his trust towards the formulas of the language. In particular:

– the label a of the transition indicates the context under which the agent s is considering the opinions of agent s';
– the numerical value $\mathscr{D}_a(s, s')$ associated to the transition from agent s to agent s' indicates how much weight is placed on the opinion of s' by s.

Therefore, the transition label $a \in A$ represents the context in which trust is estimated, while $\mathscr{D}_a(s, s')$ represents the *normalized level of expertise* of agent s' as perceived by agent s with respect to the given context a.

Based on this interpretation, we can now give an intuitive explanation of the semantics of **TEL**. If we consider the non-modal fragment of our language, the truth of formulas in a state actually expresses the beliefs of the corresponding agent. Then, the weak trust formula $wT^a_p(\phi)$ estimated in a state s establishes whether the corresponding agent weakly trusts ϕ with respect to the trustworthiness threshold p and the context a. Such an estimation depends on the opinions of the agents to which s is connected through a-labeled transitions and weighted by the values associated to such transitions. In particular, $wT^a_p(\phi)$ is true at a state s if the sum of the values that the distribution $\mathscr{D}_a(s, \_)$ associates to the states in which ϕ is true is $\geq p$.

Formally, the interpretation of **TEL** formulas is given as follows.

Definition 4 *(Truth).* Let $\phi \in \mathcal{L}_{\mathbf{TEL}}$ and $\mathfrak{M} = (S, At, A, \{\mathscr{D}_a\}_{a\in A}, v)$ be a PLSTS. We inductively define the notion of ϕ being *satisfied* (or *true*) *at state* $s \in S$ *in* $\mathfrak{M}$, written $s \vDash_{\mathfrak{M}} \phi$, as follows:

(a) $s \vDash_{\mathfrak{M}} \top$ iff true;
(b) $s \vDash_{\mathfrak{M}} \alpha$ iff $\alpha \in v(s)$, where $\alpha \in At$;
(c) $s \vDash_{\mathfrak{M}} \neg\phi$ iff $s \nvDash_{\mathfrak{M}} \phi$;
(d) $s \vDash_{\mathfrak{M}} \phi \vee \psi$ iff $s \vDash_{\mathfrak{M}} \phi$ or $s \vDash_{\mathfrak{M}} \psi$;
(e) $s \vDash_{\mathfrak{M}} wT^a_p(\phi)$ iff $\mathscr{D}_a(s, S_\phi) \geq p$, where $p \in \mathbb{B}$ and:

$$S_\phi \triangleq \{s' \in S \mid s' \vDash_{\mathfrak{M}} \phi\}. \tag{3}$$

In this case, we also say that ϕ is *satisfiable* in $\mathfrak{M}$. We say that ϕ is *true in* $\mathfrak{M}$, written $\vDash_{\mathfrak{M}} \phi$, when $s \vDash_{\mathfrak{M}} \phi$ holds for every $s \in S$; we say that ϕ is *true*, written $\vDash \phi$, when $\vDash_{\mathfrak{M}} \phi$ holds for every PLSTS $\mathfrak{M}$. A set of formulas Γ is *satisfied* (or *true*) *at state* $s \in S$ *in* $\mathfrak{M}$, written $s \vDash_{\mathfrak{M}} \Gamma$, if $s \vDash_{\mathfrak{M}} \phi$ for any $\phi \in \Gamma$. In this case we also say that Γ is *satisfiable* in $\mathfrak{M}$. We say that Γ is *true in* $\mathfrak{M}$, written $\vDash_{\mathfrak{M}} \Gamma$, when $s \vDash_{\mathfrak{M}} \Gamma$ holds for any $\phi \in \Gamma$ and any $s \in S$.

Remark 1. Some considerations about the interpretation of **TEL** formulas are in order. Firstly, since we are in a classical logical setting, we have that for any formula ϕ, either ϕ holds in s, or $\neg\phi$ holds in s. In other words, every agent s states her/his beliefs and trust estimations.[3] Taking a position is not a limitation for reputation models, where all involved agents express their beliefs and compute their trust estimations. Secondly, we recall that Equation (2) imposes that the sum of the evaluations for the expertise

[3] By virtue of Definition 4 it is easy to see that beliefs are classically constrained, e.g., if s states that ϕ and also states that ψ, s/he also states $\phi \vee \psi$. We will show in the following that analogous consistency results hold also for trust, when discussing the axiomatization for our logic.

of the different agents as perceived by any agent s is equal to 1. This suggests that s will always have a *claque* of agents that s/he evaluates as experts with respect to the given context a. However, the probabilistic transition function can be used to customize the composition of such a claque. For instance, agents' beliefs can be ruled out by assigning weight 0 to them. In practice, $\mathscr{D}_a(s,s') = 0$ expresses that agent s does not consider agent s' for trust estimations about a, for various possible reasons, e.g., s' is not accessible to s, or else s is not interested in the beliefs of s'. Moreover, function $\mathscr{D}_a$ could be reflexive, allowing an agent to take into consideration also her/his own expertise. In a limiting scenario there might be contexts in which the only relevant expertise is the one of the evaluating agent s, i.e., $\mathscr{D}_a(s,s) = 1$. This would mean, among other things, that the beliefs of agent s in context a wholly determine her/his trust.

The following are straightforward properties deriving from Definition 4.

Proposition 1. *Let $\mathfrak{M} = (S, At, A, \{\mathscr{D}_a\}_{a\in A}, v)$ be a* PLSTS. *For every $s \in S$, $a \in A$, and for every $\phi, \psi \in \mathcal{L}_{\mathbf{TEL}}$:*

1. $\mathscr{D}_a(s, S_{\neg\phi}) = 1 - \mathscr{D}_a(s, S_\phi)$;
2. $\mathscr{D}_a(s, S_{\phi\vee\psi}) + \mathscr{D}_a(s, S_{\phi\wedge\psi}) \geq \mathscr{D}_a(s, S_\phi) + \mathscr{D}_a(s, S_\psi)$.

Proof. Concerning point 1, we have $S_{\neg\phi} = S \setminus S_\phi$. Then:

$$\begin{aligned}\mathscr{D}_a(s, S_{\neg\phi}) &= \sum_{s'\in S_{\neg\phi}} \mathscr{D}_a(s,s') = \sum_{s'\in S\setminus S_\phi} \mathscr{D}_a(s,s') \\ &= \sum_{s'\in S} \mathscr{D}_a(s,s') - \sum_{s'\in S_\phi} \mathscr{D}_a(s,s') = 1 - \mathscr{D}_a(s, S_\phi).\end{aligned}$$

As for point 2, the following equation can be easily checked:

$$\sum_{\substack{s'\in S\\ s'\vDash_{\mathfrak{M}}\phi\vee\psi}} \mathscr{D}_a(s,s') + \sum_{\substack{s'\in S\\ s'\vDash_{\mathfrak{M}}\phi\wedge\psi}} \mathscr{D}_a(s,s') \geq \sum_{\substack{s'\in S\\ s'\vDash_{\mathfrak{M}}\phi}} \mathscr{D}_a(s,s') + \sum_{\substack{s'\in S\\ s'\vDash_{\mathfrak{M}}\psi}} \mathscr{D}_a(s,s') \quad \square$$

Using Definition 4 and Proposition 1, we can assign truth-conditions to the alternative modalities of trust:

Proposition 2. *Let $\phi \in \mathcal{L}_{\mathbf{TEL}}$ and $\mathfrak{M} = (S, At, A, \{\mathscr{D}_a\}_{a\in A}, v)$ be a* PLSTS. *Then, for all $s \in S$:*

- *$s \vDash_{\mathfrak{M}} T^a_p(\phi)$ iff $\mathscr{D}_a(s, S_\phi) > p$, where $p \in \mathbb{B}$;*
- *$s \vDash_{\mathfrak{M}} eT^a_p(\phi)$ iff $\mathscr{D}_a(s, S_\phi) = p$, where $p \in \mathbb{B}$;*

where S_ϕ is as in Equation (3).

Remark 2. We recall that weak distrust $wD^a_p(\phi)$ stands for $wT^a_p(\neg\phi)$. This duality leads to the following contradiction: for every $p \in \mathbb{B}$ with $p > 0.5$, it holds that $wT^a_p(\phi) \wedge wD^a_p(\phi)$ is not satisfiable for any state of every PLSTS, i.e., a statement cannot be weakly trusted and weakly distrusted at the same time with respect to a given context and a threshold test expressing majority.[4] However, uncertainty is admissible in the case of lower thresholds, which could make both trust and distrust satisfiable. In particular, notice that $p = 0.5$ admits the satisfiability of $wT^a_p(\phi) \wedge wD^a_p(\phi)$.

The duality contradiction holds also for $T^a_p(\phi) \wedge D^a_p(\phi)$, provided that $p \geq 0.5$.

PLSTSs generalize Kripke models. It is then natural to look for a suitable notion of *frame* in the above extended setting. This will allow us to define classes of frames and to provide axiomatic characterizations for them, as in standard modal logic.

Definition 5 *(Frame).* Let $\mathfrak{M} = (S, At, A, \{\mathscr{D}_a\}_{a\in A}, v)$ be a PLSTS. The *frame of* $\mathfrak{M}$, written $\mathfrak{F}_{\mathfrak{M}}$, is a pair $(S, \{\mathcal{R}_a\}_{a\in A})$ where each $\mathcal{R}_a$ is called *accessibility relation* and is defined for any $s, s' \in S$ as:

$$(s,s') \in \mathcal{R}_a \quad \text{iff} \quad \mathscr{D}_a(s,s') > 0.$$

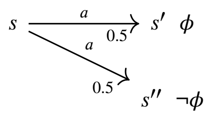
In this case we also say that $\mathfrak{M}$ is *based on* $\mathfrak{F}_{\mathfrak{M}}$. Frames range over $\mathfrak{F}$. A frame $\mathfrak{F}_{\mathfrak{M}} = (S, \{\mathcal{R}_a\}_{a\in A})$ is *reflexive* (resp. *symmetric*, *transitive*) if, for any $a \in A$, $(S, \mathcal{R}_a)$ is reflexive (resp. symmetric, transitive). We define $\mathcal{K}$, $\mathcal{T}$, $\mathcal{B}$ and $\mathcal{K}4$ as, respectively, the class of all frames, the class of all reflexive frames, the class of all symmetric frames, and the class of all transitive frames. Moreover, $\mathcal{S}4$ and $\mathcal{S}5$ denote the class of all frames whose relation is, respectively, a preorder and an equivalence.

[4] By Definition 4, $\mathscr{D}_a(s, S_\phi) + \mathscr{D}_a(s, S_{\neg\phi}) = 1$. Hence, the result follows immediately by virtue of the semantics of the modal operator.

Remark 3. Since, by Equation (2), the total mass of each distribution $\mathscr{D}_a(s,\_)$ is equal to 1, the accessibility relation $\mathcal{R}_a$ is *serial*, i.e. $\forall s \in S.\, \exists s' \in S.\, (s,s') \in \mathcal{R}_a$. This means that every frame is serial.

Definition 6 *(Validity).* Let $\phi \in \mathcal{L}_{\mathbf{TEL}}$ and let $\mathfrak{F}$ be a frame. We say that ϕ is *valid at a state* $s \in S$ *in* $\mathfrak{F}$, written $s \vDash_{\mathfrak{F}} \phi$, if $s \vDash_{\mathfrak{M}} \phi$ for every PLSTS $\mathfrak{M}$ based on $\mathfrak{F}$ (i.e. $\mathfrak{F}_{\mathfrak{M}} = \mathfrak{F}$). We say that ϕ is *valid in* $\mathfrak{F}$, written $\vDash_{\mathfrak{F}} \phi$, if ϕ is valid in $\mathfrak{F}$ at any state $s \in S$. Finally, we say that a formula ϕ is *valid* on a class of frames $\mathcal{F}$ if it is valid in every frame $\mathfrak{F} \in \mathcal{F}$. The notion of validity can be easily extended to sets of formulas.

Remark 4. By applying an opposite argument to the dualities discussed in Remark 2, for every $p \in \mathbb{B}$ with $p \leq 0.5$, it holds that $wT^a_p(\phi) \vee wD^a_p(\phi)$ is valid. Analogously, $T^a_p(\phi) \vee D^a_p(\phi)$ is valid as well, assuming $p < 0.5$. The case $p = 0.5$ is excluded by the following counterexample:



where ϕ holds in s' but not in s''. Then, neither $s \vDash T^a_{\frac{1}{2}}(\phi)$ nor $s \vDash D^a_{\frac{1}{2}}(\phi)$ hold. Instead, both $wT^a_p(\phi)$ and $wD^a_p(\phi)$ hold.

Finally, since **TEL** shares features with both PCTL and probabilistic HML, it inherits some of their semantic properties. A remarkable example is the logical characterization of bisimulation, whose standard definition is reported below.

Definition 7 *(Bisimulation).* Let $\mathfrak{M} = (S, At, A, \{\mathscr{D}_a\}_{a \in A}, v)$ be a PLSTS. An equivalence relation $\mathcal{B}$ over S is a *bisimulation* if and only if whenever $(s,t) \in \mathcal{B}$ it holds that $v(s) = v(t)$ and $\forall a \in A$, $\forall C \in S/\mathcal{B}$, which is the partition induced by the equivalence relation B:

$$\sum_{s' \in C} \mathscr{D}_a(s,s') = \sum_{t' \in C} \mathscr{D}_a(t,t').$$

As usual, we say that two states s and t in S are *bisimilar*, denoted $s \sim t$, if there exists a bisimulation $\mathcal{B}$ on S such that $s\mathcal{B}t$; two states s and t in S are *logically equivalent*, denoted $s \equiv t$, if and only if they satisfy exactly the same formulas of $\mathcal{L}_{\mathbf{TEL}}$. Then, we have the following theorem[5]:

Theorem 3 *(Logical characterization of bisimulation). For any* PLSTS, $\sim$ *coincides with* $\equiv$.

Proof. A straightforward adaptation to PLSTSs of the results in [7]. □

It is interesting to notice the trust-theoretical meaning of bisimulation, i.e., what does it imply for two agents to be bisimilar in the setting of trust? Thanks to Theorem 3, two bisimilar agents trust the same formulas. More in-depth, recalling the two conditions of Definition 7, two agents s and t belonging to the same class of equivalence share the same evidence ($v(s) = v(t)$) and, for each context, express the same trustworthiness towards every class of agents induced by bisimulation. Reasoning at the level of classes implies two facts. First, s and t could even interact with different agents and be, anyway, bisimilar. Second, as bisimulation induces aggregation of equivalent states, every agent may treat each class of agents as a unique entity whose belief is evaluated to some degree.

4. A case study in TEL

Trust is a critical factor in distributed systems like, e.g., mobile ad-hoc networks and sensor networks. In these systems, mechanisms for establishing trust are developed to support, e.g., the choice of the provider of a cloud service, or the choice of a neighbor for a forward service. For instance, the Reputation-based Framework for Sensor Networks [20] (RFSN) and the Robust Reputation System [11] (RRS) have been proposed for developing a web of trust based on the sharing of reputation information obtained by monitoring the cooperative and non-cooperative behaviors of the agents in the neighborhood. Then, both direct observations and second-hand recommendations represent parameters used to support trust based decision systems. These mechanisms can be used, e.g., within systems like the Dynamic Source Protocol [29], in which agents request message forwarding operations to trusted neighbors. Analogous trust based mechanisms are used for cloud computing and the choice of the provider of cloud services, see, e.g., [28,50].

[5] In [15] it is shown that disjunction can be discarded from the logic (and in [9] that, as an alternative to disjunction, conjunction can be discarded), without changing the logical characterization result.

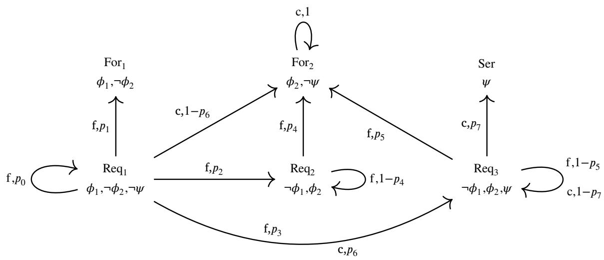


Fig. 1. PLSTS modeling a web of trust with two different contexts and six agents: a cloud service provider (*Ser*), two forwarding service nodes (For_i), and three requesting agents (Req_j).

We now show an example illustrating how our framework can model (and support the analysis of) the scenario of a distributed network, where the parameters expressing the strength of the connections between agents derive from by the trust mechanisms mentioned above.

In particular, we consider two different contexts, f identifying forwarding operations and c identifying cloud services. We have three agents that may *request* operations and services, two agents that can also provide the *forward service*, and a server delivering *cloud services*, as represented by the PLSTS in Fig. 1. The transitions express the web of opinion sharings that connect the various agents with each other. The formulas of interest express the reliability of each agent with respect to a given task. In particular, ϕ_i expresses the reliability of agent For_i as a forwarder, while ψ expresses the reliability of the server *Ser* as a cloud service provider. Hence, a formula of the form $wT^c_q(\psi)$ is satisfied by an agent s if the agent weakly trusts (with threshold q) the server as a reliable cloud service provider. For each state in Fig. 1, the related valuation reports only the formulas believed by the corresponding agent that are of interest for the following examples.

Concerning context f, Fig. 1 captures a scenario in which agent Req_1 has previous experience with agent For_1, resulting in a relation (see the f-labeled transitions with weights p_0 and p_1) enforced by the fact that Req_1 believes ϕ_1. On the other hand, Req_1 is also available to take into account the opinions of the neighborhood (see the f-labeled transitions towards agents Req_2 and Req_3, with weights p_2 and p_3, respectively), which instead supports agent For_2. Indeed, both agents Req_2 and Req_3 believe $\phi_2 \wedge \neg\phi_1$. Notice also that, by Definition 4, it must be $\sum_{i=0}^{3} p_i = 1$. Then, if Req_1 requires again a forwarding service, s/he might privilege agent For_1 if the trust based choice is governed by the formula $wT^f_q(\phi_1)$, with $(p_0 + p_1) \geq q$. Suppose that the delivery of such a service is not satisfactory because agent For_1 is poorly connected with respect to the destination asked by Req_1, and that the negative feedback causes an update of the probabilistic parameters,[6] with p_0 that is reduced in favor of p_2 and p_3. Then, for future requests, $wT^f_q(\phi_1)$ may be not satisfied anymore by agent Req_1, while, instead, $wT^f_q(\phi_2)$ could become true for Req_1, in spite of the fact that Req_1 does not believe ϕ_2. Afterwards, to keep track of the fact that agents For_1 and For_2 must be evaluated differently depending on the routing requirements of the service required by Req_1, it may be useful to refine the context f. This can be done by splitting it into, e.g., two different contexts, which would allow Req_1 to distinguish the scenarios with respect to which the forwarding agents must be evaluated.

Let us consider a slightly different example concerning context c. Suppose that Req_1 requires a cloud service, but s/he never interacted with the agent *Ser* providing such a service. Notice that what Req_1 believes about ψ in context c is irrelevant, as Req_1 does not consider herself an expert in context c. Hence, Req_1 decides to collect information from the neighborhood to understand whether other agents trust agent *Ser*. Notice that such a neighborhood is represented by agents Req_3 and For_2, to which Req_1 is connected through c-labeled transitions. Hence, by assuming a threshold q, the formula to check could be $wT^c_q(wT^c_q(\psi))$. Since agent Req_3 satisfies $wT^c_q(\psi)$, while agent For_2 satisfies $wD^c_q(\psi)$, by virtue of Definition 4, Req_1 satisfies $wT^c_q(wT^c_q(\psi))$ if and only if $p_6 \geq q$, where p_6 is the weight associated to the c-labeled transition from Req_1 to Req_3.

In general, by feeding the model with normalized values deriving from the trust-based mechanisms under investigation, through our framework it is possible to formally model the network of interest and verify the properties related to trust-based decisions. Regarding the scalability issues in the verification process, it is worth investigating the decidability properties and the model checking algorithms for **TEL**. This is discussed in the following sections.

5. Normal modal logics for TEL

This section presents results about various *trust evidence normal modal logics* (TENML), the *normal* modal logics for **TEL**. In particular, we shall focus on the systems TEK, TET, TEB, TEK4, TES4 and TES5, which can be seen as quantitative and *serial* extensions of the standard modal logics K, T, B, K4, S4 and S5, respectively.

Each such TENML is proven to be sound with respect to all those PLSTSs that are based on a specific class of frames. Moreover, we emphasize how each TENML and the related frame imply certain assumptions about the properties of trust that derive in such a framework. Then, the completeness result requires to establish a version of the canonical model theorem for **TEL** (Theorem 9), the

[6] The model proposed in this paper is purely static, and for this case study we are assuming that changes in the scenario imply the re-definition of the model, which can be obtained as a properly update of the current one. Making fully automated the dynamics of the model is left for future work.

main result of this section. We also show a counterexample to compactness and we use it to infer the failure of strong completeness. The proof of the canonical model theorem follows the lines of [13], adapting the techniques to a quantitative setting, while the counterexample to compactness is taken from [48]. Finally, we show that both TEK and TET are decidable without restrictions on the models.

All the proofs and the preparatory results are shown in the Appendix.

5.1. Trust evidence normal modal logics

We start by defining the notion of trust evidence normal modal logic.

Definition 8 *(TENML).* A *trust evidence normal modal logic* (TENML) is a set $\Lambda \subseteq \mathcal{L}_{\mathbf{TEL}}$ containing (i) all propositional tautologies, (ii) all the substitution instances of the following axiom schemata, for $\phi, \psi \in \mathcal{L}_{\mathbf{TEL}}$, $a \in A$ and $p, q \in \mathbb{B}$:

1. $T^a_p(\phi) \to wT^a_p(\phi)$;
2. $wT^a_p(\phi) \to T^a_q(\phi)$ with $q < p$;
3. $wT^a_0(\phi)$ always holds;
4. $wT^a_1(\phi \to \psi) \to (wT^a_p(\phi) \to wT^a_p(\psi))$;
5. $wT^a_1(\neg(\phi \wedge \psi)) \to (wT^a_p(\phi) \wedge wT^a_q(\psi)) \to wT^a_{p+q}(\phi \vee \psi)$ with $p + q \leq 1$;

and (iii) closed under *modus ponens* (MP) and *necessitation* (NEC_a with $a \in A$):

$$\frac{\phi}{wT^a_1(\phi)}\ \mathrm{NEC}_a$$

The above axioms state fundamental properties about $\mathscr{D}_a(s, S_\phi)$, see Equation (3). Axiom 1 and Axiom 2 formalize properties about inequalities. Axiom 3 and its dual formulation $\neg T^a_1(\neg\phi)$ state that $\mathscr{D}_a(s, S_\phi)$ is always a value in the base $\mathbb{B}$. Axiom 4 allows modalities to distribute over implication, and it can be seen as a generalization of (**K**), i.e. $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$. Finally, Axiom 5 describes compositionality for $\vee$, which holds whenever both disjuncts are "incompatible".

We shall write $\vdash_\Lambda \phi$, or simply $\vdash \phi$ (when no confusion arises), when $\phi \in \Lambda$. If Γ is a set of formulas, we write $\Gamma \vdash_\Lambda \phi$ if either $\vdash_\Lambda \phi$ or there are $\psi_1, \ldots, \psi_n \in \Gamma$ such that $\vdash_\Lambda (\psi_1 \wedge \ldots \wedge \psi_n) \to \phi$. Finally, we say that Γ is Λ-*consistent*, or simply *consistent* (when no confusion arises), when $\Gamma \nvdash_\Lambda \bot$. A set of formulas Γ is *maximal* Λ-*consistent*, or simply *maximal consistent* (when no confusion arises), if Γ is Λ-consistent and any Γ' properly containing Γ is not Λ-consistent. We shall often refer to Γ as a *mc-set*, and the set of all mc-sets is MAX.

The following proposition states some basic properties about Λ.

Proposition 4. *Let* $\phi, \psi \in \mathcal{L}_{\mathbf{TEL}}$ *and* $p, q \in \mathbb{B}$*:*

1. *If* $\vdash \phi \to \psi$ *then* $\vdash O^a_p(\phi) \to O^a_p(\psi)$*, where* $O \in \{wT, T, eT\}$*;*
2. *If* $\vdash \phi \leftrightarrow \psi$ *then* $\vdash O^a_p(\phi) \leftrightarrow O^a_p(\psi)$*, where* $O \in \{wT, T, eT\}$*;*
3. *If* $q < p$ *then* $\vdash wT^a_p(\phi) \to wT^a_q(\phi)$*;*
4. $\vdash eT^a_0(\phi \wedge \psi) \to ((eT^a_p(\phi) \wedge eT^a_q(\psi)) \to eT^a_{p+q}(\phi \vee \psi))$*.*

As usual, there is a smallest TENML Λ containing a given set of formulas Γ, called the TENML *axiomatized by* Γ. When $\Gamma = \emptyset$, Λ will be written TEK. In analogy with standard modal logic, we consider the following axioms:

$$(\mathbf{T}^+) = wT^a_1(\phi) \to \phi$$
$$(\mathbf{B}^+) = \phi \to wT^a_1(T^a_0(\phi))$$
$$(\mathbf{4}^+) = wT^a_1(\phi) \to wT^a_1(wT^a_1(\phi))$$

We shall work with extensions of TEK that combine the above axioms. In particular, TET, TEB, TEK4 are obtained by adding to TEK, respectively, ($\mathbf{T}^+$), ($\mathbf{B}^+$), and ($\mathbf{4}^+$). Moreover, TES4 extends TEK with both ($\mathbf{T}^+$) and ($\mathbf{4}^+$), and TES5 extends TES4 with ($\mathbf{B}^+$).

5.2. Soundness

The following theorem shows a series of soundness results relating the above defined TENMLs and classes of frames. In particular, TEK turns out to be sound w.r.t. all PLSTSs.

Theorem 5 *(Soundness for various TENMLs). For any $\phi \in \mathcal{L}_{\mathbf{TEL}}$:*

$\vdash_{\mathsf{TEK}} \phi$ implies $\vDash_{\mathcal{K}} \phi$; $\vdash_{\mathsf{TET}} \phi$ implies $\vDash_{\mathcal{T}} \phi$;
$\vdash_{\mathsf{TEB}} \phi$ implies $\vDash_{\mathcal{B}} \phi$; $\vdash_{\mathsf{TEK4}} \phi$ implies $\vDash_{\mathcal{K}4} \phi$;
$\vdash_{\mathsf{TES4}} \phi$ implies $\vDash_{\mathcal{S}4} \phi$; $\vdash_{\mathsf{TES5}} \phi$ implies $\vDash_{\mathcal{S}5} \phi$.

In Remark 3 we stressed that any frame is serial by construction. This condition is formalized by Axiom 2, which can be seen as a quantitative generalization of Axiom (**D**), i.e. $\Box\phi \to \Diamond\phi$. To see this, let us fix $A = \{*\}$ in Equation (1). It is not difficult to see that, by setting $\Box\phi \triangleq wT_1^*\phi$, Axioms 1-5 define exactly the serial normal modal logic D. In particular, Axiom 2 collapses to (**D**). More formally, we have:

Proposition 6. TEK *is a conservative extension of the normal modal logic* D.

This observation can be also found in [17], where a variant of Axiom 2 is considered. Let us finally notice that we could avoid seriality by defining each $\mathscr{D}_a(s, \_)$ as a sub-distribution, i.e., a distribution whose mass can be smaller than 1. However, turning the equality in Equation (2) into an inequality would break the equation in Proposition 1.1 and, as a consequence, the logical dualities between trust and distrust.

5.3. Axioms, frames, and trust

We will now show how the validities defined above are interpreted in terms of properties of trust. Specifically, we will first discuss all the axiom schemata and rules of Definition 8. Then, we will comment on the axioms $\mathbf{T}^+, \mathbf{B}^+, \mathbf{4}^+$, and the related frames.

The only interesting rule trust-wise is necessitation,[7] which states that all validities of the language must be weakly trusted in any context. Note that, according to our interpretation, a valid formula is a formula that is believed by all agents. Thus, if something is believed to be true by all agents, it is reasonable that each agent will also trust it, because there cannot be any instances of opposing evidence to the content of such formula, neither from personal experiences nor from reported cases.

As far as the five axiom schemata are concerned, statement 1 simply claims that if something is trusted, then it is also weakly trusted. Note that for something to be trusted, it must hold that the threshold for trust is completely passed, while weak trust only requires the threshold to be reached. Thus, independently from the notion of trust that is taken into consideration, whenever something is trusted, it will automatically be weakly trusted. Similar reasonings could be made for statements 2 and 3. In the former case, the left-hand side of the implication states that threshold p for trust has, at least, been reached. This would automatically imply that for all lower thresholds, both trust and weak trust would hold, i.e., the consequent of the implication is true. In the latter case, we have that any formula is weakly trusted if no threshold is actually applied. Statement 4 is a conservative extension of the principle relating logical consequence and trust [35,46]. The quantitative elements establish that full trust towards $\phi \to \psi$ (i.e., the formula is trusted even with respect to the maximum threshold) guarantees that if ϕ is weakly trusted with some threshold then ψ is weakly trusted with the same threshold. Statement 5 can be recast as $wD_1^a((\phi \wedge \psi)) \to ((wT_p^a(\phi) \wedge wT_q^a(\psi)) \to wT_{p+q}^a(\phi \vee \psi))$ with $p+q \leq 1$. If $\phi \wedge \psi$ is fully distrusted, then weak trust towards ϕ and ψ separately implies weak trust towards their disjunction. Since it is fully trusted that ϕ and ψ are mutually exclusive, the estimation of the trust towards their disjunction is simply given by the sum of the estimations of the trust towards the two disjuncts.

Let us now consider the various TENMLs and classes of frames (see, in particular, Theorem 5). In the setting of $\mathcal{T}$, by the reflexivity property we have that every agent considers also her/his own belief on ϕ to estimate the trust towards ϕ. Such a kind of relation is reasonable in settings in which no neophytes are present in the system and everybody give some consideration to their own opinion. Under such a hypothesis, axiom $\mathbf{T}^+$ states that full trust towards ϕ implies belief in ϕ, as even the personal opinion contributes to the estimation of trust. In the setting of $\mathcal{B}$, by the symmetry property we have that relations between agents are mutual (if A takes into account B's opinion, then also B takes into account A's opinion, possibly with different weights). This could be a reasonable assumption in closed and relatively small communities (e.g., a club open only to members), where mutual knowledge is a rule of thumb and every known agent is considered to be, to some extent, an expert in the context under consideration. Under such a hypothesis, axiom $\mathbf{B}^+$ states that if ϕ is believed by an agent, then it is fully trusted by the agent that some non-zero trust can be put on ϕ by other agents. Indeed, reciprocity ensures that the opinion of the agent on ϕ will be taken into consideration by the neighborhood. In the setting of $\mathcal{K}4$, by the transitivity property we have that relations among agents are transitive (if A takes into account B's opinion and B takes into account C's opinion, then A takes into account C's opinion; however, the weights attributed to such opinions are not interlaced). This is a rather strong assumption, which could be reasonable only in small communities, as relations among agents may propagate rapidly to the whole system, thus making infeasible the proper estimation of a large number of opinions. In such a scenario, axiom $\mathbf{4}^+$ means that if ϕ is fully trusted, then it is fully trusted that ϕ is fully trusted, as trust reiterates by virtue of the transitivity relation. Propagation of trust is a debated principle that, in many real-world cases, is assumed to be intransitive [16]. Finally, the remaining frames $\mathcal{S}4$ and $\mathcal{S}5$ provide combinations of the properties discussed above and do not provide novel insights about the interpretation for trust.

[7] *Modus Ponens* holds trivially, since it applies only to reasonings where trust plays no direct role.

Summarizing, while TEK and, to some extent, TET can be considered as reasonable frameworks for a general analysis of the concept of trust, the remaining extended normal modal logics are adequate only in very specific scenarios relying on strong assumptions about the properties of trust.

5.4. Completeness

In this subsection, we present the canonical model theorem for **TEL**, from which we infer a series of completeness results for TEK, TET, TEB, TEK4, TES4 and TES5. The canonical model for **TEL** can be obtained by adapting the one for graded modal logics (GML) [13] to a quantitative setting.

Definition 9 *(Canonical model).* Given any consistent TENML Λ, we define its *canonical model* $\mathfrak{M}_\Lambda = \langle S^\Lambda, At^\Lambda, A^\Lambda, \{\mathscr{D}_a^\Lambda\}_{a\in A^\Lambda}, v^\Lambda\rangle$ as follows:

- S^Λ is the set MAX of all maximally consistent sets;
- At^Λ, A^Λ are the sets of state and transition labels of $\mathcal{L}_{\mathbf{TEL}}$;
- for any $a \in A^\Lambda$ and for any $\Gamma, \Delta \in S^\Lambda$, we set:

$$\mathscr{D}_a^\Lambda(\Gamma,\Delta) = \min\{p \in \mathbb{B} \mid eT_p^a\phi \in \Gamma,\ \phi \in \Delta\}; \tag{4}$$

- for any $\Gamma \in S^\Lambda$, $v^\Lambda(\Gamma) = \{\alpha \in At^\Lambda \mid \alpha \in \Gamma\}$.

In [13], De Caro showed a key property relating the canonical model for GML to graded modalities $\Diamond_n$ ($n \in \mathbb{N}$): given a state of this model, i.e. a maximally consistent set Γ, $\Diamond_n\phi \in \Gamma$ iff the number of mc-sets containing ϕ that are "accessible" from Γ is strictly greater than n. In a similar way, by essentially transplanting De Caro's proof techniques into **TEL**, we obtain the following:

Lemma 7. *Let $\Gamma_0 \in S^\Lambda$, $\phi \in \mathcal{L}_{\mathbf{TEL}}$, $a \in A^\Lambda$, and let $p \in \mathbb{B}$. Then:*

$$\mathscr{D}_a^\Lambda(\Gamma_0, \{\Gamma \in S^\Lambda \mid \phi \in \Gamma\}) \geq p \iff wT_p^a(\phi) \in \Gamma_0$$

Before stating the Truth Lemma we show that the construction of $\mathfrak{M}_\Lambda$ yields a PLSTS. This fact follows by proving that the functions $\mathscr{D}_a^\Lambda(\Gamma, \_)$ are actually probabilistic distributions, and can be established by means of two fundamental properties of the maximally consistent sets:

1. For any $\Gamma \in S^\Lambda$ and $\phi \in \mathcal{L}_{\mathbf{TEL}}$ there exists exactly one $p \in \mathbb{B}$ such that $eT_p^a(\phi) \in \Gamma$;
2. Let $\Gamma_1, \dots, \Gamma_h \in S^\Lambda$ ($h \geq 2$) be distinct mc-sets. Then there exist $\psi_1, \dots .\psi_h \in \mathcal{L}_{\mathbf{TEL}}$ such that $\psi_i \in \Gamma_j$ iff $i = j$ and $\vdash \bigwedge_{\substack{1\leq i,j\leq h \\ \text{s.t. } i\neq j}} \neg(\psi_i \wedge \psi_j)$.

Point 1 crucially relies on the quantitative features of PLSTSs, as it requires the density property of infinite bases $\mathbb{B}$, while point 2 can be seen as a "separation result" for various modal logics (see [13]). In particular, if $\Gamma_0 \in S^\Lambda$ and $\phi \in \mathcal{L}_{\mathbf{TEL}}$, point 1 implies $wT_p^a(\phi), wT_{1-p}^a(\neg\phi) \in \Gamma_0$, for some $p \in \mathbb{B}$. By applying Lemma 7, we can easily conclude that the mass of $\mathscr{D}_a^\Lambda(\Gamma_0, \_)$ is greater than (or equal to) 1. Now, suppose towards contradiction that $\mathscr{D}_a^\Lambda(\Gamma_0, \_)$ has mass > 1. W.l.o.g. $\mathscr{D}_a^\Lambda(\Gamma_0, \_)$ can be taken with finite support $\{\Gamma_1, \dots, \Gamma_h\}$, where $h \geq 2$ by Lemma 7, so that there exist $\psi_1, \dots, \psi_h$ as in point 2. By point 1 there exists p_i such that $eT_{p_i}^a(\psi_i) \in \Gamma_0$ ($1 \leq i \leq h$), and by the fact that $\psi_1, \dots, \psi_h$ are mutually incompatible, using Proposition 4.4 we obtain $eT^a_{p_1+\dots+p_h}(\bigvee_{1\leq i\leq h}\psi_i) \in \Gamma_0$ with $p_1 + \dots + p_h \leq 1$. By definition we have $\mathscr{D}_a^\Lambda(\Gamma_0, \Gamma_i) \leq p_i$, which contradicts the assumptions.

Thanks to Lemma 7, we can easily establish the Truth Lemma by induction on formulas.

Lemma 8 *(Truth Lemma). For any $\phi \in \mathcal{L}_{\mathbf{TEL}}$ and $\Gamma \in S^\Lambda$:*

$$\Gamma \vDash_{\mathfrak{M}_\Lambda} \phi \quad \text{iff} \quad \phi \in \Gamma$$

Given the Truth Lemma, the canonical model theorem follows using a standard argument.

Theorem 9 *(Canonical model). Let Λ be a* TENML. *Then, for any $\phi \in \mathcal{L}_{\mathbf{TEL}}$:*

$$\phi \in \Lambda \qquad \text{iff} \qquad \phi \text{ is true in } \mathfrak{M}_\Lambda.$$

To show that a TENML Λ is complete with respect to a class of frames C it suffices to check that its canonical model $\mathfrak{M}_\Lambda$ is based on a frame in C. This allows us to infer in a fairly simple way the following completeness results:

Corollary 10 *(Completeness for various TENMLs). For any $\phi \in \mathcal{L}_{\mathbf{TEL}}$:*

$\vDash_{\mathcal{K}} \phi$ implies $\vdash_{\mathsf{TEK}} \phi$; $\vDash_{\mathcal{T}} \phi$ implies $\vdash_{\mathsf{TET}} \phi$;
$\vDash_{\mathcal{B}} \phi$ implies $\vdash_{\mathsf{TEB}} \phi$; $\vDash_{\mathcal{K}4} \phi$ implies $\vdash_{\mathsf{TEK4}} \phi$;
$\vDash_{\mathcal{S}4} \phi$ implies $\vdash_{\mathsf{TES4}} \phi$; $\vDash_{\mathcal{S}5} \phi$ implies $\vdash_{\mathsf{TES5}} \phi$.

Unfortunately, the compactness property does not hold for TENMLs over bases as given in Definition 1. To see this, let $\alpha \in At$ and consider the following infinite set of formulas:

$$\Gamma_\alpha = \{\neg(eT_p^a(\alpha)) \mid p \in \mathbb{B}\} \tag{5}$$

Clearly, any set $\Gamma_p \triangleq \Gamma_\alpha \setminus \{\neg(eT_p^a(\alpha))\}$ is satisfiable (and so any finite subset of Γ_α is satisfiable) but Γ_α is not. Now, by a standard argument, a TENML Λ is strongly complete with respect to a class of frames C iff any Λ-consistent set of formulas Δ is satisfiable on some $\mathfrak{F} \in C$. This means that the failure of strong completeness follows by showing that Γ_α is Λ-consistent. Formally, we have the following proposition:

Proposition 11 *(Failure of strong completeness). The following statements hold:*

1. *For any $p \in \mathbb{B}$, $eT_p^a(\alpha)$ is* TEK*-consistent;*
2. *(1) implies that Γ_α is* TEK*-consistent;*
3. *(2) implies that strong completeness fails for* TEK.

We point out that when a finite base and only a general context is considered, **TEL** can be reduced to the language proposed by Fattorosi-Barnaba and Amati in [17]. Hence, their compactness result would apply, *mutando mutandis*, to **TEL**. This allows us to recover strong completeness.

5.5. Finite model property and decidability

In this subsection, we adapt the so-called *filtration* technique (see, e.g., [10]) to prove the finite model property for TEK, i.e., taking a set of formulas Γ closed under subformulas, any formula $\phi \in \Gamma$ that is satisfiable in a PLSTS $\mathfrak{M}$ is also satisfiable in a *finite* one $\mathfrak{M}^\Gamma$ derived from $\mathfrak{M}$. To apply the filtration technique, the model satisfying any $\phi \in \Gamma$ is turned into a finite model by defining an equivalence relation $\sim_\Gamma$ that identifies all those states that verify the same set of subformulas of $\phi \in \Gamma$. However, such a technique cannot be directly applied to our quantitative setting, as two states s and t of $\mathfrak{M}$ can be associated with distinct distributions and yet satisfy $s \sim_\Gamma t$, so that there is no obvious way of defining distributions for $\mathfrak{M}^\Gamma$.

A way of adapting filtration to our framework is suggested by [48], where $\mathfrak{M}^\Gamma$ is constructed by choosing a representative for each equivalence class, and by taking the corresponding distributions.

Definition 10. Let Γ be a set of formulas closed under subformulas, and let $\mathfrak{M} = (S, At, A, \{\mathscr{D}_a\}_{a\in A}, v)$ be a PLSTS. We define $\sim_\Gamma$ as the following equivalence relation on S:

$$s \sim_\Gamma t \ := \ \text{for all } \phi \in \Gamma \ s \vDash_{\mathfrak{M}} \phi \ \text{ if and only if } \ t \vDash_{\mathfrak{M}} \phi.$$

Then, we define the PLSTS $\mathfrak{M}^\Gamma = (S^*, At, A, \{\mathscr{D}_a^*\}_{a\in A}, v^*)$, called *filtration of $\mathfrak{M}$ through* Γ, as follows:

– S^* is a subset of S such that $\biguplus_{s\in S^*}\{t \in S \mid t \sim_\Gamma s\} = S$. For any $s \in S$, with $!s$ we denote the (unique) state of S^* such that $!s \sim_\Gamma s$.
– for all $s, t \in S^*$ and $a \in A$,

$$\mathscr{D}_a^*(s,t) := \sum_{t'\in S \text{ s.t. } !t'=t} \mathscr{D}_a(s,t')$$

which satisfies $\sum_{t\in S^*} \mathscr{D}_a(s,t) = 1$.
– For any $s \in S^*$, $v^*(s) = \{\alpha \in At \mid \alpha \in v(s) \cap \Gamma\}$.

Lemma 12 *(Filtration). Let $\mathfrak{M}^\Gamma$ be a filtration of $\mathfrak{M}$ through Γ, and let S be the set of states of $\mathfrak{M}$. Then, for all $\phi \in \Gamma$ and for all $s \in S$:*

$$s \vDash_{\mathfrak{M}} \phi \quad \Longleftrightarrow \quad !s \vDash_{\mathfrak{M}^\Gamma} \phi.$$

It is straightforward to see that filtration preserves reflexivity. As a result, defining $\mathfrak{Fin}$ to be the class of frames with finite domain, we have.

Theorem 13. TEK *and* TET *have the finite model property. In particular, for any* $\phi \in \mathcal{L}_{\mathbf{TEL}}$*:*

$$\vdash_{\mathsf{TEK}} \phi \iff \models_{\mathcal{K}\cap\mathfrak{Fin}} \phi$$
$$\vdash_{\mathsf{TET}} \phi \iff \models_{\mathcal{T}\cap\mathfrak{Fin}} \phi$$

As a consequence, TEK *and* TET *are decidable.*

Since filtration does not preserve symmetry and transitivity, the same technique cannot be used to prove the decidability of the remaining TENMLs, which still remains an open issue. Finally, on the verification side for the decidable problems, we recall that the model checking techniques of [6,33] can be applied for PLSTSs by analogy of the probabilistic diamond operator with our trust operator.

6. Conclusion and future works

In this paper, we presented a formal language that can be used to analyze and reason with context-aware trust relations based on possessed and referred evidence. The language relies on probabilistic modal logics, where the modalities are simply inherited from the probabilistic variants of HML and CTL; we have shown how the ingredients of those formal frameworks could be combined in order to create a language that is suitable to formalize trust in environments where information might be unreliable and coming from different sources. In particular, our interpretation of the underlying semantic model allowed us to model networks of agents, contexts of evaluation, beliefs of agents about the statements, and personal judgments about the expertise of agents. Subsequently, we proved various interesting properties of such language.

For future work, it is worth investigating properties of trust that emerged in alternative formal frameworks, such as [4,35,36]. An example is to relax the additivity constraint on $\mathscr{D}_a$, moving to a sub-additive version, which would allow us to model scenarios in which higher-order uncertainty also plays a role.

As far as the theoretical foundations of **TEL** are concerned, we have extended soundness and completeness results of graded modal logics to our setting. As a desirable extension, it might be interesting to add dynamic components to the language in order to allow agents to modify their evaluations about experts and to use their trust in propositions to act in specific ways. This would also allow us to keep track of how trust might influence the beliefs of an agent and how those beliefs would then influence the beliefs of others. Another expansion we could explore is the possibility of moving from a classical Boolean setting to a Many-Valued Logic, in order to explicitly model the uncertainty of an agent about the truth of a proposition.

Another prospective avenue for the future development of this work is to explore its proof theoretical features. For example, the strong completeness results presented in the paper rely on a reduction theorem from a more expressive language (for finite bases). A potential future direction involves obtaining strong completeness results that do not dependent on a reduction to other languages. Moreover, the TEL framework represents a range of trust relations, therefore a promising area for further study is the exploration of proof systems that are more efficient in verifying the correctness of formulas expressing such relations. Another interesting proof-theoretical development of our language is to provide a sequent calculus for TEL in order to improve the applicability of the language in automated practical settings.

Finally, from a computational complexity perspective, the paper shows that the TEL language is decidable and that the model checking algorithms of [6,33] can be inherited as they are. The study of the decision problem for the classes of frames not considered in Theorem 13 represents one possible development in this regard. We also plan to study various model-theoretic properties - that can be derived from the bisimulation result provided - to understand whether different informational scenarios could be reduced to more simple cases (e.g., through filtration of models).

CRediT authorship contribution statement

Alessandro Aldini: Conceptualization, Formal analysis, Writing – original draft, Writing – review & editing. **Gianluca Curzi:** Formal analysis, Writing – original draft, Writing – review & editing. **Pierluigi Graziani:** Conceptualization, Formal analysis, Writing – original draft, Writing – review & editing. **Mirko Tagliaferri:** Conceptualization, Formal analysis, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix A. Proofs of Section 5

A.1. Proofs of Subsection 5.2

Proof of Proposition 4. Point 1 follows essentially by necessitation and Axiom 4. Point 2 follows by point 1. Point 3 follows by Axiom 1 and Axiom 2. Concerning point 4, by point 1, Axiom 1 and Axiom 2 we have $\vdash T^a_{p+q}(\phi \vee \psi) \to T^a_p(\phi)$. This implies $\vdash T^a_{p+q}(\phi \vee \psi) \to T^a_p(\phi) \vee T^a_q(\psi)$, and hence $\vdash wT^a_{1-p}(\neg\phi) \wedge wT^a_{1-q}(\neg\psi) \to wT^a_{1-(p+q)}(\neg(\phi \vee \psi))$. Since $\vdash (\phi_1 \wedge \phi'_1) \to ((\phi_2 \wedge \phi'_2) \to (\phi_3 \wedge \phi'_3))$ holds whenever $\vdash \phi_1 \to (\phi_2 \to \phi_3)$ and $\vdash \phi'_1 \to (\phi'_2 \to \phi'_3)$ do, and since $eT^a_r(\theta) \triangleq wT^a_r(\theta) \wedge wT^a_{1-r}(\neg\theta)$, we conclude by applying Axiom 3 and Axiom 5. □

Proof of Theorem 5. We only prove that $\vdash_{\mathsf{TEK}} \phi$ implies $\vDash_{\mathcal{K}} \phi$, and we show that the statement holds for the axioms in Definition 8 and it is preserved by modus ponens and necessitation. We just consider the (less trivial) cases of Axiom 4 and Axiom 5. Concerning the former, suppose that $\vDash_{\mathfrak{M}} wT^a_1(\phi \to \psi)$ and $\vDash_{\mathfrak{M}} wT^a_p(\phi)$ hold for any PLSTS $\mathfrak{M}$. Then, for any $s, s' \in S$, if $\mathscr{D}_a(s,s') > 0$ then $s' \vDash_{\mathfrak{M}} \phi \to \psi$. For any $s \in S$ we have:

$$\begin{aligned}\sum_{\substack{s' \in S\\ s' \vDash_{\mathfrak{M}} \psi}} \mathscr{D}_a(s,s') &\geq \sum_{\substack{s' \in S\\ s' \vDash_{\mathfrak{M}} \phi,\ s' \vDash_{\mathfrak{M}} \phi \to \psi}} \mathscr{D}_a(s,s') \\ &\geq \sum_{\substack{s' \in S\\ s' \vDash_{\mathfrak{M}} \phi,\ \mathscr{D}_a(s,s') > 0}} \mathscr{D}_a(s,s') \\ &\geq \sum_{\substack{s' \in S\\ s' \vDash_{\mathfrak{M}} \phi}} \mathscr{D}_a(s,s') \geq p\end{aligned}$$

Let us now show the case of Axiom 5. Suppose that $\vDash_{\mathfrak{M}} wT^a_1(\neg\phi \vee \neg\psi)$, $\vDash_{\mathfrak{M}} wT^a_p(\phi)$ and $\vDash_{\mathfrak{M}} wT^a_q(\psi)$ hold for any PLSTS $\mathfrak{M}$. By Axiom 3 we have $\vDash_{\mathfrak{M}} eT^a_0(\phi \wedge \psi)$. By Proposition 1.2, for any $s \in S$ we have:

$$\mathscr{D}_a(s, S_{\phi\vee\psi}) + \mathscr{D}_a(s, S_{\phi\wedge\psi}) \geq \mathscr{D}_a(s, S_\phi) + \mathscr{D}_a(s, S_\psi)$$

Since $\mathscr{D}_a(s, S_{\phi\wedge\psi}) = 0$, $\mathscr{D}_a(s, S_\phi) \geq p$ and $\mathscr{D}_a(s, S_\phi) \geq q$, we are done. □

Proof of Proposition 6. We fix $A \triangleq \{*\}$ and we set in $\mathcal{L}_{\mathbf{TEL}}$:

$$\Box\phi \triangleq wT^*_1(\phi)$$
$$\Diamond\phi \triangleq \neg\Box\neg\phi = \neg wT^*_1(\neg\phi) = T^*_0(\phi)$$

To show that TEK is a conservative extension of D we have to prove that any theorem of D is a theorem of TEK, and any theorem of TEK in the language of D is a theorem of D. This can be easily inferred from the following observations:

- NEC_* is exactly the usual necessitation rule in modal logic;
- the axioms (**K**) (i.e. $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$) and (**D**) (i.e. $\Box\phi \to \Diamond\phi$) are instances of, respectively, Axiom 2 and Axiom 4;
- the only axioms of TEK in the language of D are Axiom 2 and Axiom 4. □

A.2. Proofs of Subsection 5.4

Let Λ be any TENML, that we keep fixed throughout this subsection. The following proposition allows us to generalize Proposition 4.4.

Proposition 14. *Let $\phi_0, \ldots, \phi_k \in \mathcal{L}_{\mathbf{TEL}}$ $(k \geq 1)$. If $\vdash \bigwedge_{\substack{0 \leq i,j \leq k\\ i \neq j}} \neg(\phi_i \wedge \phi_j)$ then:*

$$\vdash \bigwedge_{0 \leq i \leq k} eT^a_{p_i}(\phi) \to eT^a_p\Big(\bigvee_{0 \leq i \leq k} \phi_i\Big)$$

for every $p_0, \ldots, p_1 \in \mathbb{B}$, where $p = p_0 + \ldots + p_k \leq 1$.

Proof. The proof is by induction on k. If $k = 1$ and $\vdash \neg(\phi_0 \wedge \phi_1)$ then we have $\vdash wT_1^a(\neg(\phi_0 \wedge \phi_1))$ by necessitation. By Axiom 3, $\vdash wT_0^a(\phi_0 \wedge \phi_1)$. By definition, $\vdash eT_0^a(\phi_0 \wedge \phi_1)$. Let $p_0, p_1 \in \mathbb{B}$. By Proposition 4.4 and modus ponens, $\vdash (eT_{p_0}^a(\phi_0) \wedge eT_{p_1}^a(\phi_1)) \rightarrow eT_{p_0+p_1}^a(\phi_0 \vee \phi_1)$. Let us now consider the case $k+1$, and assume $\vdash \bigwedge_{\substack{0\le i,j\le k+1\\ i\neq j}} \neg(\phi_i \wedge \phi_j)$. We have:

$$\vdash \bigwedge_{\substack{0\le i,j\le k\\ i\neq j}} \neg(\phi_i \wedge \phi_j) \tag{6}$$

$$\vdash \bigwedge_{0\le i\le k} \neg(\phi_i \wedge \phi_{k+1}) \tag{7}$$

By induction hypothesis, Equation (6) implies

$$\vdash \bigwedge_{0\le i\le k+1} eT_{p_i}^a(\phi) \rightarrow eT_{p'}^a\big(\bigvee_{0\le i\le k} \phi_i\big) \wedge eT_{p_{k+1}}^a(\phi_{k+1}) \tag{8}$$

for every $p_0, \ldots, p_{k+1} \in \mathbb{B}$, where $p' = p_0 + \ldots + p_k$. By using logical equivalences, Equation (7) implies:

$$\vdash \neg \bigvee_{0\le i\le k} (\phi_i \wedge \phi_{k+1})$$

which, in turn, implies:

$$\vdash \neg\big(\big(\bigvee_{0\le i\le k} \phi_i\big) \wedge \phi_{k+1}\big)$$

By applying the induction hypothesis on the base case, we have:

$$\vdash \left(eT_{p'}^a\big(\bigvee_{0\le i\le k} \phi_i\big) \wedge eT_{p_{k+1}}^a \phi_{k+1}\right) \rightarrow eT_p^a\big(\bigvee_{0\le i\le k+1} \phi_i\big)$$

where $p = p' + p_{k+1}$. Finally, by using Equation (8), we get the result. □

The next propositions and lemmas are about maximally consistent sets. To begin with, we recall some well-known basic properties that follow by maximality of mc-sets.

Proposition 15 *(Properties of mc-sets [10]). If Λ is a* TENML*, Γ is a mc-set and $\phi, \psi \in \mathcal{L}_{\mathbf{TEL}}$ then:*

- *if $\phi, \phi \rightarrow \psi \in \Gamma$ then $\psi \in \Gamma$;*
- *$\Lambda \subseteq \Gamma$;*
- *either $\phi \in \Gamma$ or $\neg\phi \in \Gamma$;*
- *$\phi \vee \psi \in \Gamma$ iff $\phi \in \Gamma$ or $\psi \in \Gamma$.*

The following result holds in any logic extending the classical propositional calculus.

Lemma 16 *([13]). Let $\Gamma_1, \ldots, \Gamma_h$ ($h \ge 2$) be distinct mc-sets. Then there exist $\psi_1, \ldots .\psi_h \in \mathcal{L}_{\mathbf{TEL}}$ such that:*

- *$\psi_i \in \Gamma_j$ iff $i = j$, and*
- *$\vdash \bigwedge_{\substack{1\le i,j\le h\\ \text{s.t. } i\neq j}} \neg(\psi_i \wedge \psi_j)$.*

The following is a remarkable property of finite bases.

Lemma 17 *(Peremptory lemma [48]). If $\mathbb{B} = \{p_0, \ldots, p_n\}$ then, for all $\phi \in \mathcal{L}_{\mathbf{TEL}}$, $\vdash_{\mathsf{TEK}} eT_{p_0}^a(\phi)\nabla \ldots \nabla eT_{p_n}^a(\phi)$, where $\phi\nabla\psi := (\phi \vee \psi) \wedge \neg(\phi \wedge \psi)$.*

We are now able to state and prove Lemma 18 and Lemma 19, which introduce some key properties about mc-sets we are interested in. In particular, point 4 of the following lemma makes crucial use of the density property of infinite bases $\mathbb{B}$: this is exactly where the quantitative features of the semantics of PLSTSs are exploited for the canonical model construction.

Lemma 18. *Let $\Gamma \in MAX$, $\phi, \psi, \phi_1, \ldots, \phi_k \in \mathcal{L}_{\mathbf{TEL}}$, and $p, q, p_0, \ldots, p_k \in \mathbb{B}$:*

1. *if $p \ge q$ and $wT_p^a(\phi) \in \Gamma$ then $wT_q^a(\phi) \in \Gamma$;*

2. *if* $\vdash \bigwedge_{\substack{0\le i,j\le k\\ i\neq j}} \neg(\phi_i \wedge \phi_j)$ *and* $eT^a_{p_i}(\phi_i) \in \Gamma$ $(i \le k)$ *then* $eT^a_p(\bigvee_{0\le i\le k} \phi_i) \in \Gamma$*, where* $p = p_0 + \ldots + p_k \le 1$*;*
3. *if* $eT^a_p(\phi), eT^a_q(\phi) \in \Gamma$ *then* $p = q$*;*
4. *for every* ϕ *there exists* $p \in \mathbb{B}$ *such that* $eT^a_p(\phi) \in \Gamma$*;*
5. *if* $\vdash \phi \to \psi$ *and* $eT^a_p(\psi) \in \Gamma$*, then there exists exactly one* q *such that* $eT^a_q(\phi) \in \Gamma$ *and moreover* $q \le p$*.*

Proof. Point 1 follows by maximality of Γ and Proposition 4.3. Point 2 follows by maximality of Γ and Proposition 14. To show point 3, assume $p > q$. Since $eT^a_p(\phi), eT^a_q(\phi) \in \Gamma$, by definition and by maximality of Γ we have $wT^a_p(\phi), \neg T^a_q(\phi) \in \Gamma$. By Axiom 2 and maximality of Γ, we have $T^a_q(\phi) \in \Gamma$, which contradicts consistency of Γ. Let us now show point 4. If the basis $\mathbb{B}$ is finite we simply apply Lemma 17 and maximality of Γ. Otherwise, $\mathbb{B}$ is infinite, and hence dense. By Axiom 3 and maximality of Γ we have $wT^a_0(\phi), \neg T^a_1(\phi) \in \Gamma$. This means that the following sets are nonempty:

$$A \triangleq \{p \in \mathbb{B} \mid wT^a_p(\phi) \in \Gamma\}$$

$$B \triangleq \{q \in \mathbb{B} \mid \neg T^a_q(\phi) \in \Gamma\}$$

and we set $p^* \triangleq \max A$ and $q^* \triangleq \min B$. Suppose now that $q^* > p^*$. This means that there exists $q^* > r > p^*$ such that $\neg T^a_r(\phi) \notin \Gamma$. By maximality of Γ, we have $T^a_r(\phi) \in \Gamma$. By Axiom 1, $wT^a_r(\phi) \in \Gamma$, which contradicts maximality of p^*. Hence, it must be that $p^* \ge q^*$. By definition, we have $wT^a_{p^*}(\phi), \neg T^a_{q^*}(\phi) \in \Gamma$. If it were the case that $p^* > q^*$ then, by Axiom 2 and by maximality of Γ, we would have $wT^a_{p^*}(\phi) \to T^a_{q^*}(\phi) \in \Gamma$, and hence $\neg wT^a_{p^*}(\phi) \in \Gamma$, contradicting consistency of Γ. Therefore, $p^* = q^*$. Let us finally prove point 5. By point 4, there exists $q \in \mathbb{B}$ such that $eT^a_q(\phi) \in \Gamma$. Suppose towards contradiction that $q > p$. By definition, we have $wT^a_q(\phi) \in \Gamma$. By Proposition 4.1 we have $wT^a_q(\psi) \in \Gamma$. Since by definition $\neg T^a_p(\psi) \in \Gamma$ and $q > p$, by Axiom 2 we have $\neg wT^a_q(\psi) \in \Gamma$. This contradicts consistency of Γ. □

Lemma 19. *Let* $p \in \mathbb{B}$, $\Gamma \in MAX$*, and let* $\phi \in \mathcal{L}_{\mathbf{TEL}}$*. If* $eT^a_p(\psi) \in \Gamma$ *then there exist* $p_0, p_1 \in \mathbb{B}$ *such that* $p = p_0 + p_1 \le 1$ *and both* $eT^a_{p_0}(\psi \wedge \phi)$ *and* $eT^a_{p_1}(\psi \wedge \neg\phi)$ *are in* Γ*.*

Proof. Since we have both $\vdash \psi \wedge \overline{\phi} \to \psi$ with $\overline{\phi} \in \{\phi, \neg\phi\}$ and $eT^a_p(\psi) \in \Gamma$, by Lemma 18.5 there exist unique $p_0, p_1 \le p$ such that $eT^a_{p_0}(\psi \wedge \phi), eT^a_{p_1}(\psi \wedge \neg\phi) \in \Gamma$. Moreover, since $\vdash \neg((\psi \wedge \phi) \wedge (\psi \wedge \neg\phi))$, by Lemma 18.2 we have $eT^a_{p_0+p_1}(\xi) \in \Gamma$, where $\xi \triangleq (\psi \wedge \phi) \vee (\psi \wedge \neg\phi)$. Clearly, $\vdash \xi \leftrightarrow \psi$, so that by Lemma 4.2, $\vdash eT^a_p(\xi) \leftrightarrow eT^a_p(\psi)$. This means that $eT^a_p(\xi) \in \Gamma$, and hence $p = p_0 + p_1$ by Lemma 18.3. □

We now need three lemmas in order to prove the Truth Lemma (Lemma 8), the crucial step for proving completeness.

Lemma 20. *Let* Λ *be a* TENML*, and let* $\Gamma, \Delta \in S^\Lambda$*. Then, the following are equivalent:*

1. $\mathscr{D}^\Lambda_a(\Gamma, \Delta) > 0$*,*
2. *for any* $\phi \in \mathcal{L}_{\mathbf{TEL}}$*,* $\phi \in \Delta$ *implies* $T^a_0(\phi) \in \Gamma$*,*
3. *for any* $\phi \in \mathcal{L}_{\mathbf{TEL}}$*,* $wT^a_1(\phi) \in \Gamma$ *implies* $\phi \in \Delta$*.*

Proof. Let us prove that point 1 implies point 2. Suppose $\mathscr{D}^\Lambda_a(\Gamma, \Delta) > 0$ and $\phi \in \Delta$. By Lemma 18.3-4 there exists a unique $p \in \mathbb{B}$ such that $eT^a_p(\phi) \in \Gamma$. Since $\mathscr{D}^\Lambda_a(\Gamma, \Delta) > 0$ we have $p > 0$, and so $T^a_0(\phi) \in \Gamma$. Let us now show that point 2 implies point 3. Suppose $\phi \notin \Delta$. By maximality of Δ we have $\neg\phi \in \Delta$. By point 2 we have $T^a_0(\neg\phi) \in \Gamma$. Since $\neg wT^a_1(\phi) \leftrightarrow T^a_0(\neg\phi)$ then $\neg wT^a_1(\phi) \in \Gamma$. By maximality of Γ we have $wT^a_1(\phi) \notin \Gamma$. Let us finally show that point 3 implies point 1. If $\phi \in \Delta$ then $\neg\phi \notin \Delta$, so that $wT^a_1(\neg\phi) \notin \Gamma$. This mean that $\neg wT^a_1(\neg\phi) \in \Gamma$, and so $T^a_0(\phi) \in \Gamma$. By the latter and Lemma 18.3-4 there exists a unique $p > 0$ such that $eT^a_p(\phi) \in \Gamma$. By definition this implies $\mathscr{D}^\Lambda_a(\Gamma, \Delta) > 0$. □

Lemma 21. *Let* $\Gamma \in S^\Lambda$ *and* $\phi \in \mathcal{L}_{\mathbf{TEL}}$*. If* $T^a_0(\phi) \in \Gamma$ *then there exists* $\Gamma^* \in S^\Lambda$ *such that* $\phi \in \Gamma^*$ *and* $\mathscr{D}^\Lambda_a(\Gamma, \Gamma^*) \neq 0$*.*

Proof. Let $\phi_1, \ldots, \phi_n, \ldots$ be an enumeration of $\mathcal{L}_{\mathbf{TEL}}$ such that $\phi_1 \triangleq \phi$. We define a chain of finite sets of formulas $\Psi_1 \subseteq \ldots \subseteq \Psi_n \subseteq \ldots$ such that, if $\psi_n = \bigwedge\{\theta \mid \theta \in \Psi_n\}$, then $T^a_0(\psi_n) \in \Gamma$. We proceed by induction. Concerning the base case, we set $\Psi_1 = \{\phi_1\}$, and $T^a_0(\psi_0) \in \Gamma$ holds by hypothesis. Suppose now that Ψ_n is such that $T^a_0(\psi_n) \in \Gamma$. By Lemma 18.3 there exists $p \in \mathbb{B}$ such that $eT^a_p(\psi_n) \in \Gamma$. Moreover, $T^a_0(\psi_n) \in \Gamma$ implies $p > 0$. By Lemma 19 there exist $p_0, p_1 \in \mathbb{B}$ such that $p = p_0 + p_1 \le 1$ and $eT^a_{p_0}(\psi_n \wedge \phi_{n+1}), eT^a_{p_1}(\psi_n \wedge \neg\phi_{n+1}) \in \Gamma$. Since $p > 0$, it must be that either $p_0 > 0$ or $p_1 > 0$. In the former case, we would have $wT^a_{p_0}(\psi_n \wedge \phi_{n+1}) \in \Gamma$ and hence, by Axiom 2, $T^a_0(\psi_n \wedge \phi_{n+1}) \in \Gamma$. So, we set $\Psi_{n+1} \triangleq \Psi_n \cup \{\phi_{n+1}\}$. In the latter case, we set $\Psi_{n+1} \triangleq \Psi_n \cup \{\neg\phi_{n+1}\}$.

Let us now show that, for any $n \ge 1$, Ψ_n is consistent. If it weren't the case, then $\vdash \neg\psi_n$, so that $\vdash wT^a_1(\neg\psi_n)$ by necessitation, and $\vdash \neg T^a_0(\psi_n)$ by definition. Since $T^a_0(\psi_n) \in \Gamma$, we would contradict consistency of Γ. Let us now set $\Gamma^* \triangleq \bigcup_{n\ge 0} \Psi_n$. Clearly, Γ^* is a mc-set and $\phi \in \Gamma^*$. Let $\chi \in \Gamma^*$. By definition, there exists $n \in \mathbb{N}$ such that $\chi \in \Psi_n$. Since $T^a_0(\psi_n) \in \Gamma$ and $\vdash T^a_q(\theta_1 \wedge \theta_2) \to T^a_q(\theta_i)$ holds by Proposition 4.1, $T^a_0(\chi) \in \Gamma$. By Lemma 20, we have $\mathscr{D}^\Lambda_a(\Gamma, \Gamma^*) \neq 0$. □

Proof of Lemma 7. Suppose firstly that $\mathscr{D}_a^\Lambda(\Gamma_0, \{\Gamma \in S^\Lambda \mid \phi \in \Gamma\}) \geq p$. We have two cases:

1. there exists $\Gamma \in S^\Lambda$ such that $\phi \in \Gamma$ and $\mathscr{D}_a^\Lambda(\Gamma_0, \Gamma) \geq p$;
2. for every $\Gamma \in S^\Lambda$, if $\phi \in \Gamma$ then $\mathscr{D}_a^\Lambda(\Gamma_0, \Gamma) < p$.

In the first case, by definition we have $wT_{p'}^a(\phi) \in \Gamma_0$, for some $p' \geq p$. By Lemma 18.1 we have $wT_p^a(\phi) \in \Gamma_0$. Let us now consider the second case. Clearly, there exist $h \geq 2$ distinct $\Gamma_1, \dots, \Gamma_h \in S^\Lambda$ such that, for each $1 \leq j \leq h$:

- $\phi \in \Gamma_j$;
- $0 < \mathscr{D}_a^\Lambda(\Gamma_0, \Gamma_j) < p$;
- $\sum_{j=1}^h p_j \geq p$, where $p_j \triangleq \mathscr{D}_a^\Lambda(\Gamma_0, \Gamma_j)$.

By Lemma 16, there exist $\psi_1, \dots . \psi_h$ formulas such that $\psi_j \in \Gamma_j$ $(1 \leq j \leq h)$ and $\vdash \bigwedge_{1 \leq i,j \leq h, i \neq j} \neg(\psi_i \wedge \psi_j)$. By definition, there exist $\theta_1, \dots, \theta_h$ such that $\theta_j \in \Gamma_j$ and $eT_{p_j}^a(\theta_j) \in \Gamma_0$ $(1 \leq j \leq h)$. We set $\xi_j \triangleq \theta_j \wedge \psi_j \wedge \phi$. We want to show the following properties:

(i) $\xi_j \in \Gamma_j$ $(1 \leq j \leq h)$;

(ii) $\vdash \bigwedge_{\substack{1 \leq i,j \leq h \\ \text{s.t. } i \neq j}} \neg(\xi_i \wedge \xi_j)$;

(iii) $eT_{p_j}^a(\xi_j) \in \Gamma_0$ $(1 \leq j \leq h)$.

Property (i) holds because $\theta_j, \psi_j, \phi \in \Gamma_j$ and because of maximality of Γ_j. Property (ii) holds because $\vdash (\xi_i \wedge \xi_j) \to (\psi_i \wedge \psi_j)$ for all $1 \leq i, j \leq h$ such that $i \neq j$, and $\vdash \bigwedge_{1 \leq i,j \leq h, i \neq j} \neg(\psi_i \wedge \psi_j)$. Concerning property (iii), since $\vdash \xi_j \to \theta_j$ and $eT_{p_j}^a(\theta_j) \in \Gamma_0$, by Lemma 18.5 there exists $r_j \leq p_j$ such that $eT_{r_j}^a(\xi_j) \in \Gamma_0$ for all $1 \leq j \leq h$. Since $\xi_j \in \Gamma_j$ by point (i), then $p_j = \mathscr{D}_a^\Lambda(\Gamma_0, \Gamma_j) \leq r_j$ by definition of $\mathscr{D}_a^\Lambda$.

Let us now define $\chi \triangleq \bigvee_{1 \leq j \leq h} \xi_j$. From point (ii)-(iii) above and Lemma 18.2, we have $eT_q^a(\chi) \in \Gamma_0$, where $q \triangleq \sum_{j=1}^h p_j$, and hence $wT_q^a(\chi) \in \Gamma_0$. Since $\vdash \xi_j \to \phi$, we also have $\vdash \chi \to \phi$ and $\vdash wT_1^a(\chi \to \phi)$ by necessitation. Then, by Axiom 4, we have $wT_q^a(\phi) \in \Gamma_0$. Finally, since $q \geq p$, Lemma 18.1 implies $wT_p^a(\phi) \in \Gamma_0$.

Concerning the other direction, suppose that $wT_p^a(\phi) \in \Gamma_0$. If $p = 0$ then we are done. Suppose $p > 0$. By Lemma 21 there exists at least one $\Gamma \in S^\Lambda$ such that $\phi \in \Gamma$ and $\mathscr{D}_a^\Lambda(\Gamma_0, \Gamma) \neq 0$. We can assume the family of these Γ to be finite. Let $\Gamma_1, \dots, \Gamma_h$ be these sets. We have to show that $q \triangleq \sum_{1 \leq j \leq h} p_j \geq p$, where $p_j \triangleq \mathscr{D}_a^\Lambda(\Gamma_0, \Gamma_j)$ $(1 \leq j \leq h)$. Suppose first that $h \geq 2$. Consider as above the formulas $\xi_j \triangleq \theta_j \wedge \psi_j \wedge \phi$ $(1 \leq j \leq h)$, and recall the properties (i)-(iii) above. Furthermore, if $\nu = \neg \bigvee_{1 \leq j \leq h}(\theta_j \wedge \psi_j)$ we have:

(a) $\vdash (\bigvee_{1 \leq j \leq h} \xi_j \vee (\nu \wedge \phi)) \leftrightarrow \phi$;
(b) $\vdash \neg(\xi_j \wedge (\nu \wedge \phi))$ $(1 \leq j \leq h)$;
(c) $eT_0^a(\nu \wedge \phi) \in \Gamma_0$.

Properties (a) and (b) hold by propositional calculus. We prove the property (c). Suppose $wT_r^a(\nu \wedge \phi) \in \Gamma_0$ for $r > 0$. By Lemma 21 there exists $\Gamma' \in S^\Lambda$ such that $\nu \wedge \phi \in \Gamma'$ and $\mathscr{D}_a^\Lambda(\Gamma_0, \Gamma') \neq 0$. So, Γ' is one of $\Gamma_1, \dots, \Gamma_h$ above. This fact, together with (i) and (b) contradicts the consistency of Γ'.

Now, let $\chi \triangleq \bigvee_{1 \leq j \leq h} \xi_j \vee (\nu \wedge \phi)$. By Lemma 18.2, the properties (ii) and (iii) imply $eT_q^a(\bigvee_{1 \leq j \leq h} \xi_j) \in \Gamma_0$. By applying once more Lemma 18.2, the properties (b) and (c) imply $eT_q^a(\chi) \in \Gamma_0$. By Lemma 4.2, from the property (a) we have $eT_q^a(\phi) \in \Gamma_0$. This implies $q \geq p$.

Let us finally discuss the case $h = 1$. The preceding argument still holds: being there no $\Gamma_1, \dots, \Gamma_h$ to be considered, we have $\xi_j = \theta_j \wedge \phi$ $(j = h = 1)$ and $\nu = \neg\theta_1$. The clause (ii) can obviously be skipped and, in the property (a), one has the formula $(\xi_1 \vee (\neg\theta \wedge \phi)) \leftrightarrow \phi$. □

Before stating the Truth Lemma we have to show that the construction of $\mathfrak{M}_\Lambda$ (see Definition 9) yields a PLSTS. This fact follows by proving that the functions $\mathscr{D}_a^\Lambda(\Gamma, \_)$ are actually probabilistic distributions. To do this we shall exploit the "separation property" introduced in Lemma 16.

Lemma 22. $\mathfrak{M}_\Lambda = \langle S^\Lambda, At^\Lambda, A^\Lambda, \{\mathscr{D}_a^\Lambda\}_{a \in A^\Lambda}, v^\Lambda \rangle$ *is a* PLSTS.

Proof. It suffices to show that, for all $a \in A^\Lambda$ and for all $\Gamma \in S^\Lambda$:

$$\sum_{\Delta \in S^\Lambda} \mathscr{D}_a^\Lambda(\Gamma, \Delta) = 1.$$

Let $\phi \in \mathcal{L}_{\mathbf{TEL}}$. By Lemma 18.3-4 there exists a unique $p \in \mathbb{B}$ such that $eT_p^a(\phi) \in \Gamma$. Using Lemma 7 we have:

$$\begin{aligned}
eT^a_p(\phi) \in \Gamma &\leftrightarrow wT^a_p(\phi) \wedge wT^a_{1-p}(\neg\phi) \in \Gamma\\
&\leftrightarrow wT^a_p(\phi) \in \Gamma \text{ and } wT^a_{1-p}(\neg\phi) \in \Gamma\\
&\leftrightarrow \mathscr{D}^\Lambda_a(\Gamma, \{\Delta \in S^\Lambda \mid \phi \in \Delta\}) \geq p, \text{ and}\\
&\quad \mathscr{D}^\Lambda_a(\Gamma, \{\Delta \in S^\Lambda \mid \neg\phi \in \Delta\}) \geq 1-p\\
&\leftrightarrow \mathscr{D}^\Lambda_a(\Gamma, \{\Delta \in S^\Lambda \mid \phi \in \Delta\}) \geq p, \text{ and}\\
&\quad \mathscr{D}^\Lambda_a(\Gamma, \{\Delta \in S^\Lambda \mid \phi \notin \Delta\}) \geq 1-p\\
&\leftrightarrow \mathscr{D}^\Lambda_a(\Gamma, \{\Delta \in S^\Lambda \mid \phi \in \Delta\}) \geq p, \text{ and}\\
&\quad \mathscr{D}^\Lambda_a(\Gamma, \{\Delta \in S^\Lambda \mid \phi \notin \Delta\}) + p \geq 1
\end{aligned}$$

This means that:

$$\begin{aligned}
1 &\leq p + \mathscr{D}^\Lambda_a(\Gamma, \{\Delta \in S^\Lambda \mid \phi \notin \Delta\})\\
&\leq \mathscr{D}^\Lambda_a(\Gamma, \{\Delta \in S^\Lambda \mid \phi \in \Delta\}) + \mathscr{D}^\Lambda_a(\Gamma, \{\Delta \in S^\Lambda \mid \phi \notin \Delta\})\\
&\leq \sum_{\Delta \in S^\Lambda} \mathscr{D}^\Lambda_a(\Gamma, \Delta).
\end{aligned}$$

Now, suppose toward contradiction that $1 < \sum_{\Delta \in S^\Lambda} \mathscr{D}^\Lambda_a(\Gamma, \Delta)$. Then, there must be a finite subset X of S^Λ such that $1 < \sum_{\Delta \in X} \mathscr{D}^\Lambda_a(\Gamma, \Delta)$. Let $X = \{\Delta_1, \dots, \Delta_n\}$, where $n \geq 2$ by Lemma 7. By Lemma 16, there exist $\psi_1, \dots, \psi_n \in \mathcal{L}_{\mathbf{TEL}}$ such that:

- $\psi_i \in \Delta_j$ iff $i = j$, and
- $\vdash \bigwedge_{\substack{1 \leq i,j \leq n\\ \text{s.t. } i \neq j}} \neg(\psi_i \wedge \psi_j)$.

By Lemma 18.3-4, for any $i \leq n$ there exists a unique $p_i \in \mathbb{B}$ such that $eT^a_{p_i}(\psi_i) \in \Gamma$. By Lemma 18.2 we have $eT^a_{p_1+\ldots+p_n}(\bigvee_{1 \leq i \leq n} \psi_i) \in \Gamma$, so that $p_1 + \ldots + p_n \leq 1$. By definition, for any $i \leq n$ we have:

$$\mathscr{D}^\Lambda_a(\Gamma, \Delta_i) = \min\{q \in \mathbb{B} \mid eT^a_q(\phi) \in \Gamma,\ \phi \in \Delta_i\} \leq p_i$$

so that $\sum_{1 \leq i \leq n} \mathscr{D}^\Lambda_a(\Gamma, \Delta_i) \leq \sum_{i=1}^n p_i \leq 1$, which contradicts our assumptions. □

Proof of Lemma 8. By induction on ϕ. If ϕ is an atomic proposition α then $\Gamma \vDash_{\mathfrak{M}_\Lambda} \alpha$ iff $\alpha \in v^\Lambda(\Gamma)$ iff $\alpha \in \Gamma$. If $\phi = \neg\psi$ then, using the induction hypothesis and maximality of Γ, $\Gamma \vDash_{\mathfrak{M}_\Lambda} \neg\psi$ iff $\Gamma \nvDash_{\mathfrak{M}_\Lambda} \psi$ iff $\psi \notin \Gamma$ iff $\neg\psi \in \Gamma$. If $\phi = \psi_1 \vee \psi_2$ then, by using the induction hypothesis and maximality of Γ, $\Gamma \vDash_{\mathfrak{M}_\Lambda} \psi_1 \vee \psi_2$ iff $\Gamma \vDash_{\mathfrak{M}_\Lambda} \psi_1$ or $\Gamma \vDash_{\mathfrak{M}_\Lambda} \psi_2$ iff $\psi_1 \in \Gamma$ or $\psi_2 \in \Gamma$ iff $\psi_1 \vee \psi_2 \in \Gamma$. Finally, if $\phi = wT^a_p(\psi)$ then:

$$\begin{aligned}
\Gamma \vDash_{\mathfrak{M}_\Lambda} wT^a_p(\psi) &\Leftrightarrow \mathscr{D}^\Lambda_a(\Gamma, \{\Delta \in S^\Lambda \mid \Delta \vDash_{\mathfrak{M}_\Lambda} \psi\}) \geq p && \text{by definition}\\
&\Leftrightarrow \mathscr{D}^\Lambda_a(\Gamma, \{\Delta \in S^\Lambda \mid \psi \in \Delta\}) \geq p && \text{by induction hypothesis}\\
&\Leftrightarrow wT^a_p(\psi) \in \Gamma && \text{Lemma 7.}
\end{aligned}$$
□

Given the Truth Lemma, the canonical model theorem follows using a standard argument.

Proof of Theorem 9. We have the following equivalences:

$$\begin{aligned}
\phi \in \Lambda \quad &\text{iff} \quad \forall \Gamma \in S^\Lambda.\ \ \phi \in \Gamma && \text{Proposition 15}\\
&\text{iff} \quad \forall \Gamma \in S^\Lambda.\ \ \Gamma \vDash_{\mathfrak{M}_\Lambda} \phi && \text{Lemma 8}\\
&\text{iff} \quad \phi \text{ is true in } \mathfrak{M}_\Lambda
\end{aligned}$$
□

Proof of Corollary 10. Let Λ be a TENML, and let C be a class of frames. If we show that $\mathfrak{F}_{\mathfrak{M}_\Lambda} \in C$ then, for any $\phi \in \mathcal{L}_{\mathbf{TEL}}$, $\vDash_C \phi$ would imply $\vDash_{\mathfrak{M}_\Lambda} \phi$, so that $\vdash_\Lambda \phi$ by Theorem 9. Now, $\vDash_{\mathcal{K}} \phi$ implies $\vdash_{\mathsf{TEK}} \phi$ is straightforward because $\mathcal{K}$ is the class of all frames. As for the remaining statements, it suffices to show the following properties:

1. $\mathfrak{F}_{\mathfrak{M}_{\mathsf{TET}}} \in \mathcal{T}$,
2. $\mathfrak{F}_{\mathfrak{M}_{\mathsf{TEB}}} \in \mathcal{B}$,
3. $\mathfrak{F}_{\mathfrak{M}_{\mathsf{TEK4}}} \in \mathcal{K}4$.

Concerning point 1, let $\Gamma \in S^{\mathsf{TET}}$ and $\phi \in \Gamma$. Since $(\mathbf{T}^{+}) \in \mathsf{TET}$, by maximality of Γ we have $\phi \to T_0^a(\phi) \in \Gamma$ and so $T_0^a(\phi) \in \Gamma$. Since $\phi \in \Gamma$ implies $T_0^a(\phi) \in \Gamma$, by Lemma 20 we have $\mathscr{D}_a^{\mathsf{TET}}(\Gamma, \Gamma) > 0$. Therefore, $\mathfrak{F}_{\mathfrak{M}_{\mathsf{TET}}} \in \mathcal{T}$. Let us now prove point 2. Let $\Gamma, \Delta \in S^{\mathsf{TEB}}$ be such that $\mathscr{D}_a^{\mathsf{TEB}}(\Gamma, \Delta) > 0$ and $\phi \in \Gamma$. Since $(\mathbf{B}^{+}) \in \mathsf{TEB}$, by maximality of Γ we have $\phi \to wT_1^a(wT_0^a(\phi)) \in \Gamma$ and so $wT_1^a(wT_0^a(\phi)) \in \Gamma$. Since $\mathscr{D}_a^{\mathsf{TEB}}(\Gamma, \Delta) > 0$, by Lemma 20 we have $T_0^a(\phi) \in \Delta$. Since $\phi \in \Gamma$ implies $T_0^a(\phi) \in \Delta$, by applying once again Lemma 20 we have $\mathscr{D}_a^{\mathsf{TEB}}(\Delta, \Gamma) > 0$. Therefore, $\mathfrak{F}_{\mathfrak{M}_{\mathsf{TEB}}} \in \mathcal{B}$. Let us finally prove point 3. Suppose $\mathscr{D}_a^{\mathsf{TEK4}}(\Gamma, \Delta) > 0$, $\mathscr{D}_a^{\mathsf{TEK4}}(\Delta, \Sigma) > 0$ and $\phi \in \Sigma$. Since $\mathscr{D}_a^{\mathsf{TEK4}}(\Delta, \Sigma) > 0$, by Lemma 20 we have $T_0^a(\phi) \in \Delta$. Since $\mathscr{D}_a^{\mathsf{TEK4}}(\Gamma, \Delta) > 0$, the same lemma implies $T_0^a(T_0^a(\phi)) \in \Gamma$. Since $(\mathbf{4}^{+}) \in \mathsf{TEK4}$ by maximality of Γ, we have $T_0^a(T_0^a(\phi)) \to T_0^a(\phi) \in \Gamma$. By modus ponens and maximality of Γ, we have $T_0^a(\phi) \in \Gamma$. Since $\phi \in \Sigma$ implies $T_0^a(\phi) \in \Gamma$, by Lemma 20 we have $\mathscr{D}_a^{\mathsf{TEK4}}(\Gamma, \Sigma) > 0$. □

Proof of Proposition 11. Concerning point 1, suppose towards contradiction that this is not the case. By definition, $\vdash_{\mathsf{TEK}} eT_p^a(\alpha) \to \bot$, that is $\vdash_{\mathsf{TEK}} \neg wT_p^a(\alpha) \vee T_p^a(\alpha)$. We prove that both $\neg wT_p^a(\alpha) \vdash_{\mathsf{TEK}} \bot$ and $T_p^a(\alpha) \vdash_{\mathsf{TEK}} \bot$, so that we can conclude $\vdash_{\mathsf{TEK}} \bot$. So assume $\neg wT_p^a(\alpha)$. By Proposition 4.3, we have $\neg wT_1^a(\alpha)$ and, since TENMLs are closed under substitutions, $\neg wT_1^a(\top)$. We derive the absurdity by applying the necessitation rule to $\top$. Now, assume $T_p^a(\alpha)$. By applying Axiom 2 and Axiom 1, we have $T_0^a(\alpha)$ and, since TENMLs are closed under substitutions, $T_0^a(\bot)$, that is $\neg wT_1^a(\top)$. We derive the absurdity by applying the necessitation rule to $\top$.

Let us now show point 2. Suppose towards contradiction that $\Gamma_\alpha \vdash_{\mathsf{TEK}} \bot$. By definition, there exist $\psi_1, \ldots, \psi_n \in \Gamma_\alpha$ such that $\vdash_{\mathsf{TEK}} (\psi_1 \wedge \ldots \wedge \psi_n) \to \bot$. Now, let $p \in \mathbb{B}$ be such that $\neg eT_p^a(\alpha) \notin \{\psi_1, \ldots, \psi_n\}$. By the Lindenbaun Lemma (see [10]), any TEK-consistent set of formulas can be extended to a mc-set. Since by point 1 $\{eT_p^a(\alpha)\}$ is TEK-consistent, a mc-set Γ^* containing $eT_p^a(\phi)$ exists. By Lemma 18.3 and maximality, $\{\psi_1, \ldots, \psi_n\} \subseteq \Gamma^*$. But then $\Gamma^* \vdash_{\mathsf{TEK}} \bot$, contradicting TEK-consistency of Γ^*.

Concerning point 3, any set $\Gamma_p \triangleq \Gamma_\alpha \setminus \{\neg eT_p^a(\alpha)\}$ is satisfiable (and so any finite subset of Γ_α is satisfiable) but Γ_α is not. Now, by a standard argument, a TENML Λ is strongly complete with respect to a class of frames C iff any Λ-consistent set of formulas Δ is satisfiable on some $\mathfrak{F} \in C$. This means that the failure of strong completeness follows by point 2. □

A.3. Proofs of Subsection 5.5

Proof of Lemma 12. The proof is by induction on ϕ. If $\phi = \alpha$ then by definition $s \vDash_{\mathfrak{M}} \alpha \iff \alpha \in v(s) \iff \alpha \in v^*(s) \iff !s \vDash_{\mathfrak{M}^\Gamma} \alpha$. The cases $\phi = \neg\psi$, $\phi = \psi_1 \vee \psi_2$ are obtained by applying the induction hypothesis. Last, if $\phi = wT_p^a(\psi)$ then, we first notice that:

$$\mathscr{D}_a(!s, \{t \in S \mid t \vDash_{\mathfrak{M}} \psi\}) = \mathscr{D}_a^*(!s, \{t \in S^* \mid t \vDash_{\mathfrak{M}^\Gamma} \psi\}) \tag{9}$$

Indeed, we have:

$$\begin{aligned}
\mathscr{D}_a(!s, \{t \in S \mid t \vDash_{\mathfrak{M}} \psi\}) &= \sum_{t \in S \text{ s.t. } t \vDash_{\mathfrak{M}} \psi} \mathscr{D}_a(!s, t) \\
&= \sum_{t \in S \text{ s.t. } !t \vDash_{\mathfrak{M}^\Gamma} \psi} \mathscr{D}_a(!s, t) && \text{ind. hyp.} \\
&= \sum_{u \in S^* \text{ s.t. } u \vDash_{\mathfrak{M}^\Gamma} \psi} \; \sum_{t \in S \text{ s.t. } !t = u} \mathscr{D}_a(!s, t) \\
&= \sum_{u \in S^* \text{ s.t. } u \vDash_{\mathfrak{M}^\Gamma} \psi} \mathscr{D}_a^*(!s, u) \\
&= \mathscr{D}_a^*(!s, \{t \in S^* \mid t \vDash_{\mathfrak{M}^\Gamma} \psi\})
\end{aligned}$$

Now, for any $s \in S$:

$$\begin{aligned}
s \vDash_{\mathfrak{M}} wT_p^a(\psi) &\iff !s \vDash_{\mathfrak{M}} wT_p^a(\psi) && \text{by definition} \\
&\iff \sum_{t \in S \text{ s.t. } t \vDash_{\mathfrak{M}} \psi} \mathscr{D}_a(!s, t) \geq p \\
&\iff \sum_{u \in S^* \text{ s.t. } u \vDash_{\mathfrak{M}^\Gamma} \psi} \mathscr{D}_a^*(!s, u) \geq p && \text{by (9)} \\
&\iff !s \vDash_{\mathfrak{M}^\Gamma} wT_p^a(\psi) \quad \square
\end{aligned}$$

Proof of Theorem 13. The left-right direction follows from Theorem 5. Concerning the right-left direction, by Corollary 10 $\nvdash_{\mathsf{TEK}} \phi$ implies $s \nvDash_{\mathfrak{M}} \phi$ such that, if $\nvdash_{\mathsf{TEK}} \phi$ then $\mathfrak{F}_{\mathfrak{M}} \in \mathcal{T}$. Let Γ_ϕ be the set of all subformulas of ϕ, and let $\mathfrak{M}^{\Gamma_\phi}$ be any filtration of $\mathfrak{M}$ through Γ_ϕ. By Lemma 12 $!s \nvDash_{\mathfrak{M}^{\Gamma_\phi}} \phi$, and since Γ_ϕ is finite $\mathfrak{M}^{\Gamma_\phi}$ has finite domain. In particular, if Γ_ϕ has n elements, then $\mathfrak{M}^{\Gamma_\phi}$ has at most 2^n states. If, moreover $\mathfrak{F}_{\mathfrak{M}} \in \mathcal{T}$, then $\mathfrak{F}_{\mathfrak{M}^{\Gamma_\phi}} \in \mathcal{T}$. □

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