# Portfolio Insurers and Constant Weight traders: who will survive?

E. Barucci[a], P. Dindo[b], F. Grassetti*[1]Corresponding author Email: francesca.grassetti@polimi.it[a]

[a]*Department of Mathematics, Politecnico di Milano, via Bonardi 9, 20133, Milano, Italy*
[b]*Department of Economics, University of Venice, Fondamenta San Giobbe, 30121 Venezia, Italy*
*(Received 22 April 2021)*

## Abstract

We consider the dynamics of asset prices and wealth for an exchange economy with long-lived assets where agents adopt different portfolio strategies: one agent allocates wealth according to the Constant Weight Strategy while the other follows a Portfolio Insurance Strategy. In a Lucas tree setting, assuming a binomial model for the endowment process, we provide conditions for survival and (relative) dominance of agents and discuss them in terms of the expected log-return of the risky asset. Both strategies survive for low expected log-returns, while both strategies dominate, but on different paths, for high expected log-returns. We show that the portfolio insurance strategy plays a stabilizing effect on the market volatility.

*Keywords:* Market Selection Hypothesis; Evolutionary Finance; Constant Weight Strategy; Constant Proportion Portfolio Insurance.
*JEL Classification:* C61, G11, G12

[1]*

## 1. Introduction

In this paper, we investigate the long run evolution of a financial market populated by two classes of agents: agents allocating wealth according to a Constant Weight Strategy (CWS) and agents adopting a POrtfolio Insurance Strategy (POIS). CWS and POIS represent the two most popular trading strategies in financial markets, the goal is to discern the emergence of homogeneous/heterogenous behaviors in the long run when prices are endogenously determined in equilibrium. We show that the final outcome depends on the asset fundamentals, the market can be populated by only one of the two classes of agents, and in some cases whether POIS or CWS dominate depends on the path of realized states of the world, or both strategies can survive.

The paper deals with heterogeneity of agents in financial markets. Up to now, the debate on the effects of heterogeneity on financial markets is not fully settled.

The 2007-2008 financial crisis showed that the financial network was too complex and that intermediaries were exposed to similar risks. [21, 20] provide evidence that the stock market performances of financial intermediaries (investment banks, commercial banks, insurers, hedge funds) were quite similar in the new millennium. Homogeneous market behavior may enhance financial instability. This claim has been confirmed theoretically in some papers. [27] shows that market volatility is not at the highest level when the economy is more heterogeneous but when the less risk averse agent owns a large quota of the wealth and therefore there is less risk sharing in the market. [11] show that the unconditional volatility of market return in the homogeneous model is higher than in a model with heterogenous agents (with respect to their coefficients of relative risk aversion), see also [4].

The effect of heterogeneity is more complex in case agents differ in beliefs. In this context, agents also trade when there is no risk to share. Considering the long run evolution of a market populated by agents with different beliefs, there are results showing that in case of persistent heterogeneity the dispersion of asset evaluations increases, and therefore, market volatility goes up, but on the other hand the unconditional evaluations of agents get closer to the one implied by fundamentals than the one of each individual investor in an homogeneous economy, see [8] for a temporary equilibrium model and [12] for a general equilibrium model.

The paper contributes to two strands of literature.

First of all we contribute to the literature on markets populated by traders employing a POIS, see [17, 10, 2, 3]. This literature has shown that the market volatility and risk premium are decreased by the presence of portfolio insurance. As far as we know, there is no equilibrium analysis on the evolution of the market populated by this type of traders. We address this topic investigating whether agents employing a POIS can persist and affect prices in a market populated also by other agents (those adopting the CWS). [25] show that a CWS dominates a POIS in case of erratic behavior of the asset price without a strong trend, the reverse is observed in case the evolution of the asset price is characterized by a trend. Their analysis considers an exogenous asset price and therefore is partial. In our analysis, we address the topic considering an endogenous price as the result of the equilibrium in the market populated by the two classes of agents.

We also contribute to the analysis of financial markets populated by heterogeneous traders. The reference is provided by the Friedman's conjecture (or Market Selection Hypothesis) according to which rational agents having more sophisticated information perform better than less informed or irrational agents, see [16]. A formal proof of the Friedman's conjecture has been obtained only recently in a general equilibrium setting showing that in the long run, among subjective expected utility agents, agents with rational expectations dominate the market notwithstanding their risk preferences, see [6, 26]. When trading strategies are not derived from expected utility maximization, results are not clear-cut. In this context dominance by one class of agents (homogeneity) is only one of the possible outcomes. Analyzing financial markets for long-lived assets in a temporary equilibrium setting, [14] show that the CWS investing proportionality to assets' expected relative dividends, the so-called Generalized Kelly Startegy (GKS), is going to dominate the market in the long run when the market is populated by other CWS. When such a strategy is not traded, long run market outcomes have only being investigated when all investors employ a CWS. [9] show that adopting a CWS close to the GKS guarantees survivorship but not dominance, and that when the CWSs are heterogenous enough, e.g., they differ from the GKS on different assets, then all investors survive and heterogeneity persists.

Differently from the existing literature on heterogeneity and financial evolution, we may have three different scenarios (almost surely) in the long run: dominance of one class of agents, coexistence of heterogenous traders on almost all path, dominance of one of the two classes of agents on different paths. This third outcome highlighting path dependency in the long run evolution is a novelty in the evolutionary finance literature and it depends on the switching mode of the POIS (constant fraction of wealth invested in the risky asset conditional on the wealth being above the floor, zero investment otherwise). To illustrate this scenario we concentrate on the most interesting case in which the POIS invests a fraction of wealth (when it is above the floor) higher than the fraction of the CWS. Consider an economy with good fundamentals (high expected returns of the risky asset). The POIS, conditional on being above the floor, is exposed to the risky asset, its dominance in the long run relies on the risky asset having good initial dividend realizations (good luck) so that the wealth of POIS agents remains above the floor. Otherwise, under adverse initial realizations (bad luck), the wealth of POIS agents touches the floor and, by exiting from the investment in the risky asset with good fundamentals, is driven out of the market by the CWS. Given the same fundamentals, these paths lead to a different dominant strategy in the long run. Instead, survival of both CWS and POIS agents is the long run outcome when the risky asset has weak fundamentals. In this case, the POIS reverting to a floor has a chance to survive instead of vanishing.

As far as the nexus between volatility and market heterogeneity is concerned, we show that there is no clear ranking, it depends on the type of strategy that dominates in the long run. A homogenous market with traders adopting a CWS is more volatile than a markets with heterogeneous traders investing through a CWS and a POIS. Instead, a homogenous market with traders adopting a POIS is less volatile than a markets with heterogeneous traders investing through a CWS and a POIS. This result confirms that POIS plays a stabilizing effect on the market.

The paper is organized as follows. In Section 2 we present the model. In Section 3 we introduce the two strategies. In Section 4 we analyze the evolution of the market and the dominance/survivorship if the two classes of agents. In Section 5 we provide a discussion and numerical simulations of the model. In Appendix Appendix A we provide the proof of our resutls.

## 2. The model

To model the financial economy we follow [9], both in terms of notation and characteristics of the asset market.

Time is discrete and indexed by $t \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. At each time $t \in \mathbb{N}$ one of the possible $S$ states of the world occurs. Given the set of all the possible sequences of states of the world $\Sigma$, a generic sequence in $\Sigma$ is denoted by

$\sigma$. The partial history of states, up to date $t$ included, is given by $\sigma_t = (s_1, s_2, \dots, s_t) \in S^t$, where $s_t \in S$ is the state of the world realized at date $t$. The natural filtration is $\{\Im_t\}$, $\Im$ is the $\sigma$-algebra generated by $\{\Im_t\}$ while $\mathcal{P}$ is the probability measure on $(\Sigma, \Im)$. Dividends, asset prices and wealth are adapted to $\{\Im_t\}$.

We consider an exchange economy populated by two agents trading two long-lived assets or Lucas' trees in a competitive market. Both assets pay as dividend units of the consumption good (apples, the numéraire of the economy). By assumption, the dividend of asset 1 is risk-free, as for a perpetual bond, while the dividend of asset 2 is risky, as for equity shares. The dividend payed by asset $k \in K = \{1, 2\}$ in $t \in \mathbb{N}_0$, possibly depending on the realization of $\sigma_t$, is denoted as $D_{k,t}(\sigma_t)$.

Let $P_t = (P_{1,t}, P_{2,t})$ the vector of asset equilibrium prices at time $t$ and $h_t^{(i)} = \left(h_{1,t}^{(i)}, h_{2,t}^{(i)}\right)$ the asset holding of agent $i$ from time $t$ to time $t+1$. Then the budget constraint of agent $i$ at time $t \geq 1$ satisfies the following equation:

$$C_t^{(i)} + P_{1,t} h_{1,t}^{(i)} + P_{2,t} h_{2,t}^{(i)} = \left(D_{1,t}(\sigma_t) + P_{1,t}\right) h_{1,t-1}^{(i)} + \left(D_{2,t}(\sigma_t) + P_{2,t}\right) h_{2,t-1}^{(i)} \quad (1)$$

where $C_t^{(i)}$ is her consumption in $t$. Notice that, in $t = 0$, the right-hand side of (1) is the value of agent $i$'s initial endowment, the agent-specific initial holding of asset shares and consumption good.

Assuming a unitary aggregate initial holding of both assets, the market clearing condition reads

$$\sum_{i=1}^{2} h_{k,t}^{(i)} = 1 \qquad \forall k \in K,\ t \in \mathbb{N}_0. \quad (2)$$

The vector of agents' wealths in $t \geq 1$ is $W_t = \left(W_t^{(1)}, W_t^{(2)}\right)$, where

$$W_t^{(i)} = \left(D_{1,t}(\sigma_t) + P_{1,t}\right) h_{1,t-1}^{(i)} + \left(D_{2,t}(\sigma_t) + P_{2,t}\right) h_{2,t-1}^{(i)}, \quad \forall i \in K \quad (3)$$

is the pre-consumption net wealth of agent $i$ at time $t$.

We denote with $\delta \in (0, 1)$ the fraction of wealth invested by each agent, implicitly assuming that they invest the same fraction, and therefore the consumption process is

$$C_t^{(i)} = (1 - \delta)\, W_t^{(i)}.$$

The assumption of a homogeneous $\delta$ is made to focus the analysis on the effect of different investment strategies on price/market evolution, see [6] for the discussion of different consumption fractions and the long run outcome.

We define the vector of investment fractions (over wealth) of agent $i$ as $\alpha_t^{(i)} = \left(\alpha_{1,t}^{(i)}, \alpha_{2,t}^{(i)}\right)$, so that

$$\alpha_{k,t}^{(i)} W_t^{(i)} = h_{k,t}^{(i)} P_{k,t}, \qquad \forall k \in K. \tag{4}$$

Note that $\sum_{k=1}^{2} \alpha_{k,t}^{(i)} = \delta$.

The wealth of the two agents and the prices of the two assets can be written as

$$W_t^{(i)} = (D_{1,t}(\sigma_t) + P_{1,t}) \frac{\alpha_{1,t-1}^{(i)} W_{t-1}^{(i)}}{P_{1,t-1}} + (D_{2,t}(\sigma_t) + P_{2,t}) \frac{\alpha_{2,t-1}^{(i)} W_{t-1}^{(i)}}{P_{2,t-1}} \quad \forall i \in K, \tag{5}$$

$$P_{k,t} = \sum_{i=1}^{2} \alpha_{k,t}^{(i)} W_t^{(i)} \quad \forall k \in K. \tag{6}$$

To conclude the description of the financial economy, we specify the stochastic process for the two asset dividends. We denote by $Y_t$ the aggregate endowment:

$$Y_t(\sigma_t) = \sum_{k=1}^{2} D_{k,t}(\sigma_t).$$

We shall assume $Y_t(\sigma_t)$ to depend on partial histories but relative dividend (dividend over endowment) to have the same distribution in all periods. In particular, given the probability measure $\mathcal{P}$ and two random variables $(d_1, d_2) = d$ on $(S, 2^s)$, there exists a probability measure $\pi > 0$ on $(S, 2^s)$ such that

**A1** $D_{k,t}(\sigma_t) = d_k(s_t) Y_t(\sigma_t), \quad \forall k \in K, \forall t \in \mathbb{N}$ and $\forall \sigma_t \in S^t$,

**A2** $s_t \in S$ are $i.i.d$ with $\mathcal{P}(s_t = s) = \pi(s) > 0$ $\forall t \in \mathbb{N}$, $\forall \sigma_t \in \Sigma$ and $\forall s \in S$.

In the following, distributions $\pi$ on $(S, 2^s)$ are identified through vectors $\pi \in \Delta^S$ by setting $\pi(s) = \pi_s$, $\forall s \in S$.

Thanks to these assumptions it follows that

$$\mathbb{E}^{\mathcal{P}}[D_{k,t} | \Im_{t-1}] = \mathbb{E}^{\pi}[d_k] \mathbb{E}^{\mathcal{P}}[Y_t | \Im_{t-1}], \quad \forall k \in K, \forall t \in \mathbb{N}.$$

Using the matrix $D$ with elements $d_{k,s} = d_k(s)$ $\forall k \in K$ and $\forall s \in S$, we also assume that dividends are non-negative, assets are non-redundant and each asset pays a positive dividend in at least one state, that is, respectively,

$$d_k(s) \geq 0, \quad \forall s \in S, \forall k \in K,$$

$$Rank(D) = K \leq S$$

and

$$\mathbb{E}^{\pi}[d_k] > 0, \quad \forall k \in K.$$

We assume $S = 2$ and therefore the aggregate endowment is a generic (geometric) random walk with the following evolution

$$Y_t = \begin{cases} g_u Y_{t-1} & \text{if } s_t = 1 \\ g_d Y_{t-1} & \text{if } s_t = 2 \end{cases} \tag{7}$$

with $g_d < g_u$.
Asset 1 is a perpetual bond with a time-varying coupon, at each time $t$ its dividend is given by

$$D_{1,t} = g_d Y_{t-1}.$$

Asset 2 pays a risky dividend

$$D_{2,t} = \begin{cases} (g_u - g_d) Y_{t-1} & \text{if } s_t = 1 \\ 0 & \text{if } s_t = 2 \end{cases} .$$

The dividend matrix $D$ can thus be written as

$$D = \begin{bmatrix} \frac{g_d}{g_u} & 1 \\ \frac{g_u - g_d}{g_u} & 0 \end{bmatrix} .$$

## 3. Trading Strategies

A CWS is such that the trader at any time splits her wealth in the assets (risky and risk free) according to time and wealth invariant weights. This strategy shows very nice properties. First of all it is the solution of the intertemporal optimal investment/consumption problem assuming a power utility function and that the assets evolve according to geometric Brownian motions in continuous time (Merton problem), see [23, 24]. The strategy provides a motivation for employing a benchmark in the asset management of long-only mutual funds. Considering an exogenous asset price dynamics (e.g., Black&Scholes or binomial model), this trading strategy turns out to be contrarian: the trader should sell the risky asset as the price goes up and should buy it as the risk asset price declines. Moreover, when the investment fractions are provided by the expected dividends/coupons of the assets, such a strategy coincides with the GKS, see [15] for a survey on the properties of this rule.

We specify the POIS as a Constant Proportion Portfolio Insurance (CPPI) strategy, see [18, 19] for alternative formulations replicating put options strategies and controlling the drawdown from the high water market. According to this strategy a floor (time varying threshold) is identified, if the wealth touches the threshold from above then the investment in the risky asset is set to zero, otherwise it is provided by a constant proportion of the cushion (difference between the wealth and the floor). Notice that this strategy can be rationalized assuming that the agent solves the classical Merton problem with a lower bound on consumption, see [5, 13]. A CPPI turned out to be difficult to be analyzed in our setting, therefore we consider a simplified version according to which a constant fraction of the wealth (and not of the cushion) is invested in the risky asset provided that the wealth is above a time varying threshold; otherwise the investment in the risky asset is set to zero. Notice that, assuming an exogenous asset price dynamics, a CPPI strategy is a momentum or trend follower strategy (buy when the asset increases and sell when the asset price decreases).

We assume that Agent 1 employs a CWS, her fraction of wealth invested in the risky asset is constant and time invariant:

$$\alpha_{1,t}^{CWS} = (1-x)\delta, \quad \alpha_{2,t}^{CWS} = x\delta, \quad x \in (0,1) \tag{8}$$

where $x$ is the investment rate for the risky asset. Notice that the CWS investing proportionally to expected dividends

$$x^{GKS} = \pi \left(1 - \frac{g_d}{g_u}\right) \in (0,1) \tag{9}$$

is the so-called GKS of [14].

Agent 2 uses the CPPI strategy. The investment strategy is built managing two accounts: a safe account protects from the downside of the risk exposure while the risky account is used to get an extra return. The asset allocation is performed rebalancing dynamically the funds between the two accounts.

More in detail, we define the *floor*

$$F_t = \lambda \delta W_0^{(2)} P_{1,t}, \quad \lambda \in (0,1),$$

as the value below which the net asset value of Agent 2 should not fall and the *cushion* $CH_t = \max[\delta W_t^{(2)} - F_t, 0]$ as the positive difference between the net asset value at time $t$ and the floor.

According to the CPPI strategy, the exposure $E_t$, i.e., the amount invested in the risky asset at time $t$, is equal to a multiple of $CH_t$: $E_t = mCH_t$, $m > 1$. Therefore, the fraction of wealth invested in the risky asset at time $t$ would be

$$\alpha_{2,t}^{POIS} = \frac{E_t}{W_t^{(2)}} \,.$$

However, the analysis considering a CPPI is difficult because the wealth invested in the risky asset at time $t$ would depend on prices at time $t$ that in turn depend on wealth at time $t$, making an explicit solution, suitable for performing an analysis of the market dynamics, not feasible.

In what follows, we consider a simplified version such that the fraction of wealth invested in the risky asset is a constant weight $m \in (0,1)$ of the wealth if it is above a threshold (one period before) and is set to zero in case it is below the threshold.

Finally, for both strategies we assume that no short-selling is allowed ($x$ and $m$ are greater than 0 and lower than 1). This assumption is standard in the finance evolutionary literature as it significantly simplifies the analysis, see [1] for a recent paper that studies the impact of short selling on the survivorship of agents.

Summarizing, the fraction of wealth invested in the risky asset at time $t$ by Agent 2 is

$$\alpha_{2,t}^{POIS} = \begin{cases} m\delta & W_t^{(2)} > F_{t-1}/\delta \\ 0 & W_t^{(2)} \leq F_{t-1}/\delta \end{cases} \tag{10}$$

and consequently we have

$$\alpha_{1,t}^{POIS} = \begin{cases} (1-m)\delta & W_t^{(2)} > F_{t-1}/\delta \\ \delta & W_t^{(2)} \leq F_{t-1}/\delta \end{cases} \quad . \tag{11}$$

## 4. Wealth dynamics and market selection

We analyze how the wealth of the two agents evolves depending on their investment rules. First, we normalize wealth and prices with respect to the aggregate endowment as follows

$$w_t^{(i)} = \frac{1-\delta}{Y_t} W_t^{(i)}, \quad p_{k,t} = \frac{1-\delta}{\delta Y_t} P_{k,t}, \quad i, k \in K, \tag{12}$$

so that

$$\sum_{i=1}^{2} w_t^{(i)} = \sum_{k=1}^{2} p_{k,t} = 1.$$

In order to study the relative wealth dynamics of the two agents, we denote $w_t = w_t^{CWS} = w_t^1$, $w_t^{POIS} = 1 - w_t = w_t^2$, and compute

$$p_{1,t} = \begin{cases} 1 - x w_t & w_t \geq f_t \\ 1 - m - (x-m) w_t & w_t < f_t \end{cases}, \tag{13}$$

$$p_{2,t} = \begin{cases} x w_t & w_t \geq f_t \\ (x-m) w_t + m & w_t < f_t \end{cases}, \tag{14}$$

$$w_t = \sum_{k=1}^{2} \frac{[(1-\delta) d_{k,t} + \delta p_{k,t}] \alpha_{k,t-1}^{CWS}}{\delta p_{k,t-1}} w_{t-1}, \tag{15}$$

where $f_t = 1 - \frac{\lambda W_0^{POIS} \delta Y_{t-1}}{Y_t} p_{1,t-1}$.

Substituting equation (13)-(14) into (15), the dynamics of the relative wealth of the agent adopting the CWS can be written as

$$w_t = \begin{cases} \frac{\{(1-\delta)[(1-x)d_{1,t}-xd_{2,t}]+(1-x)\delta\}w_{t-1}+(1-\delta)d_{2,t}}{1-\delta x+(\delta-1)xw_{t-1}} & \text{if } \begin{array}{l} w_t \geq f_t \\ w_{t-1} \geq f_{t-1} \end{array} \\ \\ \frac{(x-m)\{(1-x)[(1-\delta)d_{1,t}+\delta]-(1-\delta)xd_{2,t}\}w_{t-1}+m(1-x)[(1-\delta)d_{1,t}+\delta]+(1-m)(1-\delta)xd_{2,t}}{(x-m)\{[(1-\delta)x+m]w_{t-1}+1-\delta x-2m\}w_{t-1}+m(1-m)} w_{t-1} & \text{if } \begin{array}{l} w_t \geq f_t \\ w_{t-1} < f_{t-1} \end{array} \\ \\ \frac{\{(1-\delta)[(1-x)d_{1,t}-xd_{2,t}]+\delta(1-x-m)\}w_{t-1}+(1-\delta)d_{2,t}+\delta m}{1-\delta(x-m)+[\delta(x-m)-x]w_{t-1}} & \text{if } \begin{array}{l} w_t < f_t \\ w_{t-1} \geq f_{t-1} \end{array} \\ \\ \frac{\{(1-\delta)[(1-x)d_{1,t}-xd_{2,t}]+\delta(1-m-x)\}[m+(x-m)w_{t-1}]+[(1-\delta)d_{2,t}+\delta m]x}{(1-m)m+\{1-2m-(x-m)[\delta+(1-\delta)w_{t-1}](x-m)w_{t-1}\}} w_{t-1} & \text{if } \begin{array}{l} w_t < f_t \\ w_{t-1} < f_{t-1} \end{array} \end{cases} \tag{16}$$

We analyze the evolution of the relative wealth as in (16) to study the survival or dominance of the two classes of agents. Following the literature, see e.g. [9], we say that agent $i$ dominates on a sequence $\sigma$ if

$$\lim_{t\to\infty} w_t^{(i)}(\sigma) = 1\,.$$

She survives on $\sigma$ if

$$\limsup_{t\to\infty} w_t^{(i)}(\sigma) > 0\,,$$

and vanishes otherwise.

Survival and dominance determine the evolution of the economy: if an agent dominates, then the economy becomes homogeneous in the long run; if both agents survive, the economy exhibits long run heterogeneity and prices keep be determined by both agents. If an agent vanishes then, in relative terms, she disappears and does not have an impact on asset prices in the long run.

We characterize survivorship-dominance of the two classes of agents analyzing the difference between the conditional expected log wealth growth rate of the two agents. Given the assumptions on dividends and agents' behaviour, the expected growth rate at time $t$ depends on the relative wealth distribution

$$\begin{aligned}\mathbb{E}^{\mathcal{P}}\left[\log \frac{w_t^{CWS}}{w_{t-1}^{CWS}} - \log \frac{1-w_t^{CWS}}{1-w_{t-1}^{CWS}} \middle| \Im_{t-1}\right] &= \mathbb{E}^{\pi}\left[\log \frac{\sum_{k=1}^{2}[(1-\delta)d_{k,t}+\delta p_{k,t}]p_{k,t-1}^{-1}\alpha_{k,t-1}^{CWS}}{\sum_{k=1}^{2}[(1-\delta)d_{k,t}+\delta p_{k,t}]p_{k,t-1}^{-1}\alpha_{k,t-1}^{POIS}}\right] \\ &= \mu(w_{t-1}).\end{aligned} \tag{17}$$

where $\pi$ is the probability of $s_t = 1$, i.e., $D_{2,t} = (g_u - g_d)Y_{t-1}$.

As discussed in [9], sufficient conditions for the survival and dominance of one of the two classes of agents may be derived by studying the sign of $\mu$ for high an low levels of the relative wealth $w$:

- if $\mu(0) > 0$ and $\mu(1) < 0$, then almost surely CWS and POIS agents survive;
- if $\mu(0) > 0$ and $\mu(1) > 0$, then almost surely CWS agents dominate and POIS agents vanish;
- if $\mu(0) < 0$ and $\mu(1) < 0$, then almost surely POIS agents dominate and CWS agents vanish;
- if $\mu(0) < 0$ and $\mu(1) > 0$, then almost surely either CWS or POIS agents dominate (path-dependency).

Note that in our economy, the sign of $\mu$ depends on the relative wealth for two reasons. First, as it is typical with long-lived assets, relative portfolio returns depend on price levels. Second, a specific feature of our model, the POIS is endogenous (time/state varying).

Notice that arbitrage opportunities may occur because asset holdings are defined according to a fixed rule and not from the maximization of expected utility. In this framework, [9] prove that no arbitrage condition is granted if the vector of portfolio weights belong to the interior cone generated by the two columns of the dividend matrix, in our case if:

$$\frac{g_d}{g_u} < 1 - \max[x, m].$$

The following result can be proved.

**Proposition 1.** *Assume the no arbitrage condition* $\frac{g_d}{g_u} < 1 - \max[x, m]$ *holds true. Then:*

- *If* $x < m$*:*

  - *for $\pi$ low, both agents survive;*
  - *for $\pi$ intermediate, CWS agent dominates;*
  - *for $\pi$ high, there is path dependency.*

- *If $x > m$:*
  - *for $\pi$ low POIS agent dominates;*
  - *for $\pi$ intermediate, if $\delta$ is low and $\frac{g_d}{g_u}$ is low then there is path dependency, otherwise both agents survive;*
  - *for $\pi$ high, CWS agent dominates.*

*The exact thresholds are defined in the Proof, see Appendix Appendix A.*

## 5. Discussion of the results

The long run survival/dominance of the two classes of agents can be summarized as in Figure 1. On the left hand side, the fraction of wealth invested on the risky asset by the POIS is smaller than the fraction invested on the risky asset by the CWS ($x > m$). On the right hand side we consider the case where the fraction invested by the POIS is higher than the one invested by the CWS ($m > x$).

Results can be interpreted referring to the expected log-return of the risky asset which, conditional on $P_{2,t}$, is increasing in $\pi$, due to an increase in the expected dividend yield ($\pi$ is the probability of a high dividend). The higher is $\pi$, the higher the expected log-return of the risky asset, the larger the benefit of investing in it, the higher the (geometric) return of the agent who is the most exposed in the risky asset, the more likely that the agent dominates.

When the CWS is more aggressive than the POIS ($x > m > 0$), then the CWS dominates for high expected returns of the risky asset (high $\pi$). The POIS dominates for low expected returns (low $\pi$). Either survival of both types of strategies or path dependency occurs for intermediate levels of expected return (intermediate $\pi$). The rationale of these results is that in the first case ($\pi$ high) the CWS, being always more aggressive than the POIS on the risky assets, whose expected log-return increasing in $\pi$, is more likely to have a relatively larger portofolio return and thus dominates in the long run. In the latter case ($\pi$ low), the CWS is too much exposed to the risky asset but its return are not so favorable and therefore its performance

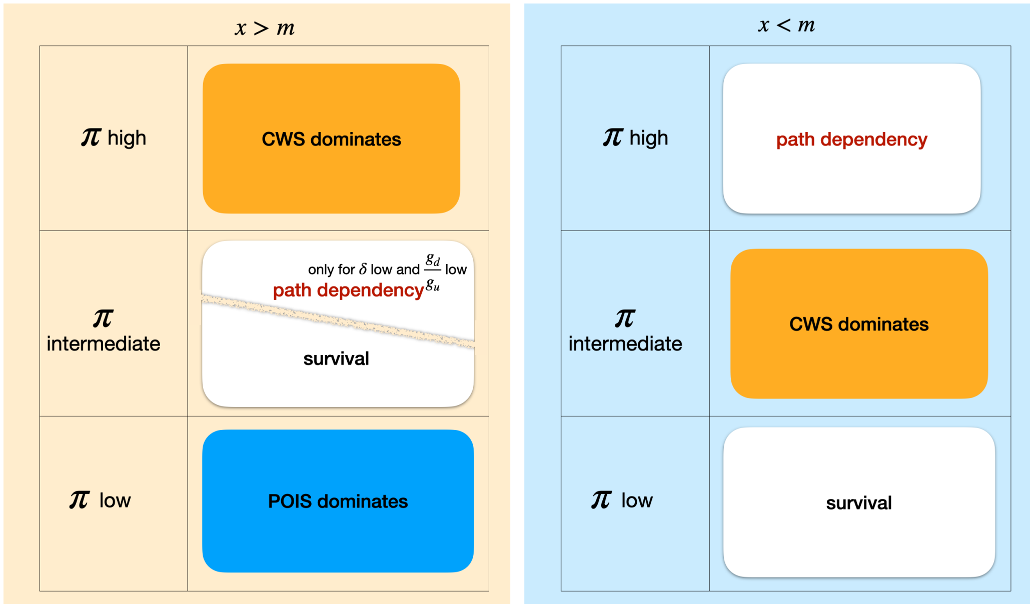


Figure 1: Survival and dominance of the two classes of agents as a function of $\pi$.

in the log-run is dominated by the POIS. Note that the result can be only partially understood in terms in distance to the $x^{GKS}$ in (9), which is also increasing in $\pi$, because being close to the GKS is sufficient for survival but not for dominance, as shown in [9].

When the POIS, conditional on being above the floor, is more aggressive than the CWS ($m > x > 0$), the long run outcomes are not the reversal of those obtained for $x > m$. The reason is that POIS is either more aggressive (when it is above the floor) or less aggressive (when it is below the floor) than the CWS. We concentrate on this case which is the more relevant from a practical point of view, in fact in the financial practice $m$ is often well above 1.

A high risky asset expected return (obtained for a high $\pi$) is associated to path dependence: either dominance of POIS or CWS. In agreement with the dictate of the GKS, dominance of the POIS is obtained only on those paths such that the initial realizations keep her wealth above the floor. Otherwise, if the paths lead the wealth below the threshold, the POIS switches to a zero weight and becomes less aggressive than the CWS which dominates in the long run. Given that both types of path occur with positive probability, the outcome is path dependent. In particular, the realizations at the beginning of the path discriminate between the dominance of the two classes of agents.

As $\pi$ and the expected return decrease, the POIS becomes less successful and the CWS dominates (intermediate case). As $\pi$ is further decreased (low $\pi$), both CWS and POIS survive because there is no dominant strategy. Notice the difference with respect to the case $x > m$, in that case the most aggressive strategy (CWS) is outperformed by the POIS which is less exposed to the risky asset and is always bounded from below by the floor. Instead, if $m > x$, both strategies turn out to be too aggressive after they have gained substantial wealth. When the wealth of the CWS is large (in relative terms), the POIS is below the floor and thus the fraction invested in the risky asset is null and the CWS turns out to be more aggressive. This feature allows the POIS to revert to higher wealth levels as time goes because the probability of a high return is low and the CWS is likely to underperform with respect to the POIS. When the wealth of the POIS is large (in relative terms), it is above the floor and more aggressive than the CWS ($m > x$). This feature allows the CWS to revert to higher wealth levels as time goes because the probability of a high return is low and the POIS is likely to underperform with respect to the CWS. In the long run neither strategy dominates, both survive and heterogeneity is persistent.

In Figure 2, we keep studying the case $m > x$ showing the long run outcomes (left panel) and the expected log-return of the risky asset (right panel) as a function of the high growth-rate probability ($\pi$) and of the low-growth rate $g_d$. The expected log-return of the risky asset is computed for the economy with a representative agent adopting an intermediate CWS, i.e., a risky-asset fraction $y \in (x, m)$.

The long run outcome confirms what is depicted in Figure 1. High expected returns (the right angle at the top left) are associated to path dependency, low expected returns (the right angle at the bottom right ) are associated to the survivorship of both classes of agents, middle expected returns are associated to dominance of the CWS.

The expected log-return positively depends on the expected return (drift) and negatively on its variance. Actually in case of a lognormal random variable the expected log-return is the logarithm of the expected return minus half its variance. This observation allows to read the results in terms of drift and volatility. High drift/low volatility is in favor to the POIS only on a subset of realizations (with positive measure). Low drift/high volatility leads to survival of both strategies. In the intermediate case, we observe the dominance of the CWS. This result sheds new light on those obtained by

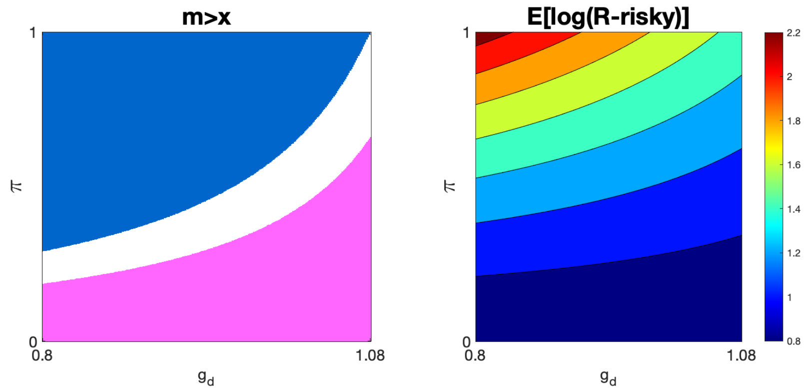


Figure 2: Parameter values: $g_u = 1.2$, $\delta = 0.8$. Left panel: long run outcomes for $m = 0.1$, $x = 0.0667$. Path dependency (blue), CWS dominates (white), survivorship of CWS and POIS (pink). Right panel: expected log-return of the risky asset when the representative agent uses $y = 0.09 \in (x, m)$

[25]. According to their partial analysis (asset price is exogenous), a CWS dominates a CPPI strategy in case of erratic behavior of the asset price without a strong trend, the reverse is observed in case the evolution of the asset prices is characterized by a trend. Their claim is partially confirmed in an equilibrium analysis context: in case of a strong trend, path dependency emerges with dominance either of the CWS or of the POIS and predominance of a CWS is obtained in case of an intermediate drift/volatility.

Figure 3 clarifies the difference between a POIS and a CWS providing further insights on Figure 2. We analyze the long run evolution of two economies: an economy populated by a CWS investing a fraction $x = 0.0667$ and a CWS investing a fraction $m = 0.1$ (left panel); an economy populated by a CWS investing a fraction $x = 0.0667$ and a CWS investing a fraction $m = 0.001$ (right panel). We denote the two trading strategies by CWS(x) and CWS(m). Notice that CWS(m) takes the two investing fractions of the POIS in Figure 2. The goal is to analyze what happens when the POIS doesn't switch but sticks to 0 (in our setting approximated by $m = 0.001$) or $m = 0.1$ with $m > x$. Notice that for a high expected return, on the left panel, it is CWS(m) that outperforms CWS(x) dominating in the long run. On the right panel it is CWS(x) that outperforms CWS(m) dominating in the long run. This outcome is also in agreement with [14] on the dominance of $x^{GKS}$. The different behavior for the two fractions of the POIS explains the path dependent result observed in Figure 2.

For a low expected return either CWS(x) dominates or both agents survive yielding the survival of both agents in Figure 2. Also in the intermediate region either CWS(x) dominates or both agents survive but in this case the CWS dominates. For high expected returns, either CWS(m) or CWS(x) dominates depending on the value of $m$, this leads to the path dependent result observed in Figure 2.

The POIS, by being aggressive when relatively wealthy and not aggressive when relatively not wealthy, differs from an aggressive strategy by surviving almost surely sure with low expected returns, instead of vanishing, but dominating only on a subset of path with high expected returns, instead of dominating almost surely. At the same time, the POIS differs from a strategy than invests very little in the risky asset by vanishing almost surely with intermediate expected returns, instead of surviving, and dominating on a subset of path with high expected returns, instead of vanishing almost surely.

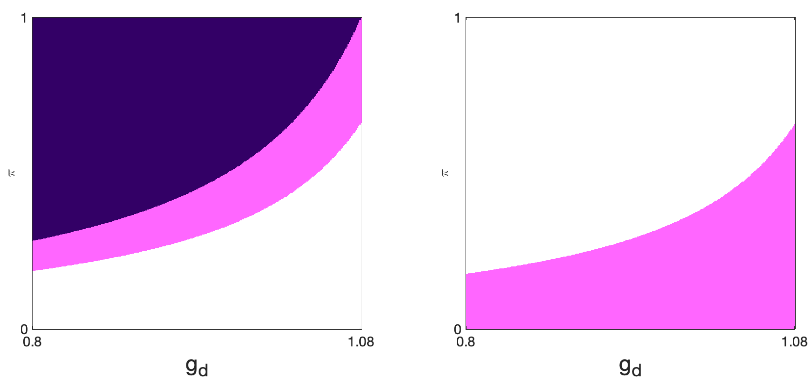


Figure 3: Left panel: long run outcomes for two CWS with $m = 0.1$ and $x = 0.0667$, respectively. Right panel: long run outcomes for to CWS with $m = 0.001$ and $x = 0.0667$. CWS(x) dominates (white), CWS(m) dominates (purple), survivorship of CWS and POIS (pink). Parameter values: $g_u = 1.2$, $\delta = 0.8$.

To conclude our investigation we provide some simulations reporting the relative wealth dynamics of a CWS and of a POIS. In the simulation we also report a measure of variability of the asset return over the entire simulation: $\overline{\sigma} = \sqrt{\sum_{t=1}^{T} \frac{1}{T} \log^2 \left( \frac{P_{2,t}+D_{2,t}}{P_{2,t-1}} \right)}$. We perform simulations using MATLAB© and the process of dividends is generated using the tool *binornd* that generates random numbers from the binomial distribution specified by the number

| Parameter/Condition | Value | Parameter/Condition | Value |
|---|---|---|---|
| $\delta$ | 0.8 | $g_d$ | 1.05 |
| $\lambda$ | 1 | $g_u$ | 1.2 |
| $W_0^{(1)}$ | 1 | $W_0^{(2)}$ | 1 |

Table 1: Baseline parameter values for simulations

of trials $n$ and the probability of success for each trial $\pi$. Unless differently specified, baseline parameter values for simulations are summarized in Table 1.

The numerical simulations confirm the theoretical results established above and showed in Figure 1 providing some interesting insights on the nexus volatility-heterogeneity and on the effects of portfolio insurers in the market.

In Figure 4 we consider the case where CWS invests a fraction of wealth in the risky asset higher than POIS ($x > m$), varying the probability of an upside move. Confirming what is observed in Figure 1, we have that CWS dominates in case of a high probability of an up move. As the probability decreases, we observe survivorship of both types of agents and in the end dominance by POIS.

In Figure 5 we consider the case where the fraction invested in the risky asset by POIS is higher than CWS ($m > x$), varying the probability of an upside move. In case of high probability of an up move path dependency emerges (for the chosen realization of dividends, the POIS dominates in the long run, see also Figure 7). As the probability decreases, CWS dominates. As the probability decreases further, we observe suvivorship of both types.

Notice that the Panel (a), (b), (c) of Figure 4 and 5 are characterized by the same probability of upside move ($\pi$) and therefore they are characterized by the same intrinsic volatility. Comparing the same panels in the two Figures we may evaluate the effect of a different investing weight for the two strategies and of the long run composition of the market on the volatility. Panel (c) in the two Figures show a similar volatility, in this case being the dominant agent large the POIS and the CWS coincide. In Panel (a) of Figure 4 we have dominance of the POIS versus survivorship of both classes of agents in Figure 5, volatility is smaller in the first Figure. In Panel (b) we have survivorship of both classes of agents versus dominance of the CWS. Despite the persistence of heterogeneity, volatility is smaller in the first Figure. This type of results is robust and is confirmed considering different set of parameters (fraction of wealth invested in the risky asset and coefficients of the binomial model). We can deduce that there is no clear cut result on volatility and homogeneity/heterogeneity, it depends on the type of strategy that dominates in the long run. A homogenous market with traders adopting a POIS is less volatile than a market populated by traders adopting a POIS and a CWS. A homogenous market with traders adopts a CWS is more volatile than a market populated by traders adopting a POIS and a CWS. As suggested by [2], the presence of POIS traders in the market plays a stabilizing effect.

In Figure 6, we consider three simulations characterized by survivorship of both classes of agents. Notice that in agreement with Figure 1 survivorship of both types of agents may occur in case $x > m$ and also in case $m > x$.

In Figure 7, we consider the case $m > x$ and provide two simulations in the region characterized by path dependency, changing the sequence of states of the world. The probability that the risk asset pays pay a positive dividend (instead of zero) remains the same but realized sequence of dividend payments changes in the two simulations. In both cases we have dominance of one of the two classes of traders but it is enough to change the sequences of the states to observe a different outcome. Confirming the above observation on the stabilizing effect of the POIS, in case of dominance of POIS the volatility is lower than in case of dominance of CWS.

Finally, in Figure 8, we increase the fraction that each strategy invests in the risky asset, for both $x > m$ and $m > x$. The volatility is decreasing in the fractions invested in the risky asset. The observation suggests that the stabilizing role of the POIS could also be related to being, when above the floor, an aggressive strategy.

## 6. Conclusions

In recent years many institutional investors started to adopt portfolio insurance strategies. The rationale goes to the features of financial products managed for customers, i.e., with profit/traditional insurance policies, or to the need of introducing protection on asset under management with a stop loss approach. The effects of these strategies on the markets have been investigated but no result exists on their capability to outperform traditional constant weight strategies and therefore to populate the market in the long run.

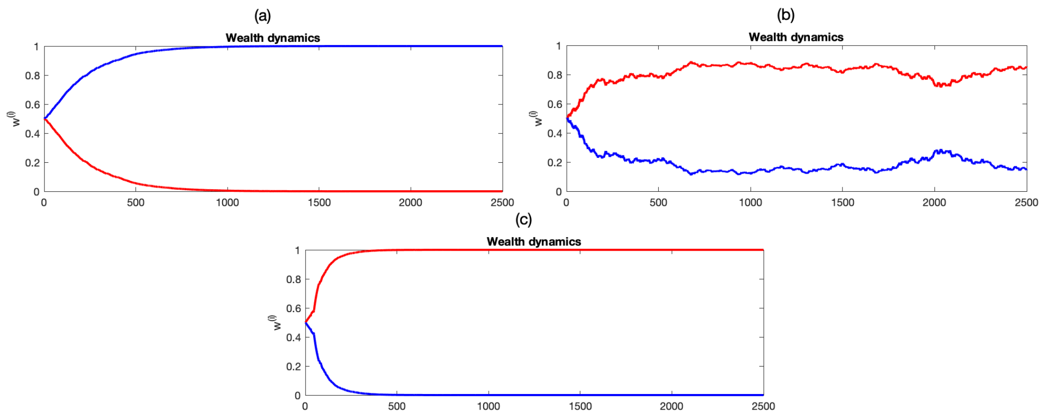


Figure 4: Varying $\pi$. Relative wealth $w^{(i)}$ for CWS (red line) and POIS (blue line) varying $\pi$. Panel ($a$): $\pi = 0.1$, $\bar{\sigma} = 0.0734$; Panel ($b$): $\pi = 0.5$, $\bar{\sigma} = 0.3354$; Panel ($c$): $\pi = 0.9$, $\bar{\sigma} = 0.5419$. Common parameter values: $x = 0.1$, $m = 0.0667$.

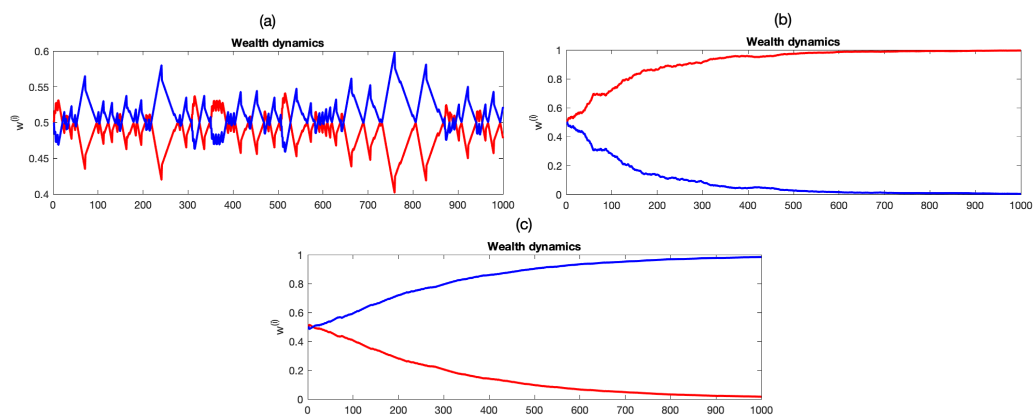


Figure 5: Relative wealth $w^{(i)}$ for CWS (red line) and POIS (blue line) varying $\pi$. Panel (a): $\pi = 0.1$, $\bar{\sigma} = 0.2123$; Panel (b): $\pi = 0.5$, $\bar{\sigma} = 0.4030$; Panel (c): $\pi = 0.9$, $\bar{\sigma} = 0.5391$. Common parameter values: $x = 0.0667$, $m = 0.1$.

In this paper, we have shown that it is rather complex to investigate the evolution of a financial market populated by traders adopting a Constant Weight Strategy and those adopting a Portfolio Insurance Strategy. In the most plausible case (the portfolio insurer is more aggressive than the trader adopting a constant weight strategy when she is above the floor), we have shown that if the expected return of the asset is high enough then both classes of traders may dominate and the final outcome is largely indeterminate/path dependent. This outcome is due to the fact that the portfolio insurer invests

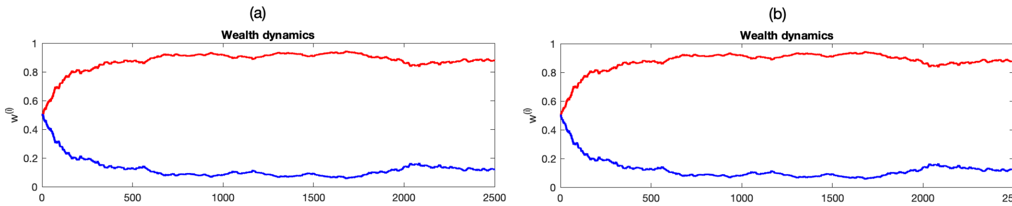


Figure 6: Relative wealth $w^{(i)}$ for CWS (red line) and POIS (blue line) varying $m$. Panel $(a)$: $m = 0.03$, $\bar{\sigma} = 0.3213$; Panel $(b)$: $m = 0.1167$, $\bar{\sigma} = 0.3222$. Common parameter values: $x = 0.0833$, $\pi = 0.45$.

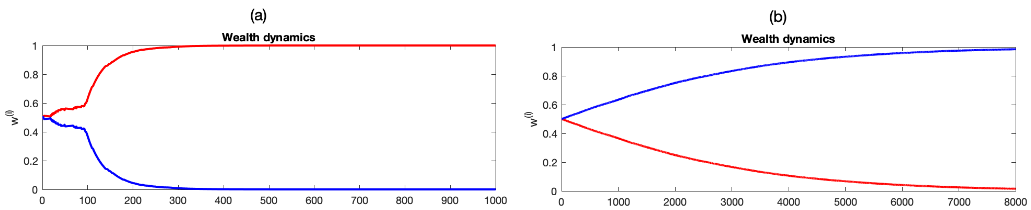


Figure 7: Relative wealth $w^{(i)}$ for CWS (red line) and POIS (blue line) in case of path dependence changing the sequence of states of the world. Panel $(a)$: $\bar{\sigma} = 0.5753$; Panel $(b)$: $\bar{\sigma} = 0.4752$. Parameter values: $m = 0.0725$, $x = 0.0687$, $\pi = 0.9$.

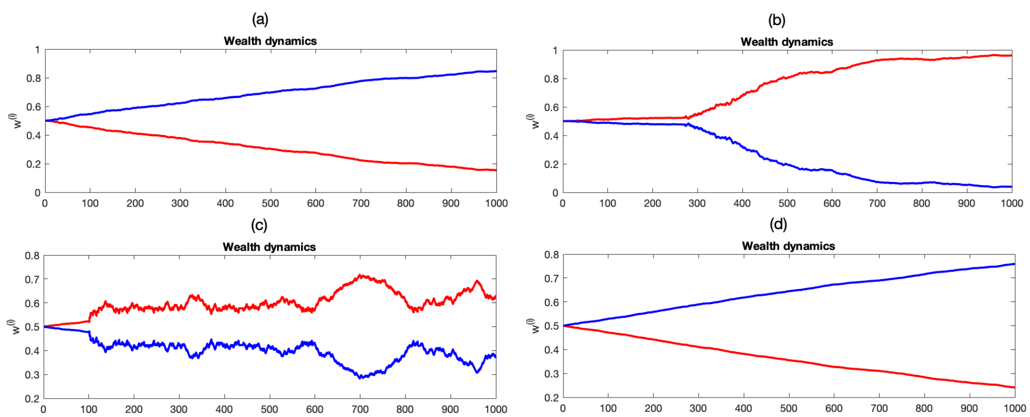


Figure 8: Relative wealth $w^{(i)}$ for CWS (red line) and POIS (blue line). Panel $(a)$: $m = 0.04$, $x = 0.03$, $\bar{\sigma} = 0.4985$; Panel $(b)$: $m = 0.05$, $x = 0.06$, $\bar{\sigma} = 0.4209$; Panel $(c)$: $m = 0.12$, $x = 0.11$, $\bar{\sigma} = 0.3572$; Panel $(c)$: $m = 0.13$, $x = 0.14$, $\bar{\sigma} = 0.2539$. Parameter value: $\pi = 0.5$.

or not in the risky asset depending on her wealth. For other set of parameters, the two trading strategies may persist in the market or traders adopting the Constant Weight Strategy dominate.

We have also shown that the relationship between market homogeneity/heterogeneity and volatility depends on the type of homogeneous agent dominating the market. If the agent adopts a Constant Weight Strategy then the market volatility is higher than in case of a market with heterogeneous traders. If the agent adopts a Portfolio Insurance Strategy then the market volatility is lower than in case of a market with heterogeneous traders. This result confirms that portfolio insurers play a stabilizing role in the market.

## Appendix A. Proof of the Proposition 1

Computing the value of $\mu(w)$ for $w \to 0$ and $w \to 1$, we obtain

$$\mu(0) = \log\left(\frac{\{(1-\delta)(m-x)\frac{g_d}{g_u} + (1-m)[x+\delta(m-x)]\}^{\pi}}{m^{\pi}(1-m)[(1-\delta)(1-x)+(1-m)\delta]^{\pi-1}}\right) \tag{A.1}$$

and

$$\mu(1) = \log\left(\frac{(1-x)(1-\delta x)^{\pi-1}}{[(1-\delta)\frac{g_d}{g_u} + \delta(1-x)]^{\pi}}\right). \tag{A.2}$$

We can consider $\mu(0)$ and $\mu(1)$ as function of $\pi$.

We first discuss the case $x > m$.

We can prove that

- $\lim_{\pi\to 0}\mu(0) < 0$, $\frac{\partial\mu(0)}{\partial\pi} > 0$, $\lim_{\pi\to 1}\mu(0) > 0$ iff $m < 1 - \frac{g_d}{g_u}$;
- $\lim_{\pi\to 0}\mu(1) < 0$, $\frac{\partial\mu(1)}{\partial\pi} > 0$, $\lim_{\pi\to 1}\mu(1) > 0$ iff $x < 1 - \frac{g_d}{g_u}$.

Therefore, if the no arbitrage condition applies, then $\mu(0)$ and $\mu(1)$ are negative for $\pi \to 0$, are increasing in $\pi$ and they assume positive values for a sufficiently high $\pi$.

We can conclude that for a low $\pi$ the POIS dominates (both functions are negative) and for a high $\pi$ the CWS dominates (both functions are positive). It remains to discuss the case of intermediate $\pi$: if $\mu(0)$ crosses 0 before $\mu(1)$ then there is survivorship of both classes of agents because for $\pi$ intermediate $\mu(0)$ is positive and $\mu(1)$ is negative; if $\mu(1)$ crosses 0 before $\mu(0)$, then there is path dependency because for $\pi$ intermediate $\mu(0)$ is negative and $\mu(1)$ is positive.

We obtain $\mu(0) = 0$ for

$$\pi = \pi_0 = \frac{\log\left(\frac{1-m}{(1-\delta)(1-x)+\delta(1-m)}\right)}{\log\left(\frac{(1-\delta)(m-x)\frac{g_d}{g_u}+(1-m)[x+\delta(m-x)]}{m[(1-\delta)(1-x)+\delta(1-m)]}\right)} \tag{A.3}$$

and $\mu(1) = 0$ for

$$\pi = \pi_1 = \frac{\log\left(\frac{1-\delta x}{1-x}\right)}{\log\left(\frac{1-\delta x}{(1-\delta)\frac{g_d}{g_u}+\delta(1-x)}\right)}. \tag{A.4}$$

We now consider $\pi_0$ and $\pi_1$ as functions of $\frac{g_d}{g_u}$.

For $\frac{g_d}{g_u} = 0$ we have $\pi_1(0) \in (0, x)$ and $\pi_0(0) \in \left(m, \frac{\log\left(\frac{1-m}{1-x}\right)}{\log\left(\frac{(1-m)x}{(1-x)m}\right)}\right)$. Nothing can be said a priori on the inequality $\pi_0(0) > \pi_1(0)$ therefore we study $\pi_0(0)$ and $\pi_1(0)$ as functions of $\delta$.

It can be shown that $\frac{\partial \pi_0(0)}{\partial \delta} < 0$, $\lim_{\delta \to 0} \pi_0(0) = \frac{\log\left(\frac{1-m}{1-x}\right)}{\log\left(\frac{(1-m)x}{(1-x)m}\right)}$ and $\lim_{\delta \to 1} \pi_0(0) = m$. As far as $\pi_1(0)$ is concerned, we have $\frac{\partial \pi_1(0)}{\partial \delta} > 0$, $\lim_{\delta \to 0} \pi_1(0) = 0$ and $\lim_{\delta \to 1} \pi_1(0) = x$. From these results we can deduce that for low values of $\delta$, $\pi_0(0) > \pi_1(0)$ and for high values of $\delta$, $\pi_0(0) < \pi_1(0)$.

$\pi_0$ and $\pi_1$ are increasing in $\frac{g_d}{g_u}$. When the no arbitrage condition applies, $\frac{g_d}{g_u}$ may assume $1-x$ as maximum value. It is easy to show that $\pi_0(1-x) < 1$ and $\pi_1(1-x) = 1$. Therefore, for a sufficiently high $\frac{g_d}{g_u}$ we have $\pi_0 < \pi_1$. Nothing can be said for lower values because $\pi_0$ and $\pi_1$, being concave functions, might intersect each other as $\frac{g_d}{g_u}$ goes up. In order to study the number of intersections, we introduce the auxiliary functions

$$L_1\left(\frac{g_d}{g_u}\right) = -\log_a\left(\frac{(1-\delta)\frac{g_d}{g_u} + \delta(1-x)}{1-\delta x}\right)$$

and

$$L_0\left(\frac{g_d}{g_u}\right) = \log_b\left(\frac{(1-\delta)(m-x)\frac{g_d}{g_u} + (1-m)[x + \delta(m-x)]}{m[(1-\delta)(1-x) + \delta(1-m)]}\right).$$

where $a = \frac{1-\delta x}{1-x}$ and $b = \frac{1-m}{(1-\delta)(1-x)+\delta(1-m)}$. Notice that we have $\pi_0 = \pi_1$ iff $L_0 = L_1$.

$L_1\left(\frac{g_d}{g_u}\right)$ is strictly decreasing and convex while $L_0\left(\frac{g_d}{g_u}\right)$ is strictly decreasing and concave therefore they might intersect each other at most twice. Being $L_1(1) = L_0(1) = 0$, at most one intersection may exist for $\frac{g_d}{g_u} < 1$.

Recall that for $\frac{g_d}{g_u} = 1 - x$, we have $\pi_0(1-x) < \pi_1(1-x)$ and both functions are increasing in $\frac{g_d}{g_u}$. Consider now the case $\frac{g_d}{g_u} = 0$. For a low $\delta$ we have $\pi_0(0) > \pi_1(0)$ and knowing that the two functions can intersect at most once we can conclude that in case of a low $\delta$, for a low $\frac{g_d}{g_u}$ we have $\pi_0 > \pi_1$ while for a high $\frac{g_d}{g_u}$ we have $\pi_0 < \pi_1$. For a high $\delta$ we have $\pi_0(0) < \pi_1(0)$ and knowing that the two functions can intersect at most once we can conclude that $\pi_0 > \pi_1$ for any feasible value of $\frac{g_d}{g_u}$.

We have shown that for $x > m$ we have $\pi_1 < \pi_0$ for a low $\delta$ and a low $\frac{g_d}{g_u}$ otherwise we have $\pi_1 > \pi_0$. In the first case $\mu(0) < 0$ and $\mu(1) > 0$ and consequently there is indeterminacy, in the second case we have survivorship of both classes of agents.

The following result can be established when the no arbitrage condition applies for $x > m$:

- for $\pi$ low POIS dominates;
- for $\pi$ intermediate, if $\delta$ and $\frac{g_d}{g_u}$ are both low then there is path dependency, in all the other cases both agents survive;
- for $\pi$ high, CWS dominates.

We now consider the case $x < m$. The following result can be established:

- $\lim_{\pi\to 0}\mu(0) > 0$, $\frac{\partial\mu(0)}{\partial\pi} < 0$, $\lim_{\pi\to 1}\mu(0) < 0$ iff $m < 1 - \frac{g_d}{g_u}$;
- $\lim_{\pi\to 0}\mu(1) < 0$, $\frac{\partial\mu(1)}{\partial\pi} > 0$, $\lim_{\pi\to 1}\mu(1) > 0$ iff $x < 1 - \frac{g_d}{g_u}$.

Therefore, when the no arbitrage condition applies, $\mu(1)$ is negative for $\pi \to 0$, is increasing in $\pi$ and it assumes positive values for sufficiently high $\pi$. Conversely $\mu(0)$ is positive for $\pi \to 0$, is decreasing in $\pi$ and it assumes positive values for sufficiently high $\pi$.

We can already conclude that for a low $\pi$ there is path dependency because $\mu(0)$ is positive and $\mu(1)$ is negative. For a high $\pi$ there is path dependency because $\mu(0)$ is negative and $\mu(1)$ is positive. It remains to discuss the case of an intermediate $\pi$: if $\mu(0)$ crosses 0 before $\mu(1)$ then POIS dominates because for $\pi$ intermediate both the function are negative, otherwise POIS dominates because for an intermediate $\pi$ both functions are positive.

We now consider the values for which the functions are equal to 0 ($\pi_0$ and $\pi_1$) as functions of $\frac{g_d}{g_u}$.

For $\frac{g_d}{g_u} = 0$ we have $\pi_1 \in (0, x)$ and $\pi_0 \in \left(\frac{\log\left(\frac{1-m}{1-x}\right)}{\log\left(\frac{(1-m)x}{(1-x)m}\right)}, m\right)$. Being $\frac{\log\left(\frac{1-m}{1-x}\right)}{\log\left(\frac{(1-m)x}{(1-x)m}\right)} > x$ we can conclude $\pi_1(0) > \pi_0(0)$. Both the functions are concave and in increasing in $\frac{g_d}{g_u}$, they might intersect each other. Recall that $\pi_0 = \pi_1$ for $L_0 = L_1$.

Assuming $x < m$, we have that $L_0\left(\frac{g_d}{g_u}\right)$ and $L_1\left(\frac{g_d}{g_u}\right)$ are strictly decreasing and convex with $L_0(1) = L_1(1) = 0$. Moreover $L_1\left(\frac{g_d}{g_u}\right)$ tends to $+\infty$ for

$\frac{g_d}{g_u} \to -\frac{\delta(1-x)}{1-\delta} = g_1$ while $L_0\left(\frac{g_d}{g_u}\right)$ tends to $+\infty$ for $\frac{g_d}{g_u} \to -\frac{(1-m)[x+\delta(m-x)]}{(1-\delta)(m-x)} < g_1$ and $\lim_{\frac{g_d}{g_u}\to 1^-} L_0\left(\frac{g_d}{g_u}\right) < \lim_{\frac{g_d}{g_u}\to 1^-} L_1\left(\frac{g_d}{g_u}\right)$.

It follows that for $x < m$, $\pi_0 \neq \pi_1$ $\forall \frac{g_d}{g_u} \in (0,1)$. Consequently $\pi_1 < \pi_0$ for all parameter values and - being $\mu(0) > 0$ and $\mu(1) > 0$ - CWS dominates.

The following results can be established for $x < m$, when the no arbitrage condition applies:

- for $\pi$ low, both agents survive;
- for $\pi$ intermediate, CWS dominates;
- for $\pi$ high, there is path dependency.

## References


[1] Rabah Amir, Sergei Belkov, Igor V. Evstigneev, and Thorsten Hens. An evolutionary finance model with short selling and endogenous asset supply. *Economic Theory*, 2020. doi: 10.1007/s00199-020-01269-x.

[2] S. Basak. A general equilibrium model of portfolio insurance. *Review of Financial Studies*, 8:1059–1090, 1995. doi: 10.1007/s00199-017-1066-8.

[3] S. Basak. A comparative study of portfolio insurance. *Journal of Economic Dynamics and Control*, 26:1217–1241, 2002. doi: 10.1007/s00199-017-1066-8.

[4] H. Bhamra and R. Uppal. Asset prices with heterogeneity in preferences and beliefs. *Review of Financial Studies*, 27:518–580, 2014.

[5] F. Black and A. Perold. Theory of constant proportion portfolio insurance. *Journal of Economic Dynamics and Control*, 16:403–426, 1992. doi: 10.1007/s00199-017-1066-8.

[6] L. Blume and D. Easley. If you are so smart why aren't you rich? belief selection in incomplete and complete markets. *Econometrica*, 74:929–966, 2006. doi: 10.1007/s00199-017-1066-8.

[7] K. Borch. Equilibrium in a reinsurance market. *Econometrica*, 30:424–444, 1962.

[8] G. Bottazzi and D. Giachini. Far from the madding crowd: Collective wisdom in prediction markets. *Quantitative Finance*, 19:1461–1471, 2019.

[9] G. Bottazzi, P. Dindo, and D. Giachini. Long-run heterogeneity in an exchange economy with fixed-mix traders. *Economic Theory*, 66(2):407–447, 2018. doi: 10.1007/s00199-017-1066-8.

[10] M. Brennan and E. Shwartz. Portfolio insurance and financial market equilibrium. *The Journal fo Business*, 62(4):455–472, 1989.

[11] Y.L: Chan and L. Kogan. catching up with the joneses: Heterogeneous preferences and the dynamics of asset prices. *The Journal of Political Economy*, 110(6):1255–1285, 2002.

[12] P. Dindo and F. Massari. The wisdom of the crowd in dynamic economies. *Theoretical Economics*, 15:1627–1668, 2020.

[13] P. Dybvig. Dusenberry's ratcheting of consumption: Optimal dynamic consumption and investment given intolerance for any decline in standard of living. *The Review of Economic Studies*, 62(2):287–313, 1995.

[14] I. Evstigneev, T. Hens, and K.R. Schenk-Hoppé. Globally evolutionary stable portfolio rules. *Journal of Economic Theory*, 140:197–228, 2008.

[15] I. Evstigneev, T. Hens, and K.R. Schenk-Hoppé. Evolutionary finance. In T. Hens and K.R. Schenk-Hoppé, editors, *Handbook of Financial Markets: Dynamics and Evolution*. North-Holland (Handbooks in Economics Series), 2009.

[16] M. Friedman. *Essays in Positive Economics*. Chicago University Press, Chicago, 1953.

[17] G. Gennotte and H. Leland. Market liquidity, hedging and crashes. *The American Economic Review*, 80:999–1021, 1990.

[18] S. Grossman and J.L. Vila. Portfolio insurance in complete markets: a note. *The Journal fo Business*, 62(4):473–476, 1989.

[19] S. Grossman and A. Zhou. Optimal investment strategies for controlling drawdowns. *Mathematical Finance*, 3(3):241–276, 1993.

[20] A. Haldane. Rethinking the financial network. *Speech at the Financial Student Association, Amsterdam*, 2009.

[21] A. Haldane and R. May. Systemic risk in banking ecosystems. *Nature*, 469:351–355, 2011.

[22] J.L: Kelly. A new interpretation of information rate. *Bell Syst. Tech. J.*, 35:917–926, 1956.

[23] R. Merton. Lifetime portfolio selection under uncertainty: the continuous case. *Review of Economics and Statistics*, 51:247–257, 1969. doi: 10.1007/s00199-017-1066-8.

[24] R. Merton. Optimum consumption and portfolio rules in a continuous time model. *Journal of Economic Theory*, 3:373–413, 1971. doi: 10.1007/s00199-017-1066-8.

[25] A. Perold and W. Sharpe. Dynamic strategies for asset allocation. *Financial Analysts Journal*, pages 16–27, 1988. doi: 10.1007/s00199-017-1066-8.

[26] A. Sandroni. Do markets favor agents able to make accurate predictions. *Econometrica*, 68(6):1303–1341, 2000. doi: 10.1007/s00199-017-1066-8.

[27] D. Weinbaum. Investor heterogeneity, asset pricing and volatility dynamics. *Journal of Economic Dynamics and Control*, 33:1379–1397, 2009.

[28] R. Wilson. The theory of syndacates. *Econometrica*, 36:119–132, 1968.