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Nonlinear asset-price dynamics and stabilization policies

Noemi Schmitt · Fabio Tramontana · Frank Westerhoff

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Abstract We first present a brief review of nonlinear asset-pricing models and contributions in which such models have been used as benchmarks to evaluate the effectiveness of a number of regulatory policy measures. We then illustrate the functioning of one particular asset-pricing model—the seminal framework by Brock and Hommes (J Econ Dyn Control 22:1235–1274, 1998)—and its possible stabilization via a central authority that seeks to counter the destabilizing trading behavior of speculators. Our paper underlines that tools from the field of nonlinear dynamical systems may foster our understanding of the functioning of asset markets, thereby enabling policymakers to design better trading environments in the future.

Keywords Boom-bust cycles · Asset-pricing models · Stabilization policies · Nonlinear dynamical systems · Steady states · Stability and bifurcation analysis · Chaos

JEL classification G12 · G18 · G41

N. Schmitt · F. Westerhoff (✉)
Department of Economics, University of Bamberg, Feldkirchenstrasse 21, 96045 Bamberg, Germany
e-mail: frank.westerhoff@uni-bamberg.de

F. Tramontana
Department of Mathematical Sciences, Mathematical Finance and Econometrics, Catholic University of Sacred Heart, Milan, Italy

1 Introduction

Financial markets regularly display severe bubbles and crashes. The detailed historical accounts compiled by Galbraith [38], Kindleberger and Aliber [52] and Shiller [76] highlight the fact that the instability of financial markets may also harm the real economy. The Great Depression, triggered by the stock market crash of 1929, and the Great Recession, caused by the financial havoc of 2007, are just two such examples of notoriety. Nonlinear asset-pricing models, featuring interactions between heterogeneous interacting speculators that can result in complex (chaotic) price dynamics, provide important insights into the functioning of financial markets and may therefore help policymakers derive strategies that are conductive to improving market stability. To illustrate the power of this research approach, we discuss one particular asset-pricing model—the seminal framework by Brock and Hommes [16]—and explore its possible stabilization via a central authority seeking to offset the destabilizing trading behavior of speculators. Our paper shows that tools from the field of nonlinear dynamical systems may foster our understanding of the functioning of financial markets, thereby enabling policymakers to design better trading environments in the future.

To set the stage for our paper, a few remarks are in order. In the aforementioned asset-pricing literature, speculators follow simple behavioral trading rules to determine their investment positions, an assumption that is in line with numerous empirical observations. In particular, Menkhoff and Taylor [59] and Hommes [46] present empirical and experimental evidence showing that speculators believe in two contrasting trading concepts. First, they use technical trading rules [60], assuming the continuation of the current market trend. Second, they use fundamental trading rules [39], believing that asset prices will return toward their fundamental values. While technical trading rules add a destabilizing positive feedback effect to the dynamics, fundamental trading rules entail a stabilizing negative feedback. Since there is widespread empirical evidence that speculators switch between technical and fundamental trading rules (see, e.g., [2, 4, 12, 36], most models in this field possess at least one nonlinearity, giving rise to the following stylized boom-bust generating mechanism. Near the model's fundamental value, technical trading rules tend to dominate the market, and their use may initiate a bubble process. As the price runs away from its fundamental value, however, fundamental trading rules become more influential and eventually bring asset prices back to values that are more moderate. Unfortunately, this is often the starting point for the next bubble, as technical trading rules gain in popularity again.

The interplay between destabilizing technical trading rules and stabilizing fundamental trading rules can produce complex (chaotic) dynamics that closely resembles the dynamics of actual financial markets. Since these models possess a high degree of realism—they are based on empirical observations, display a plausible internal functioning, and match important statistical properties of financial markets—policymakers can use them as artificial laboratories to evaluate the effects of regulatory policies. In general, we can expect at least two different types of result from such exercises: a positive one and a negative one. This is also the case for our experiments. As we will see, our analysis reveals that a central authority may stabilize the dynamics of the model by Brock and Hommes [16] by adopting a targeting long-run fundamentals strategy, i.e., by buying assets when they are undervalued and selling them when they are overvalued. However, our analysis also demonstrates that well-intended intervention strategies that are plausible, at least at first sight, may fail. For instance, a central authority that applies a leaning against the wind strategy by selling assets when the market is increasing and buying assets when the market is decreasing does not necessarily bring prices back toward fundamental values.

Elementary insights offered by the field of nonlinear dynamical systems may help us to comprehend this puzzling outcome. The model by Brock and Hommes [16] creates complex (chaotic) boom-bust asset-price dynamics via a bifurcation route that first displays a pitchfork bifurcation and then a Neimark–Sacker bifurcation. Our analysis indicates that a leaning against the wind strategy may suppress or reverse a Neimark–Sacker bifurcation, but has no real power against a pitchfork bifurcation. In contrast, a targeting long-run fundamentals strategy may prevent a pitchfork bifurcation, therefore allowing policymakers to establish efficient markets. Simulations reveal that the stabilizing effect of the latter strategy is quite robust, e.g., with respect to exogenous noise. We would like to stress that the realization of the negative result is quite important because it reveals that policymakers must plan and execute their interventions very carefully, as remarked by Baumol [8]. Of course, positive results are also relevant because they may offer policymakers powerful instruments to stabilize financial markets. In this respect, we would like to stress that insights from the theory of chaos control, summarized by Schöll and Schuster [75], may be worthy of greater attention from the economic profession.

The rest of the paper is organized as follows. In Sect. 2, we sketch the literature that addresses nonlinear asset-pricing models and a number of related policy papers. In Sect. 3, we introduce a popular specification of the model by Brock and Hommes [16], and discuss how interventions by a central authority may affect its dynamics. In Sect. 4, we conclude our paper. A number of derivations and discussions are presented in Appendix A.

2 Literature review

In this section, we provide a brief review of nonlinear asset-pricing models and policy insights derived from them. For more detailed surveys about nonlinear asset-pricing models, see, for instance, Chiarella et al. [22], Hommes and Wagener [45] and Dieci and He [32].[1] Westerhoff [85] and Westerhoff and Franke [87] cover some of the progress that has been made in recent years with respect to applying such models to conduct policy experiments.

2.1 Nonlinear asset-pricing models

Day and Huang [25] develop one of the first nonlinear asset-pricing models.[2] In their seminal work, interactions between chartists (following a linear rule) and fundamentalists (following a nonlinear trading rule) may create complex (chaotic) bull and bear market dynamics. Chiarella [19] studies the equally reasonable opposite scenario, i.e., chartists adhere to a nonlinear trading rule, while the trading behavior of fundamentalists is linear. De Grauwe et al. [27] propose a model in which the market impact of fundamentalists is nonlinear since they disagree with the market's true fundamental value. In Lux [56] and Bischi et al. [10], speculators switch between technical and fundamental trading rules because of herding effects, while Brock and Hommes [16] assume that speculators' rule-selection behavior depends on past realized profits. He and Westerhoff [40] introduce a framework in which speculators pick trading rules with a view toward market circumstances. The market impact of chartists and fundamentalists may also vary over time due to different wealth dynamics, as discussed in Chiarella and He [23], Chiarella et al. [21] or Anufriev and Dindo [3]. Branch and Evans [13, 14] introduce models in which speculators learning about an asset's risk-return profile may cause bubbles and crashes.

Rosser et al. [78] consider that the evolution of a market's fundamental value may follow a chaotic process. Taking this idea a step further, de Grauwe and Grimaldi [29, 30] study a financial market framework with nonlinear repercussions from the real economy, while Westerhoff [86], Naimzada and Pireddu [61, 62] and Lengnick and Wohltmann [55] capture nonlinear interactions between financial markets and the real economy. Moreover, Westerhoff [84] and Chiarella et al. [20] study the case in which speculators switch across different markets. Schmitt and Westerhoff [71] and Dieci et al. [33] show that market entry and exit waves may also lead to endogenous boom-bust cycles. Similar ideas and modeling concepts have recently been used to capture the dynamics of housing markets, see, e.g., Dieci and Westerhoff [34], Diks and Wang [35], Campisi et al. [18], Bolt et al. [11] and Schmitt and Westerhoff [73]. Moreover, Huang and Day [48] develop a piecewise-linear version of the model by Day and Huang [25]. Studies by Tramontana et al. [79, 80], Huang et al. [49] and Huang and Zheng [50] reveal that such models, often deeply analytically tractable, may give rise to many surprising economic phenomena. See also the survey by Tramontana and Westerhoff [81] and the contribution by Avrutin et al. [7] for in-depth mathematical background information.

2.2 Policy insights

Let us start with the contributions that are most closely related to our work: central bank interventions. Hung [51] and Neely [63, 64] report that central banks have intervened quite frequently in foreign exchange markets in the past, using two different intervention strategies. Central banks may conduct so-called leaning against the wind interventions, that is, they buy (sell) foreign currency when the exchange rate decreases (increases) to diminish or break the momentum of the current exchange rate trend. Alternatively, they conduct targeting long-run fundamentals interventions, that is, they intervene in support of a target exchange rate, e.g., the market's fundamental value. Of course, such behavior is not limited to foreign exchange markets; it can also be applied by policymakers to other asset markets such as stock or commodity markets. This is precisely what we do in our paper.

Szpiro [77] is one of the first to use a nonlinear asset-pricing model to explore the effects of such interventions. In particular, he shows that the targeting long-run fundamentals strategy may inadvertently induce chaos. In contrast, Wieland and Westerhoff [88] show that the targeting long-run fundamentals strategy is related to certain chaos control methods and may, if properly executed, stabilize asset markets. See Westerhoff [85] and Westerhoff and Franke [87] for further applications. All these contributions are based

[1] Our main attention focusses on nonlinear asset-pricing models that are analytically tractable. However, there also exist more elaborate nonlinear asset-pricing models, see, e.g., the inspiring contributions by Palmer et al. [66], Arthur et al. [6] and LeBaron et al. [54]. See LeBaron [53] for a survey.

[2] Some predecessors include Zeeman [90], Beja and Goldman [9] and Frankel and Froot [37].

on models in which a market maker adjusts prices with respect to speculators' order flow. Within the Brock and Hommes [16] model—the workhorse for our analysis—asset prices adjust such that the demand for the risky asset is equal to the supply of the risky asset. Moreover, we conduct our analysis taking a dynamical system perspective, highlighting the role of the model's bifurcation structure.

Nonlinear asset-pricing models have also been used to explore the effects of transaction taxes [57, 83], trading halts [82, 89], price limits [24, 40], short-selling constraints [5, 26] and interest rate rules [74]. Scalas et al. [67] addressed insider trading and fraudulent behavior, while Hermsen et al. [43] explore the effects of disclosure requirements. Brock et al. [17] examine problems that may arise due to the increasing number of hedging instruments. We briefly remark that similar experiments exist for other markets that entail nonlinearities. For instance, Schmitt and Westerhoff [70] show that the nonlinear cobweb model by Brock and Hommes [15] may be stabilized if firms have to pay a profit tax. However, Schmitt et al. [69] warn that profit taxes may also harbor a number of surprising and possibly undesirable side effects.

3 Stabilizing nonlinear asset-price dynamics: an illustrative example

In the following, we recap a popular specification of the seminal asset-pricing model by Brock and Hommes [16] to illustrate how nonlinear interactions between heterogeneous speculators may create complex (chaotic) boom-bust asset-price dynamics. However, we extend their setup by considering a central authority seeking to stabilize such dynamics via two different countercyclical intervention strategies.

3.1 Basic model setup

Let us turn to the details of the model. Market participants can invest in a safe asset, paying the risk-free interest rate r, and in a risky asset, paying an uncertain dividend D_t. The dividend process of the risky asset is specified by

$$D_t = \bar{D} + \delta_t, \tag{1}$$

where $\delta_t \sim N(0, \sigma_\delta^2)$. While the price of the safe asset is constant, the price of the risky asset depends on the trading behavior of the market participants, comprising (heterogeneous) speculators, a central authority, long-term investors and liquidity trades. Our modeling of speculators' demand for the risky asset follows Brock and Hommes [16].[3] Let P_t be the price of the risky asset (ex-dividend) at time t. The end-of-period wealth of speculator i can be expressed as

$$W_{t+1}^i = (1+r)W_t^i + Z_t^i(P_{t+1} + D_{t+1} - (1+r)P_t), \tag{2}$$

where Z_t^i represents speculator i's demand for the risky asset. Note that variables indexed with $t+1$ are random. Speculators are myopic mean–variance maximizers. Their demand for the risky asset follows from

$$\max_{Z_t^i} \left[E_t^i\left[W_{t+1}^i\right] - \frac{\alpha^i}{2} V_t^i\left[W_{t+1}^i\right] \right], \tag{3}$$

where $E_t^i\left[W_{t+1}^i\right]$ and $V_t^i\left[W_{t+1}^i\right]$ denote speculator i's belief about the conditional expectation and conditional variance of his wealth, and parameter $\alpha^i > 0$ stands for his risk aversion. Accordingly, speculator i's optimal demand for the risky asset is

$$Z_t^i = \frac{E_t^i[P_{t+1}] + E_t^i[D_{t+1}] - (1+r)P_t}{\alpha^i V_t^i[P_{t+1} + D_{t+1}]}. \tag{4}$$

To achieve a convenient expression of speculators' aggregate demand for the risky asset, Brock and Hommes [16] introduce the following simplifying assumptions. There are N speculators in total, believing that $E_t^i[D_{t+1}] = \bar{D}$. Moreover, all speculators hold the same constant variance beliefs, i.e., $V_t^i[P_{t+1} + D_{t+1}] = \sigma_R^2$, and possess the same degree of risk aversion, i.e., $\alpha^i = \alpha > 0$. We can therefore express speculators' aggregate demand for the risky asset as $Z_t^S = \sum_{i=1}^N Z_t^i = \frac{\sum_{i=1}^N E_t^i[P_{t+1}] + N\bar{D} - N(1+r)P_t}{\alpha\sigma_R^2}$. Denoting speculators' average expectation about the risky asset's next-period price by $E_t[P_{t+1}] = \frac{1}{N}\sum_{i=1}^N E_t^i[P_{t+1}]$ and normalizing the mass of speculators to $N = 1$ yield

$$Z_t^S = \frac{E_t[P_{t+1}] + \bar{D} - (1+r)P_t}{\alpha\sigma_R^2}. \tag{5}$$

[3] See also the insightful presentations by Hommes and Wagener [45] and Hommes [47].

Note that speculators' demand for the risky asset increases with their price and dividend expectations and decreases with the risk-free interest rate, the current price of the risky asset, their risk aversion and variance beliefs.

Speculators may use a technical or a fundamental expectation rule to forecast the price of the risky asset. The market shares of speculators following the technical and fundamental expectation rule are labeled N_t^C and $N_t^F = 1 - N_t^C$. Speculators' average price expectations are defined by

$$E_t[P_{t+1}] = N_t^C E_t^C[P_{t+1}] + N_t^F E_t^F[P_{t+1}]. \tag{6}$$

Speculators compute the fundamental value of the risky asset price by discounting future dividend payments, that is, $F = \bar{D}/r$. Speculators applying the technical expectation rule, also called chartists, expect the deviation between the price of the risky asset and its fundamental value to increase. Their expectations are formalized by

$$E_t^C[P_{t+1}] = P_{t-1} + \chi(P_{t-1} - F), \tag{7}$$

where $\chi > 0$ denotes the strength of speculators' extrapolation behavior. Speculators using the fundamental expectation rule, also called fundamentalists, believe that the price of the risky asset will approach its fundamental value. Their expectations can be written as

$$E_t^F[P_{t+1}] = P_{t-1} + \phi(F - P_{t-1}), \tag{8}$$

where $0 < \phi \leq 1$ indicates speculators' expected mean reversion speed. Note that both expectation rules forecast the price of the risky asset for period $t + 1$ at the beginning of period t, based on information available in period $t - 1$.

Speculators switch between the technical and fundamental expectation rule with respect to their evolutionary fitness, measured in terms of past realized profits. Accordingly, the attractiveness of the two expectation rules is computed as

$$A_t^C = (P_{t-1} + D_{t-1} - (1 + r)P_{t-2})Z_{t-2}^C, \tag{9}$$

and

$$A_t^F = (P_{t-1} + D_{t-1} - (1 + r)P_{t-2})Z_{t-2}^F - \kappa, \tag{10}$$

where

$$Z_{t-2}^C = \frac{E_{t-2}^C[P_{t-1}] + \bar{D} - (1 + r)P_{t-2}}{\alpha\sigma_R^2}, \tag{11}$$

and

$$Z_{t-2}^F = \frac{E_{t-2}^F[P_{t-1}] + \bar{D} - (1 + r)P_{t-2}}{\alpha\sigma_R^2}. \tag{12}$$

Brock and Hommes [16] consider that the use of the fundamental expectation rule may be costly, so they subtract constant per period information costs $\kappa \geq 0$ from [10]. However, we may also regard parameter k as a behavioral bias in favor of the simpler technical expectation rule. See Anufriev et al. [1] for empirical evidence.

The market shares of chartists and fundamentalists are due to the discrete choice approach, i.e.,

$$N_t^C = \frac{\exp\left[\beta A_t^C\right]}{\exp\left[\beta A_t^C\right] + \exp\left[\beta A_t^F\right]}, \tag{13}$$

and

$$N_t^F = \frac{\exp\left[\beta A_t^F\right]}{\exp\left[\beta A_t^C\right] + \exp\left[\beta A_t^F\right]}. \tag{14}$$

The intensity of choice parameter $\beta > 0$ measures how quickly the mass of speculators switches to the more successful trading rule. The higher parameter β is, the more speculators opt for the more profitable trading rule. In the limit, as parameter β approaches infinity, all speculators opt for the trading rule that produces the highest fitness. In this sense, speculators display a boundedly rational learning behavior.

Let us now turn to the central authority seeking to offset the destabilizing behavior of speculators by following two different intervention strategies [51, 63, 85]. According to the first strategy, called the leaning against the wind strategy, the central authority acts against current price trends by buying the risky asset if its price decreases and selling the risky asset if its price increases. According to the second strategy, called the targeting long-run fundamentals strategy, the central authority seeks to guide the price of the risky asset toward its fundamental value by buying the risky asset if the market is undervalued and selling the risky asset if the market is overvalued. Here, we focus on simple linear feedback strategies and express the central authority's demand for the risky asset as

$$Z_t^G = -m(P_{t-1} - P_{t-2}) - d(P_{t-1} - F), \tag{15}$$

where $m \geq 0$ and $d \geq 0$ are control parameters, capturing the central authority's intervention strength with respect to the market's momentum and distortion.[4] Note that the central authority computes the asset's fundamental value in the same way as speculators do.[5]

The demand for the risky asset by long-term investors, following a buy-and-hold strategy, is constant and set to

$$Z_t^I = \bar{Z}^I. \tag{16}$$

Moreover, the demand for the risky asset by liquidity traders is random and given as follows:

$$Z_t^L = \lambda_t, \tag{17}$$

with $\lambda_t \sim N(0, \sigma_L^2)$. The total demand for the risky asset by all market participants is

$$Z_t = Z_t^S + Z_t^G + Z_t^I + Z_t^L. \tag{18}$$

Market equilibrium requires that the total demand for the risky asset by all market participants equals the total supply of the risky asset, that is

$$Z_t = Y_t. \tag{19}$$

The total supply of the risky asset, i.e., the number of shares offered by firms, is constant and given by

$$Y_t = \bar{Y}. \tag{20}$$

Brock and Hommes [16] assume that there is a zero supply of outside shares. For simplicity, we therefore assume that

$$\bar{Y} = \bar{Z}^I, \tag{21}$$

i.e., the number of shares offered by firms is identical to the number of shares requested by long-term investors.

Combining [5] with [15–21] reveals that the price of the risky asset is determined by

$$P_t = \frac{E_t[P_{t+1}] + \bar{D} + \alpha\sigma_R^2\left(Z_t^G + Z_t^L\right)}{1+r}. \tag{22}$$

Note that [21] implies that P_t increases if speculators have bullish price expectations. As suggested by the leaning against the wind strategy, the central authority sells the risky asset, which should depress its price. Likewise, the targeting long-run fundamentals strategy recommends that the central authority sells the risky asset if the market is overvalued, which should also bring its price back to more moderate levels. At least at first sight, both the leaning against the wind strategy and the targeting long-run fundamentals strategy sound plausible. As we will see, however, the model's nonlinear price formation process is less trivial to stabilize as our intuition may suggest.[6]

3.2 Analytical and numerical results

Before we conduct a detailed numerical analysis of the impact of the central authority's intervention strategies on the model's dynamics, preliminary remarks are in order. In the absence of exogenous shocks, the dynamics of the model is driven by the iteration of a three-dimensional nonlinear deterministic map. In Appendix A, we show the following results (an overbar denotes steady-state quantities):

1. Our model possesses a fundamental steady state according to which $\bar{P}_1 = F = \bar{D}/r$, implying, among other things, that $\bar{N}_1^C = (1 + \exp[-\beta\kappa])^{-1}$ and $\bar{N}_1^F = (1 + \exp[\beta\kappa])^{-1}$. Note that neither the leaning against the wind strategy nor the targeting long-run fundamentals strategy influences the model's fundamental steady state. Moreover, speculators' distribution among expectation rules is also independent of parameters m and d. Of course, the reason for this is that the central authority is inactive at $\bar{P}_1$, i.e., $\bar{Z}_1^G = 0$.
2. The fundamental steady state undergoes a pitchfork bifurcation if the stability condition $\bar{N}_1^C\chi - \bar{N}_1^F\phi < r + \alpha\sigma_R^2 d$ is violated. Such a bifurcation may occur if the extrapolation parameter of the technical expectation rule increases. While the leaning against the wind strategy does not affect the stability domain of the model's fundamental

[4] As pointed out by one of the referees, the effect of the first term of [15] is similar to having an additional group of contrarians and that of the second term is like a stronger reaction from fundamentalists.

[5] To prevent any kinds of strategic trading, we assume that the central authority intervenes secretly in the risky asset market, as central banks usually do in real foreign exchange markets [64, 65].

[6] By excluding the demand of the central authority and of liquidity traders, we arrive at $P_t = (E_t[P_{t+1}] + \bar{D})/(1+r)$, which governs the dynamics of the model by Brock and Hommes [16].

steady state, the targeting long-run fundamentals strategy may suppress or reverse a pitchfork bifurcation.

3. The pitchfork bifurcation gives rise to two further steady states. These nonfundamental steady states are given by $\bar{P}_{2,3} = \bar{P}_1 \pm \sqrt{\frac{2\alpha\sigma_R^2\left(\frac{\beta}{2}\kappa + arctanh\left[\frac{\chi-\phi-2r-2\alpha\sigma_R^2 d}{\chi+\phi}\right]\right)}{r\beta(\chi+\phi)}}$, implying among others, that $\bar{N}_{2,3}^C = \frac{\phi+r+\alpha\sigma_R^2 d}{\chi+\phi}$ and $\bar{N}_{2,3}^F = \frac{\chi-r-\alpha\sigma_R^2 d}{\chi+\phi}$. Importantly, the targeting long-run fundamentals strategy allows the central authority to cut the gap between $\bar{P}_1$ and $\bar{P}_{2,3}$, i.e., to reduce mispricing of the risky asset. Unfortunately, this increases speculators' use of the technical expectation rule and necessitates permanent interventions, given by $\bar{Z}_{2,3}^G = \pm d \sqrt{\frac{2\alpha\sigma_R^2\left(\frac{\beta}{2}\kappa + 2arctanh\left[\frac{\chi-\phi-2r-2\alpha\sigma_R^2 d}{\chi+\phi}\right]\right)}{r\beta(\chi+\phi)}}$..[7] Since the leaning against the wind strategy does not generate interventions when the price of the risky asset is at rest, $\bar{P}_{2,3}$, $\bar{N}_{2,3}^C$, $\bar{N}_{2,3}^F$ and $\bar{Z}_{2,3}^G$ are independent of parameter m.

4. The nonfundamental steady states may become unstable due to a Neimark–Sacker bifurcation. For instance, a Neimark–Sacker bifurcation occurs if the extrapolation parameter of the technical expectation rule becomes sufficiently large. Further numerical explorations suggest that both intervention strategies may prevent the emergence of complex (chaotic) dynamics.

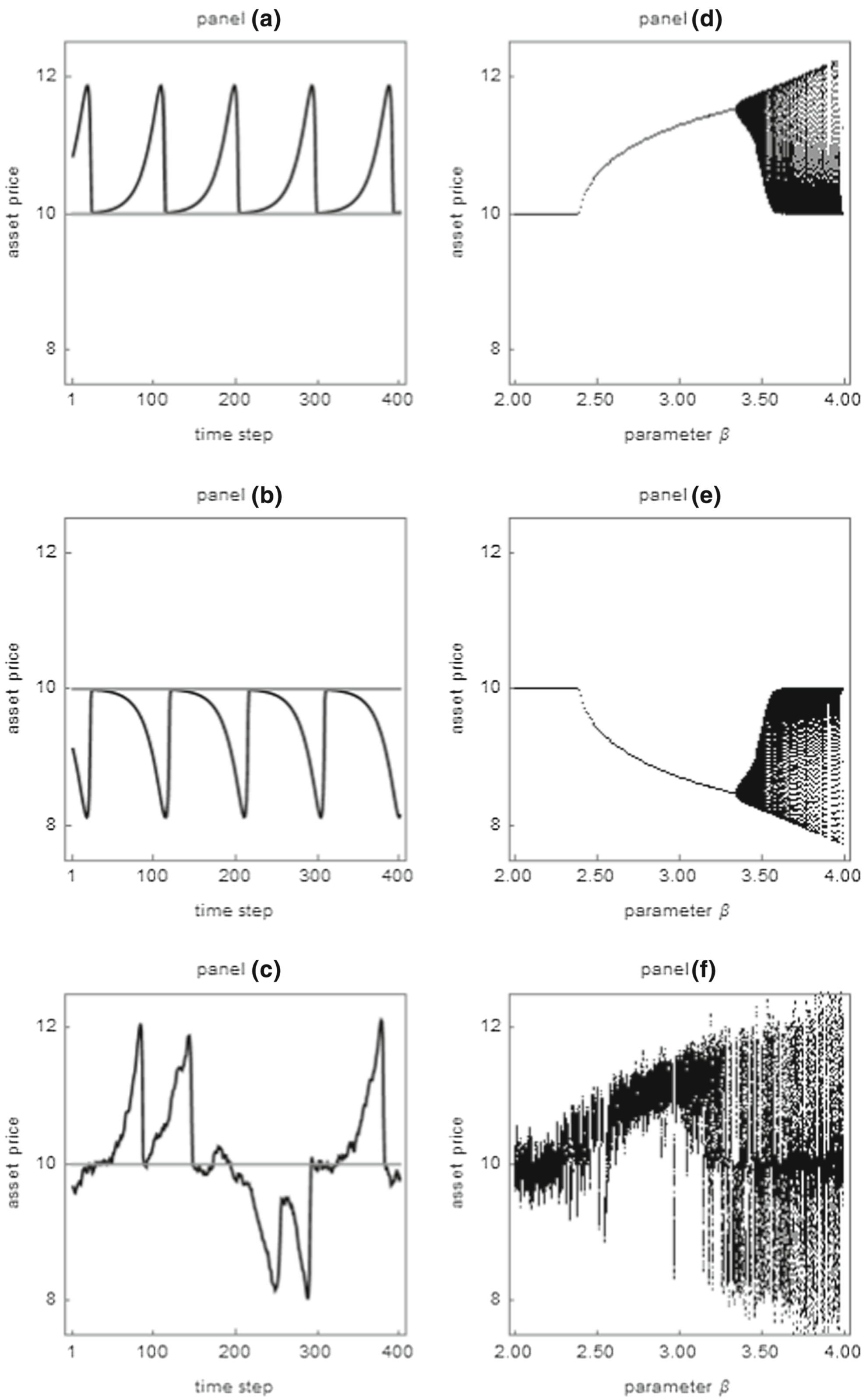
Armed with these insights, we are now ready to conduct a systematic numerical investigation of our setup. In doing so, our goal is to show that speculators' behavior may cause endogenous boom-bust dynamics. Moreover, we describe how the central authority's intervention strategies fare against their destabilizing trading behavior. The base parameter setting we use for our simulations closely follows Brock and Hommes [16]. To be precise, we assume that $r = 0.1$, $\bar{D} = 1$, $\sigma_\delta^2 = 0$, $\alpha = 1$, $\sigma_R^2 = 1$, $\chi = 0.2$, $\phi = 1$, $\kappa = 1$, $\beta = 3.6$, $\sigma_L^2 = 0$, $m = 0$ and $d = 0$. Since $\bar{N}_1^C = 0.973$ and $\bar{N}_1^F = 0.027$, the fundamental steady state $\bar{P}_1 = F = 10$ is unstable. For instance, the critical value for the intensity of choice that would just ensure the local asymptotic stability of the fundamental steady state is given by $\beta_{\text{crit}}^{\text{PF}} = 2.4$. As we will see in the sequel, the nonfundamental steady states are also unstable, and the model's dynamics is characterized by two coexisting limit cycles. Numerically, we can compute that the Neimark–Sacker bifurcation occurs at about $\beta_{\text{crit}}^{\text{NS}} = 3.3$. Our stochastic simulations always rely on $\sigma_L^2 = 0.0025$.

Figure 1 provides an overview of the dynamics of the unregulated market, i.e., of the dynamics of the original model by Brock and Hommes [16]. Panels (a) and (b) of Fig. 1 present the evolution of the price of the risky asset for 400 periods in the time domain; they differ only with respect to their initial conditions. Apparently, the model is able to generate endogenous bull or bear market dynamics. Depending on the initial conditions, we observe endogenous fluctuations either above or below the fundamental value. Panel (c) of Fig. 1 reveals that minimal random demand shocks, induced by liquidity traders, are sufficient to create erratic transitions between bull and bear market dynamics. Moreover, the dynamics appears less regular. Panels (d) and (e) of Fig. 1 show routes to complex asset-price dynamics via a pitchfork and Neimark–Sacker bifurcation. The intensity of choice parameter β is varied between 2 and 4; the panels rely on different initial conditions.[8] Panel (f) of Fig. 1 repeats these experiments for a stochastic environment. Consistent with panel (c) of Fig. 1, we observe intricate attractor switching dynamics.[9]

[7] The reason why speculators rely more strongly on the technical expectation rule as the gap between $\bar{P}_1$ and $\bar{P}_{2,3}$ decreases is as follows. The more the nonfundamental steady states deviate from the fundamental steady state, the higher (lower) the profitability of the fundamental (technical) expectation rule. At the upper nonfundamental steady state, for instance, going short, as suggested by the fundamental expectation rule, is more profitable than going long, as suggested by the technical expectation rule. (A long position is associated with a relatively unfavorable dividend–price relation, while a short position benefits from relatively favorable interest rate payments.) Hence, if the central authority manages to reduce the market's mispricing, the relative fitness of the technical expectation rule improves, and thus, more speculators employ this rule.

[8] Brock and Hommes [16] demonstrate in more detail that their model exhibits a rational route to randomness, that is, a bifurcation route to chaos as the intensity of choice parameter increases. See also Hommes [47].

[9] A cautionary note is in order. Due to the interplay of coexisting attractors, exogenous noise and, possibly, complex basins of attraction, the visual appearance of panel (f) of Fig. 1 depends on the number of iterations plotted. The same is true for panels (c) and (d) of Fig. 3 and for panels (c) and (d) of Fig. 6.


panel (a)
asset price
time step
panel (d)
asset price
parameter β
panel (b)
asset price
time step
panel (e)
asset price
parameter β
panel (c)
asset price
time step
panel (f)
asset price
parameter β

◀**Fig. 1** The functioning of the unregulated model. Panels **a** and **b** show the deterministic dynamics of asset prices for our base parameter setting and different sets of initial conditions. Panel **c** shows the same for a stochastic environment. The bifurcation diagrams depicted in panels **d** and **e** show how asset prices react to an increase in parameter β for different sets of initial conditions. Panel **f** shows the same for a stochastic environment

Figure 2 provides examples of time series for the effectiveness of the leaning against the wind strategy with $m = 0.25$ in the deterministic setting, panels (a) and (b), and $m = 1$ in the stochastic setting, panels (c) and (d). The top panels present the evolution of the

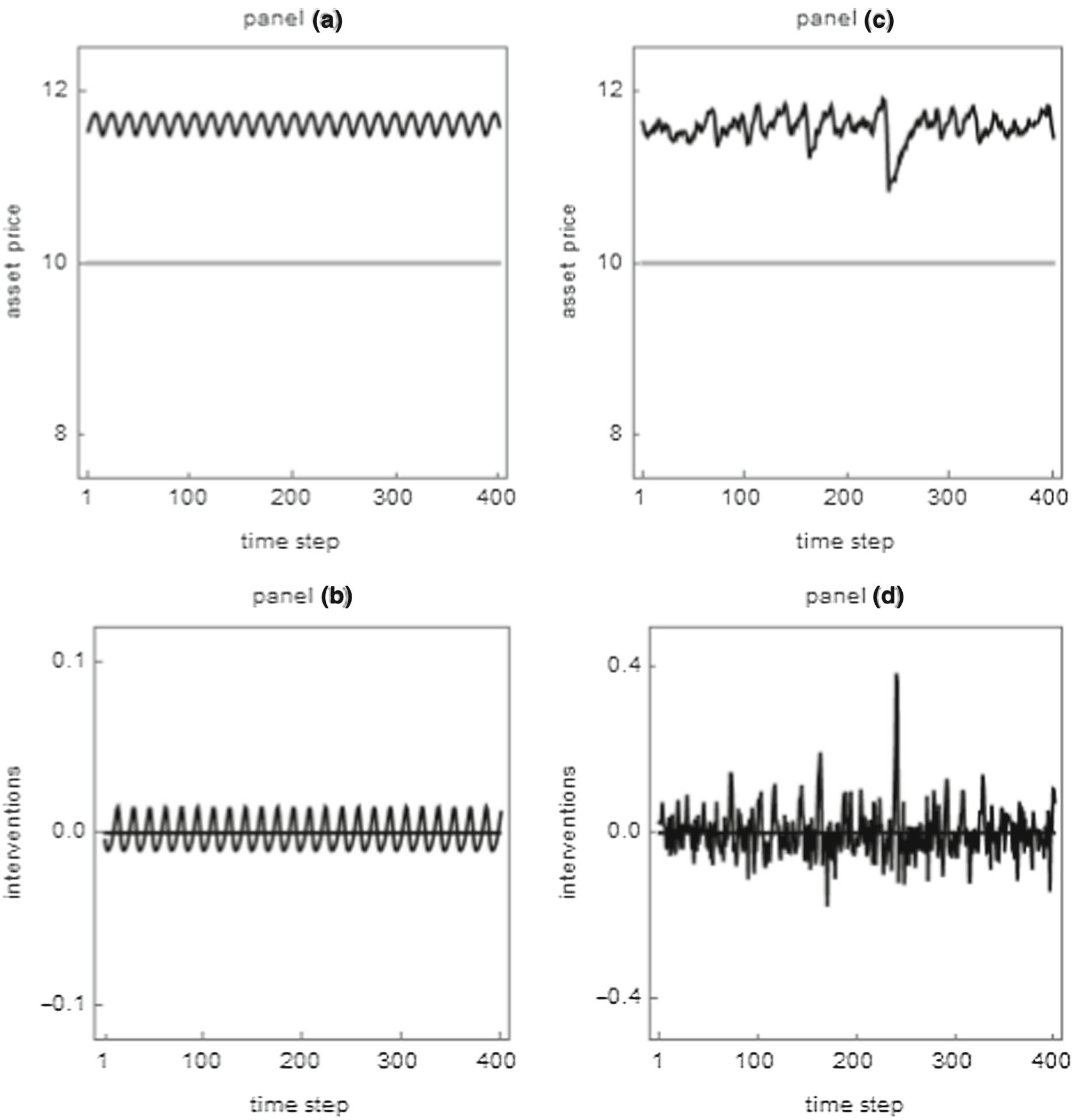


Fig. 2 Examples of time series for the leaning against the wind strategy. Panels **a** and **b** show the deterministic evolution of asset prices and interventions for our base parameter setting, except that $m = 0.25$. Panels **c** and **d** show the same for a stochastic environment and $m = 1$

price of the risky asset, while the bottom panels report the level of the central authority's interventions. A comparison of panel (a) of Fig. 1 with panel (a) of Fig. 2 reveals that the leaning against the wind strategy manages to reduce fluctuations of the price of the risky asset. But while a stabilization effect in

Fig. 4 Summary statistics for the leaning against the wind ▶ strategy in a stochastic environment. Panels **a**, **b**, **c**, **d**, **e** and **f** show how volatility, mispricing, the average absolute level of interventions, the profitability of interventions, the average market share of fundamentalists and their average absolute position react to an increase in parameter m. Base parameter setting, except that parameter m is varied as indicated on the axis

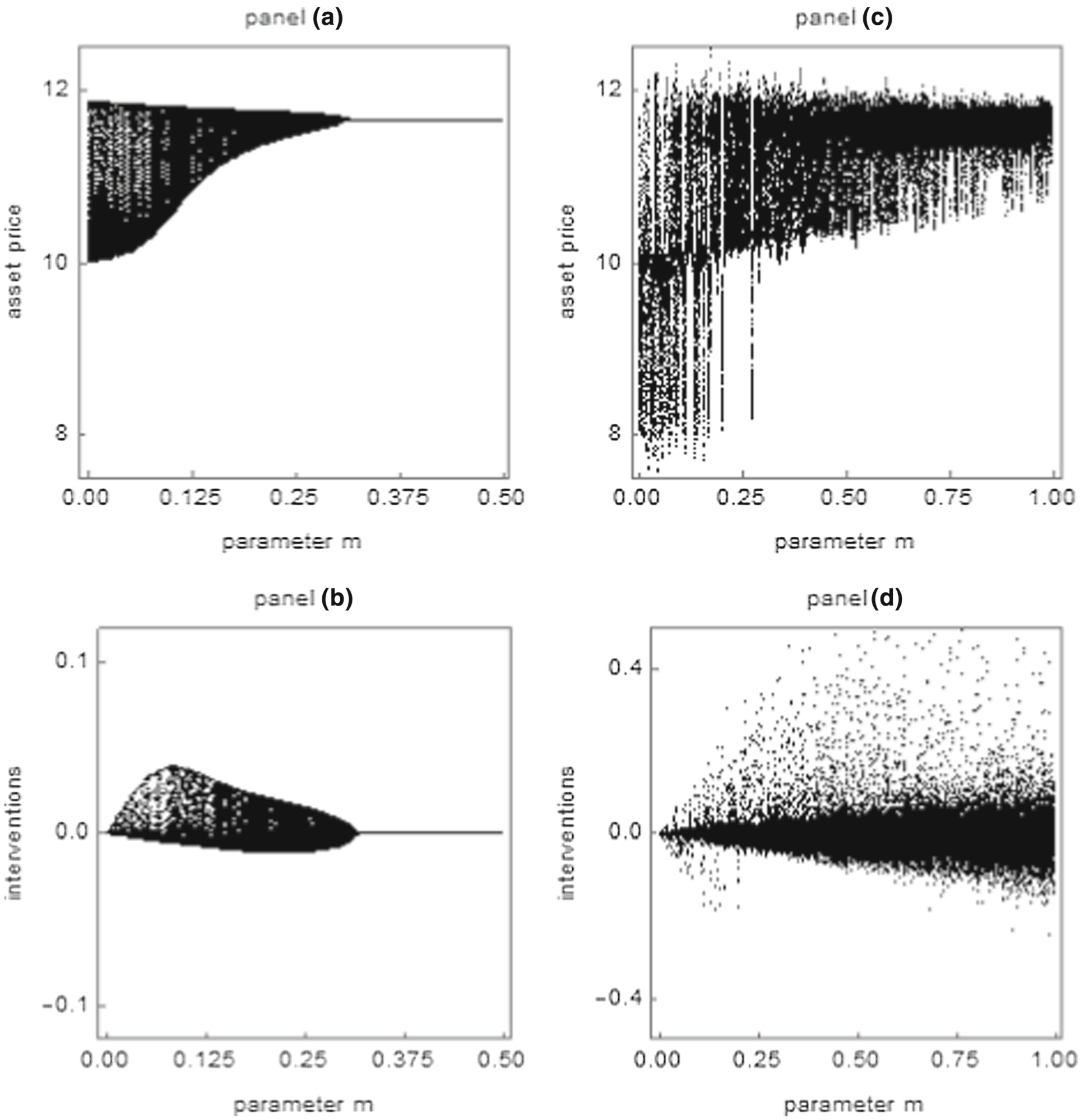
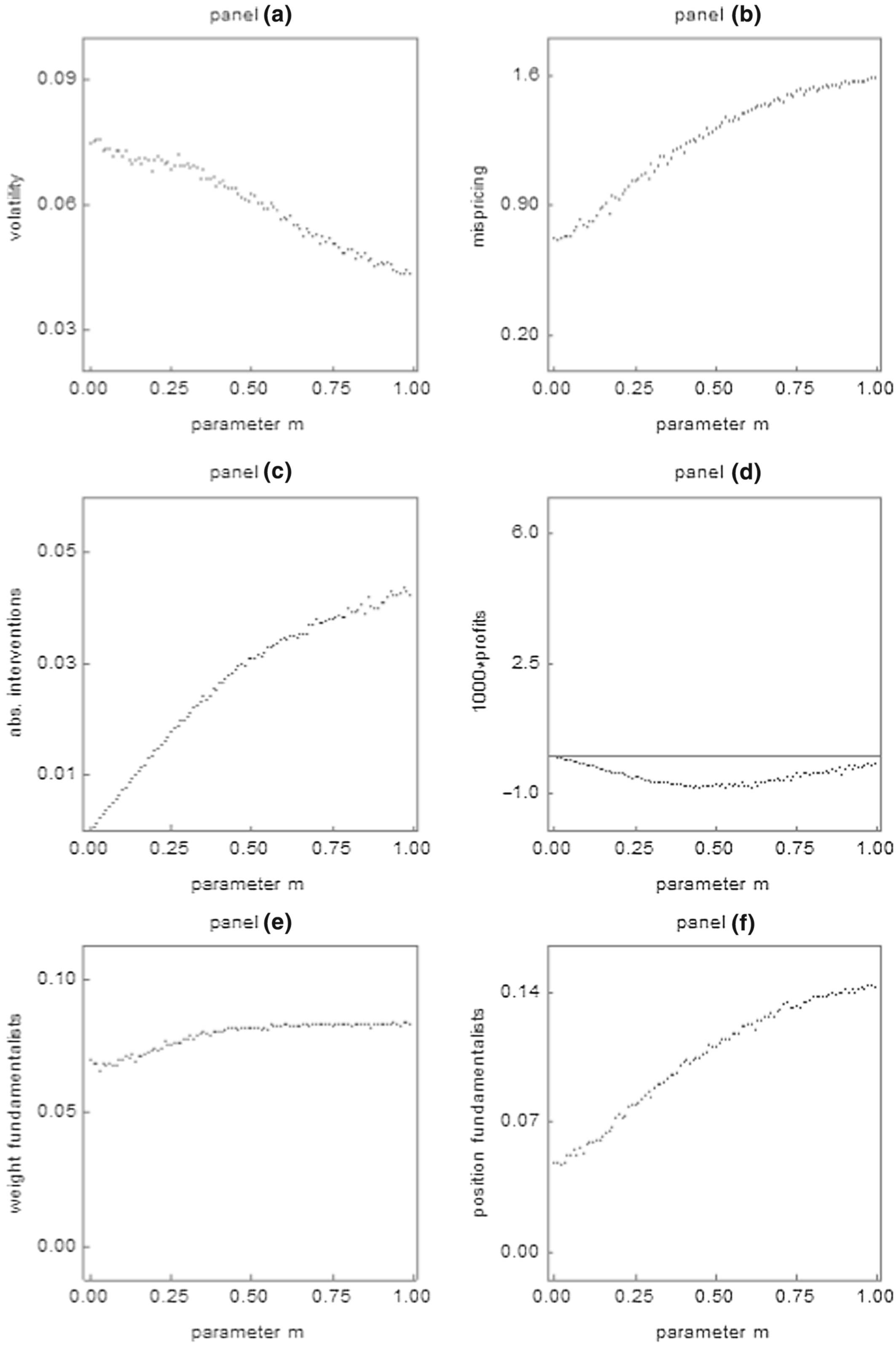


Fig. 3 Bifurcation diagrams for the leaning against the wind strategy. Panels **a** and **b** show how asset prices and interventions react to an increase in parameter m. Panels **c** and **d** show the same for a stochastic environment. Base parameter setting, except that parameter m is varied as indicated on the axis


panel (a)
0.09
0.06
0.03
volatility
0.00 0.25 0.50 0.75 1.00
parameter m
panel (b)
1.6
0.90
0.20
mispricing
0.00 0.25 0.50 0.75 1.00
parameter m
panel (c)
0.05
0.03
0.01
abs. interventions
0.00 0.25 0.50 0.75 1.00
parameter m
panel (d)
6.0
2.5
−1.0
1000∗profits
0.00 0.25 0.50 0.75 1.00
parameter m
panel (e)
0.10
0.05
0.00
weight fundamentalists
0.00 0.25 0.50 0.75 1.00
parameter m
panel (f)
0.14
0.07
0.00
position fundamentalists
0.00 0.25 0.50 0.75 1.00
parameter m

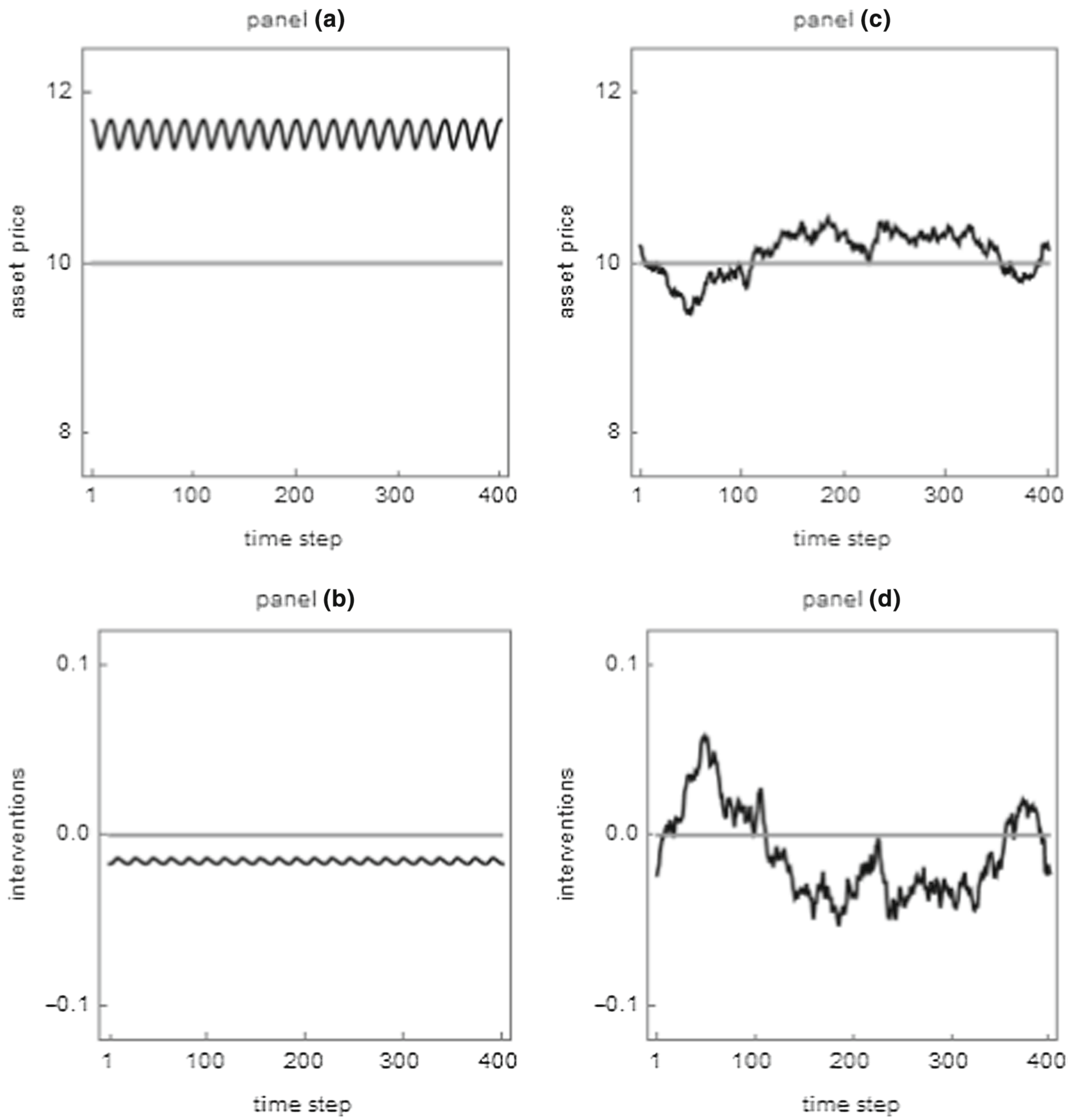


Fig. 5 Examples of time series for the targeting long-run fundamentals strategy. Panels **a** and **b** show the deterministic evolution of asset prices and interventions for our base parameter setting, except that $d = 0.01$. Panels **c** and **d** show the same for a stochastic environment and $d = 0.1$

terms of a lower price variability is clearly visible, the price of the risky asset keeps a distance to its fundamental value. Panel (b) of Fig. 2 indicates that the central authority has to intervene in the risky asset market in each time step. Panels (c) and (d) of Fig. 2 suggest that these observations are robust with respect to exogenous shocks.

Figure 3 provides a more systematic analysis of the leaning against the wind strategy. Panel (a) of Fig. 3 shows a bifurcation diagram for parameter m and reveals that the leaning against the wind strategy

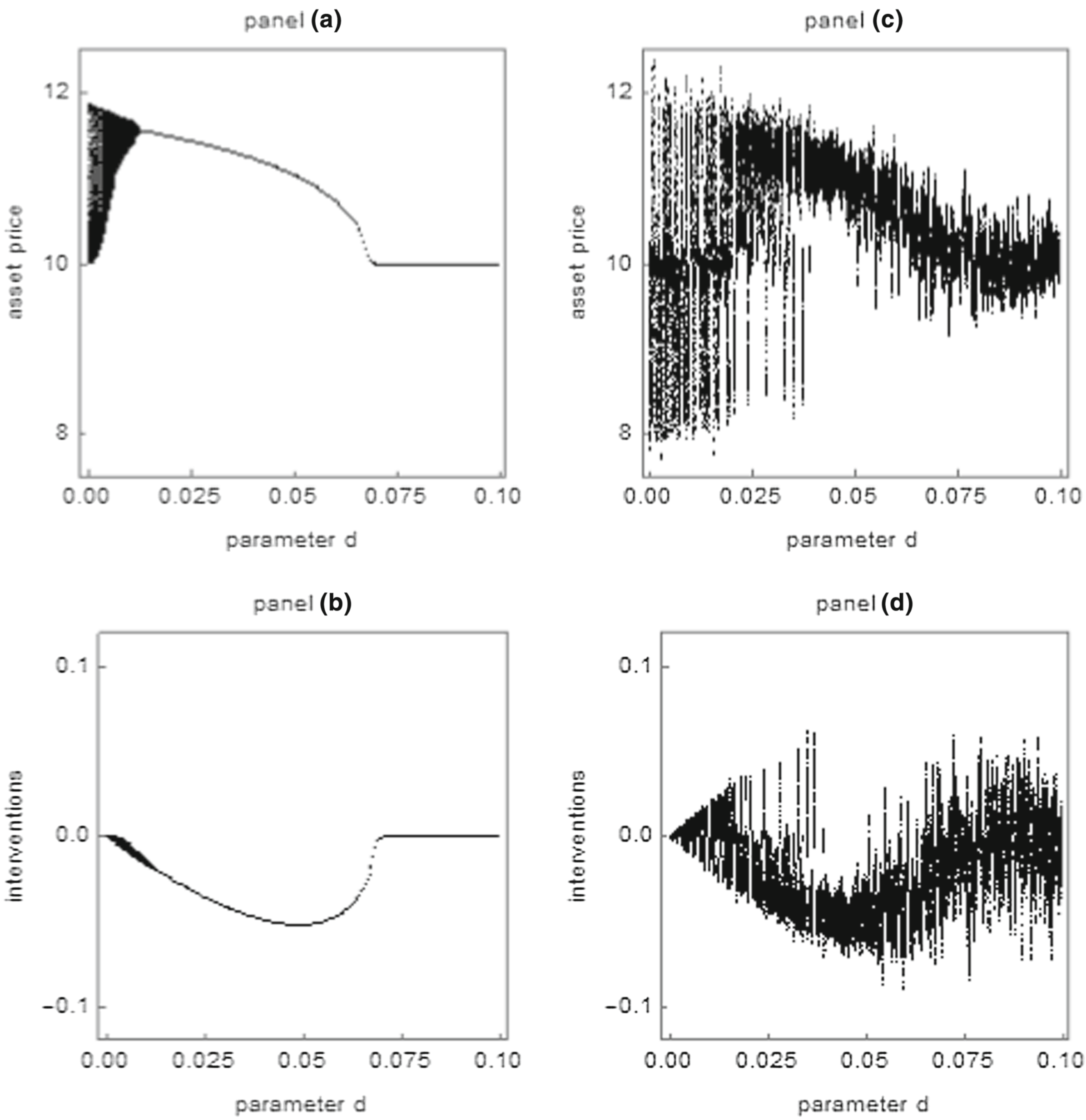
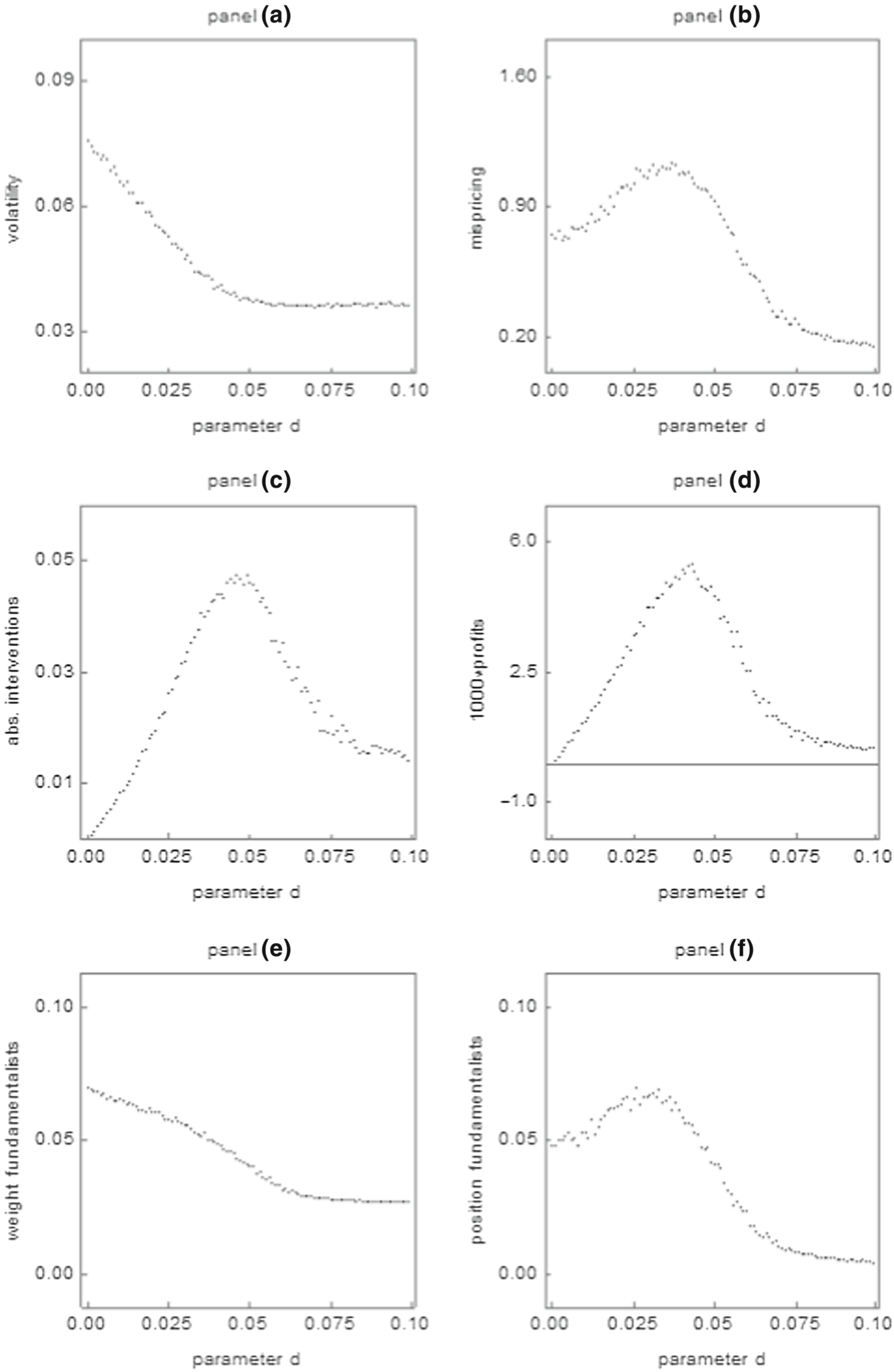


Fig. 6 Bifurcation diagrams for the targeting long-run fundamentals strategy. Panels **a** and **b** show how asset prices and interventions react to an increase in parameter d. Panels **c** and **d** show the same for a stochastic environment. Base parameter setting, except that parameter d is varied as indicated on the axis

stabilizes the dynamics around the upper nonfundamental steady state if parameter m exceeds a value of about 0.32.[10] Such an outcome should be regarded as a mixed blessing: While the leaning against the wind strategy enables the central authority to reduce the risky asset's price fluctuations, it fails to reduce the market's mispricing. Panel (b) of Fig. 3 depicts the

[10] For other initial conditions, we may observe that applying the leaning against the wind strategy leads to a stabilization of the price of the risky asset around the lower nonfundamental steady state.


panel (a)
volatility
0.09
0.06
0.03
0.00
0.025
0.05
0.075
0.10
parameter d
panel (b)
mispricing
1.60
0.90
0.20
parameter d
panel (c)
abs. interventions
0.05
0.03
0.01
parameter d
panel (d)
1000∗profits
6.0
2.5
−1.0
parameter d
panel (e)
weight fundamentalists
0.10
0.05
0.00
parameter d
panel (f)
position fundamentalists
0.10
0.05
0.00
parameter d

◀ **Fig. 7** Summary statistics for the targeting long-run fundamentals strategy in a stochastic environment. Panels **a**, **b**, **c**, **d**, **e** and **f** show how volatility, mispricing, the average absolute level of interventions, the profitability of interventions, the average market share of fundamentalists and their average absolute position react to an increase in parameter d. Base parameter setting, except that parameter d is varied as indicated on the axis

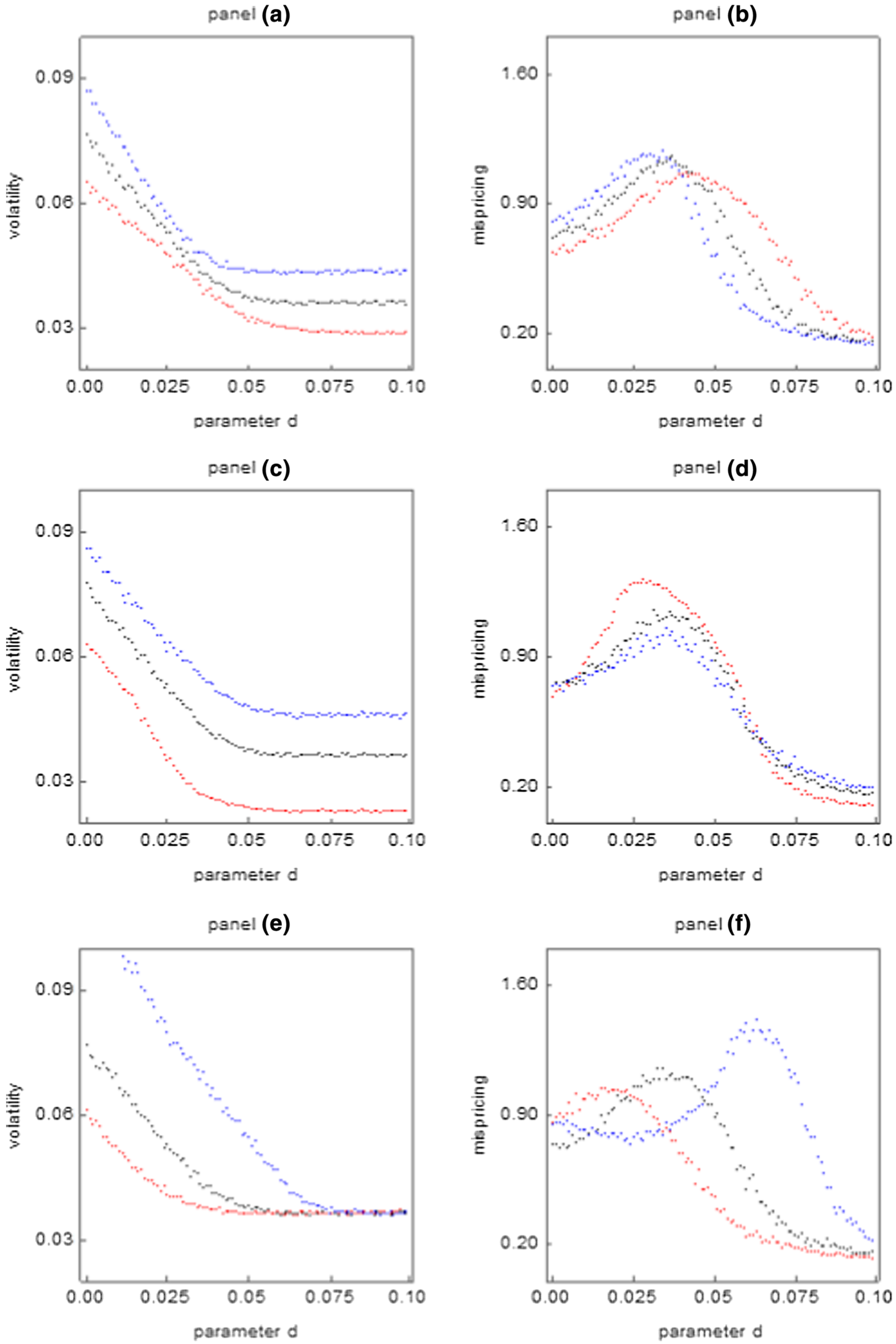
central authority's associated interventions. Note that if parameter m is set high enough, interventions can be low or even zero. Panel (c) of Fig. 3 confirms the stabilizing potential of the leaning against the wind strategy within a stochastic environment. However, panel (d) of Fig. 3 suggests that the central authority's intervention intensity increases with parameter m. This is clearly different to what we see in panel (b) of Fig. 3, where the central authority's intervention intensity eventually goes to zero. (There are no price changes at the nonfundamental steady state.) Of course, the stochastic environment is the relevant environment for judging the effectiveness of interventions.

To be able to quantify the success of the central authority's interventions, we introduce six summary statistics. Let T be the sample length used for computing these statistics. We capture the risky asset's volatility by the average absolute price change of the risky asset $\frac{1}{T}\sum_{t=1}^{T}|P_t - P_{t-1}|$, the risky asset's mispricing by the average absolute distance between the price of the risky asset and its fundamental value $\frac{1}{T}\sum_{t=1}^{T}|P_t - F|$, the central authority's intervention intensity by the average absolute level of interventions $\frac{1}{T}\sum_{t=1}^{T}|Z_t^G|$, the average profitability of their interventions by $\frac{1}{T}\sum_{t=1}^{T}(P_t + D_t - (1+r)P_{t-1})Z_{t-1}^G$, the average market share of fundamentalists by $\frac{1}{T}\sum_{t=1}^{T}N_t^F$ and their average absolute position by $\frac{1}{T}\sum_{t=1}^{T}N_t^F|Z_t^F|$. Note that we measure the central authority's profits in the same way as we do for speculators. For simplicity, we ignore the costs associated with conducting these interventions. There may be fixed costs, for instance, that could simply be subtracted from our profit measure. To obtain reasonable estimates of these statistics, we focus on the stochastic environment and use $T = 10,000$ observations.

Based on these statistics, we can now quantify the success of the leaning against the wind strategy. Overall, the results are mixed. According to panels (a) and (b) of Fig. 4, volatility declines when this strategy is applied, while mispricing increases. As discussed above, the leaning against the wind strategy manages to stabilize the dynamics, albeit around the model's nonfundamental steady states. Moreover, panel (c) of Fig. 4 indicates that the central authority has to intervene more and more aggressively if it wants to reduce volatility. Apparently, the leaning against the wind strategy also produces losses, as depicted in panel (d) of Fig. 4. Interestingly, the average market share of fundamentalists increases (slightly) with parameter m. The same is true for the average absolute position of fundamentalists, an outcome that is driven by an increase in fundamentalists' average market share and an increase in the market's mispricing. See panels (e) and (f) of Fig. 4. So far, we can thus conclude that the leaning against the wind strategy may offset endogenous fluctuations, engendered by a Neimark–Sacker bifurcation, but has no real power to fight deviations from fundamental values, as created by a pitchfork bifurcation.

Figures 5 and 6 show the effects of the targeting long-run fundamentals strategy. Most importantly, the bifurcation diagram depicted in panel (a) of Fig. 6 shows that the targeting long-run fundamentals strategy may allow the central authority to reduce volatility and mispricing of the risky asset. In line with our analytical results, the price of the risky asset approaches its fundamental value for $d > 0.68$. To diminish mispricing connected with a bull (bear) market, however, the central authority needs to go permanently short (long), as visible from panel (b) of Fig. 6. Importantly, this strategy also seems to work in a stochastic environment. Panel (c) of Fig. 6 reveals that asset prices fluctuate much closer around the fundamental value as parameter d increases. Moreover, we can conclude from panel (d) of Fig. 6 that the central authority's interventions oscillate around zero if parameter d is set high enough. To illustrate these findings, panels (a) and (c) of Fig. 5 show examples of time series for the price of the risky asset when the targeting long-run fundamentals strategy is applied with $d = 0.01$ (deterministic setting) and $d = 0.1$ (stochastic setting). Panels (b) and (d) of Fig. 5 depict the central authority's corresponding interventions.

Figure 7 shows how the targeting long-run fundamentals strategy may affect our four policy measures. As can be seen from panels (a) and (b) of Fig. 7, the targeting long-run fundamentals strategy allows the


panel (a)
volatility
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0.05
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0.10
parameter d
panel (b)
mispricing
1.60
0.90
0.20
0.00
0.025
0.05
0.075
0.10
parameter d
panel (c)
volatility
0.09
0.06
0.03
0.00
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0.05
0.075
0.10
parameter d
panel (d)
mispricing
1.60
0.90
0.20
0.00
0.025
0.05
0.075
0.10
parameter d
panel (e)
volatility
0.09
0.06
0.03
0.00
0.025
0.05
0.075
0.10
parameter d
panel (f)
mispricing
1.60
0.90
0.20
0.00
0.025
0.05
0.075
0.10
parameter d

◀ **Fig. 8** Simple robustness checks. The top, central and bottom lines of panels show how volatility and mispricing react to an increase in parameter d, varying speculators' variance beliefs (red: $\sigma_R^2 = 0.8$, black: $\sigma_R^2 = 1$, blue: $\sigma_R^2 = 1.2$), the impact of liquidity traders (red: $\sigma_L^2 = 0.001$, black: $\sigma_L^2 = 0.0025$, blue: $\sigma_L^2 = 0.004$) and information costs (red: $\kappa = 0.9$, black: $\kappa = 1$, blue: $\kappa = 1.25$). See Sect. 3.3 for more details. (Color figure online)

central authority to reduce volatility and mispricing of the risky asset, provided that it executes its interventions forcefully enough. Interestingly, the absolute level of interventions shrinks if parameter d becomes large enough. The reason for this is as follows. For intermediate values of parameter d, the targeting long-run fundamentals strategy stabilizes the dynamics around one of the two nonfundamental steady states. If parameter d is set high enough, this strategy stabilizes the dynamics around the risky asset's fundamental value. Panel (d) of Fig. 7 suggests that this strategy may even be profitable. In line with our analytical results, we can conclude from panel (e) of Fig. 7 that the targeting long-run fundamentals strategy diminishes the market impact of fundamentalists. According to panel (f) of Fig. 7, this also holds for the average absolute position of fundamentalists, at least if this strategy is applied forcefully. Nevertheless, and this is the important message, the targeting long-run fundamentals strategy achieves to reduce the market's volatility and mispricing.

3.3 Model extensions and robustness checks

Robustness checks are important for policy analysis. Here, we consider two kinds of robustness checks. As a first and rather simple robustness check, we explore the effectiveness of the targeting long-run fundamentals strategy in a stochastic environment for alternative parameter settings. In panels (a) and (b) of Fig. 8, we plot how volatility and mispricing react to an increase in parameter d, assuming different values for speculators' variance beliefs (red dots: $\sigma_R^2 = 0.8$, black dots: $\sigma_R^2 = 1$, blue dots: $\sigma_R^2 = 1.2$). Apparently, the targeting long-run fundamentals strategy is able to stabilize the dynamics when speculators' variance beliefs change. In panels (c) and (d) of Fig. 8, we assume that the market impact of liquidity traders is, say, low (red dots: $\sigma_L^2 = 0.001$), medium (black dots: $\sigma_L^2 = 0.0025$) or high (blue dots: $\sigma_L^2 = 0.004$). As can be seen, an increase in σ_L^2, reflecting more aggressive liquidity traders, shifts volatility and mispricing upwards, yet does not affect the effectiveness of the targeting long-run fundamentals strategy. In panels (e) and (f) of Fig. 8, we vary the costs associated with conducting fundamental analysis (red dots: $\kappa = 0.9$, black dots: $\kappa = 1$, blue dots: $\kappa = 1.25$). For all three values of parameter k, the central authority manages to reduce volatility and mispricing. Note that by providing better information about the fundamental value of the risky asset, the central authority may be able to reduce information costs. Such a policy has a stabilizing effect too.

As a second and slightly more involved robustness check, let us assume that the risky asset's dividend process follows a random walk. Accordingly, we specify the dividend process by $D_t = D_{t-1} + D_{t-1}\delta_t$, where $\delta_t \sim N(0, \sigma_\delta^2)$, implying that $E_t[D_{t+1}] = D_{t-1}$ and, consequently, that $F_{t-1} = D_{t-1}/r$, i.e., the fundamental value of the risky assets follows a random walk, too. Moreover, the price of the risky asset now obeys $P_t = \frac{E_t[P_{t+1}] + D_{t-1} + \alpha\sigma_R^2 (Z_t^G + Z_t^L)}{1+r}$, the technical expectation rule turns into $E_t^C[P_{t+1}] = P_{t-1} + \chi(P_{t-1} - F_{t-1})$, and the fundamental expectation rule reads $E_t^F[P_{t+1}] = P_{t-1} + \phi(F_{t-1} - P_{t-1})$. The attractiveness of the two expectation rules is due to $A_t^C = (P_{t-1} + D_{t-1} - (1+r)P_{t-2})Z_{t-2}^C$ and $A_t^F = (P_{t-1} + D_{t-1} - (1+r)P_{t-2})Z_{t-2}^F - \kappa$, where $Z_{t-2}^C = \frac{E_{t-2}^C[P_{t-1}] + D_{t-3} - (1+r)P_{t-2}}{\alpha\sigma_R^2}$ and $Z_{t-2}^F = \frac{E_{t-2}^F[P_{t-1}] + D_{t-3} - (1+r)P_{t-2}}{\alpha\sigma_R^2}$. Finally, the central authority's demand for the risky asset is given by $Z_t^G = -m(P_{t-1} - P_{t-2}) - d(P_{t-1} - F_{t-1})$. The model's other building blocks remain as before.

The simulation results depicted in Fig. 9 rely on our stochastic parameter setting, except that the variance of the dividend shocks is given by $\sigma_\delta^2 = 0.0001$. Panel (a) of Fig. 9 shows the dynamics of the asset price (black line) and its fundamental value (red line) when the central authority is inactive ($d = 0$). As can be seen, the asset price fluctuates widely around it time-varying fundamental value. Panel (b) of Fig. 9 depicts the behavior of the asset price and its fundamental

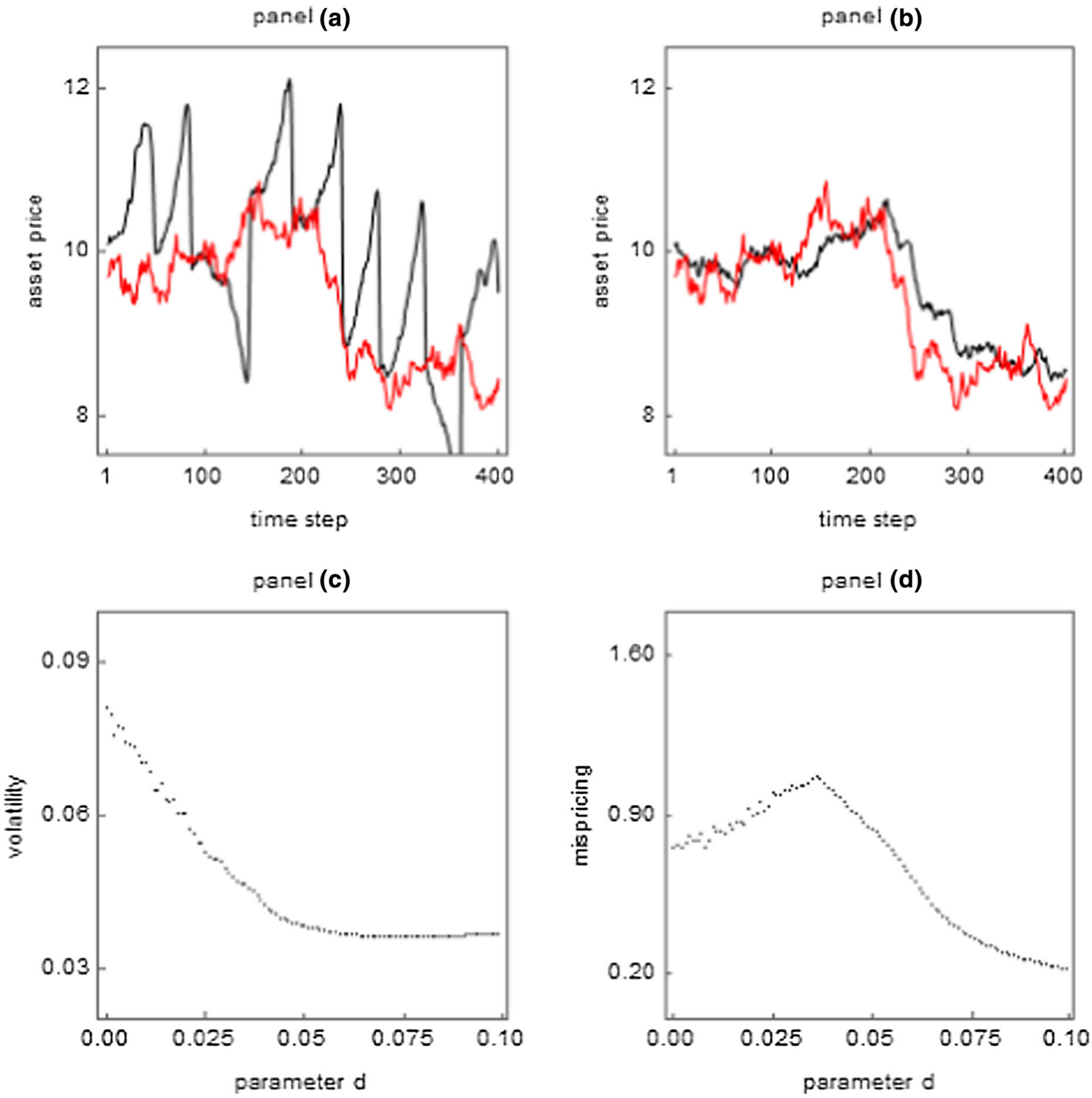

Fig. 9 Effectiveness of the targeting long-run fundamentals strategy when dividends follow a random walk. Panel **a** shows the dynamics of asset prices (black line) and its fundamental value (red line) when the central authority is inactive ($d = 0$). Panel **b** shows the same for the case when the central authority is active ($d = 0.1$). Panels **c** and **d** show how volatility and mispricing react to an increase in parameter d. See Sect. 3.3 for more details

value when the central authority is active ($d = 0.1$). Without question, the targeting long-run fundamentals strategy manages to push the price of the risky asset closer toward its fundamental value. Finally, panels (c) and (d) of Fig. 9 report how volatility and mispricing react to an increase in parameter d. Once again, we can conclude that the central authority may stabilize the risky asset market by using the targeting long-run fundamentals strategy.[11]

[11] Hommes [44] studies the case in which the dividend process follows a random walk with drift. See also Dieci et al. [31], He and Li [41, 42] and Schmitt and Westerhoff [72] for studies that consider the intricate interplay between exogenous noise, fundamental shocks and nonlinear market interactions.

4 Conclusions

The dynamics of financial markets, in particular their boom-bust nature, may be quite harmful to the real economy. Since nonlinear asset-pricing models, involving the trading behavior of heterogeneous interacting speculators, have improved our understanding of the functioning of financial markets, they may be used by policymakers to stress-test the effectiveness of regulatory measures. To illustrate the potential of this research field, we show that a central authority may stabilize the boom-bust dynamics of the seminal asset-pricing model by Brock and Hommes [16] if it follows a so-called targeting long-run fundamentals strategy. In fact, by buying (selling) the risky asset when the market is undervalued (overvalued), a central authority may guide prices toward fundamental values. As it turns out, however, a leaning against the wind strategy, which recommends buying (selling) the asset when its price decreases (increases), fails to do so. These results may be understood as follows. The model by Brock and Hommes [16] generates complex (chaotic) boom-bust cycles via a pitchfork and subsequent Neimark–Sacker bifurcation. While the leaning against the wind strategy may cope with the Neimark–Sacker bifurcation, it cannot offset a pitchfork bifurcation. Note that the leaning against the wind strategy becomes inactive at any steady state and therefore also at the model's nonfundamental steady states. The targeting long-run fundamentals strategy, in turn, is able to prevent a pitchfork bifurcation. In fact, applying this strategy properly ensures that the model's fundamental steady state is at least locally stable. In addition, simulations suggest that our results are quite robust, e.g., with respect to exogenous noise.

We conclude our paper by pointing out a few avenues for future research. In our paper, we study the effects of simple linear feedback rules. Future work may explore whether nonlinear feedback rules may do a better job. For instance, a central authority may only take action when the asset's price trend or its mispricing exceeds a critical threshold value. Avrutin et al. [7] provide mathematical tools for piecewise-defined maps. Related to this, the central authority may encounter speculators who follow nonlinear and possibly piecewise-defined trading rules. It should be noted that the results presented in our paper rely on the assumption that the technical expectation rule predicts a continuation of the current mispricing. Alternatively, one may develop a model in which the technical expectation rule extrapolates past price changes. In such an environment, the leaning against the wind strategy may fare much better. In this respect, it also seems worthwhile to explore in more detail whether a combined application of the leaning against the wind strategy and the targeting long-run fundamentals strategy may improve the effectiveness of the central authority's market interventions. Moreover, we focus on the case where the central authority computes the fundamental value in the same way as speculators do—and both groups are right. Future work may consider the case where speculators and the central authority have different perceptions of the asset's fundamental value. For instance, one may assume that speculators are too optimistic or too pessimistic when computing the asset's fundamental value, as studied by de Grauwe and Kaltwasser (2012). Such a model extension would also provide a framework in which the central authority may convey private information with respect to the asset's fundamental value to the market participants via its interventions. Of course, nonsecret interventions open up the door for strategic trading, an aspect that seems to be worth investigating as well. Despite allowing for some random disturbances, in our paper we focus on nonlinear forces that may create complex (chaotic) boom-bust dynamics. Future work may use more developed stochastic models that are able to match the stylized facts of financial markets. An agent-based version of the model by Brock and Hommes [16], as proposed by Schmitt [68], may serve as an ideal starting point for such an endeavor. We hope that our contribution stimulates more work in this important research direction.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Appendix A

In this appendix, we derive the model's law of motion, compute its steady states and demonstrate that the model's fundamental steady state may lose its local asymptotic stability via a pitchfork bifurcation, while the model's nonfundamental steady states may become unstable due to a Neimark–Sacker bifurcation. Finally, we present a number of simulations which illustrate the role played by parameters β, m and d in this context.

We are interested in the model's deterministic behavior. Excluding random disturbances [21] reads

$$P_t = \frac{E_t[P_{t+1}] + \bar{D} + \alpha\sigma_R^2 Z_t^G}{1+r}, \tag{23}$$

where $E_t[P_{t+1}] = \frac{P_{t-1}+\chi(P_{t-1}-F)}{1+\exp[\beta(A_t^F - A_t^C)]} + \frac{P_{t-1}+\phi(F-P_{t-1})}{1+\exp[-\beta(A_t^F - A_t^C)]}$, $A_t^F - A_t^C = \frac{P_{t-1}+\bar{D}-(1+r)P_{t-2}}{\alpha\sigma_R^2}(\chi+\phi)(F-P_{t-3}) - \kappa$, and $Z_t^G = -m(P_{t-1}-P_{t-2}) - d(P_{t-1}-F)$. As can be seen, [22] corresponds to a third-order nonlinear difference equation that can easily be transformed into a system of three third-order difference equations.

In order to facilitate the analysis, however, we follow Brock and Hommes [16] and rewrite the model in deviations from the fundamental value, i.e., $x_t = P_t - F$, and introduce the difference in fractions, i.e., $N_t^F - N_t^C = z_t = \tanh\left[\frac{\beta}{2}\left(A_t^F - A_t^C\right)\right]$.[12] Our model can then be expressed by the following dynamical system

$$S: \begin{cases} x_t = \dfrac{1}{1+r}\left\{\dfrac{1-z_{t-1}}{2}(1+\chi)x_{t-1} + \dfrac{1+z_{t-1}}{2}(1-\phi)x_{t-1} - \alpha\sigma_R^2(m(x_{t-1}-y_{t-1}) + dx_{t-1})\right\} \\ y_t = x_{t-1} \\ z_t = \tanh\left[\dfrac{\beta}{2}\left\{(x_t - (1+r)x_{t-1})\dfrac{-(\chi+\phi)y_{t-1}}{\alpha\sigma_R^2} - \kappa\right\}\right] \end{cases} \tag{24}$$

where $y_t = x_{t-1}$ is an auxiliary variable.

To derive the model's steady states, we apply the equilibrium conditions to the pricing equation and obtain

$$(1+r)x^* = \frac{1-z^*}{2}(1+\chi)x^* + \frac{1+z^*}{2}(1-\phi)x^* - \alpha\sigma_R^2 dx^*, \tag{25}$$

yielding $x^* = 0$ or $z^* = \frac{\chi-\phi-2r-2\alpha\sigma_R^2 d}{\chi+\phi}$. For $x^* = 0$, we have $y^* = 0$ and $z^* = \tanh\left[-\frac{\beta}{2}\kappa\right]$. We call the steady state $\bar{S}_1 = \left(0, 0, \tanh\left[-\frac{\beta}{2}\kappa\right]\right)$ a fundamental steady state of our model since $x^* = 0$ implies that $\bar{P}_1 = F$. Note also that $\bar{N}_1^C = \frac{1}{1+\exp[-\beta\kappa]}$ and $\bar{N}_1^F = \frac{1}{1+\exp[\beta\kappa]}$. From

$$z^* = \tanh\left[\frac{\beta}{2}\left\{(x^* - (1+r)x^*)\frac{-(\chi+\phi)x^*}{\alpha\sigma_R^2} - \kappa\right\}\right] \tag{26}$$

and $z^* = \frac{\chi-\phi-2r-2\alpha\sigma_R^2 d}{\chi+\phi}$, we obtain $x^* = y^* = \pm\sqrt{\frac{2\alpha\sigma_R^2\left(\frac{\beta}{2}\kappa + \operatorname{arctan}h\left[\frac{\chi-\phi-2r-2\alpha\sigma_R^2 d}{\chi+\phi}\right]\right)}{r\beta(\chi+\phi)}}$.

Note that [25] only has two solutions $\pm x^*$ if $\frac{\beta}{2}\kappa + \operatorname{arctan}h\left[\frac{\chi-\phi-2r-2\alpha\sigma_R^2 d}{\chi+\phi}\right] \geq 0$, implying that $\frac{1}{1+\exp[-\beta\kappa]}\chi - \frac{1}{1+\exp[\beta\kappa]} \geq r + \alpha\sigma_R^2 d$. If this condition holds, the two nonfundamental steady states $\bar{S}_{2,3} = \left(\pm x^*, \pm y^*, \frac{\chi-\phi-2r-2\alpha\sigma_R^2 d}{\chi+\phi}\right)$ exist. From these expressions, we can conclude that $\bar{P}_{2,3} = \bar{P}_1 \pm\sqrt{\frac{2\alpha\sigma_R^2\left(\frac{\beta}{2}\kappa + \operatorname{arctan}h\left[\frac{\chi-\phi-2r-2\alpha\sigma_R^2 d}{\chi+\phi}\right]\right)}{r\beta(\chi+\phi)}}$, $\bar{N}_{2,3}^C = \frac{\phi+r+\alpha\sigma_R^2 d}{\chi+\phi}$ and $\bar{N}_{2,3}^F = \frac{\chi-r-\alpha\sigma_R^2 d}{\chi+\phi}$.

[12] Note that $N_t^C = (1-z_t)/2$, $N_t^F = (1+z_t)/2$ and $\bar{D} = rF$.

To study the local asymptotic stability properties of the model's fundamental steady state, we must evaluate the Jacobian matrix of (23) at $\bar{S}_1$, that is,

$$J(\bar{S}_1) = \begin{bmatrix} \frac{2 - 2\alpha\sigma_R^2(m+d) + \chi - \phi + (\chi + \phi)\tanh\left[\frac{\beta}{2}\kappa\right]}{2(1+r)} & \frac{\alpha\sigma_R^2 m}{1+r} & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \tag{27}$$

from which we obtain the characteristic polynomial

$$\lambda\left(\lambda^2 - \lambda tr + \det\right) = 0, \tag{28}$$

where $tr = \frac{2-2\alpha\sigma_R^2(m+d)+\chi-\phi+(\chi+\phi)\tanh\left[\frac{\beta}{2}\kappa\right]}{2(1+r)}$ and $\det = -\frac{\alpha\sigma_R^2 m}{1+r}$ represent the trace and determinant of the top left 2×2 sub-matrix of $J(\bar{S}_1)$. Since one eigenvalue of (A6) is always equal to zero, the stability of $\bar{S}_1$ depends on the two eigenvalues resulting from the remaining second-degree characteristic polynomial. A set of necessary and sufficient conditions [58] guaranteeing that the remaining two eigenvalues are less than one in modulus are given by (1) $1 + tr + \det > 0$, (2) $1 - tr + \det > 0$ and (3) $1 - \det > 0$. Condition (3), implying that $1 + \frac{\alpha\sigma_R^2 m}{1+r} > 0$, is obviously always true. Rewriting inequalities (1) and (2) using $\bar{m}_1 = \bar{N}_1^F - \bar{N}_1^C = \tanh\left[-\frac{\beta}{2}\kappa\right]$ and $\bar{N}_1^F + \bar{N}_1^C = 1$ yields

$$\bar{N}_1^C\chi - \bar{N}_1^F\phi < 2 + r + \alpha\sigma_R^2(2m + d), \tag{29}$$

and

$$\bar{N}_1^C\chi - \bar{N}_1^F\phi < r + \alpha\sigma_R^2 d. \tag{30}$$

Since [29] is more stringent than [28], it characterizes the stability domain of the fundamental steady state. Note that any violation of (2) constitutes a necessary condition for the emergence of a pitchfork bifurcation. Together with the observation of the birth of the two nonfundamental steady states and the numerical evidence provided below, we can conclude that a violation of [29] is indeed associated with a pitchfork bifurcation.

The characteristic polynomial of the Jacobian matrix of [23] computed at the nonfundamental steady states $\bar{S}_{2,3}$ is given by

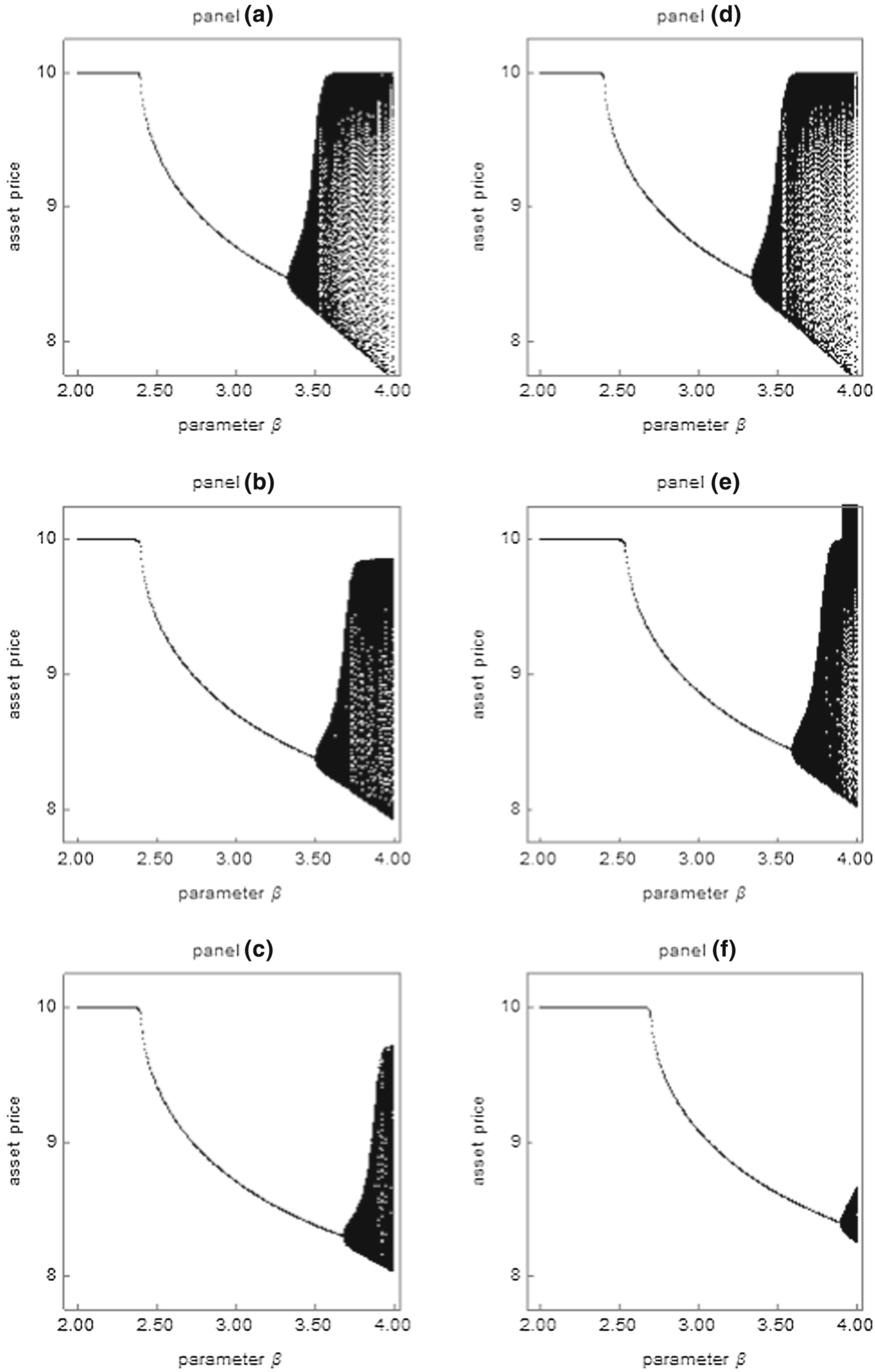
$$p(\lambda) = \lambda^3 + \lambda^2\left(Z(1+r) - 1 + \frac{\alpha\sigma_R^2 m}{1+r}\right) - \lambda\left(Z\left(1 + 2r + r^2\right) + \frac{\alpha\sigma_R^2 m}{1+r}\right) - r(1+r)Z, \tag{31}$$

where $Z = -\frac{2\left(r+\alpha\sigma_R^2 d+\phi\right)\left(r+\alpha\sigma_R^2 d-\chi\right)\left(-\frac{\beta}{2}\kappa+\operatorname{arctanh}\left[\frac{2r+2\alpha\sigma_R^2 d+\phi-\chi}{\chi+\phi}\right]\right)}{r(1+r)(\chi+\phi)}$.

At the pitchfork bifurcation, we have $\frac{1}{1+\exp[-\beta\kappa]}\chi - \frac{1}{1+\exp[\beta\kappa]}\phi = r + \alpha\sigma_R^2 d$ and $Z = 0$ for which (30) yields $\lambda_1 = 0$, $\lambda_2 = 1$ and $\lambda_3 = -\frac{\alpha\sigma_R^2 m}{1+r}$.[13] Now, let us suppose that parameter χ slightly increases. Then, Z becomes slightly negative and (30) yields three eigenvalues inside the unit circle, i.e., $\bar{S}_{2,3}$ are initially stable. For simplicity, recall next that we are back in the original model by Brock and Hommes [16] if the central authority is inactive. For $m = d = 0$, the proof of their Lemma 3 implies the following. For $\chi \to \infty$, we have $Z = -\infty$ and at least one of the eigenvalues must cross the unit circle at some critical value for χ. Moreover, $p(1) = -2Zr(1+r) > 0$ and $p(-1) = 2(-1 + Z(1+r)) < 0$ and, consequently, two eigenvalues must be complex. Accordingly, the two nonfundamental steady states become unstable by a Neimark–Sacker bifurcation. Numerical evidence suggests that this line of reasoning carries over to the case in which the central authority is active.

With the help of Figs. 10 and 11, relying on our base parameter setting, we numerically explore how parameters β, m and d affect the model's dynamics. Panel (a) of Fig. 10 shows that the model by Brock and Hommes [16] first produces a pitchfork bifurcation and then a Neimark–Sacker bifurcation as parameter β increases. Panels (b) and (c) of Fig. 10 repeat these simulations by increasing parameter m from $m = 0$ to $m = 0.2$ and $m = 0.4$. The following results are apparent. First, the values of the two nonfundamental steady states are independent of parameter m. Second, parameter m has no effect on the pitchfork bifurcation value of parameter β. Third, parameter m influences the Neimark–Sacker bifurcation value of parameter β in the following sense: the larger parameter m is, the larger the bifurcation value of parameter β is. Fourth, the amplitude of the oscillations decreases with parameter m. Note that the oscillatory part of the

[13] For obvious economic reasons, we only consider parameter values for which $m < \frac{1+r}{\alpha\sigma_R^2}$ is satisfied.


panel (a)
panel (d)
panel (b)
panel (e)
panel (c)
panel (f)
asset price
parameter β
10
9
8
2.00
2.50
3.00
3.50
4.00

◀**Fig. 10** The effects of parameters m, d and $\boldsymbol{\beta}$. Panels **a**–**c** show bifurcation diagrams for our base parameter setting using $m = 0$, $m = 0.2$ and $m = 0.4$, respectively. Panels **d**–**f** show the same, except that $d = 0$, $d = 0.012$ and $d = 0.024$, respectively

bifurcation diagram is not only moved to the right, but also characterized by a lower scale. Overall, these results seem to be qualitatively robust, at least with respect to some moderate deviations from our base parameter setting. Panels (d)–(f) of Fig. 10 show the

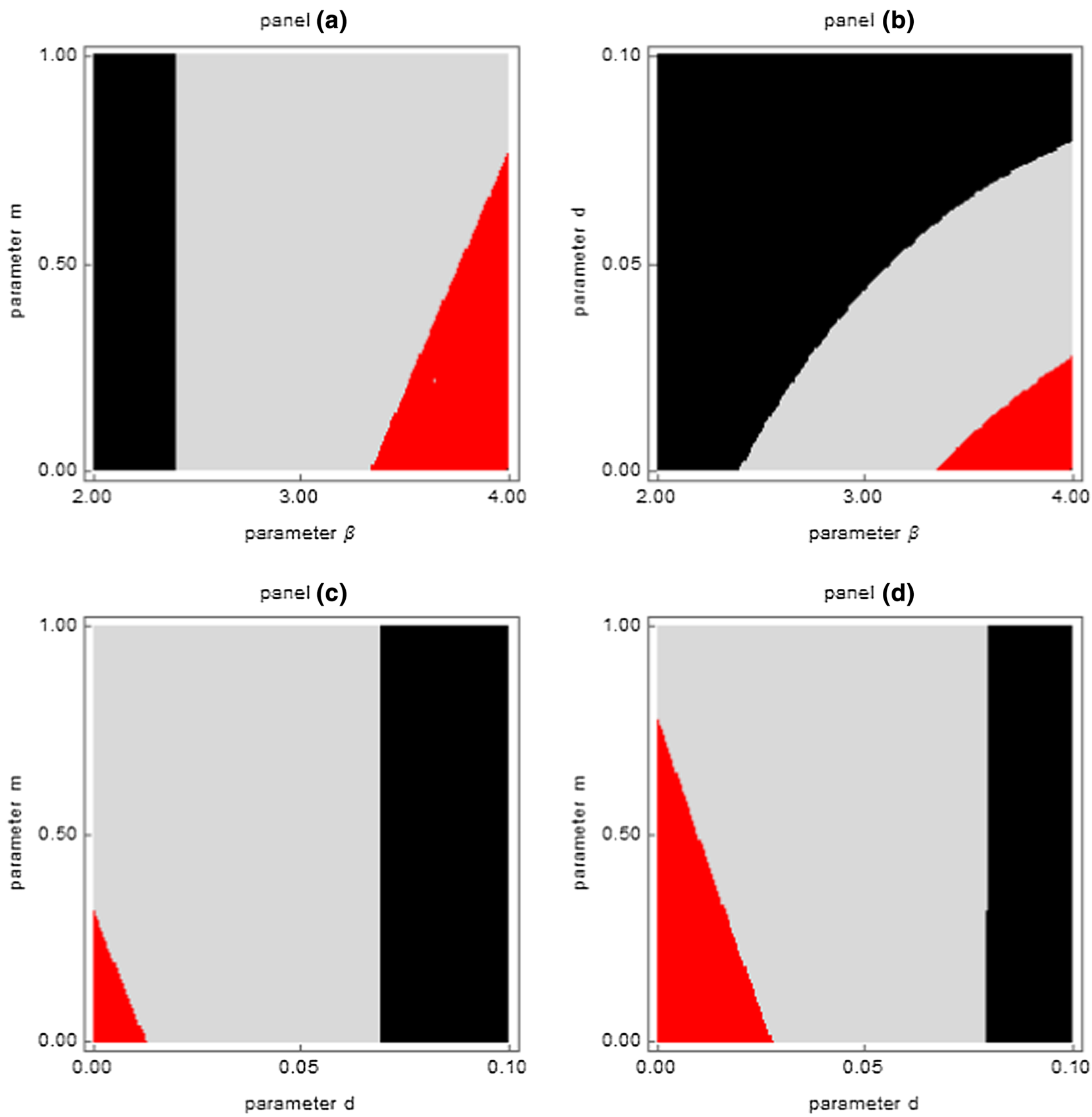


Fig. 11 Pitchfork and Neimark–Sacker bifurcations. Panels **a**–**c** show two-dimensional bifurcation diagrams for our base parameter setting. Parameter combinations depicted in black and gray yield a convergence toward the fundamental steady state or to one of the two nonfundamental steady states, respectively. Parameter combinations marked in red yield endogenous dynamics. Panel **d** is based on $\beta = 4$

same, except that the simulations are based on $d = 0$, $d = 0.012$ and $d = 0.024$, respectively. As can be seen, parameter d has a stabilizing effect on the dynamics. First, it delays the occurrence of the pitchfork bifurcation. Second, it pushes the nonfundamental steady states closer toward the fundamental steady states. Third, it also delays the Neimark–Sacker bifurcation. In contrast to the role played by parameter m, however, the region of oscillations is not characterized by a smaller scale. There are even parameter combinations that generate irregular switching between bull and bear market dynamics. In fact, interventions by the central authority may induce an overshooting of the fundamental value. Further simulations reveal that these chaotic dynamics are highly unpredictable.

Panels (a)–(c) of Fig. 11 show two-dimensional bifurcation diagrams. Parameter combinations depicted in black and gray yield a convergence toward the fundamental steady state or to one of the two nonfundamental steady states, respectively. Parameter combinations marked in red yield endogenous dynamics. Panel (a) of Fig. 11 reveals that an increase in parameter β first creates a pitchfork bifurcation and then a Neimark–Sacker bifurcation. While the pitchfork bifurcation value of parameter β is independent of parameter m, higher values of parameter m increase the Neimark–Sacker bifurcation value of parameter β. Panel (b) of Fig. 11 shows that the same is true for parameter d, except that an increase in parameter d also increases the pitchfork bifurcation value of parameter β as predicted by our analytical results. Panel (c) of Fig. 11 confirms that an increase in parameters m and/or d can prevent a Neimark–Sacker bifurcation. However, the emergence of a pitchfork bifurcation can only be prevented by higher values of parameter d. Panel (d) of Fig. 11 is based on $\beta = 4$ (instead of $\beta = 3.6$). A higher value of parameter β increases the parameter space that yields oscillations. An increase in parameters m and/or d can stabilize the dynamics, but only parameter d allows a pitchfork bifurcation to be countered.

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