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A Logical Modeling of Severe Ignorance

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Abstract
In the logical context, ignorance is traditionally defined recurring to epistemic logic. In particular, ignorance is essentially interpreted as "lack of knowledge". This received view has - as we point out - some problems, in particular we will highlight how it does not allow to express a type of content-theoretic ignorance, i.e. an ignorance of φ that stems from an unfamiliarity with its meaning. Contrarily to this trend, in this paper, we introduce and investigate a modal logic having a primitive epistemic operator **I**, modeling ignorance. Our modal logic is essentially constructed on the modal logics based on weak Kleene three-valued logic introduced by Segerberg (*Theoria, 33*(1):53–71, 1997). Such non-classical propositional basis allows to define a Kripke-style semantics with the following, very intuitive, interpretation: a formula φ is ignored by an agent if φ is neither true nor false in every world accessible to the agent. As a consequence of this choice, we obtain a type of content-theoretic notion of ignorance, which is essentially different from the traditional approach. We dub it *severe ignorance*. We axiomatize, prove completeness and decidability for the logic of reflexive (three-valued) Kripke frames, which we find the most suitable candidate for our novel proposal and, finally, compare our approach with the most traditional one.

Keywords Ignorance · Three-valued modal logic · Bochvar external logic · Weak Kleene logic

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1 Introduction

The study of ignorance is certainly as old as the study of knowledge; however the formal study of the logic of ignorance is still a young area of research. In the epistemological studies of ignorance the standard view is to define it as lack of knowledge (see for example the debate in [29–32]). In logic, it is not easy to reconstruct this tradition (see [14, 17] and [23]). However, in our view an important step in the history is Hintikka's seminal work [26], where he distinguishes two notions of lack of knowledge relative to an agent, namely "a (an agent) does not know that φ" ($\varphi \wedge \neg \mathbf{K}_a \varphi$) and "$a$ does not know whether φ" ($\neg \mathbf{K}_a \varphi \wedge \neg \mathbf{K}_a \neg \varphi$). Such regimentations have become standard in the logical literature on ignorance [14, 17, 23]. Throughout this article, we will refer to the standard view by the expression "ignorance as lack/absence of knowledge". In particular, we will use the expression "whether view" to address the second notion of lack of knowledge, i.e. "a does not know whether φ". From psychology to education studies, passing through philosophy and many other disciplines, a plenitude of deep analyses of knowledge and ignorance have been put forward [2, 17, 25, 33, 34] and the standard view in the literature describes ignorance in terms of lack, or absence, of knowledge. Therefore, it is not surprising that this is also the standard view in the logical treatment of ignorance. However, in more recent times, van der Hoek and Lomuscio [44] introduced a modal logic (**Ig**) where ignorance is modeled by a primitive modal operator, unrelated to (lack of) knowledge. The spirit behind **Ig** is expressing "ignorance as a first class citizen" [44, p.3]. However, despite their intention, their solution does not seem too far from "not knowing whether". Indeed, in their semantics for the operator **I** – for ignorance – an agent ignores φ if s/he has access to two (different) worlds, where φ is evaluated differently (true in one and false in the other). In their own words (again): "[the] formula $\mathbf{I}\varphi$ is to be read as «the agent is ignorant about φ, i.e. s/he is not aware of whether or not φ is true»". The semantics of **I** reflects that of absence of knowledge, with the only difference that **Ig** "can not speak" about knowledge.

Similarly, the *Logic of Unknown Truths* (LUT) and the subsequent logics of ignorance proposed by Steinsvold [40] subordinate the concept of ignorance to that of knowledge. In these logics the black box (■) in fact stands for $\varphi \wedge \neg \mathbf{K}\varphi$; if the latter formula is true, and $\varphi \to \neg \mathbf{K} \neg \varphi$ holds, then also $\neg \mathbf{K}\varphi \wedge \neg \mathbf{K} \neg \varphi$ holds, which is again the "whether view" of ignorance.

Following the research trend opened in Fano and Graziani [15] (see also [1]), this article intends to discuss the fact that *lack of knowledge* is just one way to look at ignorance and, taking up van der Hoek and Lomuscio's challenge, to introduce a logic which addresses the purpose of defining "ignorance as a first class citizen". In this paper, after discussing the consequences of defining ignorance as lack of knowledge (in the epistemic logic S_4), we introduce and investigate a modal logic having a primitive epistemic operator **I**, modeling ignorance. In particular, the idea we have in mind is that of modelling a type of *content-theoretic ignorance*, so to say an ignorance of something that stems from an unfamiliarity with its meaning, i.e. a *severe* notion of ignorance that implies a lack of awareness[1] with respect to a subject-matter. In our view, this type of ignorance constantly affects the practice of science. For instance, consider the following situation: Max Planck, in approaching the black body radiation problem, knew that, in the theoretical predictions of the black body, there was a divergence for high frequencies, in contrast with experimental data. However, he did not simply ignore which physical phenomena constituted the cause, but, more importantly, he did not have any idea (was ignorant) of what could be a bundle of causes. In logical terms, it is not *merely* the case that Planck does not know the truth value of a physical statement (that could be the cause), but he does not know which kind of event could be a cause. In other terms, when thinking about severe ignorance, we have in mind situations where scientists are ignorant of the bundle of causes that might be at the root of a phenomenon. Contrarily, the "whether view" of ignorance appears related to the lack of knowledge of single agents, such as, for instance, a physical statement that is, perhaps, known in the community of trained physicists but, possibly, ignored by a non-physicist or a first-year student. To achieve the goal of modeling severe ignorance, we base the semantics of our (modal) logic on the presence of a third truth-value, whose behaviour is infectious, as severe ignorance ultimately is. Returning to the example about Planck's ignorance, the infectivity of his ignorance depends on the fact that every scientific issue whose content is theoretically connected to the explanation of the black body is ignored severely at the same way that the explanation is. The most natural examples of infectious logics are the so-called weak Kleene logics, which can be intuitively introduced via a matrix where truth-values $\{0, 1\}$ are joined by a third truth-value $1/2$ whose behaviour is infectious in the sense that a complex formula φ is evaluated to the third value $1/2$ whenever any of its atomic formulas is evaluated to $1/2$ (independently of the structure of φ). Our modal logic will be essentially constructed following the ideas of the modal logics based on (one of the) weak Kleene logics introduced by Prior [36] and Segerberg [39]. Our philosophical approach keeps fixed the classical account that ignorance, as well as knowledge, is an *epistemic* notion and, for this reason, the logical modeling we primarily purse is an epistemic (modal) logic, whose privileged semantics is a relational one (Kripke-style). As a byproduct of our analysis, we discover that the non-classical propositional basis chosen (Bochvar external logic) indeed already incorporates (some) connectives that can be interpreted as modalities, to be used (also) for the formal representation of severe ignorance. Therefore, we will highlight the coincidence between the Kripke-style interpretation of the modality for ignorance and that of one connective in the enriched language of Bochvar logic.

[1] In this article, we will not address a comparison of our logical model of severe ignorance with the logical foundations of theories of unawareness. We will reserve this comparison for a separate article. We wish to thank Valentin Goranko for focusing our attention on this comparison.

The paper is organized into four parts: in Section 2, we introduce the standard (logical) approach to ignorance as “lack of knowledge”. In Section 3, we outline the key features Bochvar external logic which we will use in order to give a modal approach to severe ignorance. In Section 4, it is introduced the logic **SI** of severe ignorance; an axiomatization with relative completeness is proved in Section 4.2. We conclude the paper with Section 5 where we make some remarks on the validity of certain formulas relevant to capture a severe notion of ignorance, and compare the differences between the standard view and the proposed logic for severe ignorance.

2 Ignorance as Lack of Knowledge

As mentioned, the traditional logical approach to ignorance is based on the idea of defining ignorance as “lack of knowledge” (see [17] and [12]). This translates ignorance into a modal operator in the epistemic logic S_4 defined as follows:

$$\mathbf{I}\varphi := \neg\mathbf{K}\varphi \wedge \neg\mathbf{K}\neg\varphi, \tag{1}$$

where **K** stands for the knowledge operator. It follows from Eq. 1 that, in the standard Kripke-style semantics for S_4, the formula $\mathbf{I}\varphi$ is true in a world w (under a certain evaluation v, in symbols $v(w, \mathbf{I}\varphi) = 1$) if and only if there exist two worlds w', w'' related to w, such that φ is true (false, respectively) in w' (under v) and φ is false (true, resp.) in w'' (under v). In words, an agent ignores a formula φ, in a world w, if (and only if) s/he has access to two worlds each of which assigns a different truth value to φ. Roughly speaking, “do not knowing whether φ'' – ignoring φ according to the “whether view” – means seeing (at least two different) worlds where φ is assigned with different truth-values. We might say that this view models ignorance as a *truth-theoretic* notion: “ignoring whether φ” is translated as “being unsure” about the truth value of φ, due to the existing conflict of evaluation in the related worlds.

It is useful to underline that the semantics for the ignorance modality **I** is the same, as above, also in the logic **Ig**, introduced by Van der Hoek and Lomuscio (see [44, Definition 2.1]) with the aim of treating ignorance as a primitive notion, not subordinated to knowledge. Indeed, the main difference, with respect to the “whether view” approach, is that, in **Ig**, ignorance is not defined as “lack of knowledge” as **Ig** does not contain any primitive knowledge operator: it is modeled via the primitive operator **I**.

We wonder whether modeling ignorance as lack of knowledge (or via **I** in **Ig**) is the only way to logically address the notion of ignorance. Far from saying that it is not the correct way to analyze the concept, we simply claim that “lack of knowledge” is only *one way* to approach ignorance, whose features are exemplified by the logical laws in which **I** actually occurs. We recap the logical laws and the notable failures involving **I** in the following remarks.

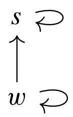
Remark 1 It is immediate to check that the following formulas are logical truths in S_4 – where $\mathbf{I}$ is defined according to Eq. 1:

(1) $\models_{S_4} \mathbf{I}\varphi \leftrightarrow \mathbf{I}\neg\varphi$;
(2) $\models_{S_4} \mathbf{I}(\varphi \vee \psi) \rightarrow \mathbf{I}\varphi \vee \mathbf{I}\psi$;
(3) $\models_{S_4} \mathbf{I}(\varphi \wedge \psi) \rightarrow \mathbf{I}\varphi \vee \mathbf{I}\psi$;
(4) $\models_{S_4} \mathbf{II}\varphi \rightarrow \mathbf{I}\varphi$.

Remark 2 The following formulas do not hold, in general, in S_4 – where $\mathbf{I}$ is defined according to Eq. 1:

a) $\not\models_{S_4} \mathbf{I}\varphi \wedge \mathbf{I}\psi \rightarrow \mathbf{I}(\varphi \wedge \psi)$;
b) $\not\models_{S_4} \mathbf{I}(\varphi \wedge \psi) \rightarrow \mathbf{I}\varphi \wedge \mathbf{I}\psi$;
c) $\not\models_{S_4} \mathbf{I}\varphi \vee \mathbf{I}\psi \rightarrow \mathbf{I}(\varphi \vee \psi)$;
d) $\not\models_{S_4} \mathbf{I}(\varphi \rightarrow \psi) \rightarrow (\mathbf{I}\varphi \rightarrow \mathbf{I}\psi)$;
e) $\not\models_{S_4} \mathbf{I}\varphi \rightarrow \mathbf{II}\varphi$;
f) $\not\models_{S_4} \mathbf{I}\varphi \rightarrow \mathbf{I}(\varphi \vee \neg\varphi)$;
g) $\not\models_{S_4} \mathbf{I}\varphi \rightarrow \mathbf{I}(\varphi \wedge \neg\varphi)$;
h) $\not\models_{S_4} \mathbf{I}\varphi \rightarrow \varphi$.

We just show a simple counterexample for a). Consider a Kripke model $\mathcal{M} = \langle W, R, v\rangle$ with $W = \{w, s\}$, $R = \{(w, w), (s, s), (w, s)\}$,



where evaluation v is defined as follows: $p \in w$, $p \notin s$ and $q \notin w$, $q \in s$. It then follows that $v(w, \mathbf{I}p) = 1$, $v(w, \mathbf{I}q) = 1$. On the other hand, $v(w, p \wedge q) = v(s, p \wedge q) = 0$, thus $v(w, \mathbf{I}(p \wedge q)) = 0$.

Intuitively, to falsify a) it is sufficient to consider a model with two different related worlds, each of which makes one formula true and the other false, respectively. In this way, each formula is ignored but the conjunction is not, since is false in every world.

We are convinced that the set of formulas listed in Remarks 1 and 2 – although they might not constitute an exhaustive list – tells something relevant about the notion of ignorance that the supporters of the "whether view" had in mind (more detailed comments on this can be found in Section 5). Let us analyze, through some examples, the applicability (as well as limits of applicability) of this interpretation of ignorance.

Suppose that Magnus and Jan[2] are about to play a single chess match. It is plausible to think that a rational agent ignores (does not know) whether Magnus is going

[2]Names refer to real professional players: the (current) world-number one Magnus Carlsen and our colleague (and Grandmaster) Jan Michael Sprenger.

to win (although it is very unlikely to happen, he might lose or the match could end in a draw); similarly, s/he ignores whether Jan is going to win. On the other hand, our rational agent *does not* ignore whether Magnus *and* Jan is going to win, as s/he knows that the same chess match can not have two different winners. This shows a case in which the ignorance of two conjuncts does not translate in the ignorance of their conjunction, as it happens to be the case in S_4 (see Remark 2-a).

Observe, however, that ignorance as lack of knowledge behaves according to the principle that ignoring a conjunction implies ignoring both the conjuncts and the disjuncts (Remark 1), which shades some confusion between "and" and "or" when referring to notions that are ignored.

Nevertheless, we believe that, sometimes, lack of knowledge is understood in a way which is not exemplified by the behaviour of **I** in S_4 (and **Ig**). We try to clarify what we mean, giving some examples relative to the behaviour of **I** with respect to the conjunction.

Suppose that one of the authors of this paper has just concluded to examine a student, who aimed to pass his/her exam in modal logic. During the exam, s/he was asked to answer some questions (obviously, in a finite number!), each of which with the precise goal to verify whether s/he *is ignorant* – hopefully, is not ignorant – of the main topics which, together, form the program of the entire course. Unfortunately, due to her deficient answers, the examiner has collected enough evidence to conclude that s/he is ignorant of all the main topics, say $\varphi_1, \ldots, \varphi_n$, characterizing the course. The rational examiner is so brought to conclude that the student is ignorant of the whole subject of the exam, which can be exemplified as the conjunction $\varphi_1 \wedge \cdots \wedge \varphi_n$, and thus can not pass the exam. In other words, s/he is ignorant of the program $\varphi_1 \wedge \cdots \wedge \varphi_n$ of the exam. It seems reasonable to think that the above exemplified notion of ignorance is indeed lack of knowledge (the examiner is ultimately testing if the student "knows $\varphi_1, \ldots, \varphi_n$") and it seems reasonable also to think that the ignorance of each of the statements $\varphi_1, \ldots, \varphi_n$ implies the ignorance of the conjunction $\varphi_1 \wedge \cdots \wedge \varphi_n$ (how could this not be the case?!). However, ignorance as lack of knowledge, modeled in S_4, and closure with respect to conjunction can not stand together.
Another weakness regarding the standard view (as discussed in recent literature, see [22, 28]) is that the so called *Factivity Principle* (usually intended relative to knowledge as $\mathbf{K}\phi \rightarrow \phi$), does not work in the standard view framework, i.e. it does not hold that if an agent ignores ϕ then ϕ is true. This fact is also highlighted in our Remark 2, where we prove that factivity of ignorance does not hold in S_4 (in contrast with the factivity of knowledge which clearly holds).

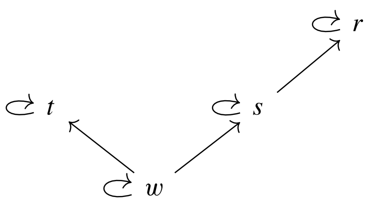
It is also possible to design other examples allowing us to stress that there are cases where ignorance is severe and does not coincide with lack of knowledge. Let us consider the discovery, happened at the beginning of November 2021, of the new Omicron variant of Coronavirus. The group of South-African scientists who isolated the variant communicated immediately their discovery, however it is reasonable to think that the sentence "Omicron is a variant of concern" was ignored by everyone at the time (and perhaps in the following days). This kind of ignorance is severe (in our sense), since it is natural that it spreads over sentences containing the previous. For instance, also any implication of the form "if the Omicron variant is of concern then there will be more deaths due to it" is genuinely to be ignored. This example seems to be convincing on the infectiousness of severe ignorance. More precisely, the lesson to learn from the above discussion is that the notion of ignorance is more subtle and problematic than it might appear at first look. Modeling it as "lack of knowledge" is surely a possibility, which has both qualities and flaws, depending on the context of applicability.

The aim of the present work is to propose a logical modeling of severe ignorance, a notion that differs from standard lack of knowledge ("whether view"). This change of perspective significantly impacts on the formulas holding/not holding in this new logic with respect to S_4 (see Section 5 for a comparison and further discussion). Indeed, when ignorance is conceived as severe, then the failure of certain formulas, such as (2) and (4) in Remark 1, comes with no surprise; similarly, it is not surprising that a formula like a) in Remark 2 holds in this new system. Intuitively, one could think that a way to address a severe ignorance is possible also in S_4, by recurring to the so-called "second-order ignorance" [17], rendered by applying $\mathbf{I}$ twice to a formula. However, applying $\mathbf{II}$ does not validate the fact that ignoring two formulas implies ignoring their conjunction, as witnessed by the following.

Remark 3 The following formulas are not logical truths of S_4:

(1) $\not\models_{S_4} \mathbf{II}\varphi \wedge \mathbf{II}\psi \rightarrow \mathbf{II}(\varphi \wedge \psi)$;
(2) $\not\models_{S_4} \mathbf{II}(\varphi \wedge \psi) \rightarrow \mathbf{II}\varphi \wedge \mathbf{II}\psi$.

The same counterexample introduced in Remark 2 serves also for (1). Indeed, observe that $v(w, \mathbf{I}p) = v(w, \mathbf{I}q) = 1$ and $v(s, \mathbf{I}p) = v(s, \mathbf{I}q) = 0$, thus $v(w, \mathbf{II}p) = v(w, \mathbf{II}q) = 1$, hence $v(w, \mathbf{II}p \wedge \mathbf{II}q) = 1$. However, since $v(w, \mathbf{I}(p \wedge q)) = v(s, \mathbf{I}(p \wedge q)) = 0$ and there exists no world $x \in W$ such that wRx and $v(x, \mathbf{I}(p \wedge q)) = 1$, then $v(w, \mathbf{II}(p \wedge q)) = 0$.
A simple counterexample to (2) is given by the following. Consider a Kripke model $\mathcal{M} = \langle W, R, v \rangle$ with $W = \{w, r, s, t\}$, $R = \{(w, w), (r, r), (s, s), (t, t), (w, s), (s, r), (w, r), (w, t)\}$



Evaluation is defined as follows: $p, q \in \{w, t, r\}$ and $q \in s$, $p \notin s$. It follows that $v(s, \mathbf{I}(p \wedge q)) = 1$ and $v(t, \mathbf{I}(p \wedge q)) = 0$, thus $v(w, \mathbf{II}(p \wedge q)) = 1$. However, $v(x, \mathbf{I}q) = 0$, for every $x \in W$, thus $v(w, \mathbf{II}q) = 0$.

The possibility of nesting the modality $\mathbf{I}$ (i.e. having formulas such as $\mathbf{II}\varphi$, $\mathbf{III}\varphi$, etc.), which is allowed in S_4, as we just saw, presents also a remarkable disadvantage. Although $\mathbf{I}\varphi \rightarrow \mathbf{II}\varphi$ is not a theorem of S_4, it is not difficult to check that the formula $\mathbf{II}\varphi \rightarrow \mathbf{III}\varphi$ is a theorem. More in general, abbreviating with $\mathbf{I}^n$ the n-times application of the modality $\mathbf{I}$, in S_4 the formula $\mathbf{I}^n\varphi \rightarrow \mathbf{I}^{n+1}\varphi$ holds, for $n \geq 2$. This quite problematic phenomenon is usually referred to as the *black hole* of ignorance (see [17]).

3 Bochvar External Logic B_e

Given a similarity type ν, the absolutely free algebra $\mathbf{Fm}$ of type ν over a countably infinite set X of generators will be called the *formula algebra* of type ν; its members will be called *formulas*. Members of X will be called (propositional) *variables* and referred to by the symbols $p, q, \ldots$. We denote algebras by $\mathbf{A}, \mathbf{B}, \mathbf{C} \ldots$ and the respective universes by $A, B, C \ldots$ We understand a *logic* (of type ν) as a pair $\mathrm{L} = \langle \mathbf{Fm}, \vdash_{\mathrm{L}} \rangle$, where $\mathbf{Fm}$ is the formula algebra (of type ν), and $\vdash_{\mathrm{L}}$ is a substitution-invariant consequence relation over $\mathbf{Fm}$ ($\vdash \subseteq \mathcal{P}(Fm) \times Fm$).

Kleene's three-valued logics – introduced by Kleene in his *Introduction to Metamathematics* [27] – are traditionally divided into two families, depending on the meaning given to the connectives: *strong Kleene* logics – counting strong Kleene and the logic of paradox [35] – and *weak Kleene* logics, namely Bochvar logic [4] and paraconsistent weak Kleene – PWK in brief (sometimes referred to as Hallden's logic [24]).

Kleene logics are traditionally defined in the algebraic language $\mathcal{L}_K : \neg, \vee$ of type $(1, 2)$; $\varphi \wedge \psi$, $\varphi \rightarrow \psi$ and $\varphi \leftrightarrow \psi$ are abbreviations for $\neg(\neg\varphi \vee \neg\psi)$, $\neg\varphi \vee \psi$ and $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$, respectively. The language $\mathcal{L}_K$ is usually referred to as the *internal language* (and $\neg, \vee$ as the *internal connectives*). Enlarging $\mathcal{L}_K$ with a new unary connective (and the constants 0,1), one obtains $\mathcal{L}_{K^e} : \neg, \wedge, J_2, 0, 1$ (of type $(1, 2, 1, 0, 0)$). Let us denote, with an abuse of notation (that hopefully does not create confusion) by $\mathbf{Fm}$ the formula algebra over the algebraic language $\mathcal{L}_{K^e}$.

Semantics of the language $\mathcal{L}_{K^e}$ is given by the three-elements algebra $\mathbf{WK}^e = \langle \{0, 1, 1/2\}, \neg, \wedge, J_2, 0, 1 \rangle$ displayed in Fig. 1 (semantics of $\vee$ and $\rightarrow$ is recalled in Fig. 2).

	$\neg$
1	0
1/2	1/2
0	1

$\vee$	0	1/2	1
0	0	1/2	1
1/2	1/2	1/2	1/2
1	1	1/2	1

	J_2
1	1
1/2	0
0	0

Fig. 1 The algebra $\mathbf{WK}^e$

$\wedge$	0	1/2	1
0	0	1/2	0
1/2	1/2	1/2	1/2
1	0	1/2	1

$\to$	0	1/2	1
0	1	1/2	1
1/2	1/2	1/2	1/2
1	0	1/2	1

Fig. 2 Semantics of $\vee$ and $\to$ in the algebra $\mathbf{WK}^e$

Despite recent attempts (see [3, 5, 13, 41]) to provide an epistemic reading to the truth-value 1/2, its most common interpretation is as "meaningless" (see e.g. [16] and [42]).

The language $\mathcal{L}_{K^e}$ is significantly richer than $\mathcal{L}_K$ and allows to define the so-called *external* formulas.[3] Intuitively, a formula α is *external* when it is evaluated to $\{0, 1\}$ (which is the universe of a Boolean subalgebra of $\mathbf{WK}^e$) from any homomorphism $h\colon \mathbf{Fm} \to \mathbf{WK}^e$. In other words, an external formula is one such that can not be evaluated to 1/2 (see [18, p. 208]).

Via J_2, it is possible to define more connectives (which will be very useful for our analysis): $J_3\varphi := \neg J_2\neg\varphi \to J_2\varphi$, $J_1\varphi := \neg J_3\varphi$ and $J_0\varphi := \neg(J_1\varphi \vee J_2\varphi)$ interpreted (in $\mathbf{WK}^e$) as follows.

φ	$J_3\varphi$	φ	$J_1\varphi$	φ	$J_0\varphi$
1	1	1	0	1	0
1/2	0	1/2	1	1/2	0
0	1	0	0	0	1

Intuitively, connectives J_0, J_1, J_2, J_3 allow to speak not only about a statement φ but also about its truthfulness, falseness and more.

Bochvar (external) logic B_e is the logic induced by the matrix $\langle \mathbf{WK}^e, \{1\}\rangle$, i.e.

$$\Gamma \models_{\mathsf{B}_e} \varphi \text{ if and only if, for every homomorphism } h\colon \mathbf{Fm} \to \mathbf{WK}^e,$$

$$\text{if } h[\Gamma] \subseteq \{1\} \text{ then } h(\varphi) = 1.$$

In words, B_e is the logic preserving only the truth-value 1 ("true").[4]

Definition 4 A variable p is *open* in a formula φ when there is at least one occurrence of it which does not fall under the scope of J_i, with $i \in \{0, 1, 2, 3\}$. It is *covered* if it is not open, namely it occurs in φ and all occurrences fall under the scope of J_i, for some $i \in \{0, 1, 2, 3\}$.

The intuition behind external formulas is made precise by the following.

[3]The idea of considering the external language is originally due to Bochvar [4], who wanted to move non-classical to get rid of set-theoretic and semantic paradoxes (by interpreting them to 1/2) but without losing the expressiveness of classical logic. Unfortunately, it can be shown that paradoxes resurfaces (see [43]).

[4]The different choice (on the same formula algebra) of the truth set $\{1, 1/2\}$ defines the logic H_0 studied by Segerberg [38].

Definition 5 A formula $\varphi \in Fm$ is called *external* if all the variables occurring in φ are covered.

Examples of external formulas are: $J_1 p \vee J_2 q$, $J_1(p \vee q)$, etc.

A Hilbert-style axiomatization of B_e has been introduced by Finn and Grigolia [18, p. 236]. In order to present it, let

$$\varphi \equiv \psi := \bigwedge_{i=0}^{2} J_i\varphi \leftrightarrow J_i\psi,$$

and α, β, γ denote external formulas.

Axioms

(A1) $(\varphi \vee \varphi) \equiv \varphi$;
(A2) $(\varphi \vee \psi) \equiv (\psi \vee \varphi)$;
(A3) $((\varphi \vee \psi) \vee \chi) \equiv (\varphi \vee (\psi \vee \chi))$;
(A4) $(\varphi \wedge (\psi \vee \chi) \equiv ((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$;
(A5) $\neg(\neg\varphi) \equiv \varphi$;
(A6) $\neg 1 \equiv 0$;
(A7) $\neg(\varphi \vee \psi) \equiv (\neg\varphi \wedge \neg\psi)$;
(A8) $0 \vee \varphi \equiv \varphi$;
(A9) $J_2\alpha \equiv \alpha$;
(A10) $J_0\alpha \equiv \neg\alpha$;
(A11) $J_1\alpha \equiv 0$;
(A12) $J_i\neg\varphi \equiv J_{2-i}\varphi$, for any $i \in \{0, 1, 2\}$;
(A13) $J_i\varphi \equiv \neg(J_j\varphi \vee J_k\varphi)$, with $i \neq j \neq k \neq i$;
(A14) $(J_i\varphi \vee \neg J_i\varphi) \equiv 1$, with $i \in \{0, 1, 2\}$;
(A15) $((J_i\varphi \vee J_k\psi) \wedge J_i\varphi) \equiv J_i\varphi$, with $i, k \in \{0, 1, 2\}$;
(A16) $(\varphi \vee J_i\varphi) \equiv \varphi$, with $i \in \{1, 2\}$;
(A17) $J_0(\varphi \vee \psi) \equiv J_0\varphi \wedge J_0\psi$;
(A18) $J_2(\varphi \vee \psi) \equiv (J_2\varphi \wedge J_2\psi) \vee (J_2\varphi \wedge J_2\neg\psi) \vee (J_2\neg\varphi \wedge J_2\psi)$;
(A19) $\alpha \to (\beta \to \alpha)$;
(A20) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$;
(A21) $\alpha \wedge \beta \to \alpha$;
(A22) $\alpha \wedge \beta \to \beta$;
(A23) $(\alpha \to \beta) \to ((\alpha \to \gamma) \to (\alpha \to \beta \wedge \gamma))$;
(A24) $\alpha \to \alpha \vee \beta$;
(A25) $\beta \to \alpha \vee \beta$;
(A26) $(\alpha \to \gamma) \to ((\beta \to \gamma) \to (\alpha \vee \beta \to \gamma))$;
(A27) $(\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha)$;
(A28) $\alpha \to (\neg\alpha \to \beta)$;
(A29) $\neg\neg\alpha \to \alpha$.

Deductive rule

$$[\mathrm{MP}]\ \frac{\varphi \quad \varphi \to \psi}{\psi}$$

Observe that the axiomatization contains a set of axioms (A19-A29), which, together with the rule of *modus ponens*, yields a complete axiomatization for classical logic (relative to external formulas). Upon defining the notion of derivation ($\vdash_{\mathsf{B}_e} \varphi$) in the usual way, Finn and Grigolia proved weak completeness for B_e.

Theorem 6 [18, Theorem 3.4] $\vdash_{\mathsf{B}_e} \varphi$ *if and only if* $\models_{\mathsf{B}_e} \varphi$.

It is natural to wonder whether B_e can be provided with a more synthetic Hilbert-style axiomatization and/or with a different style axiomatization (natural deduction, Gentzen-style, etc.). Actually a stronger version of Theorem 6 can be proved (the details of the proof are displayed in the Appendix, where we also show that B_e is algebraizable).

Theorem 7 $\Gamma \vdash_{\mathsf{B}_e} \varphi$ *if and only if* $\Gamma \models_{\mathsf{B}_e} \varphi$.

Theorem 8 (Deduction Theorem) $\Gamma \vdash_{\mathsf{B}_e} \varphi$ *if and only if there exist formulas* $\gamma_1, \dots, \gamma_n \in \Gamma$ *such that* $\vdash_{\mathsf{B}_e} J_2\gamma_1 \wedge \cdots \wedge J_2\gamma_n \to J_2\varphi$.

Proof ($\Rightarrow$) By induction on the length of the derivation of φ (from Γ).

($\Leftarrow$) We reason by contraposition and suppose that $\Gamma \nvdash_{\mathsf{B}_e} \varphi$, thus $\Gamma \not\models_{\mathsf{B}_e} \varphi$ (by Theorem 7), i.e. there is a homomorphism $h\colon \mathbf{Fm} \to \mathbf{WK}^e$ such that $h(\gamma) = 1$, for every $\gamma \in \Gamma$ and $h(\varphi) \neq 1$. Then, for every subset of formulas $\{\gamma_1, \dots, \gamma_n\} \subseteq \Gamma$, $h(J_2\gamma_i) = 1$, for every $i \in \{1, \dots, n\}$ and $h(J_2\varphi) = 0$, hence $h(J_2\gamma_1 \wedge \cdots \wedge J_2\gamma_n \to J_2\varphi) = 0$, i.e. $\not\models_{\mathsf{B}_e} J_2\gamma_1 \wedge \cdots \wedge J_2\gamma_n \to J_2\varphi$, thus $\nvdash_{\mathsf{B}_e} J_2\gamma_1 \wedge \cdots \wedge J_2\gamma_n \to J_2\varphi$ (by Theorem 6). □

4 The Logic SI of Severe Ignorance

The logic of (severe) ignorance we are going to introduce consists of a modal logic, whose propositional basis is B_e.

Let $\mathbf{Fm_I}$ be the formula algebra constructed over a numerable infinite set of propositional variables Var in the language $\mathcal{L}_\mathbf{I} : \neg, \vee, J_2, \mathbf{I}, 0, 1, 1/2$ of type $\langle 1, 2, 1, 1, 0, 0, 0\rangle$.

In the expanded language $\mathcal{L}_\mathbf{I}$, we generalize the definition of covered variable (see Definition 4) as follows: a variable p is *covered* in a formula φ if it occurs in φ and all occurrences fall under the scope of J_i, for some $i \in \{0, 1, 2, 3\}$, or under the scope of $\mathbf{I}$. In other words, we are stipulating that formulas like $\mathbf{I}\varphi$, $\mathbf{I}\psi$, ... are external (for any $\varphi, \psi \in Fm_\mathbf{I}$), in the sense of Definition 5.

We introduce the logic $\vdash_{\mathsf{SI}}$ as the one induced by the Hilbert-style axiomatization given by the following.

Axioms

- the axioms of B_e;

(1/2) $J_1 1/2$;

($\mathbf{I}J_1$) $\mathbf{I}\varphi \to J_1\varphi$.

Deduction rules

- The rule [MP].
- The rule [I]: $J_1\varphi \vdash_{\mathbf{SI}} \mathbf{I}\varphi$.

The intuition behind the axiomatization is that ignoring φ implies that $J_1\varphi$ is true, i.e. φ takes the third value. Moreover, the rule [I] states that the formula $\mathbf{I}\varphi$ is derived from $J_1\varphi$: intuitively, the ignorance of φ can be inferred from the assumption that $J_1\varphi$ is the case (i.e. semantically, J_1 takes the third value). Observe that this is different with respect to the rule of necessitation for standard modal logic (where $\Box\varphi$ can be inferred from any theorem φ).

By $\vdash_{\mathbf{SI}}$ we intend the derivability relation of the deductive system defined by the above axioms and inference rules. We now introduce the intended Kripke-style semantics for the logic $\vdash_{\mathbf{SI}}$.

4.1 Semantics

The semantics of the logic of ignorance consists of a relational (Kripke-style) structure where formulas, in each world, are evaluated into $\mathbf{WK}^e$. We introduce these structures according to the current terminology adopted in many-valued modal logics (see, for instance, [11, 19, 20]).

Definition 9 A *weak three-valued Kripke model* $\mathcal{M}$ is a structure $\langle W, R, v\rangle$ such that:

(1) W is a non-empty set (of possible worlds);
(2) R is a binary relation over W ($R \subseteq W \times W$);
(3) v is a map, called *valuation*, assigning to each world and each variable, an element in $\mathbf{WK}^e$ ($v\colon W \times \mathbf{Fm_I} \to \mathbf{WK}^e$).

Non-modal formulas will be interpreted as in B_e, i.e. we assume that v is a homomorphism, in its second component, with respect to $\neg, \vee, J_2, 1, 0, 1/2$. The reduct $\mathcal{F} = \langle W, R\rangle$ of a model $\mathcal{M}$ is called *frame*.
Notation: for ordered pairs of related elements, we equivalently write $(w, s) \in R$ or wRs.

The semantical interpretation of the epistemic modality **I** in a weak three-valued Kripke model is given in the following.

Definition 10 Let $\langle W, R, v\rangle$ be a weak three-valued Kripke model, and $w \in W$. Then

(1) $v(w, \mathbf{I}\varphi) = 1$ if $v(s, \varphi) = 1/2$ for every s $\in W$ such that wRs.
(2) $v(w, \mathbf{I}\varphi) = 0$ otherwise, i.e. there exists $s \in W$ such that wRs and $v(s, \varphi) \neq 1/2$.

The interpretation of the operator **I** is defined according to our intuition behind the notion of severe ignorance: a formula φ is ignored (in a world) when it is indeterminate (i.e. evaluated to $1/2$) in all the related worlds.

Observe that, in accordance with the syntactic stipulations, we are establishing that is an external formula (in the sense that it can assume classical truth-value only). In other words, the recurse to the third truth-value is used only at the propositional level and does not affect modal formulas. Moreover, there is no special assumption behind the accessibility relation R in weak three-valued Kripke structures: it is simply interpreted as an *epistemic accessibility relation*. Accordingly, the rationale behind the interpretation of $\mathbf{I}$ is that a formula is being ignored in case it is neither true nor false – it is indeterminate – in every world s an agent has epistemic access to from w. Recall that the notion of ignorance we aim at modeling with this semantics is severe. To further clarify our goal imagine, for instance, the following situation. Charles Darwin was aware, in 1859, of the existence of a form of hereditariness; however he did not know exactly the functioning mechanism of such process. Moreover, in every scenario accessible to his mind in that period, the cause of hereditariness was not determined. So, if we think a formula φ exemplifying the mechanism of hereditariness, then, in 1859, it held that Darwin was (severely) ignorant of φ, because φ was indeterminated in every possibile scenario. Other mechanisms, although not entirely known, were not (severely) ignored by Darwin himself at that time. For instance, we can not say that he was ignorant of the so called "missing links". Although he could not find them, he had an idea of how to search them, thanks to the analysis of fossils.

Accordingly, it is false that a formula φ is being ignored (in a world w) when there is a (related) world where φ is either true or false.

We say that a formula φ is *valid* in a model $\mathcal{M} = \langle W, R, v\rangle$ – we will write $\mathcal{M} \models \varphi$ – if, for every $w \in W$, $v(w, \varphi) = 1$.

A comment on this choice is in order. The introduced semantics of $\mathbf{I}$ relies on the presence of the third truth-value $1/2$ to be read as "indeterminate". In particular, severe ignorance, thought as a *content-theoretic* notion (in contrast with the truth-theoretic notion modeled by the standard view in S_4), is rendered thanks to the infectious behaviour of $1/2$. For this reason, it is natural to take 1/2 as not designated, since the evaluation of a formula to $1/2$ (in every related world) is a good reason for its ignorance.

We say that a formula φ is *valid* in a frame $\mathcal{F} = \langle W, R\rangle$ (and write $\mathcal{F} \models \varphi$) if it is valid in every model having $\mathcal{F}$ as frame. A frame (accordingly, a model) will be called *reflexive* if its accessibility relation is reflexive. From now on, we will write Kripke model instead of weak three-valued Kripke model. We define $\models_{\mathbf{SI}}$ as the *global* modal logic on the class of all *reflexive* Kripke frames (see e.g. [11]), i.e.

- $\Gamma \models_{\mathbf{SI}} \varphi$ iff, for every reflexive Kripke model $\mathcal{M}$,

  if $\mathcal{M} \models \gamma$, for every $\gamma \in \Gamma$, then $\mathcal{M} \models \varphi$.

Remark 11 Notice that, given a model $\mathcal{M} = \langle W, R, v\rangle$ and a world $w \in W$, the truth of the formula $\mathbf{I}\varphi$ at the world w ($v(w, \mathbf{I}\varphi) = 1$) is equivalent to the fact that, for every world s related to w, there exists a propositional variable p occurring open in φ such that $v(s, p) = 1/2$.

The above consideration is due to the peculiar behavior of the truth-value $1/2$ in weak Kleene and gives already a gist of the severity of ignorance obtained via the introduced semantics of $\mathbf{I}$. Indeed a (complex) formula φ is being ignored when a part of it (occurring open) is actually being ignored (as evaluated to $1/2$ in every related world), independently of the logical form of φ (exceptions hold for external formulas).

The choice of defining the logic $\models_{\mathrm{SI}}$ as that of all reflexive frames is mainly motivated by the fact that accessibility is interpreted in epistemic sense, thus is natural to think that every world is (epistemically) accessible to itself.

The following provides the behaviour of the epistemic modality $\mathbf{I}$ in $\models_{\mathrm{SI}}$.

Proposition 12 *The following formulas are valid in* $\models_{\mathrm{SI}}$:

(1) $\models_{\mathrm{SI}} \mathbf{I}\varphi \leftrightarrow \mathbf{I}\neg\varphi$;
(2) $\models_{\mathrm{SI}} \mathbf{I}\varphi \wedge \mathbf{I}\psi \rightarrow \mathbf{I}(\varphi \wedge \psi)$;
(3) $\models_{\mathrm{SI}} \mathbf{I}(\varphi \wedge \psi) \rightarrow \mathbf{I}\varphi \vee \mathbf{I}\psi$;
(4) $\models_{\mathrm{SI}} \mathbf{I}\varphi \vee \mathbf{I}\psi \leftrightarrow \mathbf{I}(\varphi \vee \psi)$;
(5) $\models_{\mathrm{SI}} \mathbf{I}\varphi \rightarrow \mathbf{I}(\varphi \vee \neg\varphi)$;
(6) $\models_{\mathrm{SI}} \mathbf{I}\varphi \rightarrow \mathbf{I}(\varphi \wedge \neg\varphi)$;
(7) $\models_{\mathrm{SI}} \mathbf{I}\varphi \rightarrow J_1\varphi$;
(8) $\models_{\mathrm{SI}} \mathbf{II}\varphi \rightarrow \mathbf{I}\varphi$;
(9) $\models_{\mathrm{SI}} \mathbf{II}\varphi \rightarrow \mathbf{I}^n\varphi$, *for* $n \geq 2$.

Proof We just show the validity of some of the listed formulas.

(1) We verify only one direction (as the other is analog). Let $\mathcal{M} = \langle W, R, v\rangle$ be a model such that $v(w, \mathbf{I}\varphi) = 1$, for some $w \in W$; then $v(s, \varphi) = 1/2$, for every $s \in W$ such that wRs, hence $v(s, \neg\varphi) = 1/2$, i.e. $v(s, \mathbf{I}\neg\varphi) = 1$.

(3) Let $\mathcal{M} = \langle W, R, v\rangle$ be a model such that $v(w, \mathbf{I}(\varphi \wedge \psi)) = 1$, for some $w \in W$. Then $v(s, \varphi \wedge \psi) = 1/2$, for every $s \in W$ such that wRs, which implies that $v(s, \varphi) = 1/2$ or $v(s, \psi) = 1/2$, whence $v(w, \mathbf{I}\varphi) = 1$ or $v(w, \mathbf{I}\psi) = 1$.

(7) Suppose that $\mathcal{M} = \langle W, R, v\rangle$ be a model such that $v(w, \mathbf{I}\varphi) = 1$ for some $w \in W$ and, in view of a contradiction, that $v(w, J_1\varphi) = 0$, i.e. $v(w, \varphi) \neq 1/2$. Then $v(s, \varphi) = 1/2$, for every $s \in W$ such that wRs and, since R is reflexive, $v(w, \varphi) = 1/2$, a contradiction. □

Observe that the validity $\mathbf{I}\varphi \rightarrow J_1\varphi$ is strictly related with the reflexivity of the models. It is immediate to check that the formula is not valid in non-reflexive models (think, for instance, to a model with only one world with the empty relation). Indeed the formula characterizes the class of reflexive frames.

Proposition 13 *Let* $\mathcal{F} = \langle W, R\rangle$ *be a frame. Then* $\mathcal{F} \models \mathbf{I}\varphi \rightarrow J_1\varphi$ *if and only if* R *is reflexive.*

Proof ($\Rightarrow$). Let $\mathcal{F} \models \mathbf{I}\varphi \rightarrow J_1\varphi$. Let $w \in W$ and consider the set $X = \{x \in W \mid (w, x) \in R\}$. We have to show that $w \in X$. Consider the valuation $v(s, p) = 1/2$ if and only if $s \in X$ (for every propositional variable p). Then, by Definition 10,

$v(w, \mathbf{I}p) = 1$. By assumption, the model $\mathcal{M} = \langle W, R, v\rangle$ validates $\mathbf{I}\varphi \to J_1\varphi$, thus $v(w, J_1 p) = 1$, whence $v(w, p) = 1/2$, therefore $w \in X$.
(2) $\Rightarrow$ (1) follows from Proposition 12. □

As noticed in [44], the essential feature of any notion of ignorance is captured by formulas (1) and (3) in Proposition 12. It is indeed very reasonable to think that an agent is ignorant about a formula if and only if is about its negation. Moreover, ignorance transfers from a conjunction to at least one constitutive part of it. Severe ignorance meets the minimal desiderata.

As a remarkable difference with S_4 (and **Ig**), in this new semantics for **I**, being ignorant of two (or more) formulas implies being ignorant also of their conjunction.

Not surprisingly, the converse (which holds in S_4) does not characterize severe ignorance (see Remark 14). It is indeed reasonable to think that being ignorant of a book (a conjunction of statements), does not mean to be ignorant of any single statements in the book, but, perhaps, some relevant parts of it. Moreover, (4) holds in virtue of the infectivity of the third truth-value.

We will discuss the significance of all the mentioned logical laws in Section 5.

Some notable failures are collected in the following.

Remark 14 The following formulas are not valid in $\models_{\mathsf{SI}}$:

(1) $\not\models_{\mathsf{SI}} \mathbf{I}(\varphi \wedge \psi) \to \mathbf{I}\varphi \wedge \mathbf{I}\psi$;
(2) $\not\models_{\mathsf{SI}} \mathbf{I}(\varphi \to \psi) \to (\mathbf{I}\varphi \to \mathbf{I}\psi)$;
(3) $\not\models_{\mathsf{SI}} (\mathbf{I}\varphi \to \mathbf{I}\psi) \to \mathbf{I}(\varphi \to \psi)$;
(4) $\not\models_{\mathsf{SI}} \mathbf{I}\varphi \to \mathbf{II}\varphi$;
(5) $\not\models_{\mathsf{SI}} \mathbf{I}\varphi \to \varphi$;
(6) $\not\models_{\mathsf{SI}} J_1\varphi \to \mathbf{I}\varphi$.

In Section 5, we will argue that it is not a problem for the severe notion of ignorance not to have distribution of **I** over conjunction and implication (formulas (2) and (3)). On the other hand, since severe ignorance is here conceived as a content-theoretic notion, it is obvious to expect the failure of the *factivity* (5).

4.2 Completeness and Decidability

When no danger of confusion is occurring we will drop the subscripts **SI** and B_e to $\vdash$.

The following result, whose form resembles a weakened version of the (classical) deduction theorem, can be proven for $\vdash_{\mathsf{SI}}$.

Lemma 15 (1) *Let $\Gamma \vdash_{\mathsf{SI}} \varphi$, with φ deductively equivalent to no formula $\mathbf{I}\psi$ (i.e. $\varphi \nvdash \mathbf{I}\psi$ or $\mathbf{I}\psi \nvdash \varphi$), for every $\psi \in Fm_{\mathbf{I}}$. Then there exist $\gamma_1, \ldots, \gamma_n \in \Gamma$ (or the formula 1 if $\Gamma = \emptyset$) such that $\vdash_{\mathsf{SI}} J_2\gamma_1 \wedge \cdots \wedge J_2\gamma_n \to J_2\varphi$ (or $\vdash_{\mathsf{SI}} 1 \to J_2\varphi$ if $\Gamma = \emptyset$);*
(2) *If $\vdash_{\mathsf{SI}} \gamma_1 \wedge \cdots \wedge \gamma_n \to \varphi$, for any $\gamma_1, \ldots, \gamma_n, \varphi \in Fm_{\mathbf{I}}$ then $\gamma_1, \ldots, \gamma_n \vdash_{\mathsf{SI}} \varphi$.*

Proof (1). By induction on the length n of the derivation of φ from Γ.

Basis. $n = 1$, i.e. φ is (the instance of) an axiom. Thus φ is an external formula, therefore $\vdash 1 \leftrightarrow \varphi \leftrightarrow J_2\varphi$, from which $\vdash 1 \rightarrow J_2\varphi$.
Inductive step. For $n > 1$, φ can be obtained only by applying [MP] as last rule (since by assumption $\varphi \neq \mathbf{I}\psi$, for every $\psi \in Fm_{\mathbf{I}}$), therefore φ is derived from two premisses of the form $\psi, \psi \rightarrow \varphi$, for some $\psi \in Fm_{\mathbf{I}}$. By inductive hypothesis, there exist formulas $\gamma_1, \dots, \gamma_n, \delta_1, \dots, \delta_m \in \Gamma$ such that $\vdash J_2\gamma_1 \wedge \dots \wedge J_2\gamma_n \rightarrow J_2\psi$ and $\vdash J_2\delta_1 \wedge \dots \wedge J_2\delta_m \rightarrow J_2(\psi \rightarrow \varphi)$, whence $\vdash J_2\gamma_1 \wedge \dots \wedge J_2\gamma_n \wedge J_2\delta_1 \wedge \dots \wedge J_2\delta_m \rightarrow J_2\psi \wedge J_2(\psi \rightarrow \varphi)$, thus $\vdash J_2\gamma_1 \wedge \dots \wedge J_2\gamma_n \wedge J_2\delta_1 \wedge \dots \wedge J_2\delta_m \rightarrow J_2\varphi$ (since $\vdash J_2\psi \wedge J_2(\psi \rightarrow \varphi) \rightarrow J_2\varphi$).
(2) Immediate. □

Our proof of completeness consists in an adaptation of the strategy, devised by Segerberg in [39], for modal logics based on the external version of Paraconsistent Weak Kleene (H_0).

Definition 16 A set of formulas $\Sigma \subset Fm_{\mathbf{I}}$ is *maximal* iff for all formulas $\varphi \in Fm_{\mathbf{I}}$, either $\varphi \in \Sigma$, or $\neg\varphi \in \Sigma$, or $J_1\varphi \in \Sigma$.

Recall that a set of formulas Σ is inconsistent in case $\Sigma \vdash \varphi$, for every $\varphi \in Fm_{\mathbf{I}}$. Σ is consistent if it is not inconsistent.

Remark 17 A useful operative notion of consistency (for sets of formulas) is given by the following: a set of formulas $\Sigma \subset Fm_{\mathbf{I}}$ is *consistent* iff there is no formula $\varphi \in Fm_{\mathbf{I}}$ such that $\Sigma \vdash J_i\varphi$ and $\Sigma \vdash \neg J_i\varphi$, for any $i \in \{0, 1, 2\}$ (it is immediate to check that this corresponds to the above notion of consistency).

We denote by $\mathcal{X}$ the set of all maximal and consistent sets of formulas, whose basic properties are recalled in the following.

Lemma 18 *For every $X \in \mathcal{X}$, the following hold*:

(1) *If $\varphi \rightarrow \psi \in X$ and $\varphi \in X$ then $\psi \in X$*;
(2) *$\varphi \wedge \psi \in X$ if and only if $\varphi, \psi \in X$*;
(3) *$\varphi \in X$ if and only if $J_2\varphi \in X$*;
(4) *$J_2\varphi \in X$ if and only if $\neg J_2\varphi \notin X$.*

Proof Immediate. □

Lemma 19 *Let Σ be a consistent set of formulas. $\Sigma \cup \{\varphi\}$ is inconsistent if and only if $\Sigma \vdash \neg\varphi$ or $\Sigma \vdash J_1\varphi$.*

Proof ($\Rightarrow$) Let $\Sigma \cup \{\varphi\}$ be inconsistent and that $\Sigma \nvdash \neg\varphi$. By assumption, $\Sigma \cup \{\varphi\} \vdash 0$. By Lemma 15-(1), there exist formulas $\gamma_1, \dots, \gamma_n \in \Sigma$ such that $\vdash J_2\gamma_1 \wedge \dots \wedge J_2\gamma_n \wedge J_2\varphi \rightarrow J_2 0$, hence $\vdash J_2\gamma_1 \wedge \dots \wedge J_2\gamma_n \rightarrow \neg J_2\varphi$, thus $\Sigma \vdash \neg J_2\varphi$. Since $\vdash_{\mathsf{B}_e} \neg J_2\varphi \leftrightarrow J_1\varphi \vee J_0\varphi$, $\Sigma \vdash J_1\varphi \vee J_0\varphi$. By assumption, $\Sigma \nvdash \neg\varphi$ which implies $\Sigma \nvdash J_0\varphi$, hence $\Sigma \vdash J_1\varphi$.
($\Leftarrow$) Let $\Sigma \vdash \neg\varphi$ or $\Sigma \vdash J_1\varphi$. Suppose $\Sigma \vdash \neg\varphi$ is the case, hence $\Sigma \vdash J_0\varphi$ (since

$\neg\varphi \vdash_{\mathsf{B}_e} J_0\varphi$). On the other hand, $\Sigma \cup \{\varphi\} \vdash J_2\varphi$, hence $\Sigma \cup \{\varphi\} \vdash J_0\varphi \wedge J_2\varphi$, and it is immediate to check that $\vdash J_0\varphi \wedge J_2\varphi \to 0$, thus it is inconsistent. The proof is analog in case $\Sigma \vdash J_1\varphi$. □

Remark 20 Observe that the content of Lemma 19 can not be simplified by deleting the second disjunct (as in [39, Lemma 4.9]) as, for instance, $\{J_1\varphi, \varphi\}$ is inconsistent but $J_1\varphi \nvdash \neg\varphi$.

Lemma 21 *Let Σ be a consistent set of formulas. The following are equivalent*:

(1) $\Sigma \vdash \varphi$;
(2) *for every $X \in \mathcal{X}$ such that $\Sigma \subseteq X$, $\neg\varphi \notin X$ and $J_1\varphi \notin X$.*

Proof (1) $\Rightarrow$ (2). Suppose $\Sigma \vdash \varphi$. Let $X \in \mathcal{X}$ such that $\Sigma \subseteq X$ and, by contradiction, that $\neg\varphi \in X$ or $J_1\varphi \in X$. Observe that $X \vdash \varphi$, so $X \vdash J_2\varphi$ (since $\varphi \vdash_{\mathsf{B}_e} J_2\varphi$). Suppose $\neg\varphi \in X$ is the case. Then $X \vdash \neg J_2\varphi$ (since $\neg\varphi \vdash_{\mathsf{B}_e} \neg J_2\varphi$), in contradiction with the fact that X is consistent (see Remark 17). Differently, $J_1\varphi \in X$ is the case. Thus $X \vdash J_1\varphi \wedge J_2\varphi$, again in contradiction with the consistency of X (since $\vdash J_2\varphi \wedge J_1\varphi \to 0$).
(2) $\Rightarrow$ (1). We reason by contraposition and suppose that $\Sigma \nvdash \varphi$. Consider an enumeration $\psi_1, \psi_2, \psi_3, \ldots$ of the formulas in $Fm_{\mathbf{I}}$. Define:

$$\Sigma_0 = \begin{cases} \Sigma \cup \{\neg\varphi\} \text{ if consistent,} \\ \Sigma \cup \{J_1\varphi\} \text{ otherwise.} \end{cases}$$

$$\Sigma_{i+1} = \begin{cases} \Sigma_i \cup \{\psi_i\} \text{ if consistent, else} \\ \Sigma_i \cup \{\neg\psi_i\} \text{ if consistent, else} \\ \Sigma_i \cup \{J_1\psi_i\}. \end{cases}$$

$$\Sigma^* = \bigcup_{i \in \mathbb{N}} \Sigma_i.$$

Observe that Σ^* is maximal, by construction. We want to show that Σ^* is also consistent. We first claim that Σ_0 is consistent. If $\Sigma_0 = \Sigma \cup \{\neg\varphi\}$ then it is consistent by construction. Differently, $\Sigma_0 = \Sigma \cup \{J_1\varphi\}$, which means that $\Sigma \cup \{\neg\varphi\}$ is inconsistent. Hence, by Lemma 19, $\Sigma \vdash \neg\neg\varphi$ or $\Sigma \vdash J_1\neg\varphi$. However, $\Sigma \nvdash \neg\neg\varphi$ (since, by assumption, $\Sigma \nvdash \varphi$), so $\Sigma \vdash J_1\neg\varphi$, which implies $\Sigma \vdash J_1\varphi$. By Lemma 19, $\Sigma_0 = \Sigma \cup \{J_1\varphi\}$ is consistent if and only if $\Sigma \nvdash \neg J_1\varphi$ and $\Sigma \nvdash J_1J_1\varphi$. Now, since Σ is consistent and $\Sigma \vdash J_1\varphi$, then $\Sigma \nvdash \neg J_1\varphi$. Moreover, since Σ is consistent $\Sigma \nvdash J_1J_1\varphi$ (as $\vdash J_1J_1\varphi \leftrightarrow 0$). This shows that Σ_0 is consistent.
We claim that Σ_{i+1} is consistent, given that Σ_i is. So, suppose that $\Sigma_i \cup \{\varphi\}$ and $\Sigma_i \cup \{\neg\varphi\}$ are inconsistent. Then, by Lemma 19, $\Sigma_i \vdash \neg\varphi$ or $\Sigma_i \vdash J_1\varphi$, and, $\Sigma_i \vdash \neg\neg\varphi$ or $\Sigma_i \vdash J_1\neg\varphi$. By consistency of Σ_i, the only possible case is that $\Sigma_i \vdash J_1\varphi$ and $\Sigma_i \vdash J_1\neg\varphi$, from which follows the consistency of $\Sigma \cup \{J_1\varphi\}$ (indeed, if it is not consistent then, by Lemma 19, $\Sigma_i \vdash \neg J_1\varphi$, in contradiction with the consistency of Σ_i). This shows that Σ^* is maximal and consistent ($\Sigma^* \in \mathcal{X}$) and, by construction, $\neg\varphi \in \Sigma^*$ or $J_1\varphi \in \Sigma^*$. □

We now define the accessibility relation between elements in $\mathcal{X}$.

Definition 22 Let $X, Y \in \mathcal{X}$. We define a relation $\mathcal{R}$ as follows:

$X\mathcal{R}Y$ if and only if for every formula $\varphi \in Fm_{\mathbf{I}}$, if $\mathbf{I}\varphi \in X$ then $J_1\varphi \in Y$.

Lemma 23 *Let $X \in \mathcal{X}$. Then, for every formula $\varphi \in Fm_{\mathbf{I}}$, $\mathbf{I}\varphi \in X$ if and only if $J_1\varphi \in Y$, for every $Y \in \mathcal{X}$ such that $X\mathcal{R}Y$.*

Proof The left to right direction is obvious. For the other, assume that $J_1\varphi \in Y$, for every $Y \in \mathcal{X}$ such that $X\mathcal{R}Y$. Consider the set $\Sigma = \{J_1\psi \mid \mathbf{I}\psi \in X\}$. Observe that $\Sigma \neq \emptyset$, since $\mathbf{I}1/2 \in X$; we claim that $\Sigma \vdash J_1\varphi$. To this end, let $\Sigma \subseteq Z$, for some $Z \in \mathcal{X}$ and observe that this implies that $J_1\varphi \in Z$. Indeed, for $\mathbf{I}\psi \in X$ then $J_1\psi \in \Sigma$, thus $J_1\psi \in Z$; so $X\mathcal{R}Z$ (by Definition 22), whence, by hypothesis, $J_1\varphi \in Z$. Now, since Z is consistent, $\neg J_1\varphi \notin Z$ and $J_1 J_1\varphi \notin Z$ then, by Lemma 21 lemma: Lemma 6 (4.10 di Segerberg), we have $\Sigma \vdash J_1\varphi$. By Lemma 15-(1), there exist formulas $J_1\psi_1, \ldots, J_1\psi_n \in \Sigma$ such that $\vdash J_2 J_1\psi_1 \wedge \cdots \wedge J_2 J_1\psi_n \rightarrow J_2 J_1\varphi$, i.e. $\vdash J_1\psi_1 \wedge \cdots \wedge J_1\psi_n \rightarrow J_1\varphi$ (as $\vdash J_2\alpha \leftrightarrow \alpha$, for every external formula α). By axiom ($\mathbf{I}J_1$), we have $\vdash \mathbf{I}\psi_1 \wedge \cdots \wedge \mathbf{I}\psi_n \rightarrow J_1\psi_1 \wedge \cdots \wedge J_1\psi_n$ hence $\vdash \mathbf{I}\psi_1 \wedge \cdots \wedge \mathbf{I}\psi_n \rightarrow J_1\varphi$, and by Lemma 15-(2), $\mathbf{I}\psi_1 \ \ldots, \mathbf{I}\psi_n \vdash J_1\varphi$, thus $X \vdash J_1\varphi$, and applying the rule [I][5], $X \vdash \mathbf{I}\varphi$. Since X is maximal and consistent, we conclude $\mathbf{I}\varphi \in X$. □

We are ready to define the concept of *canonical model* (keeping the usual nomenclature in modal logic).

Definition 24 A *canonical model* is a weak three-valued Kripke model $\mathcal{M} = \langle \mathcal{Y}, \mathcal{R}, v \rangle$ such that $\mathcal{Y} \subseteq \mathcal{X}$ ($\mathcal{Y} \neq \emptyset$), $\mathcal{R}$ is as in Definition 22, and v is defined as follows:

- $v(X, p) = 1$ if and only if $p \in X$;
- $v(X, p) = 0$ if and only if $\neg p \in X$;
- $v(X, p) = 1/2$ if and only if $J_1 p \in X$,

for every $X \in \mathcal{Y}$.

Remark 25 Observe that any canonical model $\mathcal{M} = \langle \mathcal{Y}, \mathcal{R}, v \rangle$ is reflexive. Indeed, let $X \in \mathcal{Y}$ such that $\mathbf{I}\varphi \in X$ (for some φ), then, by Axiom ($\mathbf{I}J_1$), $\mathbf{I}\varphi \rightarrow J_1\varphi \in X$, thus $J_1\varphi \in X$, which implies that $X\mathcal{R}X$.

The following result extends the construction of canonical models to all reflexive frames.

[5] We are strongly relying on the fact that the rule [I] is essentially different from the necessitation rule.

Lemma 26 *Let* $\mathcal{M} = \langle \mathcal{Y}, \mathcal{R}, v\rangle$ *be a canonical model. Then, for every formula* $\varphi \in Fm_{\mathbf{I}}$ *and every* $X \in \mathcal{Y}$*, the following hold*:

(1) $v(X, \varphi) = 1$ *if and only if* $\varphi \in X$;
(2) $v(X, \varphi) = 0$ *if and only if* $\neg\varphi \in X$;
(3) $v(X, \varphi) = 1/2$ *if and only if* $J_1\varphi \in X$.

Proof The claim is proved by induction on the complexity of φ. The basis follows from Definition 24. As for the inductive step, we show only the cases of $\varphi = J_2\psi$ and $\varphi = \mathbf{I}\psi$, for some $\psi \in Fm_{\mathbf{I}}$ (the others are routine). As regards the former, suppose that $\varphi = J_2\psi$, for some $\psi \in Fm_{\mathbf{I}}$. For any valuation v (and any $X \in \mathcal{Y}$), $v(X, J_2\psi) \neq 1/2$ (in accordance with the fact that $J_1 J_2\varphi \notin X$), thus we only have to consider two cases:
(a) $v(X, J_2\psi) = 1$ iff $v(X, \psi) = 1$, thus, by induction hypothesis, $\psi \in X$ and, by Lemma 18, $J_2\psi \in X$. (b) $v(X, J_2\psi) = 0$ iff either $v(X, \psi) = 0$ or $v(X, \psi) = 1/2$. Consider, first, the case $v(X, \psi) = 0$; by induction hypothesis, $\neg\psi \in X$ and, since X is consistent, $J_2\psi \notin X$, thus by Lemma 18, $\neg J_2\psi \in X$. In the second (sub)case, $v(X, \psi) = 1/2$, thus, by induction hypothesis, $J_1\psi \in X$. Since X is maximal (and consistent) then $\psi \notin X$, thus, by Lemma 18, $J_2\psi \notin X$, whence $\neg J_2\psi \in X$. Consider now the case of $\varphi = \mathbf{I}\psi$, for some $\psi \in Fm_{\mathbf{I}}$. We only have to consider the following two cases.
(i) $v(X, \mathbf{I}\psi) = 1$ if and only if $v(Y, \psi) = 1/2$, for every $Y \in \mathcal{Y}$ such that $X\mathcal{R}Y$. By induction hypothesis, $J_1\psi \in Y$ thus, by Lemma 23, $\mathbf{I}\psi \in X$.

(ii) $v(X, \mathbf{I}\psi) = 0$ if and only if $v(Y, \psi) \neq 1/2$, for some $Y \in \mathcal{Y}$ such that $X\mathcal{R}Y$. By induction hypothesis, $J_1\psi \notin Y$, hence, by Lemma 23, $\mathbf{I}\psi \notin X$, hence $\neg\mathbf{I}\psi \in X$ or $J_1\mathbf{I}\psi \in X$. But the latter is never the case, since X is consistent, whence $\neg\mathbf{I}\psi \in X$. □

We are now ready to prove (strong) completeness, i.e. that $\vdash_{\mathbf{SI}}$ and $\models_{\mathbf{SI}}$ are the same logic.

Theorem 27 (Completeness) $\vdash_{\mathbf{SI}} = \models_{\mathbf{SI}}$.

Proof ($\Rightarrow$) It is immediate to check that all axioms and rules are sound. We just exemplify the case of [I]. So, let $\mathcal{M} = \langle W, R, v\rangle$ be a (reflexive) model such that $\mathcal{M} \models J_1\varphi$, i.e. $v(w, \varphi) = 1/2$, for every $w \in W$; then $v(w, \mathbf{I}\varphi) \neq 0$, hence $\mathcal{M} \models \mathbf{I}\varphi$.

($\Leftarrow$) We reason by contraposition and suppose that $\Gamma \nvdash_{\mathbf{SI}} \varphi$; this implies that Γ is consistent. Let $\mathcal{X}^{\Gamma} \subseteq \mathcal{X}$ the set of maximal and consistent sets extending Γ ($\Gamma \subseteq Y$, for every $Y \in \mathcal{X}^{\Gamma}$); observe that, by Lemma 21, $\mathcal{X}^{\Gamma} \neq \emptyset$; in particular, there exists $X \in \mathcal{X}$ such that $\Gamma \subseteq X$ and $\neg\varphi \in X$ or $J_1\varphi \in X$. Consider now the canonical model $\mathcal{M} = \langle \mathcal{X}^{\Gamma}, \mathcal{R}, v\rangle$. By Lemma 26, $\mathcal{M} \models \gamma$, for every $\gamma \in \Gamma$. On the other hand, since $\neg\varphi \in X$ or $J_1\varphi \in X$, then $v(X, \varphi) \neq 1$, i.e. $\mathcal{M} \not\models \gamma$, hence $\Gamma \not\models_{\mathbf{SI}} \varphi$. □

From now on, we will write **SI** to indicate both $\vdash_{\mathbf{SI}}$ and $\models_{\mathbf{SI}}$ (since they are equal). The completeness strategy applied insofar allows to prove decidability for **SI** (Theorem 30).

Definition 28 Let $\varphi \in Fm_{\mathbf{I}}$. The set Sub($\varphi$) of subformulas of φ is the smallest set of formulas such that:

(1) $\varphi \in \text{Sub}(\varphi)$;
(2) if $\varphi = \neg\psi$, or $\varphi = J_2\psi$, or $\varphi = \mathbf{I}\psi$ (for some $\psi \in Fm_{\mathbf{I}}$) then $\psi \in \text{Sub}(\varphi)$;
(3) if $\varphi = \psi \vee \chi \in \text{Sub}(\varphi)$, then $\psi, \chi \in \text{Sub}(\varphi)$.

Let $\mathcal{M} = \langle W, R, v\rangle$ be a model. We say that a model has cardinality n (with $n \in \mathbb{N}$), if W has cardinality n ($\mid W \mid = n$).

Lemma 29 *Let φ a formula such that $\mid \text{Sub}(\varphi) \mid = n$. The following are equivalent:*

(1) $\vdash_{\mathbf{SI}} \varphi$;
(2) $\mathcal{M} \models_{\mathbf{SI}} \varphi$ *for all models with cardinality* 2^n.

Proof (i) $\Rightarrow$ (ii) is obvious.
(ii) $\Rightarrow$ (i). We reason by contraposition and suppose that $\nvdash_{\mathbf{SI}} \varphi$. Define the following relation on the set $\mathcal{X}$: $X \equiv Y$ if and only if, for all $\psi \in \text{Sub}(\varphi)$, $\psi \in X$ iff $\psi \in Y$. It is immediate to check that $\equiv$ is an equivalence relation on $\mathcal{X}$. Since $\varphi \in \text{Sub}(\varphi)$, then clearly $\mid \mathcal{X}_{/\equiv} \mid \leq 2^n$. Define the binary relation ρ on the set $\mathcal{X}_{/\equiv}$ (whose elements are denoted by $[X], [Y], [Z], \dots$) as follows:

$[X]\rho[Y]$ if and only if, for all ψ such that $\mathbf{I}\psi \in \text{Sub}(\varphi)$, if $\mathbf{I}\psi \in X$, then $J_1\psi \in Y$.

Consider the structure $\mathcal{N} = \langle \mathcal{X}_{/\equiv}, \rho, w\rangle$, where w is defined as follows:

- $w([X], p) = 1$ if and only if $p \in \text{Sub}(\varphi)$ and $p \in [X]$;
- $w([X], p) = 0$ if and only if $p \in \text{Sub}(\varphi)$ and $\neg p \in [X]$;
- $w([X], p) = 1/2$ if and only if $p \in \text{Sub}(\varphi)$ and $J_1 p \in [X]$,

for every $[X] \in \mathcal{X}_{/\equiv}$. It is immediate to check that $\mathcal{N}$ is a model of **SI**. Moreover, let $\mathcal{M} = \langle \mathcal{X}, \mathcal{R}, v\rangle$ be a canonical model: it is not difficult to prove that $v(X, \psi) = w([X], \psi)$, for every $X \in \mathcal{X}$ and for every formula $\psi \in \text{Sub}(\varphi)$ (the proof of this claim runs by induction on the length of the formula ψ). Since $\nvdash_{\mathbf{SI}} \varphi$, then there exists $X \in \mathcal{X}$ and some v (in the canonical model $\mathcal{M}$) such that $v(X, \varphi) \neq 1$ (this follows from the proof of Theorem 27). Then, by the previous claim, $w([X], \varphi) \neq 1$ and the cardinality of the model $\mathcal{N}$ is at most 2^n. □

As a direct consequence of Lemma 29 one gets:

Theorem 30 *The logic* **SI** *is decidable.*

5 Conclusion and Comparison with Other Approaches

We have introduced *severe* ignorance as a *content-theoretic* notion. In particular, we have focused on the logical modeling of such notion, assumed as primitive ("as a first class citizen"), i.e. disconnected from knowledge, via a modal logic based on a three-valued propositional logic. The intuition behind our proposal is that being ignorant of φ means that φ is indeterminate (is assigned with the third value 1/2) in all the

Fig. 3 A comparison of the logical truths in the logics **SI** and the standard view of **I** S_4 (see Definition Eq. 1)

	Formulae	SI	S_4
1	$\mathbf{I}\varphi \leftrightarrow \mathbf{I}\neg\varphi$	✓	✓
2	$\mathbf{I}\varphi \wedge \mathbf{I}\psi \rightarrow \mathbf{I}(\varphi \wedge \psi)$	✓	✗
3	$\mathbf{I}(\varphi \wedge \psi) \rightarrow \mathbf{I}\varphi \wedge \mathbf{I}\psi$	✗	✗
4	$\mathbf{I}\varphi \vee \mathbf{I}\psi \rightarrow \mathbf{I}(\varphi \vee \psi)$	✓	✗
5	$\mathbf{I}(\varphi \vee \psi) \rightarrow \mathbf{I}\varphi \vee \mathbf{I}\psi$	✓	✓
6	$\mathbf{I}(\varphi \rightarrow \psi) \rightarrow (\mathbf{I}\varphi \rightarrow \mathbf{I}\psi)$	✗	✗
7	$\mathbf{I}\varphi \rightarrow \mathbf{II}\varphi$	✗	✗
8	$\mathbf{II}\varphi \rightarrow \mathbf{I}\varphi$	✓	✓
9	$\mathbf{I}(\varphi \wedge \psi) \rightarrow \mathbf{I}\varphi \vee \mathbf{I}\psi$	✓	✓
10	$\mathbf{I}\varphi \rightarrow \mathbf{I}(\varphi \vee \neg\varphi)$	✓	✗
11	$\mathbf{I}\varphi \rightarrow \mathbf{I}(\varphi \wedge \neg\varphi)$	✓	✗
12	$\mathbf{I}\varphi \rightarrow J_1\varphi$	✓	n.d.[a]
13	$\mathbf{I}\varphi \rightarrow \varphi$	✗	✗
14	$\mathbf{II}\varphi \rightarrow \mathbf{I}^n\varphi\ (n \geq 2)$	✓	✓

[a] n.d. stands for not defined

worlds accessible to an agent. To the best of our knowledge, the unique existing system considering **I** as a primitive modality is the logic **Ig**, by Van der Hoek and Lomuscio [44]. However, as discussed in Section 2, in **Ig** the semantics of **I** coincides with the interpretation of ignorance as "lack of knowledge" in S_4, although no (term-definable) modality expressing knowledge can be defined in **Ig**. Being conscious of this relevant difference between **Ig** and the standard view in S_4, we will identify them with respect to the behaviour of the modality for ignorance **I** in the following discussion.

We make a comparison, in Fig. 3, between **SI** and S_4 (and thus also **Ig**) in terms of logical truths explicitly involving **I** (all the listed formulas have been mentioned in the previous sections). The aim is to show the existing difference between approaching ignorance as lack of knowledge (standard view in S_4) and the type of content-theoretic ignorance analyzed here, according to the three-valued modal logic **SI**.

As already discussed in Section 2, **SI** and S_4 present remarkable differences, with respect to the behaviour of the modality **I**. Regarding, for instance, conjunctive statements, in our proposal, an agent who is ignorant of all the chapters of a book then is ignorant of the whole book (formula 2), which does not happen to be the case in S_4. In the latter, perhaps, it does not make sense to express sentences like "ignoring a book". Indeed, one could say that "an agent does not know the content of a book", and not that "an agent does not know *whether* the content of a book". The converse implication (3) does not hold neither in S_4 nor in **SI**.

A remarkable difference distinguishes S_4 and **SI** relatively to the behaviour of **I** with respect to disjunctive statements, too. Severe ignorance is characterized by the principle that a disjunction is ignored if and only if one of the two disjuncts are ignored. This shall not appear strange in scientific contexts that inspire our notion of severe ignorance. Indeed, to make an example, Kepler, before investigating the astronomical data collected by Tycho Brahe, was ignorant of (as anyone else) the laws that today go under his name. After he discovers the first law, we might think that he still was ignorant of the others, and we might say that he also was ignorant of the disjunction (of the three laws), because such disjunction contains scientific terms (the second and third law) which Kepler could not imagine nor understand.

This does not happen to be the case in S_4, because lack of knowledge is different from severe ignorance. In a toy example: suppose that I do not know whether my aunt yesterday went to the cinema but I know that s/he went out for dinner. Thus, I do not know whether s/he went our for dinner (only) or also to the cinema (maybe before or after cinema), but surely I do not ignore that s/he went out to dinner or to the cinema.

The distribution of **I** over implication (6) fails in both **SI** and S_4. Remarkably, this gives the occasion, once more, to illustrate the sense of severe ignorance (in the scientific context). To exemplify the failure of (6), we might reasonably think that in 1914, Einstein was ignorant of the fact that the curvature of space-time is the cause of the anomaly affecting the perihelion shift of Mercury. At the time, the implication is scientifically ignored, however scientists were conscious of the anomaly in Mercury's perihelion. This is a good reason why **I** should not distribute over implication, in case it models severe ignorance.

Formulas 7 and 8 witness that the two logics have the same behaviour with respect to the relationship between first-order ($\mathbf{I}\varphi$) and second-order ignorance ($\mathbf{II}\varphi$). Not surprisingly, the latter implies the former but not the other way round.

Formula 9 is also in common between S_4 and **SI**. As we already commented in Section 4, it expresses the very intuitive principle that being ignorant of a conjunction implies being ignorant of at least one of the conjuncts, a principle that must be common (together with 1) to any possible notion of ignorance.

Formulas 10-12 witness the main difference due to the choice of different propositional basis. Indeed $\varphi \vee \neg\varphi$ ($\varphi \wedge \neg\varphi$, respectively) is true in every world, of every model of S_4 (false, respectively), hence can not be ignored. On the contrary, in a three-valued setting, those formulas can be indeterminate (when φ is indeterminate) and, consequently ignored. The validity of this formula tells us that the agent who is ignorant of φ is ignorant also that $\varphi \vee \neg\varphi$ coincides with the truth (something that is possible only in non-classical cases). This confirms that the notion of severe ignorance in **SI** stands quite far from lack of knowledge.

Formula 12 (which is not expressible in the language of S_4) states that ignoring a formula implies that the formula takes the value $1/2$ and this characterizes the class of reflexive models (see Proposition 13).

Formula 13 expresses the "factivity of ignorance" (the analog of the usual notion of factivity for the modality **K** for knowledge). The importance of this property for the notion of ignorance has been recently discussed in literature, where some authors look for logics of ignorance where it holds (see [28] and [22]). In the context of severe ignorance, as a content-theoretic notion, we are not surprised that the formula does not hold. However, we highlight that a modal approach, based on a three-valued logic, can be adopted also for logics of ignorance admitting the factivity, by choosing a different set of designated values ($\{1, 1/2\}$).

Finally, both logics suffer the phenomenon that Fine [17] calls the "black hole of ignorance". In his paper, Fine shows that second-order ignorance and higher-orders of ignorance are tightly tied together: once second-order ignorance is present, an agent is doomed to the black hole of higher-order levels of ignorance. This is captured by formula 14.

We are conscious that much logical and epistemological work remains to be done and that also the choice of **SI** to model severe ignorance presents some difficulties. For instance, it could be argued that it is quite odd that being ignorant of a formula φ implies being ignorant of also $\varphi \wedge \psi$, when φ and ψ are totally unrelated formulas (this happens to be the case in **SI**). Nevertheless, the present exploration highlights that interesting aspects of ignoring are not successfully captured by the standard logical approach to ignorance, based on lack of knowledge. Interestingly, disconnecting ignorance from knowledge allows for the logical modelling of severe ignorance, a notion which is common in the everyday practice of science. We have decided to introduce a modal logic grounded on a peculiar non-classical propositional basis. A choice essentially motivated by the willingness of modeling a severe notion of ignorance. Clearly, many other options are available, in the realm of non-classical logics: a possibility that we leave for further research.

Appendix

In this Appendix we provide the details of the proof of strong completeness for Bochvar (external) logic B_e (Theorem 7). Moreover, we also show that B_e is algebraizable. We start with a preliminary lemma.

Lemma 31 *The following elementary facts holds in* B_e:

(1) $\vdash_{\mathsf{B}_e} J_0(\varphi \vee \psi) \leftrightarrow J_0\varphi \wedge_0 \psi$;
(2) $\vdash_{\mathsf{B}_e} J_1(\varphi \vee \psi) \leftrightarrow J_1\varphi \vee J_1\psi$;
(3) $\vdash_{\mathsf{B}_e} J_2(\varphi \vee \psi) \leftrightarrow (J_2\varphi \wedge J_2\psi) \vee (J_2\varphi \wedge J_2\neg\psi) \vee (J_2\neg\varphi \wedge J_2\psi)$;
(4) *if* α *is an external formula, then* $\vdash_{\mathsf{B}_e} \alpha \leftrightarrow J_2\alpha$;
(5) $\vdash_{\mathsf{B}_e} \neg J_2\psi \leftrightarrow (J_0\psi \vee J_1\psi)$.

Proof Immediate by using Theorem 6. □

Lemma 32 *Let* $\Gamma \cup \{\varphi\} \vdash_{\mathsf{B}_e} \psi$. *Then* $\Gamma \vdash_{\mathsf{B}_e} J_2\varphi \rightarrow J_2\psi$.

Proof By induction on the length of the derivation of ψ (from Γ). □

The class of *Bochvar algebras* is introduced by Finn and Grigolia [18, pp. 233-234] as algebraic semantics for B_e.

Definition 33 A Bochvar algebra $\mathbf{A} = \langle A, \vee, \wedge, \neg, J_0, J_1, J_2, 0, 1\rangle$ is an algebra of type $\langle 2, 2, 1, 1, 1, 1, 0, 0\rangle$ satisfying the following identities and quasi-identities:

(1) $x \vee x \approx x$;
(2) $x \vee y \approx y \vee x$;
(3) $(x \vee y) \vee z \approx x \vee (y \vee z)$;
(4) $(x \wedge (y \vee z) \approx ((x \wedge y) \vee (x \wedge z))$;
(5) $\neg(\neg x) \approx x$;
(6) $\neg 1 \approx 0$;
(7) $\neg(x \vee y) \approx \neg x \wedge \neg y$;
(8) $0 \vee x \approx x$;
(9) $J_2 J_i x \approx J_i x$, for every $i \in \{0, 1, 2\}$;
(10) $J_0 J_i x \approx \neg J_i x$, for every $i \in \{0, 1, 2\}$;
(11) $J_1 J_i x \approx 0$, for every $i \in \{0, 1, 2\}$;
(12) $J_i(\neg x) \approx J_{2-i} x$, for every $i \in \{0, 1, 2\}$;
(13) $J_i x \approx \neg(J_j x \vee J_k x)$, for $i \neq j \neq k \neq i$;
(14) $J_i x \vee \neg J_i x \approx 1$, for every $i \in \{0, 1, 2\}$;
(15) $((J_i x \vee J_k x) \wedge J_i x) \approx J_i x$, for $i, k \in \{0, 1, 2\}$;
(16) $x \vee J_i x \approx x$, for $i \in \{1, 2\}$;
(17) $J_0(x \vee y) \approx J_0 x \wedge J_0 y$;
(18) $J_2(x \vee y) \approx (J_2 x \wedge J_2 y) \vee (J_2 x \wedge J_2 \neg y) \vee (J_2 \neg x \wedge J_2 y)$;
(19) $J_0 x \approx J_0 y \;\&\; J_1 x \approx J_1 y \;\&\; J_2 x \approx J_2 y \;\Rightarrow\; x \approx y$.

We denote by $\mathcal{BA}_3$ the class of Bochvar algebras. $\mathcal{BA}_3$ forms a quasi-variety which is not a variety [18]. Recall that a class $\mathcal{K}$ of algebras is an algebraic semantics for a logic L provided that: $\Gamma \vdash_L \varphi$ iff $\{\tau(\gamma) : \gamma \in \Gamma\} \models_{Eq(\mathcal{K})} \tau(\varphi)$, where $\tau = \{\varphi_i(p) \approx \psi_i(p)\}$ is a formula-equation transformer and $Eq(\mathcal{K})$ denotes the usual equational consequence relation relative to the class $\mathcal{K}$.

Theorem 34 *$\mathcal{BA}_3$ is an algebraic semantics for B_e. In particular, $\Gamma \vdash_{\mathsf{B}_e} \varphi$ iff $\{\gamma \approx 1 : \gamma \in \Gamma\} \models_{Eq(\mathcal{BA}_3)} \varphi \approx 1$.*

Proof ($\Rightarrow$) By induction on the length of the derivation of φ (from Γ), by checking that axioms (A1)-(A29) are evaluated to 1 in every Bochvar algebra $\mathbf{A}$ and that the rule (MP) preserves this property.

($\Leftarrow$) We reason by contraposition. Suppose $\Gamma \nvdash_{\mathsf{B}_e} \varphi$ and provide a counterexample to such inference by constructing the Lindenbaum-Tarski algebra. Let Δ be the smallest set of formulas including Γ and closed under $\vdash_{\mathsf{B}_e}$ (from now on we will simply write $\vdash$ instead of $\vdash_{\mathsf{B}_e}$). For any pair of formulas, define

$$\varphi \sim \psi \text{ if and only if } \varphi \equiv \psi \in \Delta.$$

We claim that:

(1) $\sim$ is a congruence on $\mathbf{Fm}$;
(2) $[1]_\sim = \Delta$;
(3) The quotient algebra $\mathbf{Fm}_{/\sim}$ is a Bochvar algebra.

(1) It is easy to check that $\sim$ is an equivalence relation. To show that it is a congruence, we check the compatibility of $\sim$ with the operations in the type of $\mathcal{L}_{K^e}$.

[¬] Suppose $\varphi \sim \psi$, then $\varphi \equiv \psi \in \Delta$, i.e. $\bigwedge_{i=0}^{2} J_i\varphi \leftrightarrow J_i\psi \in \Delta$, which is equivalent to $\bigwedge_{i=0}^{2} J_{2-i}\varphi \leftrightarrow J_{2-i}\psi \in \Delta$. Hence, in virtue of (A12), $\bigwedge_{i=0}^{2} J_i(\neg\varphi) \leftrightarrow J_i(\neg\psi) \in \Delta$, i.e. $\neg\varphi \equiv \neg\psi \in \Delta$, showing that $\neg\varphi \sim \neg\psi$.

[J_2] Suppose $\varphi \sim \psi$, thus $\varphi \equiv \psi \in \Delta$, i.e. $\bigwedge_{i=0}^{2} J_i\varphi \leftrightarrow J_i\psi \in \Delta$. In particular $\Delta \vdash J_2\varphi \leftrightarrow J_2\psi$. In virtue of (A9), we have $\vdash J_2\varphi \leftrightarrow J_2J_2\varphi$ and $\vdash J_2\psi \leftrightarrow J_2J_2\psi$, from which $\Delta \vdash J_2J_2\varphi \leftrightarrow J_2J_2\psi$, i.e. $J_2J_2\varphi \leftrightarrow J_2J_2\psi \in \Delta$ (as Δ is closed under consequences of $\vdash$). Analog reasoning, using (A10) and (A11), shows that $J_0J_2\varphi \leftrightarrow J_0J_2\psi \in \Delta$ and $J_1J_2\varphi \leftrightarrow J_1J_2\psi \in \Delta$, from which $J_2\varphi \sim J_2\psi$.

[∨] Suppose $\varphi_1 \sim \psi_1$ and $\varphi_2 \sim \psi_2$. Then $\Delta \vdash J_0\varphi_1 \leftrightarrow J_0\psi_1$ and $\Delta \vdash J_0\varphi_2 \leftrightarrow J_0\psi_2$, hence $\Delta \vdash (J_0\varphi_1 \wedge J_0\varphi_2) \leftrightarrow (J_0\psi_1 \wedge J_0\psi_2)$. Applying Lemma 31-(1), we have $\Delta \vdash J_0(\varphi_1 \vee \varphi_2) \leftrightarrow J_0(\psi_1 \vee \psi_2)$, from which $J_0(\varphi_1 \vee \varphi_2) \leftrightarrow J_0(\psi_1 \vee \psi_2) \in \Delta$. Analog reasoning (using Lemma 31-(2,3)) allows to conclude $\varphi_1 \vee \varphi_2 \sim \psi_1 \vee \psi_2$.

(2) [⊆] Let $\psi \in [1]_\sim$. Then $\psi \equiv 1 \in \Delta$, i.e. $\bigwedge_{i=0}^{2} J_i\psi \leftrightarrow J_i 1 \in \Delta$. In particular, $J_2\psi \in \Delta$ (since $\vdash J_2 1$) and $J_1\psi \leftrightarrow 0 \in \Delta$ (as $\vdash J_1 1 \leftrightarrow 0$), from which we deduce that ψ is an external formula, so, by Lemma 31-(4), $\psi \in \Delta$.

[⊇] Let $\psi \in \Delta$. Observe that $1 \in \Delta$ (since $\vdash 1$), hence, by Lemma 32, $\Delta \vdash J_2\psi \leftrightarrow J_2 1$, thus $J_2\psi \leftrightarrow J_2 1 \in \Delta$. Moreover, $\Delta \vdash \neg J_2\psi \leftrightarrow 0$ and, by Lemma 31, $\vdash \neg J_2\psi \leftrightarrow (J_0\psi \vee J_1\psi)$, so $\Delta \vdash (J_0\psi \vee J_1\psi) \leftrightarrow 0$, from which $\Delta \vdash J_0\psi \leftrightarrow 0$ and $\Delta \vdash J_1\psi \leftrightarrow 0$; therefore $\Delta \vdash J_0\psi \leftrightarrow J_0 1$ and $\Delta \vdash J_1\psi \leftrightarrow J_1 1$ (as $\vdash J_0 1 \leftrightarrow 0$ and $\vdash J_1 1 \leftrightarrow 0$). This shows that $\psi \equiv 1 \in \Delta$, i.e. $\psi \in [1]_\sim$.

(3) It is routine to check that $\mathbf{Fm}_{/\sim}$ is indeed a Bochvar algebra.

To provide a counterexample to the inference $\Gamma \nvdash_{\mathsf{B}_e} \varphi$, consider the Bochvar algebra $\mathbf{A} = \mathbf{Fm}_{/\sim}$ and the homomorphism $h\colon \mathbf{Fm} \to \mathbf{A}$, $h(\varphi) = [\varphi]_\sim$. Since $\Gamma \subseteq \Delta$ and $\Delta = [1]_\sim$, then $h(\gamma) = 1^{\mathbf{A}}$, for each $\gamma \in \Gamma$, but $h(\varphi) \neq 1^{\mathbf{A}}$ (since $\varphi \notin \Delta$).

□

Theorem 7 follows from Theorem 34 by observing that $\mathcal{BA}_3$ is the quasi-variety generated by $\mathbf{WK}^e$ ([18, Theorem 3.3]).

It is natural to wonder whether the quasi-variety of Bochvar algebras is simply *an* algebraic semantics for B_e. Actually the relationship between B_e and the class $\mathcal{BA}_3$ is tighter. Recall that a logic $\vdash$ is algebraizable with $\mathcal{K}$ as equivalent algebraic semantics (where $\mathcal{K}$ is a class of algebras of the same language as the logic $\vdash$) if there exists a map τ from formulas to sets of equations, and a map ρ from equations to sets of formulas such that the following conditions hold, for any pair of formulas φ, ψ and set of equations E.

(ALG1) $\Gamma \vdash \varphi$ iff $\tau[\Gamma] \vDash_{Eq(\mathcal{K})} \tau(\varphi)$;
(ALG2) $E \vDash_{Eq(\mathcal{K})} \varphi \approx \psi$ iff $\rho(E) \vDash_{Eq(\mathcal{K})} \rho(\varphi, \psi)$;
(ALG3) $\varphi \dashv\vdash \rho(\tau(\varphi))$;
(ALG4) $\varphi \approx \psi \mathrel{=\mid\vDash}_{Eq(\mathcal{K})} \tau(\rho(\varphi, \psi))$.

Examples of algebraizable logics include, among many others, classical, intuitionistic logic, all substructural logics and global modal logics. Not all logics though are algebraizable: examples of non-algebraizable logics can be found in the realm of Kleene logics, such as the Logic of Paradox (see [37]), Paraconsistent weak Kleene (see [6]) and Bochvar internal logic (see [7–9]). The above definition of algebraizable logic can be drastically simplified: $\vdash$ is algebraizable with equivalent algebraic semantics $\mathcal{K}$ if and only if it satisfies either ALG1 and ALG4 (or, else ALG2 and ALG3).[6]

Theorem 35 *The logic* B_e *is algebraizable with* $\mathcal{BA}_3$ *as equivalent algebraic semantics.*

Proof Consider $\tau = \{\varphi \approx 1\}$ and $\rho = \{\varphi \equiv \psi\}$. □

The usefulness of the above result will be explored in a fore-coming paper, focused on a deeper understanding of the properties of Bochvar algebras [10].

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[6] We refer the interested reader directly to the introductory textbook [21].

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Declarations

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