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Infinitely many homoclinic solutions for fractional discrete Kirchhoff–Schrödinger equations

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Abstract

In the present paper, we consider a fractional discrete Schrödinger equation with Kirchhoff term. Through the fountain theorem and the dual fountain theorem, we obtain two different conclusions about infinitely many homoclinic solutions to this equation.

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Keywords: Kirchhoff equation; Fractional discrete Schrödinger equation; Fountain theorem; Dual fountain theorem

1 Introduction and main results

The fractional Laplace operator as a classical nonlocal operator has many applications in many fields of mathematics, such as harmonic analysis, finance, game theory, and so on. Especially, it has become a popular research object in partial differential equations in the past decade. The definition of the fractional Laplace operator in $\mathbb{R}^N$ for $0 < s < 1$ and $\mu \in C_0^\infty(\mathbb{R}^N)$ is given as follows:

$$(-\Delta)^s \mu(x) = C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{\mu(x) - \mu(y)}{|x - y|^{N+2s}}\, dy, \quad x \in \mathbb{R}^N,$$

where $C_{N,s} > 0$ is an explicit constant, and $P.V.$ denotes the Cauchy principle value; see [16]. As far as we know, the Lévy process is one of the most classical applications of this type of operator, see [5, 23] and their references. There are many studies devoted to replacing the Laplacian with the fractional Laplacian or other more general calculus operators, and these results can better describe various phenomena in nature compared with the previous ones; we refer to [6, 10, 13, 16, 36].

In particular, numerical approximation of the fractional Laplace operator is becoming very tricky because of its nonlocality and singularity, which makes it necessary to exploit more effective approaches to investigate the existence of solutions; see, e.g., [1, 12, 19, 22] and references therein.

Let

$$\ell_s = \left\{ \mu : \mathbb{Z}_{\mathcal{G}} \to \mathbb{R} \,\Big|\, \sum_{\xi \in \mathbb{Z}} \frac{|\mu(\xi)|}{(1+|\xi|)^{1+2s}} < \infty \right\}, \quad \mathbb{Z}_{\mathcal{G}} = \{\mathcal{G}\xi : \xi \in \mathbb{Z}\},$$

where $\mathcal{G} > 0$ is a fixed positive constant. Ciaurri et al. [15] gave a definition of the fractional discrete Laplace operator on $\mathbb{Z}_{\mathcal{G}}$ as

$$(-\Delta_{\mathcal{G}})^s \mu(\zeta) = \sum_{\xi \in \mathbb{Z}, \xi \neq \zeta} \big(\mu(\zeta) - \mu(\xi)\big) K_s^{\mathcal{G}}(\zeta - \xi),$$

where $0 < s < 1$, $\mu \in \ell_s$, and

$$K_s^{\mathcal{G}}(\zeta) = \begin{cases} \frac{4^s \Gamma(1/2+s)}{\sqrt{\pi}|\Gamma(-s)|} \cdot \frac{\Gamma(|\zeta|-s)}{\mathcal{G}^{2s}\Gamma(|\zeta|+1+s)}, & \zeta \in \mathbb{Z}\backslash\{0\}, \\ 0, & \zeta = 0. \end{cases}$$

The kernel $K_s^{\mathcal{G}}(\zeta)$ has the following property:

$$\frac{c_s}{\mathcal{G}^{2s}|\zeta|^{1+2s}} \le K_s^{\mathcal{G}}(\zeta) \le \frac{C_s}{\mathcal{G}^{2s}|\zeta|^{1+2s}}, \quad \zeta \in \mathbb{Z}\backslash\{0\},$$

where $0 < c_s \le C_s$ are two constants. Let $\Delta_{\mathcal{G}}$ be the discrete Laplace operator on $\mathbb{Z}_{\mathcal{G}}$ defined as

$$\Delta_{\mathcal{G}} \mu(\zeta) = \frac{1}{\mathcal{G}^2}\big(\mu(\zeta+1) - 2\mu(\zeta) + \mu(\zeta-1)\big).$$

Furthermore, if μ is bounded, then $\lim_{s\to 1^-}(-\Delta_{\mathcal{G}})^s\mu(\zeta) = -\Delta_{\mathcal{G}}\mu(\zeta)$. Ciaurri et al. also proved that the fractional discrete Laplacian can approximate the fractional Laplacian as $\mathcal{G} \to 0$ under certain conditions.

Next, we elaborate some results of different fractional discrete Laplacian equations through several references. Xiang and Zhang [33] studied the following discrete fractional Laplacian equation:

$$\begin{cases} (-\Delta_1)^s \mu(\xi) + V(\xi)\mu(\xi) = \lambda f(\xi, \mu(\xi)) & \text{for } \xi \in \mathbb{Z}, \\ \mu(\xi) \to 0 & \text{as } |\xi| \to \infty, \end{cases} \tag{1.1}$$

where

$$(-\Delta_1)^s \mu(\zeta) = 2 \sum_{\xi \in \mathbb{Z}, \xi \neq \zeta} \big(\mu(\zeta) - \mu(\xi)\big) K_s(\zeta - \xi), \quad \zeta \in \mathbb{Z},$$

$V : \mathbb{Z} \to (0, \infty)$, $\lambda > 0$, and $f(\zeta, \cdot) \in C(\mathbb{R}, \mathbb{R})$ for all $\zeta \in \mathbb{Z}$. They obtained at least two homoclinic solutions of Eq. (1.1) by the mountain pass theorem and Ekeland's variational principle under appropriate assumptions. Ju et al. [21] investigated the following discrete fractional p-Laplacian equation:

$$\begin{cases} (-\Delta_1)_p^s \mu(\xi) + V(\xi)|\mu(\xi)|^{p-2}\mu(\xi) & \\ \quad = \lambda a(\xi)|\mu(\xi)|^{q-2}\mu(\xi) + b(\xi)|\mu(\xi)|^{r-2}\mu(\xi) & \text{for } \xi \in \mathbb{Z}, \\ \mu(\xi) \to 0, & \text{as } |\xi| \to \infty, \end{cases} \tag{1.2}$$

where $(-\Delta_1)_p^s$ is the fractional discrete p-Laplace operator (defined later), $V:\mathbb{Z}\to(0,\infty)$, $\lambda>0$, $1<q<p<r<\infty$, $a\in\ell^{\frac{p}{p-q}}$, and $b\in\ell^{\infty}$. They detected at least two homoclinic solutions of Eq. (1.2) via the Nehari manifold method under suitable hypotheses. Using different kinds of Clark's theorem, Ju and Zhang [20] gained multiple solutions of fractional discrete Laplacian equations with different nonlinear terms. Through the results of the above literature, we know that Eq. (1.1) can be reduced to the well-known discrete form of Schrödinger equation

$$-\Delta\mu(\xi)+V(\xi)\mu(\xi)=\lambda f\bigl(\xi,\mu(\xi)\bigr),\quad \xi\in\mathbb{Z}. \tag{1.3}$$

Agarwal et al. [2] first studied a discrete Laplacian equation similar to (1.3) by variational methods. There are a number of recent papers that use critical point theory to study second-order difference equations; see [24–26, 28, 37]. At the same time, we note that the Kirchhoff–type problems have attracted wide attention in recent years. Specifically, Kirchhoff built a well-known model via the following equation:

$$\rho\frac{\partial^2\mu}{\partial t^2}-\left(\frac{p_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial\mu}{\partial x}\right|^2dx\right)\frac{\partial^2\mu}{\partial x^2}=0, \tag{1.4}$$

where ρ, p_0, h, E, and L are constants with some physical background. Equation (1.4) is regarded as an extension of the classical D'Alembert wave equation. Fiscella and Valdinoci [18] deduced a continuous expression of the fractional Kirchhoff model. Since then, there have been a lot of papers considering qualitative properties of solutions for Kirchhoff–type fractional Laplacian problems; see, e.g., [4, 11, 29–32, 34, 35]. To the beest of our knowledge, there are no results on the existence of solutions for Kirchhoff-type fractional discrete Laplacian problems.

In the present paper, we use the fountain theorem and the dual fountain theorem to investigate the multiplicity of homoclinic solutions for the following fractional discrete Kirchhoff–Schrödinger equation:

$$\bigl(a+b[\mu]_{s,p}^p\bigr)(-\Delta_1)_p^s\mu(\kappa)+V(\kappa)\bigl|\mu(\kappa)\bigr|^{p-2}\mu(\kappa)=f\bigl(\kappa,\mu(\kappa)\bigr)\quad\text{for }\kappa\in\mathbb{Z}, \tag{1.5}$$

where

$$[\mu]_{s,p}^p=\sum_{\zeta\in\mathbb{Z}}\sum_{\xi\in\mathbb{Z}}\bigl|\mu(\zeta)-\mu(\xi)\bigr|^pK_{s,p}(\zeta-\xi),$$

$a,b>0$ and $0<s<1<p<\infty$ are constants, $V(\kappa):\mathbb{Z}\to\mathbb{R}^+, f(\kappa,\cdot)\in C(\mathbb{R},\mathbb{R})$ for all $\kappa\in\mathbb{Z}$, and the fractional discrete p-Laplace operator $(-\Delta_1)_p^s$ is defined as

$$(-\Delta_1)_p^s\mu(\zeta)=2\sum_{\xi\in\mathbb{Z},\xi\neq\zeta}\bigl|\mu(\zeta)-\mu(\xi)\bigr|^{p-2}\bigl(\mu(\zeta)-\mu(\xi)\bigr)K_{s,p}(\zeta-\xi),\quad \zeta\in\mathbb{Z},$$

with discrete kernel $K_{s,p}$ such that

$$\begin{cases}\frac{c_{s,p}}{|\xi|^{1+ps}}\le K_{s,p}(\xi)\le\frac{C_{s,p}}{|\xi|^{1+ps}}, & \xi\in\mathbb{Z}\setminus\{0\},\\ K_{s,p}(0)=0,\end{cases} \tag{1.6}$$

where $0<c_{s,p}\le C_{s,p}$ are two positive constants.

Note that the operator $(-\Delta_1)_p^s$ goes back to $(-\Delta_1)^s$ with $p = 2$. The counting measure of interval $\mathcal{T}$ is denoted by $\mathcal{M}(\mathcal{T})$. The neighborhood in $\mathbb{Z}$ with center δ and radius γ is denoted by $\mathcal{U}_\gamma(\delta)$. If $\mu(i) \to 0$ as $|i| \to \infty$, then we call the solution μ of problem (1.5) homoclinic. Assume that $V(\kappa) : \mathbb{Z} \to \mathbb{R}^+$ and $f(\kappa, \cdot) \in C(\mathbb{R}, \mathbb{R})$ for all $\kappa \in \mathbb{Z}$ satisfies:

- (V_1) V_0 is a positive constant such that $V(\kappa) \geq V_0$ for all $\kappa \in \mathbb{Z}$;
- (V_2) For all $\sigma > 0$, there is a positive integer γ such that $\lim_{|\delta|\to\infty} \mathcal{M}(\{\kappa \in \mathbb{Z} | V(\kappa) \leq \sigma\} \cap \mathcal{U}_\gamma(\delta)) = 0$;
- (f_1) $|f(\kappa, \mu)| \leq C(|\mu|^{p-1} + |\mu|^{q-1})$ for any $\kappa \in \mathbb{Z}$ and $\mu \in \mathbb{R}$, where C is a positive constant, and $p < q < \infty$;
- (f_2) $\lim_{|\mu|\to\infty} \frac{F(\kappa,\mu)}{\mu^{2p}} = +\infty$ uniformly for all $\kappa \in \mathbb{Z}$, where $F(\kappa, \mu) = \int_0^\mu f(\kappa, \zeta)\, d\zeta$;
- (f_3) There exist $R > 0$, $\varphi \geq 2p$, and $\alpha \geq 0$ such that $F(\kappa, \mu) \leq \frac{1}{\varphi} f(\kappa, \mu)\mu + \alpha|\mu|^p + \varpi(\kappa)$ for all $\kappa \in \mathbb{Z}$ and $\mu \geq R$, where $\varpi \in \ell^1 \cap \ell^\infty$ and $\varpi \geq 0$ (ℓ^1 and ℓ^∞ are defined in Sect. 2);
- (f_4) $f(\kappa, \mu)$ is odd in μ.

Assumption (V_2) is weaker than the coercivity, and the former is a discrete version of the continuous form proposed by Bartsch and Wang [8] to overcome the lack of compactness. Moreover, hypothesis (f_3) is weaker than the general Ambrosetti–Rabinowitz condition [3]

- (AR) For any $\kappa \in \mathbb{Z}$ and $\mu \geq R$,

$$0 < \varphi F(\kappa, \mu) \leq f(\kappa, \mu)\mu.$$

We give the following example satisfying (f_1)–(f_4).

Example 1.1

$$f(\xi, \mu) = \mathcal{R}\big|\mu(\xi)\big|^{\gamma-2}\mu(\xi) + \mathcal{T}\big|\mu(\xi)\big|^{\tau-2}\mu(\xi), \quad p \leq \gamma < 2p < \tau,$$

where $\mathcal{R} \in \mathbb{R}$ and $\mathcal{T} > 0$ are two constants.

Now we state the following main results by the fountain theorem and the dual fountain theorem. The space E and functional J will be defined in Sect. 2.

Theorem 1.1 *Assume that* (V_1)–(V_2) *and* (f_1)–(f_4) *are satisfied. Then problem* (1.5) *possesses infinitely many nontrivial homoclinic solutions* $\{\mu_n\} \subset E$ *with energy* $J(\mu_n) \to \infty$ *as* $n \to \infty$.

Theorem 1.2 *Assume that* (V_1)–(V_2) *and* (f_1)–(f_4) *are satisfied. Then problem* (1.5) *possesses infinitely many nontrivial homoclinic solutions* $\{\mu_n\} \subset E$ *with energy* $J(\mu_n) < 0$ *such that* $J(\mu_n) \to 0$ *as* $n \to \infty$.

Remark 1.1 We briefly summarize the highlights of this paper:

- (a) The fractional discrete Schrödinger equations with Kirchhoff term is discussed for the first time.
- (b) Under the same hypotheses, Theorems 1.1 and 1.2 acquire two sequences of diverse homoclinic solutions for Eq. (1.5).
- (c) The fountain theorem and the dual fountain theorem are used for the first time to study fractional discrete Kirchhoff–Schrödinger equations.

Ultimately, we point out that the case of concave and convex nonlinearities $f(\xi,\mu) = |\mu(\xi)|^{r-2}\mu(\xi) + |\mu(\xi)|^{t-2}\mu(\xi)$ with $1 < r < p < 2p < t < \infty$, as a classical application of the fountain theorem in a bounded domain (see, e.g., [34]), is not covered by Theorems 1.1 and 1.2, because the compact embedding is not valid when $1 < r < p$ (the compact embedding lemma is introduced in Sect. 2). Also, we point out that the case of $f(\xi,\mu(\xi)) = c(\xi)|\mu(\xi)|^{r-2}\mu(\xi) + d(\xi)|\mu(\xi)|^{t-2}\mu(\xi)$ with $1 < r < p < t < \infty$, as the nonlinearity of a non-Kirchhoff-type fractional discrete p-Laplacian problem, has been investigated by a new version of Clark's theorem presented by Liu and Wang [27]. However, the excessive limitations of the coefficients $c(\xi)$ and $d(\xi)$ make the obtained result imperfect. Therefore, how to solve Eq. (1.5) with combined effect of concave and convex nonlinearities in the Kirchhoff setting by a new approach is an interesting problem, which we will investigate in the near future.

The rest of this paper consists of the following: Sect. 2 presents the variational structure of Eq. (1.5). Section 3 verifies the compactness condition and describes the related lemmas used later. Section 4 proves infinite solutions to problem (1.5) through the fountain theorem [7]. Section 5 is devoted to verifying the existence of infinitely many homoclinic solutions to problem (1.5) via the dual fountain theorem [9].

2 Variational framework

First, we revisit some fundamental definitions.

For $1 \le \varrho < \infty$, the space $(\ell^\varrho, \|\cdot\|_\varrho)$ is defined as

$$\ell^\varrho := \left\{\nu : \mathbb{Z} \to \mathbb{R} \,\middle|\, \sum_{\zeta\in\mathbb{Z}} \left|\nu(\zeta)\right|^\varrho < \infty\right\}, \qquad \|\nu\|_\varrho = \left(\sum_{\zeta\in\mathbb{Z}} \left|\nu(\zeta)\right|^\varrho\right)^{1/\varrho}.$$

In addition, the space $(\ell^\infty, \|\cdot\|_\infty)$ is defined as

$$\ell^\infty := \left\{\nu : \mathbb{Z} \to \mathbb{R} \,\middle|\, \sup_{\zeta\in\mathbb{Z}} \left|\nu(\zeta)\right| < \infty\right\}, \qquad \|\nu\|_\infty = \sup_{\zeta\in\mathbb{Z}} \left|\nu(\zeta)\right|.$$

By the corresponding results in [17] the spaces $(\ell^\varrho, \|\cdot\|_\varrho)$ and $(\ell^\infty, \|\cdot\|_\infty)$ are Banach spaces. Evidently, $\ell^{\varrho_1} \hookrightarrow \ell^{\varrho_2}$ if $1 \le \varrho_1 \le \varrho_2 \le \infty$.

Now we present a variational framework and relevant theorems to study Eq. (1.5). Set

$$E = \left\{\nu : \mathbb{Z} \to \mathbb{R} \,\middle|\, \sum_{\zeta\in\mathbb{Z}}\sum_{\xi\in\mathbb{Z}} \left|\nu(\zeta) - \nu(\xi)\right|^p K_{s,p}(\zeta - \xi) + \sum_{\tau\in\mathbb{Z}} V(\tau)\left|\nu(\tau)\right|^p < \infty\right\}.$$

The norm of space E is given by

$$\begin{aligned}\|\nu\|_E^p &= [\nu]_{s,p}^p + \sum_{\tau\in\mathbb{Z}} V(\tau)\left|\nu(\tau)\right|^p \\ &= \sum_{\zeta\in\mathbb{Z}}\sum_{\xi\in\mathbb{Z}} \left|\nu(\zeta) - \nu(\xi)\right|^p K_{s,p}(\zeta - \xi) + \sum_{\tau\in\mathbb{Z}} V(\tau)\left|\nu(\tau)\right|^p.\end{aligned}$$

Lemma 2.1 (See [21]) *$\psi \in \ell^\sigma \Rightarrow [\psi]_{s,\sigma} < \infty$. Besides, $[\psi]_{s,\sigma} \le C(s,\sigma)\|\psi\|_\sigma$ for all $\psi \in \ell^\sigma$, where $C(s,\sigma)$ is a positive constant.*

Lemma 2.2 (See [20, 21]) *Suppose that* (V_1) *is satisfied. Then there is the following equivalent norm in* $(E, \|\cdot\|_E)$:

$$\|\psi\| := \left(\sum_{\xi\in\mathbb{Z}} V(\xi)\left|\psi(\xi)\right|^p\right)^{1/p}.$$

Besides, $(E, \|\cdot\|)$ *is a reflexive and separable Banach space.*

Proof For the proof of equivalent norm and Banach space, see [21]. For the proof of reflexivity, see [20]. Here we only give the proof of separability. Let

$$\mathcal{A} := \bigcup_{n=0}^{\infty}\left\{\nu(i) \in E, i \in \mathbb{Z} | \nu(i) \in \mathbb{Q} \text{ for } |i| \le n, \nu(i) = 0 \text{ for } |i| \ge n+1\right\}.$$

Then the set $\mathcal{A}$ is countably infinite as a countably infinite union of countably infinite sets. Given any $u \in E$ and any $\varepsilon > 0$, there exists $n_0(u, \varepsilon) \ge 1$ such that

$$\sum_{|i|\ge n_0+1} V(i)\left|u(i)\right|^p \le \frac{\varepsilon^p}{2}. \tag{2.1}$$

For $1 \le i \le n_0$, there exists $\nu(i) \in \mathbb{Q}$ such that

$$\sum_{|i|\le n_0} V(i)\left|u(i) - \nu(i)\right|^p \le \frac{\varepsilon^p}{2}. \tag{2.2}$$

By (2.1) and (2.2) we know that $\nu(i) \in \mathcal{A}$ and $\|\nu - u\| \le \varepsilon$. So $\mathcal{A}$ is dense in E. This proves the separability of E. □

Lemma 2.3 (See [20]) *Suppose that* (V_1)–(V_2) *are satisfied, then for all* $p \le \iota \le \infty$, *the embedding* $E \hookrightarrow \ell^\iota$ *is compact.*

Lemma 2.4 (See [21]) *Let* $\mathcal{D} \subset E$ *be a compact subset. Then for all* $\theta > 0$, *there is* $\tau_0 \in \mathbb{N}$ *such that*

$$\left[\sum_{|\zeta|>\tau_0} V(\zeta)\left|\mu(\zeta)\right|^p\right]^{1/p} < \theta \quad \textit{for all } \mu \in \mathcal{D}.$$

Next, we define the energy functional $J : E \to \mathbb{R}$ associated with problem (1.5) as

$$J(\mu) = A(\mu) - B(\mu),$$

where

$$A(\mu) = \frac{1}{p}\left(a[\mu]_{s,p}^p + \|\mu\|^p\right) + \frac{b}{2p}\left([\mu]_{s,p}^p\right)^2$$

and

$$B(\mu) = \sum_{\zeta\in\mathbb{Z}} F\left(\zeta, \mu(\zeta)\right).$$

Lemma 2.5 *Suppose that (V_1) is satisfied. Then $A(\mu) \in C^1(E,\mathbb{R})$ with*

$$\begin{aligned}\langle A'(\mu),\nu\rangle = {}& \big(a + b[\mu]_{s,p}^p\big)\sum_{\zeta\in\mathbb{Z}}\sum_{\xi\in\mathbb{Z}}\big|\mu(\zeta)-\mu(\xi)\big|^{p-2} \\ & \times \big(\mu(\zeta)-\mu(\xi)\big)\big(\nu(\zeta)-\nu(\xi)\big)K_{s,p}(\zeta-\xi) \\ & + \sum_{\zeta\in\mathbb{Z}} V(\zeta)\big|\mu(\zeta)\big|^{p-2}\mu(\zeta)\nu(\zeta)\end{aligned}$$

for all $\mu,\nu \in E$.

Proof By Lemmas 2.1 and 2.3 we can easily deduce that A is well–defined on E. Fix $\mu,\nu\in E$. By an argument similar to that of [21] we have

$$\begin{aligned}&\lim_{\tau\to 0^+} a\frac{[\mu+\tau\nu]_{s,p}^p - [\mu]_{s,p}^p}{p\tau} \\ &\quad = a\sum_{\zeta\in\mathbb{Z}}\sum_{\xi\in\mathbb{Z}}\big|\mu(\zeta)-\mu(\xi)\big|^{p-2}\big(\mu(\zeta)-\mu(\xi)\big)\big(\nu(\zeta)-\nu(\xi)\big)K_{s,p}(\zeta-\xi)\end{aligned} \tag{2.3}$$

and

$$\lim_{\tau\to 0^+}\frac{\|\mu+\tau\nu\|^p - \|\mu\|^p}{p\tau} = \sum_{\zeta\in\mathbb{Z}} V(\zeta)\big|\mu(\zeta)\big|^{p-2}\mu(\zeta)\nu(\zeta). \tag{2.4}$$

By (2.3) we can derive that

$$\begin{aligned}&\lim_{\tau\to 0^+} b\frac{([\mu+\tau\nu]_{s,p}^p)^2 - ([\mu]_{s,p}^p)^2}{2p\tau} \\ &\quad = b\lim_{\tau\to 0^+}\frac{([\mu+\tau\nu]_{s,p}^p + [\mu]_{s,p}^p)([\mu+\tau\nu]_{s,p}^p - [\mu]_{s,p}^p)}{2p\tau} \\ &\quad = b\left(\frac{1}{2}\lim_{\tau\to 0^+}\big([\mu+\tau\nu]_{s,p}^p + [\mu]_{s,p}^p\big)\cdot\lim_{\tau\to 0^+}\frac{[\mu+\tau\nu]_{s,p}^p - [\mu]_{s,p}^p}{p\tau}\right) \\ &\quad = b[\mu]_{s,p}^p\sum_{\zeta\in\mathbb{Z}}\sum_{\xi\in\mathbb{Z}}\big|\mu(\zeta)-\mu(\xi)\big|^{p-2}\big(\mu(\zeta)-\mu(\xi)\big)\big(\nu(\zeta)-\nu(\xi)\big)K_{s,p}(\zeta-\xi).\end{aligned} \tag{2.5}$$

With the help of (2.3), (2.4), and (2.5) we get

$$\begin{aligned}\langle A'(\mu),\nu\rangle = {}& \big(a + b[\mu]_{s,p}^p\big)\sum_{\zeta\in\mathbb{Z}}\sum_{\xi\in\mathbb{Z}}\big|\mu(\zeta)-\mu(\xi)\big|^{p-2} \\ & \times \big(\mu(\zeta)-\mu(\xi)\big)\big(\nu(\zeta)-\nu(\xi)\big)K_{s,p}(\zeta-\xi) \\ & + \sum_{\zeta\in\mathbb{Z}} V(\zeta)\big|\mu(\zeta)\big|^{p-2}\mu(\zeta)\nu(\zeta).\end{aligned}$$

So A is Gâteaux differentiable.

Finally, we prove that A' is continuous. Let $\{\mu_n\}_n$ is a sequence in E such that $\mu_n \to \mu$ in E as $n\to\infty$. For convenience, we denote $\lambda(\mu) = |\mu|^{p-2}\mu$. By Lemma 2.4 and Lemma 2.5

in [21] we can get that for any $\varepsilon > 0$,

$$\left|\sum_{\zeta\in\mathbb{Z}}\sum_{\xi\in\mathbb{Z}}\left[\lambda\bigl(\mu_n(\zeta)-\mu_n(\xi)\bigr)-\lambda\bigl(\mu(\zeta)-\mu(\xi)\bigr)\right]\bigl(\nu(\zeta)-\nu(\xi)\bigr)K_{s,p}(\zeta-\xi)\right| \le C\varepsilon\|\nu\|$$

and

$$\left|\sum_{\zeta\in\mathbb{Z}}V(\zeta)\left[\lambda\bigl(\mu_n(\zeta)\bigr)-\lambda\bigl(\mu(\zeta)\bigr)\right]\nu(\zeta)\right| \le C\varepsilon\|\nu\|$$

as $n\to\infty$, where C is a positive constant. From this we obtain that

$$\begin{aligned}\left\|A'(\mu_n)-A'(\mu)\right\| &= \sup_{\|\nu\|\le 1}\left|\left\langle A'(\mu_n)-A'(\mu),\nu\right\rangle\right| \\ &\le \sup_{\|\nu\|\le 1}\left[\left(a+b[\mu_n-\mu]_{s,p}^p\right)C\varepsilon\|\nu\|+C\varepsilon\|\nu\|\right]\to 0\end{aligned}$$

as $n\to\infty$. Therefore $A(\mu)\in C^1(E,\mathbb{R})$. □

Lemma 2.6 *Suppose that* (V_1) *and* (f_1) *are satisfied. Then* $B(\mu)\in C^1(E,\mathbb{R})$ *with*

$$\left\langle B'(\mu),\nu\right\rangle = \sum_{\tau\in\mathbb{Z}}f\bigl(\tau,\mu(\tau)\bigr)\nu(\tau)$$

for all $\mu,\nu\in E$.

Proof By (f_1) and Lemma 2.3 we have

$$\begin{aligned}\sum_{\kappa\in\mathbb{Z}}F(\kappa,\mu) &\le C_1\sum_{\kappa\in\mathbb{Z}}|\mu|^p + C_2\sum_{\kappa\in\mathbb{Z}}|\mu|^q \\ &\le C_1C_p\|\mu\|^p + C_2C_q\|\mu\|^q < \infty,\end{aligned}$$

where $C_1, C_2, C_p, C_q > 0$ are constants. So B is well–defined on E. For given $\mu,\nu\in E$, we show that

$$\lim_{\tau\to 0^+}\frac{B(\mu+\tau\nu)-B(\mu)}{\tau} = \sum_{\zeta\in\mathbb{Z}}f\bigl(\zeta,\mu(\zeta)\bigr)\nu(\zeta). \tag{2.6}$$

The proof is analogous to that of Lemma 2.6 in [20]. We give it for the convenience of the reader.

Choose a positive constant W such that $\|\mu\|\le W$ and $\|\nu\|\le W$. For all $\varepsilon > 0$, there exists $\kappa_1\in\mathbb{N}$ such that

$$\sum_{|\zeta|>\kappa_1}V(\zeta)\bigl|\nu(\zeta)\bigr|^p < \left(\frac{\varepsilon}{6CC_p^{2(p-1)}(2W)^{p-1}}\right)^p. \tag{2.7}$$

Moreover, there is $0<\tau_0<1$ such that for all $0<\tau\le\tau_0$,

$$\sum_{|\zeta|\le\kappa_1}\left|\frac{F(\zeta,\mu(\zeta)+\tau\nu(\zeta))-F(\zeta,\mu(\zeta))}{\tau} - f\bigl(\zeta,\mu(\zeta)\bigr)\nu(\zeta)\right| < \frac{\varepsilon}{3}. \tag{2.8}$$

Fix $0 < \tau < \tau_0$. By the mean value theorem, for any $|\zeta| \geq \kappa_1$, there is $0 \leq \tau_\zeta \leq \tau$ such that

$$\frac{F(\zeta,\mu(\zeta)+\tau\nu(\zeta))-F(\zeta,\mu(\zeta))}{\tau}=f\big(\zeta,\mu(\zeta)+\tau_\zeta\nu(\zeta)\big)\nu(\zeta). \tag{2.9}$$

Let $\omega \in E$ and

$$\omega(\zeta)=\begin{cases}0 & \text{if } \zeta \leq \kappa_1,\\ \mu(\zeta)+\tau_\zeta\nu(\zeta) & \text{if } \zeta > \kappa_1.\end{cases}$$

Thus $\|\omega\| \leq \|\mu\|+\|\nu\| \leq 2W$. Hence by the Hölder inequality, (2.7), (2.8), (2.9), Lemma 2.3, and (f_1) we get

$$\begin{aligned}
&\left|\frac{B(\mu+\tau\nu)-B(\mu)}{\tau}-\sum_{\zeta\in\mathbb{Z}}f\big(\zeta,\mu(\zeta)\big)\nu(\zeta)\right|\\
&\quad\leq\sum_{\zeta\in\mathbb{Z}}\left|\frac{F(\zeta,\mu(\zeta)+\tau\nu(\zeta))-F(\zeta,\mu(\zeta))}{\tau}-f\big(\zeta,\mu(\zeta)\big)\nu(\zeta)\right|\\
&\quad\leq\frac{\varepsilon}{3}+\sum_{|\zeta|>\kappa_1}\big|f\big(\zeta,\omega(\zeta)\big)\nu(\zeta)\big|+\sum_{|\zeta|>\kappa_1}\big|f\big(\zeta,\mu(\zeta)\big)\nu(\zeta)\big|\\
&\quad\leq\frac{\varepsilon}{3}+C\bigg(\sum_{|\zeta|>\kappa_1}\big|\omega(\zeta)\big|^{p-1}\big|\nu(\zeta)\big|+\sum_{|\zeta|>\kappa_1}\big|\mu(\zeta)\big|^{p-1}\big|\nu(\zeta)\big|\bigg)\\
&\quad\leq\frac{\varepsilon}{3}+C\bigg[\bigg(\sum_{|\zeta|>\kappa_1}|\omega|^p\bigg)^{\frac{p-1}{p}}+\bigg(\sum_{|\zeta|>\kappa_1}|\mu|^p\bigg)^{\frac{p-1}{p}}\bigg]\bigg(\sum_{|\zeta|>\kappa_1}|\nu|^p\bigg)^{\frac{1}{p}}\\
&\quad\leq\frac{\varepsilon}{3}+C\big(C_p^{(p-1)}\|\omega\|^{p-1}+C_p^{(p-1)}\|\mu\|^{p-1}\big)\cdot C_p^{(p-1)}\|\nu\|\\
&\quad\leq\frac{\varepsilon}{3}+CC_p^{2(p-1)}\big[(2W)^{p-1}+(W)^{p-1}\big]\frac{\varepsilon}{6CC_p^{2(p-1)}(2W)^{p-1}}\\
&\quad<\varepsilon.
\end{aligned}$$

Thus (2.6) is established, and hence B is Gâteaux differentiable.

Eventually, we verify the continuity of B'. Assume that $\mu_n \to \mu$ in E as $n \to \infty$. Then by Lemma 2.3 we know that $\mu_n(\xi) \to \mu(\xi)$ for all $\xi \in \mathbb{Z}$. By Lemma 2.4 there exists $\kappa_2 \in \mathbb{N}_+$ such that for small $\varepsilon > 0$,

$$\bigg(\sum_{|\xi|>\kappa_2}V(\xi)\big|\mu_n(\xi)\big|^p\bigg)^{\frac{1}{p}}<\frac{\varepsilon}{6}\quad\text{for all } n\in\mathbb{N}$$

and

$$\bigg(\sum_{|\xi|>\kappa_2}V(\xi)\big|\mu(\xi)\big|^p\bigg)^{\frac{1}{p}}<\frac{\varepsilon}{6}.$$

In addition, we can derive that

$$\big|\mu(\xi)\big|<\frac{\varepsilon}{6V_0^{\frac{1}{p}}},\qquad\big|\mu_n(\xi)\big|<\frac{\varepsilon}{6V_0^{\frac{1}{p}}}\quad\text{for all } n\in\mathbb{N}\text{ and } |\xi|>\kappa_2.$$

From (f_1) we can get that

$$\begin{aligned}
&\sum_{|\xi|>\kappa_2} \left|f\big(\xi,\mu_n(\xi)\big)-f\big(\xi,\mu(\xi)\big)\right|^{\frac{p}{p-1}} \\
&\quad\le C^{\frac{p}{p-1}} \sum_{|\xi|>\kappa_2} \big(|\mu_n|^{p-1}+|\mu|^{p-1}\big)^{\frac{p}{p-1}} \\
&\quad\le \big(2C^p\big)^{\frac{1}{p-1}} \sum_{|\xi|>\kappa_2} \big(|\mu_n|^{p}+|\mu|^{p}\big) \\
&\quad\le \frac{(2C^p)^{\frac{1}{p-1}}}{V_0} \sum_{|\xi|>\kappa_2} V(\xi)\big(|\mu_n|^{p}+|\mu|^{p}\big) \\
&\quad< \frac{(2C^p)^{\frac{1}{p-1}}}{V_0} \left(\left(\frac{\varepsilon}{6}\right)^p+\left(\frac{\varepsilon}{6}\right)^p\right) \\
&\quad= \frac{(2C)^{\frac{p}{p-1}}\varepsilon^p}{6^p V_0}.
\end{aligned} \tag{2.10}$$

Besides, since $\mu_n(\xi) \to \mu(\xi)$ for each $\xi \in \mathbb{Z}$ as $n \to \infty$, by the continuity of $f(\xi,\cdot)$ we have that there exists an integer n_0 such that

$$\sum_{|\xi|\le\kappa_2} \left|f\big(\xi,\mu_n(\xi)\big)-f\big(\xi,\mu(\xi)\big)\right|^{\frac{p}{p-1}} \le \frac{(6^p-2^{\frac{p}{p-1}})C^{\frac{p}{p-1}}\varepsilon^p}{6^p V_0} \quad \text{for all } n \ge n_0. \tag{2.11}$$

By (2.10), (2.11), and the Hölder inequality, for any $\nu \in E$ with $\|\nu\| \le 1$ and $n \ge n_0$, we get

$$\begin{aligned}
&\left|\sum_{\xi\in\mathbb{Z}} \big[f\big(\xi,\mu_n(\xi)\big)-f\big(\xi,\mu(\xi)\big)\big]\nu(\xi)\right| \\
&\quad\le \left(\sum_{\xi\in\mathbb{Z}} \left|f\big(\xi,\mu_n(\xi)\big)-f\big(\xi,\mu(\xi)\big)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \left(\sum_{\xi\in\mathbb{Z}} \left|\nu(\xi)\right|^p\right)^{\frac{1}{p}} \\
&\quad\le \frac{1}{V_0^{\frac{1}{p}}} \left(\frac{C^{\frac{p}{p-1}}\varepsilon^p}{V_0}\right)^{\frac{p-1}{p}} \left(\sum_{\xi\in\mathbb{Z}} V(\xi)\left|\nu(\xi)\right|^p\right)^{\frac{1}{p}} \\
&\quad\le \frac{C\varepsilon^{p-1}}{V_0}.
\end{aligned}$$

Then $B'(\mu_n) \to B'(\mu)$ in E^* as $n \to \infty$. Therefore we have verified that $B(\mu) \in C^1(E, \mathbb{R})$. □

By Lemmas 2.5 and 2.6 we get $J \in C^1(E,\mathbb{R})$.

Lemma 2.7 *Assume that (V_1) and (f_1) are satisfied. If $\mu \in E$ is a critical point of J, then μ is a homoclinic solution of Eq.* (1.5).

Proof Let $\mu \in E$ be a critical point of J, i.e. $J'(\mu) = 0$. Then

$$\big(a+b[\mu]_{s,p}^p\big)\sum_{\zeta\in\mathbb{Z}}\sum_{\xi\in\mathbb{Z}} \left|\mu(\zeta)-\mu(\xi)\right|^{p-2}\big(\mu(\zeta)-\mu(\xi)\big)\big(\nu(\zeta)-\nu(\xi)\big)K_{s,p}(\zeta-\xi)$$

$$+\sum_{\zeta\in\mathbb{Z}}V(\zeta)\left|\mu(\zeta)\right|^{p-2}\mu(\zeta)\nu(\zeta)=\sum_{\tau\in\mathbb{Z}}f\bigl(\tau,\mu(\tau)\bigr)\nu(\tau) \tag{2.12}$$

for all $\nu\in E$. For any $\zeta\in\mathbb{Z}$, we define $e_\eta\in E$ as

$$e_\eta(\zeta):=\begin{cases}1 & \text{if } \zeta=\eta,\\ 0 & \text{if } \zeta\neq\eta.\end{cases}$$

Letting $\nu=e_\eta$ in (2.12), we obtain

$$\begin{aligned}&2\bigl(a+b[\mu]_{s,p}^{p}\bigr)\sum_{\xi\in\mathbb{Z},\xi\neq\eta}\left|\mu(\eta)-\mu(\xi)\right|^{p-2}\bigl(\mu(\eta)-\mu(\xi)\bigr)K_{s,p}(\eta-\xi)+V(\eta)\left|\mu(\eta)\right|^{p-2}\mu(\eta)\\ &\quad=f\bigl(\eta,\mu(\eta)\bigr).\end{aligned}$$

Consequently, μ is a solution of Eq. (1.5). Additionally, by $\mu\in E$ and Lemma 2.3 we easily get that $\mu(\xi)\to0$ as $|\xi|\to\infty$. Eventually, we show μ is a homoclinic solution of Eq. (1.5). □

3 Auxiliary results

In this section, we recall some definitions, lemmas, and their proofs to reveal the main results.

By Lemma 2.2, E is a reflexive and separable Banach space. Then there are $\{e_n\}\subset E$ and $\{f_n^*\}\subset E^*$ such that

$$E=\overline{\operatorname{span}\{e_n|n=1,2,\ldots\}},\qquad E^*=\overline{\operatorname{span}\bigl\{f_n^*|n=1,2,\ldots\bigr\}},$$

and

$$\langle f_\zeta^*,e_\xi\rangle=\begin{cases}1 & \text{if } \zeta=\xi,\\ 0 & \text{if } \zeta\neq\xi.\end{cases}$$

For brevity, we define $E_\kappa=\operatorname{span}\{e_\kappa\}$, $Y_\kappa=\bigoplus_{i=1}^{\kappa}E_i$, $Z_\kappa=\overline{\bigoplus_{i=\kappa}^{\infty}E_i}$ for $\kappa\in\mathbb{N}_+$.

Lemma 3.1 *Suppose that* (V_1) *is satisfied. Then for all* $p\le q\le\infty$,

$$\beta_\kappa(q):=\sup\bigl\{\|\mu\|_q|\mu\in Z_\kappa,\|\mu\|=1\bigr\}\to0\quad\text{as }\kappa\to\infty.$$

Proof Obviously, $0\le\beta_{\kappa+1}(q)\le\beta_\kappa(q)$, so there is $\beta(q)\ge0$ such that $\beta_\kappa(q)\to\beta(q)$ as $\kappa\to\infty$. For every $\kappa\in\mathbb{N}_+$, there exists $\mu_\kappa\in Z_\kappa$ with $\|\mu_\kappa\|=1$ such that

$$\|\mu_\kappa\|_q>\frac{\beta_\kappa(q)}{2}. \tag{3.1}$$

By the definition of Z_κ, Lemma 2.2, and the boundedness of $\{\mu_\kappa\}$ there exists a subsequence of $\{\mu_\kappa\}$ (still denoted by $\{\mu_\kappa\}$) such that $\mu_\kappa\rightharpoonup\mu$ as $\kappa\to\infty$ in E. Next, we show that $\mu=0$. For any $f_m^*\in\{f_n^*|n=1,2,\ldots\}$, we have

$$\langle f_m^*,\mu\rangle=\lim_{\kappa\to\infty}\langle f_m^*,\mu_\kappa\rangle=0.$$

So $\mu_\kappa \rightharpoonup 0$ in E. From Lemma 2.3 we know that $\mu_\kappa \to 0$ in ℓ^q. Let $\kappa \to \infty$ in (3.1). Then we get $\beta_\kappa(q) \to 0$. □

Next, we introduce the Cerami condition ((C) for short), which is provided in [14]. Let E be a reflexive and separable Banach space. For $J \in C^1(E,\mathbb{R})$, we say that J satisfies (C) if any sequence $\{\mu_n\} \subset E$ such that $\{J(\mu_n)\}$ is bounded and $(1+\|\mu_n\|)\|J'(\mu_n)\| \to 0$ as $n \to \infty$ includes a convergent subsequence. Then we introduce condition $(C)^*$ (with respect to Y_n). We say that J satisfies $(C)^*$ if any sequence $\{\mu_n\} \subset E$ such that $\mu_n \in Y_n$, $\{J(\mu_n)\}$ is bounded, and $(1+\|\mu_n\|)\|J'|_{Y_n}(\mu_n)\| \to 0$ as $n \to \infty$ contains a convergent subsequence. Clearly, condition $(C)^*$ implies condition (C).

Lemma 3.2 *Let (V_1)–(V_2) and (f_1)–(f_3) hold. Then the functional J satisfies condition $(C)^*$.*

Proof Suppose $\{\mu_n\} \subset E$ is a $(C)^*$ sequence, that is, $\mu_n \in Y_n$ for some n,

$$\left|J(\mu_n)\right| \le M, \quad \text{and} \quad \left(1+\|\mu_n\|\right)\left\|J'|_{Y_n}(\mu_n)\right\| \to 0 \quad \text{as } n \to \infty. \tag{3.2}$$

First, we prove that $\{\mu_n\}$ is bounded in E. By contradiction assume that $\|\mu_n\| \to \infty$ as $n \to \infty$. Let $\nu_n = \frac{\mu_n}{\|\mu_n\|}$. By Lemmas 2.2 and 2.3, up to a subsequence (still denoted by $\{\nu_n\}$), we can get $\nu_n \rightharpoonup \nu$ in E and $\nu_n \to \nu$ in ℓ^p. Consider two cases: $\nu = 0$ or $\nu \neq 0$. If $\nu = 0$, then by Lemma 2.2, (f_1), (f_3), and (3.2), we obtain

$$\begin{aligned}
0 &= \lim_{n\to\infty} \frac{1}{\|\mu_n\|^p}(M+1) \\
&\ge \lim_{n\to\infty} \frac{1}{\|\mu_n\|^p}\left(J|_{Y_n}(\mu_n) - \frac{1}{\varphi}\langle J'|_{Y_n}(\mu_n), \mu_n\rangle\right) \\
&\ge \lim_{n\to\infty} \frac{1}{\|\mu_n\|^p}\left(\left(\frac{1}{p}-\frac{1}{\varphi}\right)\min\{a,1\}\|\mu_n\|_E^p - \sum_{\zeta\in\mathbb{Z}} F(\zeta,\mu_n) + \frac{1}{\varphi}\sum_{\zeta\in\mathbb{Z}} f(\zeta,\mu_n)\mu_n\right) \\
&\ge \left(\frac{1}{p}-\frac{1}{\varphi}\right)C_E\min\{a,1\} + \lim_{n\to\infty}\frac{1}{\|\mu_n\|^p}\sum_{\zeta\in\mathbb{Z}}\left(\frac{1}{\varphi}f(\zeta,\mu_n)\mu_n - F(\zeta,\mu_n)\right) \\
&= \left(\frac{1}{p}-\frac{1}{\varphi}\right)C_E\min\{a,1\} + \lim_{n\to\infty}\frac{1}{\|\mu_n\|^p}\sum_{\{\zeta\,||\mu|\ge R\}}\left(\frac{1}{\varphi}f(\zeta,\mu_n)\mu_n - F(\zeta,\mu_n)\right) \\
&\quad + \lim_{n\to\infty}\frac{1}{\|\mu_n\|^p}\sum_{\{\zeta\,||\mu|<R\}}\left(\frac{1}{\varphi}f(\zeta,\mu_n)\mu_n - F(\zeta,\mu_n)\right) \\
&\ge \left(\frac{1}{p}-\frac{1}{\varphi}\right)C_E\min\{a,1\} - \lim_{n\to\infty}\alpha\sum_{\{\zeta\,||\mu|\ge R\}}|\nu_n|^p - \lim_{n\to\infty}\frac{1}{\|\mu_n\|^p}\sum_{\{\zeta\,||\mu|\ge R\}}\varpi(\zeta) \\
&\quad + \lim_{n\to\infty}\frac{1}{\|\mu_n\|^p}\sum_{\{\zeta\,||\mu|<R\}}\left[-\left(\frac{C}{\varphi}+C_1\right)|\mu_n|^p\right] \\
&\ge \left(\frac{1}{p}-\frac{1}{\varphi}\right)C_E\min\{a,1\} - \alpha\lim_{n\to\infty}\|\nu_n\|_p^p - \lim_{n\to\infty}\frac{1}{\|\mu_n\|^p}\left\|\varpi(\zeta)\right\|_1 \\
&\quad - \left(\frac{C}{\varphi}+C_1\right)\lim_{n\to\infty}\|\nu_n\|_p^p \\
&= \left(\frac{1}{p}-\frac{1}{\varphi}\right)C_E\min\{a,1\},
\end{aligned}$$

where $C_E > 0$ is a constant. This is a contradiction. If $\nu \neq 0$, then we set

$$\Omega := \{\zeta \in \mathbb{Z} | \nu(\zeta) \neq 0\} \neq \emptyset.$$

For all $\xi \in \Omega$, we have

$$|\mu_n| = |\nu_n| \cdot \|\mu_n\| \to \infty \quad \text{as } n \to \infty.$$

From (f_2) we obtain

$$\lim_{n\to\infty} \frac{F(\xi,\mu_n)}{|\mu_n|^{2p}} |\nu_n|^{2p} = +\infty.$$

Together with the Fatou lemma, this means that

$$\lim_{n\to\infty} \sum_{\xi\in\Omega} \frac{F(\xi,\mu_n)}{|\mu_n|^{2p}} |\nu_n|^{2p} = \lim_{n\to\infty} \sum_{\xi\in\Omega} \frac{F(\xi,\mu_n)}{\|\mu_n\|^{2p}} = +\infty. \tag{3.3}$$

By (f_2) there exists $S \in (0,1)$ such that

$$F(\xi,\mu) > 0 \quad \text{for all } \xi \in \mathbb{Z} \text{ and } |\mu| > S. \tag{3.4}$$

For fixed S, by (f_1) we have

$$\left|F(\xi,\mu)\right| \le C_1 |\mu|^p \quad \text{for all } \xi \in \mathbb{Z} \text{ and } |\mu| \le S. \tag{3.5}$$

With the help of (3.4) and (3.5), we acquire

$$F(\xi,\mu) \ge -C_1 |\mu|^p \quad \text{for all } \xi \in \mathbb{Z} \text{ and } \mu \in \mathbb{R}.$$

By Lemma 2.3 from this inequality we have

$$\begin{aligned} \lim_{n\to\infty} \sum_{\xi\in\mathbb{Z}\setminus\Omega} \frac{F(\xi,\mu_n)}{|\mu_n|^{2p}} |\nu_n|^{2p} &= \lim_{n\to\infty} \sum_{\xi\in\mathbb{Z}\setminus\Omega} \frac{F(\xi,\mu_n)}{\|\mu_n\|^{2p}} \\ &\ge \lim_{n\to\infty} -\frac{C_1}{\|\mu_n\|^{2p}} \sum_{\xi\in\mathbb{Z}\setminus\Omega} |\mu_n|^p \\ &\ge \lim_{n\to\infty} -C_1 \frac{\|\mu_n\|_p^p}{\|\mu_n\|^{2p}} \\ &\ge \lim_{n\to\infty} -\frac{C_1 C_p}{\|\mu_n\|^p} = 0. \end{aligned} \tag{3.6}$$

By the definition of J and Lemma 2.2 there exists a positive constant C_e such that

$$\begin{aligned} J(\mu_n) + \sum_{\xi\in\mathbb{Z}} F(\xi,\mu_n) &= \frac{1}{p}\left(a[\mu_n]_{s,p}^p + \|\mu_n\|^p\right) + \frac{b}{2p}\left([\mu_n]_{s,p}^p\right)^2 \\ &\le \frac{1}{p} \max\{a,1\} \|\mu_n\|_E^p + \frac{b}{2p} \|\mu_n\|_E^{2p} \end{aligned}$$

$$\leq \frac{C_e}{p} \max\{a, 1\} \|\mu_n\|^p + \frac{bC_e^2}{2p} \|\mu_n\|^{2p}.$$

Dividing both sides of this inequality by $\|\mu_n\|^{2p}$ and taking the limit as $n \to \infty$, by (3.3) and (3.6) we deduce that

$$\frac{bC_e^2}{2p} \geq \lim_{n\to\infty} \sum_{\xi\in\mathbb{Z}} \frac{F(\xi,\mu_n)}{\|\mu_n\|^{2p}} = \lim_{n\to\infty} \left(\sum_{\xi\in\Omega} \frac{F(\xi,\mu_n)}{\|\mu_n\|^{2p}} + \sum_{\xi\in\mathbb{Z}\setminus\Omega} \frac{F(\xi,\mu_n)}{\|\mu_n\|^{2p}} \right) = +\infty.$$

This is also a contradiction. So $\{\mu_n\}$ is bounded in E.

Now we verify that $\mu_n \to \mu$ in E. By the above discussion and Lemma 2.2, up to a subsequence (still denoted by $\{\mu_n\}$), we assume that $\mu_n \rightharpoonup \mu$ in E. Because $E = \overline{\operatorname{span}\{e_n | n = 1, 2, \ldots\}} = \overline{\bigcup_n Y_n}$, where Y_n are finite-dimensional spaces, we can choose $\nu_n \in Y_n$ such that $\nu_n \to \mu$ in E. Hence we acquire

$$\begin{aligned}
&\langle J'(\mu_n) - J'(\mu), \mu_n - \mu \rangle \\
&\quad = \langle J'(\mu_n), \mu_n - \mu \rangle - \langle J'(\mu), \mu_n - \mu \rangle \\
&\quad = \langle J'|_{Y_n}(\mu_n), \mu_n - \nu_n \rangle - \langle J'(\mu_n), \mu - \nu_n \rangle - \langle J'(\mu), \mu_n - \mu \rangle \\
&\quad \leq \| J'|_{Y_n}(\mu_n) \| \cdot \|\mu_n - \nu_n\| - \langle J'(\mu_n), \mu - \nu_n \rangle \\
&\qquad - \langle J'(\mu), \mu_n - \mu \rangle \to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{3.7}$$

Besides, by Lemma 2.3 we have

$$\begin{aligned}
&\mu_n \to \mu \quad \text{in } \ell^p, \\
&\mu_n \to \mu \quad \text{a.e. in } \mathbb{Z}
\end{aligned} \tag{3.8}$$

as $n \to \infty$, and there is a function $\vartheta \in \ell^p$ such that

$$|\mu_n| \leq \vartheta \quad \text{a.e. for all } n \in \mathbb{Z}. \tag{3.9}$$

By (f_1), $f(\kappa, \cdot) \in C(\mathbb{R}, \mathbb{R})$ for all $\kappa \in \mathbb{Z}$, (3.8), (3.9), and Lebesgue's dominated convergence theorem we have

$$\sum_{\xi\in\mathbb{Z}} f(\xi,\mu_n)\mu_n \to \sum_{\xi\in\mathbb{Z}} f(\xi,\mu_n)\mu \tag{3.10}$$

and

$$\sum_{\xi\in\mathbb{Z}} f(\xi,\mu)\mu_n \to \sum_{\xi\in\mathbb{Z}} f(\xi,\mu)\mu \tag{3.11}$$

as $n \to \infty$. Combining (3.10) and (3.11), we infer that

$$\sum_{\xi\in\mathbb{Z}} \big(f(\xi,\mu_n) - f(\xi,\mu)\big)(\mu_n - \mu) \to 0 \quad \text{as } n \to \infty. \tag{3.12}$$

Then we define the functional $D(\mu): E \to \mathbb{R}$ by

$$\langle D(\mu), \upsilon \rangle = \sum_{\zeta\in\mathbb{Z}} \sum_{\xi\in\mathbb{Z}} \left|\mu(\zeta) - \mu(\xi)\right|^{p-2} \left(\mu(\zeta) - \mu(\xi)\right)\left(\upsilon(\zeta) - \upsilon(\xi)\right) K_{s,p}(\zeta - \xi), \quad \upsilon \in E.$$

We claim that $D(\mu)$ is a continuous linear functional. Indeed, by the Hölder inequality we obtain

$$\begin{aligned}
\left|\langle D(\mu), \upsilon \rangle\right| &\le \sum_{\zeta\in\mathbb{Z}} \sum_{\xi\in\mathbb{Z}} \left|\mu(\zeta) - \mu(\xi)\right|^{p-1} \left|\upsilon(\zeta) - \upsilon(\xi)\right| K_{s,p}(\zeta - \xi) \\
&\le \left(\sum_{\zeta\in\mathbb{Z}} \sum_{\xi\in\mathbb{Z}} \left|\mu(\zeta) - \mu(\xi)\right|^{(p-1)\frac{p}{p-1}} K_{s,p}(\zeta - \xi) \right)^{\frac{p-1}{p}} \\
&\quad \times \left(\sum_{\zeta\in\mathbb{Z}} \sum_{\xi\in\mathbb{Z}} \left|\upsilon(\zeta) - \upsilon(\xi)\right|^{p} K_{s,p}(\zeta - \xi) \right)^{\frac{1}{p}} \\
&\le \|\mu\|_E^{p-1} \|\upsilon\|_E.
\end{aligned}$$

We know from the above argument that

$$\langle D(\mu), \mu_n - \mu \rangle \to 0 \quad \text{as } n \to \infty. \tag{3.13}$$

Restate the previous definition and set some new definitions:

$$\begin{aligned}
&\lambda(\mu) = |\mu|^{p-2}\mu, \qquad \Theta(\zeta) = \mu_n(\zeta) - \mu(\zeta), \\
&\Delta_n = \mu_n(\zeta) - \mu_n(\xi), \qquad \Delta = \mu(\zeta) - \mu(\xi).
\end{aligned}$$

Recall the fundamental inequality

$$\left(|x|^{p-2}x - |y|^{p-2}y\right)(x-y) \ge \begin{cases} c_p|x-y|^2(|x|+|y|)^{p-2}, & 1 < p < 2, \\ C_p|x-y|^p, & p \ge 2, \end{cases}$$

where c_p and C_p are two positive constants depending only on p. By the fundamental inequality we can deduce that

$$\begin{aligned}
&\langle J'(\mu_n) - J'(\mu), \mu_n - \mu \rangle \\
&\quad = \langle J'(\mu_n), \mu_n - \mu \rangle - \langle J'(\mu), \mu_n - \mu \rangle \\
&\quad = \left(a + b[\mu_n]_{s,p}^p\right) \sum_{\zeta\in\mathbb{Z}} \sum_{\xi\in\mathbb{Z}} \lambda(\Delta_n)\left(\Theta(\zeta) - \Theta(\xi)\right) K_{s,p}(\zeta - \xi) \\
&\qquad - \left(a + b[\mu]_{s,p}^p\right) \sum_{\zeta\in\mathbb{Z}} \sum_{\xi\in\mathbb{Z}} \lambda(\Delta)\left(\Theta(\zeta) - \Theta(\xi)\right) K_{s,p}(\zeta - \xi) \\
&\qquad + \sum_{\zeta\in\mathbb{Z}} V(\zeta)\left(\lambda(\mu_n) - \lambda(\mu)\right)\Theta(\zeta) \\
&\qquad - \sum_{\zeta\in\mathbb{Z}} \left(f\left(\zeta, \mu_n(\zeta)\right) - f\left(\zeta, \mu(\zeta)\right)\right)\left(\mu_n(\zeta) - \mu(\zeta)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \left(a + b[\mu_n]_{s,p}^p\right) \sum_{\zeta\in\mathbb{Z}} \sum_{\xi\in\mathbb{Z}} \left(\lambda(\Delta_n) - \lambda(\Delta)\right)\left(\Theta(\zeta) - \Theta(\xi)\right) K_{s,p}(\zeta - \xi) \\
&\quad - b\left([\mu]_{s,p}^p - [\mu_n]_{s,p}^p\right) \sum_{\zeta\in\mathbb{Z}} \sum_{\xi\in\mathbb{Z}} \lambda(\Delta)\left(\Theta(\zeta) - \Theta(\xi)\right) K_{s,p}(\zeta - \xi) \\
&\quad + \sum_{\zeta\in\mathbb{Z}} V(\zeta)\left(\lambda(\mu_n) - \lambda(\mu)\right)\Theta(\zeta) \\
&\quad - \sum_{\zeta\in\mathbb{Z}} \left(f\left(\zeta, \mu_n(\zeta)\right) - f\left(\zeta, \mu(\zeta)\right)\right)\left(\mu_n(\zeta) - \mu(\zeta)\right) \\
&\geq a \sum_{\zeta\in\mathbb{Z}} \sum_{\xi\in\mathbb{Z}} \left(\lambda(\Delta_n) - \lambda(\Delta)\right)(\Delta_n - \Delta) K_{s,p}(\zeta - \xi) \\
&\quad + \sum_{\zeta\in\mathbb{Z}} V(\zeta)\left(\lambda(\mu_n) - \lambda(\mu)\right)\Theta(\zeta) \\
&\quad - b\left([\mu]_{s,p}^p - [\mu_n]_{s,p}^p\right) \sum_{\zeta\in\mathbb{Z}} \sum_{\xi\in\mathbb{Z}} \lambda(\Delta)(\Delta_n - \Delta) K_{s,p}(\zeta - \xi) \\
&\quad - \sum_{\zeta\in\mathbb{Z}} \left(f\left(\zeta, \mu_n(\zeta)\right) - f\left(\zeta, \mu(\zeta)\right)\right)\left(\mu_n(\zeta) - \mu(\zeta)\right) \\
&\geq C_f \min\{a, 1\} \|\mu_n - \mu\|_E^p \\
&\quad - b\left([\mu]_{s,p}^p - [\mu_n]_{s,p}^p\right) \sum_{\zeta\in\mathbb{Z}} \sum_{\xi\in\mathbb{Z}} \lambda(\Delta)(\Delta_n - \Delta) K_{s,p}(\zeta - \xi) \\
&\quad - \sum_{\zeta\in\mathbb{Z}} \left(f\left(\zeta, \mu_n(\zeta)\right) - f\left(\zeta, \mu(\zeta)\right)\right)\left(\mu_n(\zeta) - \mu(\zeta)\right),
\end{aligned}
\tag{3.14}
$$

where C_f is a positive constant. Together with (3.7), (3.12), (3.13), and (3.14), we can deduce that $\|\mu_n - \mu\|_E \to 0$ as $n \to \infty$ and get $\mu_n \to \mu$ in E. So J satisfies condition $(C)^*$. □

4 Proof of Theorem 1.1

In this section, we use the fountain theorem to prove Theorem 1.1. Let us first recall this theorem.

Theorem 4.1 (See [7]) *Let H be a Banach space, and let $\Psi \in C^1(H, \mathbb{R})$ be an even functional. Assume that $\forall \kappa \in \mathbb{N}$, $\exists r_\kappa > \gamma_\kappa > 0$ such that*

(T_1) $\inf\{\Psi(\mu) | \mu \in Z_\kappa, \|\mu\|_H = \gamma_\kappa\} \to \infty$ *as* $\kappa \to \infty$;

(T_2) $\max\{\Psi(\mu) | \mu \in Y_\kappa, \|\mu\|_H = r_\kappa\} \leq 0$;

(T_3) Ψ *satisfies condition* (C).

Then there exists $\{\mu_d\} \subset H$ such that $\Psi'(\mu_d) = 0$ and $\Psi(\mu_d) \to \infty$ as $d \to \infty$.

Proof of Theorem 1.1 By Lemma 3.2 and the definition of J, J is even and satisfies (T_3). Next, we just need to verify conditions (T_1) and (T_2) of Theorem 4.1.

Verification of (T_1): For $\mu \in Z_\kappa$, by (f_1) we get

$$
\begin{aligned}
J(\mu) &= \frac{1}{p}\left(a[\mu]_{s,p}^p + \|\mu\|^p\right) + \frac{b}{2p}\left([\mu]_{s,p}^p\right)^2 - \sum_{\zeta\in\mathbb{Z}} F(\zeta, \mu) \\
&\geq \frac{C_E \min\{a, 1\}}{p} \|\mu\|^p - \sum_{\zeta\in\mathbb{Z}} F(\zeta, \mu)
\end{aligned}
$$

$$\geq \frac{C_E \min\{a,1\}}{p}\|\mu\|^p - C_1\|\mu\|_p^p - C_2\|\mu\|_q^q. \tag{4.1}$$

In terms of Lemma 3.1, there is a sufficiently large $m \in \mathbb{N}_+$ such that

$$\|\mu\|_p^p = \left\|\frac{\mu}{\|\mu\|}\right\|_p^p \|\mu\|^p \leq \frac{C_E \min\{a,1\}}{2pC_1}\|\mu\|^p \quad \text{for any } \mu \in Z_m. \tag{4.2}$$

Combining (4.1), (4.2), and Lemma 3.1, we obtain

$$\begin{aligned} J(\mu) &\geq \frac{C_E \min\{a,1\}}{p}\|\mu\|^p - \frac{C_E \min\{a,1\}}{2p}\|\mu\|^p - C_2\left\|\frac{\mu}{\|\mu\|}\right\|_q^q \|\mu\|^q \\ &\geq \frac{C_E \min\{a,1\}}{2p}\|\mu\|^p - C_2\beta_\kappa^q(q)\|\mu\|^q \qquad (4.3) \\ &= \|\mu\|^p\left(\frac{C_E \min\{a,1\}}{2p} - C_2\beta_\kappa^q(q)\|\mu\|^{q-p}\right), \qquad (4.4) \end{aligned}$$

where $\kappa > m$ is large enough. Choose

$$\gamma_\kappa = \left(\frac{C_E \min\{a,1\}}{2qC_2\beta_\kappa^q(q)}\right)^{\frac{1}{q-p}}. \tag{4.5}$$

Thanks to Lemma 3.1 and $1 < p < q$, it is easy to see that $\gamma_\kappa \to \infty$ as $\kappa \to \infty$. Together with (4.4) and (4.5), for any $\mu \in Z_\kappa$ and $\|\mu\| = \gamma_\kappa$, we have

$$\begin{aligned} J(\mu) &\geq \|\mu\|^p\left(\frac{C_E \min\{a,1\}}{2p} - C_2\beta_\kappa^q(q)\frac{C_E \min\{a,1\}}{2qC_2\beta_\kappa^q(q)}\right) \\ &= \frac{C_E \min\{a,1\}}{2}\left(\frac{1}{p} - \frac{1}{q}\right)\gamma_\kappa^p \to \infty \end{aligned}$$

as $\kappa \to \infty$. Hence (T_1) is established.

Verification of (T_2)*:* By the definition of Y_κ we know that Y_κ is finite-dimensional, so there exists a positive constant C_F such that

$$\|\mu\|_E^{2p} \leq C_F\|\mu\|_{2p}^{2p}. \tag{4.6}$$

For $M > \frac{bC_F}{2p} > 0$, by (f_2) there exists $L \in (0,1)$ such that

$$F(\xi,\mu) > M\mu^{2p} \quad \text{for all } \xi \in \mathbb{Z} \text{ and } |\mu| > L. \tag{4.7}$$

For fixed L, by (f_1), we obtain

$$F(\xi,\mu) \geq -C_1|\mu|^p \quad \text{for all } \xi \in \mathbb{Z} \text{ and } |\mu| \leq L. \tag{4.8}$$

With the help of (4.7) and (4.8), we acquire

$$F(\xi,\mu) \geq M\mu^{2p} - C_1|\mu|^p \quad \text{for all } \xi \in \mathbb{Z} \text{ and } \mu \in \mathbb{R}. \tag{4.9}$$

For $\mu \in Y_\kappa$, by Lemma 2.3, (4.6), and (4.9) we have

$$\begin{aligned} J(\mu) &= \frac{1}{p}\left(a[\mu]_{s,p}^p + \|\mu\|^p\right) + \frac{b}{2p}\left([\mu]_{s,p}^p\right)^2 - \sum_{\zeta\in\mathbb{Z}} F(\zeta,\mu) \\ &\le \frac{\max\{a,1\}}{p}\|\mu\|_E^p + \frac{b}{2p}\|\mu\|_E^{2p} - M\|\mu\|_{2p}^{2p} + C_1\|\mu\|_p^p \\ &\le \frac{\max\{a,1\}}{p}\|\mu\|_E^p + \frac{b}{2p}\|\mu\|_E^{2p} - \frac{M}{C_F}\|\mu\|_E^{2p} + C'\|\mu\|_E^p \\ &= C''\|\mu\|_E^p - \left(\frac{M}{C_F} - \frac{b}{2p}\right)\|\mu\|_E^{2p}, \end{aligned} \tag{4.10}$$

where $C', C'' > 0$ are two constants. For any $\mu \in Y_\kappa$ and $\|\mu\| = r_\kappa$, we get

$$J(\mu) \le 0,$$

provided that $r_\kappa > \gamma_\kappa > 1$ are sufficiently large. Therefore (T_2) is proved.

By applying Lemma 2.7 and Theorem 4.1, we obtain that problem (1.5) possesses infinite nontrivial homoclinic solutions with unbounded energy. □

5 Proof of Theorem 1.2

In the last section, we show Theorem 1.2 with the aid of the dual fountain theorem, which is given below for the reader's convenience.

Theorem 5.1 (See [9]) *Let H be a Banach space, and let $\Psi \in C^1(H,\mathbb{R})$ be an even functional. Assume that $\forall\kappa \ge \kappa_0$, $\exists r_\kappa > \gamma_\kappa > 0$, such that*

(D_1) $\inf\{\Psi(\mu) | \mu \in Z_\kappa, \|\mu\|_H = r_\kappa\} \ge 0$;

(D_2) $a_\kappa := \max\{\Psi(\mu) | \mu \in Y_\kappa, \|\mu\|_H = \gamma_\kappa\} < 0$;

(D_3) $b_\kappa := \inf\{\Psi(\mu) | \mu \in Z_\kappa, \|\mu\|_H \le r_\kappa\} \to 0$ *as* $\kappa \to \infty$;

(D_4) Ψ *satisfies condition* $(C)^*$.

Then there exists $\{\mu_d\} \subset H$ such that $\Psi'(\mu_d) = 0$, $\Psi(\mu_d) < 0$, and $\Psi(\mu_d) \to 0$ as $d \to \infty$.

Proof of Theorem 1.2 By Lemma 3.2 and the definition of J, J is even and satisfies (D_4). Next, we just need to verify conditions (D_1), (D_2), and (D_3) of Theorem 4.1.

Verification of (D_1): For $\mu \in Z_\kappa$ and (4.3), we can choose

$$r_\kappa = \left(\frac{4pC_2}{C_E \min\{a,1\}}\beta_\kappa^q(q)\right)^{\frac{1}{p-q}}. \tag{5.1}$$

Clearly, by Lemma 3.1 we know that $\lim_{\kappa\to\infty} r_\kappa = \infty$. Then there exists $\kappa_0 \in \mathbb{N}_+$ such that $r_\kappa > 1$ for $\kappa \ge \kappa_0$. Letting $\|\mu\| = r_\kappa$, by (4.3) we can derive that

$$J(\mu) \ge \frac{C_E \min\{a,1\}}{2p}\|\mu\|^p - C_2\beta_\kappa^q(q)\|\mu\|^q = \frac{C_E \min\{a,1\}}{4p}\|\mu\|^p \ge 0. \tag{5.2}$$

So condition (D_1) is satisfied.

Verification of (D_2): For $\mu \in Y_\kappa$ and (4.10), we can also find $M > \frac{bC_F}{2p} > 0$ large enough. Then we deduce that

$$J(\mu) \to -\infty \quad \text{as } \|\mu\|_E \to +\infty.$$

Hence there exists $1 < \gamma_\kappa < \infty$ such that

$$J(\mu) < 0 \quad \text{for all } \mu \in Y_\kappa \text{ with } \|\mu\|_E = \gamma_\kappa. \tag{5.3}$$

Then we can find $\kappa_1 > \kappa_0$ such that $r_\kappa > \gamma_\kappa > 1$ for all $\kappa \geq \kappa_1$. Hence we can conclude that (D_2) is fulfilled.

Verification of (D_3)*:* By means of $Y_\kappa \cap Z_\kappa \neq \emptyset$, $1 < \gamma_\kappa < r_\kappa$, and (5.3) we get

$$\begin{aligned} b_\kappa &= \inf_{\mu \in Z_\kappa, \|\mu\| \leq r_\kappa} J(\mu) \\ &\leq \inf_{\mu \in Z_\kappa, \|\mu\| \leq \gamma_\kappa} J(\mu) \\ &\leq \inf_{\mu \in Y_\kappa \cap Z_\kappa, \|\mu\| \leq \gamma_\kappa} J(\mu) \\ &\leq \inf_{\mu \in Y_\kappa \cap Z_\kappa, \|\mu\| = \gamma_\kappa} J(\mu) \\ &\leq \max_{\mu \in Y_\kappa \cap Z_\kappa, \|\mu\| = \gamma_\kappa} J(\mu) \\ &\leq \max_{\mu \in Y_\kappa, \|\mu\| = \gamma_\kappa} J(\mu) = a_\kappa < 0. \end{aligned} \tag{5.4}$$

Therefore, for $\nu \in Z_\kappa$ with $\|\nu\| = 1$, $\mu = t\nu$ with $0 < t \leq r_\kappa$, and (4.3), we obtain

$$\begin{aligned} J(\mu) &= J(t\nu) \\ &\geq \frac{C_E \min\{a, 1\}}{2p} \|t\nu\|^p - C_2 \beta_\kappa^q(q) \|t\nu\|^q \\ &\geq -C_2 t^q \beta_\kappa^q(q) \|\nu\|^q \geq -C_2 r_\kappa^q \beta_\kappa^q(q) \|\nu\|^q. \end{aligned} \tag{5.5}$$

Combining (5.4) and (5.5), we derive that

$$0 > b_\kappa \geq -C_2 r_\kappa^q \beta_\kappa^q(q) \|\nu\|^q \quad \text{for all } \kappa \geq \kappa_0.$$

By Lemma 3.1 we know that $b_\kappa \to 0$ as $\kappa \to \infty$. Consequently, (D_3) also holds.

By means of Theorem 5.1 and Lemma 2.7 we get that Eq. (1.5) has infinitely many non-trivial homoclinic solutions with negative energy converging to 0. □

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Declarations

Competing interests
The authors declare no competing interests.

Author contributions
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