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Enriched Płonka sums

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Abstract. Płonka sums are a powerful technique for the representation of algebras in regular varieties. However, certain representations of algebras in irregular varieties—like Polin's variety or the variety of pseudocomplemented semilattices—bear striking similarities to Płonka sums, although they differ from them in some important respects. We aim at finding a convenient umbrella under which these constructions, as well as other ones of a similar kind, can be subsumed. Inspired by Grätzer and Sichler's work on *Agassiz sums*, we appropriately enrich the structure of semilattice direct systems and we modify the attendant definition of a sum, while still encompassing Płonka sums as a special case. We prove that the above-mentioned representations of Polin algebras and pseudocomplemented semilattices can be recast in terms of this new framework. Finally, we investigate the problem as to which identities are preserved by the construction.

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1. Introduction

Płonka sums are a simple yet powerful technique for building new algebras out of semilattice direct systems of algebras of a fixed type. First introduced by Płonka [36] in 1967, this construction immediately proved to be essential in establishing representation theorems for regularisations of strongly irregular varieties [37] and paved the way for the fundamental theory of semilattice sums of algebras [39]. The aptness of this construction to provide information about the structure of regular varieties can be seen, at the same time, as a shortcoming, since Płonka sums are of little avail outside this particular domain of application.

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Remarkably, however, the literature teems with "Płonka-type" constructions used for the representation of algebras in *ir*regular varieties, or in varieties having certain properties stronger than regularity. All these constructions bear striking similarities to Płonka sums, but differ from them in some important respects. Examples include the representation of Polin algebras [35], Katriňák and Guričan's representation of pseudocomplemented semilattices [28], or the recently introduced construction of De Morgan–Płonka sums [40]. The purpose of the present paper is to find a convenient umbrella under which Płonka sums, as well as the other mentioned constructions, can be subsumed.

We take our cue from a 1974 paper by Grätzer and Sichler [25], who aimed at generalising Płonka sums to the case of *transitive* systems of summands with *arbitrary* indexing algebras. Shortly thereafter, Agassiz sums were thoroughly studied by Graczyńska and Wroński [23], who obtained conspicuously different results as regards the identities preserved by the construction. Although it is clear that the Agassiz sum construction encompasses Płonka sums as a special case, the mentioned papers fail to exhibit other significant examples covered by their generalisation. In particular, none of the above Płonka-type representations can be successfully formulated within this framework.

If we aim to properly address these examples, it becomes clear that the notion of an Agassiz system is, in some respects, too general, while in others, still too restrictive. On one hand, we do not require the full generality of allowing indexing algebras to be arbitrary—semilattices with an additional unary operation suffice for this purpose. On the other hand, instead of the single transitive relation that determines the homomorphisms in Agassiz systems, we need to include *two* distinct relations and, accordingly, two different classes of functions that connect the summands. We show that Płonka sums can be viewed as sums over appropriate modifications of semilattice direct systems, while the above-cited representations of Polin algebras and pseudocomplemented semilattices, as well as certain representations of algebras as sums over involutive semilattice systems of algebras, can also be recast in terms of this framework. Finally, we investigate the problem as to which identities are preserved by our construction.

The paper is structured as follows. In Section 2 we review some basic notions concerning Płonka sums and we revisit a generalisation of the concept of a regular variety proposed in [25]. In Section 3 we introduce the main concept about which the paper is centred, namely, that of an enriched Płonka sum over an E-system (enriched semilattice direct system) of algebras. In Section 4 we show that the notion is flexible enough to encompass Płonka sums and the other constructions mentioned above. In Section 5, we consider varieties of algebras $\mathfrak{V}$, whose members can be represented in terms of enriched Płonka sums of algebras in a given subvariety $\mathfrak{W}$, that are generated by algebras having "well-behaved" representations. Whenever this condition is satisfied, the identities holding in $\mathfrak{W}$ that are preserved in $\mathfrak{V}$ are precisely those that satisfy a certain generalisation of the regularity condition. In Section 6, we examine some consequences of the preceding results in the case of Polin's variety. We conclude in Section 7.

2. Preliminaries

Our terminology and notation is, for the most part, standard. The reader is referred to [5] for any undefined notions or symbols, while deviations from the established usage will be appropriately flagged. In particular, all similarity types considered hereafter will contain at least one operation symbol of arity $\geq$ 2. When they contain no constant, such types will be called *plural*.

Let τ be a type, and let X be a denumerable set of variables. We denote by $\mathbf{T}_\tau(X)$ the absolutely free algebra of τ-terms, with universe $T_\tau(X)$. If $p, q \in T_\tau(X)$ and $x \in X$, $Var(p)$ will denote the set of all variables in X contained in p, and $p(x/q)$ will denote the result of uniformly replacing all occurrences of x in p by q.

If $\mathbf{A}$ is a τ-algebra, $\mathrm{Id}(\mathbf{A})$ will denote the set of all τ-identities satisfied by $\mathbf{A}$, and if $\mathfrak{V}$ is a variety of type τ, $\mathrm{Id}(\mathfrak{V})$ will denote the set of all τ-identities satisfied by all $\mathbf{A} \in \mathfrak{V}$. Finally, $\mathfrak{SL}$ and $\mathfrak{SL}_0$ respectively denote the varieties of join-semilattices and join-semilattices with zero.

2.1. Płonka sums and regular varieties

As recalled in our introduction, Płonka sums (for which see e.g. [36, 37, 41, 3]) are an extremely useful construction in universal algebra, especially designed for the investigation of varieties satisfying only regular identities. We recap hereafter the main definitions and concepts concerning them.

Definition 2.1. (*Semilattice direct system*) Let $\tau \colon \Omega \to \mathbb{N}$ be a similarity type. A *semilattice direct system* of τ-algebras is a triple

$$\mathbb{S} = \left\langle \mathbf{S}, \{\mathbf{A}_s\}_{s\in S}, \mathrm{Hom}_{\leq}(\mathbb{S}) \right\rangle$$

where:

- (s$_1$) $\mathbf{S}$ is a join-semilattice $\langle S, \vee \rangle$ if $\tau^{-1}[0] = \emptyset$, a join-semilattice with zero $\langle S, \vee, 0 \rangle$ otherwise;
- (s$_2$) $\{\mathbf{A}_s\}_{s\in S}$ is an S-indexed family of τ-algebras with pairwise disjoint universes;
- (s$_3$) $\mathrm{Hom}_{\leq}(\mathbb{S}) = \{\varphi_{st} \in \mathrm{Hom}(\mathbf{A}_s, \mathbf{A}_t) \mid s \leq t\}$ is a family of homomorphisms such that:
  - for all $s \in S$, $\varphi_{ss} = \mathrm{id}_{\mathbf{A}_s}$;
  - for all $s, t, u \in S$, $\varphi_{tu} \circ \varphi_{st} = \varphi_{su}$ whenever $s \leq t \leq u$.

If $\mathbb{S} = \left\langle \mathbf{S}, \{\mathbf{A}_s\}_{s\in S}, \mathrm{Hom}_{\leq}(\mathbb{S}) \right\rangle$ is a semilattice direct system of τ-algebras, a new τ-algebra $\mathbf{Pł}(\mathbb{S})$ can be defined as follows:

Definition 2.2. (*Płonka sum*) The *Płonka sum* over $\mathbb{S}$ is the τ-algebra

$$\mathbf{Pł}(\mathbb{S}) = \left\langle \bigcup_{s\in S} A_s, \left\{ \omega^{\mathbf{Pł}(\mathbb{S})} \right\}_{\omega\in\Omega} \right\rangle$$

where:

(ps_1) for every n-ary $\omega \in \Omega$ (with $n \geq 1$) and any $a_1, \ldots, a_n \in \bigcup_{s \in S} A_s$,

$$\omega^{\mathbf{P}\mathbf{ł}(\mathbb{S})}(a_1, \ldots, a_n) = \omega^{\mathbf{A}_s}(\varphi_{s_1 s}(a_1), \ldots, \varphi_{s_n s}(a_n)),$$

where $a_1 \in A_{s_1}, \ldots, a_n \in A_{s_n}$ and $s = s_1 \vee \cdots \vee s_n$;

(ps_2) for every 0-ary $\omega \in \Omega$, $\omega^{\mathbf{P}\mathbf{ł}(\mathbb{S})} = \omega^{\mathbf{A}_0}$.

We sometimes refer to the algebras in $\{\mathbf{A}_s\}_{s \in S}$ as the *fibres* of $\mathbf{Pł}(\mathbb{S})$. By extension, the same term will be used in the context of the various generalisations of Płonka sums considered below.

Recall that an identity $p \approx q$ is regular whenever $Var(p) = Var(q)$. It is natural to wonder which of the identities that are satisfied in all the fibres are preserved by a Płonka sum. It turns out that, among such identities, the construction preserves precisely the regular ones. This is the content of the following:

Theorem 2.3. [36, Thm. I] *If* $\mathbb{S}$ *is a semilattice direct system of algebras containing at least two algebras, then all regular identities satisfied in all algebras of* $\mathbb{S}$ *are satisfied in* $\mathbf{Pł}(\mathbb{S})$, *whereas any other identity is not satisfied in* $\mathbf{Pł}(\mathbb{S})$.

We call *regular* a variety of algebras that satisfies only regular identities, and *irregular* a variety that is not regular. A variety $\mathfrak{V}$ is *strongly irregular* if there is an essentially binary term $p(x, y)$ such that $\mathfrak{V} \models p(x, y) \approx x$. In other words, a variety is strongly irregular if there is an essentially binary term realising the projection operation on the first component in all algebras in the variety. Obviously, any strongly irregular variety is also irregular. To each variety $\mathfrak{V}$ is associated a variety $R(\mathfrak{V})$ that satisfies all and only the regular identities holding in $\mathfrak{V}$. $R(\mathfrak{V})$ is called the *regularisation* of $\mathfrak{V}$. Elements of the regularisation of a strongly irregular variety can always be represented as Płonka sums:

Theorem 2.4. [37, Thm. I] *Let* $\mathfrak{V}$ *be a strongly irregular variety of type* τ, *and let* $\mathbf{A}$ *be a* τ*-algebra. Then* $\mathbf{A} \in R(\mathfrak{V})$ *iff* $\mathbf{A}$ *is decomposable as a Płonka sum over a semilattice direct system of algebras in* $\mathfrak{V}$.

2.2. Naming functions and N-regularity

In a semilattice direct system of algebras, as the name itself suggests, the indexing algebra is always a semilattice, or a semilattice with zero if the type contains constants. Suppose that we want to compute $\omega^{\mathbf{P}\mathbf{ł}(\mathbb{S})}(a_1, \ldots, a_n)$, where ω is n-ary and each a_k is in the fibre A_{s_k}. According to the stipulations of Definition 2.2, this is done by homomorphically mapping $a_1, \ldots, a_n$ to the fibre whose index is the semilattice join $s_1 \vee \cdots \vee s_n$. Recall, moreover, that two fibres with respective indices s and t are linked by a homomorphism just in case $s \leq t$.

In [25], Grätzer and Sichler investigate more general systems where the indexing algebra can be arbitrary, and where two fibres are linked by a homomorphism just in case they stand in a fixed transitive binary relation on the universe of such an indexing algebra. The interpretation of complex terms continues to be dictated by such homomorphisms, as in the Płonka case, but the fibre where each term is computed, given that semilattice joins need no longer be available, must be determined in a more roundabout way, via a *naming function* that translates terms in the type of the fibres to terms in the type of the indexing algebra. Since this tool will be crucial in the paper's discourse, we introduce it explicitly, together with a few attendant notions that will be used later on.

Most of the concepts and results in this subsection are taken either from [25] or from [23]. Hereafter, let $\tau : \Omega \to \mathbb{N}$ and $\rho : \Gamma \to \mathbb{N}$ be similarity types and let X be a denumerable set of variables.

Definition 2.5. (*Naming function*) A *naming function* is a total map

$$N \colon T_\tau(X) \to T_\rho(X)$$

satisfying $Var(p) = Var(N(p))$, for every $p \in T_\tau(X)$.

Note that, as we are assuming that both τ and ρ have at least one operation symbol of arity ≥ 2, a necessary and sufficient condition for the existence of a naming function $N \colon T_\tau(X) \to T_\rho(X)$ is that $0 \in \tau[\Omega]$ only if $0 \in \rho[\Gamma]$.

Definition 2.6. (*Structurality class*) Let $N \colon T_\tau(X) \to T_\rho(X)$ be a naming function. A ρ-algebra $\mathbf{A}$ belongs to the *structurality class* of N ($\mathbf{A} \in \mathfrak{SC}(N)$) whenever for each n-ary $p \in T_\tau(X)$ (with $n > 0$) and all $q_1, \ldots, q_n \in T_\tau(X)$, the following conditions are satisfied:

(sc_1) $N(p(x_1/q_1, \ldots, x_n/q_n)) \approx N(p)(x_1/N(q_1), \ldots, x_n/N(q_n)) \in \mathrm{Id}(\mathbf{A})$;

(sc_2) If $p \in T_\tau(X) \cap T_\rho(X)$, then $N(p) \approx p \in \mathrm{Id}(\mathbf{A})$ if and only if $p \in X$.

Naming functions yield the following interesting generalisation of a regular identity:

Definition 2.7. (N-*regular identity*) For $N \colon T_\tau(X) \to T_\rho(X)$ a naming function and $\mathfrak{B}$ a class of ρ-algebras, let $\mathrm{Id}_N(\mathfrak{B})$ be the class of all τ-identities $p \approx q$ such that $\mathfrak{B} \models N(p) \approx N(q)$. The identities belonging to $\mathrm{Id}_N(\mathfrak{B})$ are said to be N-*regular* in $\mathfrak{B}$.

Observe that an N-regular identity need not to be regular. Accordingly, we also obtain the following generalisation of the notion of a regular variety:

Definition 2.8. (N-*regular variety*) Let $N \colon T_\tau(X) \to T_\rho(X)$ be a naming function, $\mathfrak{B}$ be a class of ρ-algebras, and $\mathfrak{V}, \mathfrak{W}$ be varieties of type τ. $\mathfrak{V}$ is said to be the N-*regularisation* in $\mathfrak{B}$ of $\mathfrak{W}$ if $\mathrm{Id}(\mathfrak{V}) = \mathrm{Id}_N(\mathfrak{B}) \cap \mathrm{Id}(\mathfrak{W})$. A variety that is the N-regularisation in $\mathfrak{B}$ of some variety is said to be N-*regular* in $\mathfrak{B}$.

Example 2.9. Let $N \colon T_\tau(X) \to T_{\langle 2,0 \rangle}(X)$ be the naming function defined by:

- $N(x) = x$, for all $x \in X$;
- $N(\omega(p_1, \ldots, p_n)) = N(p_1) \vee \cdots \vee N(p_n)$, for all $p_1, \ldots, p_n \in T_\tau(X)$ and all n-ary $\omega \in \Omega$ (with $n > 1$);
- $N(\omega(p)) = N(p)$, for all $p \in T_\tau(X)$ and all unary $\omega \in \Omega$;
- $N(\omega) = 0$, for all constants $\omega \in \Omega$ (if any).

Then $\mathrm{Id}_N(\mathfrak{SL}_0)$ is the class of regular τ-identities, and a variety $\mathfrak{V}$ is the regularisation of a variety $\mathfrak{W}$ just in case it is its N-regularisation in $\mathfrak{SL}_0$.

3. Enriched Płonka sums

We aim to build on Grätzer and Sichler's idea that allowing *any* algebra to serve as an indexing algebra provides the flexibility to encompass a wider range of examples than those covered by Płonka sums. Nevertheless, their Agassiz sums turn out not to be sufficiently versatile. Indeed, as already observed, none of the examples mentioned in the introduction makes an instance of an Agassiz sum. The reason of this state of affairs will be clarified below, when we examine each individual example. But we can already say this much: all such representations somehow seem to require an even greater flexibility in determining how the result of each term operation is computed, according to the syntactic structure of the corresponding term.

We borrow from Grätzer and Sichler the use of naming functions to obtain this result. Indeed, if we fix a system $\mathbb{X}$ and ω is an n-ary operation symbol of type τ, we determine the index $s \in S$ of the *fibre* in which the value of $\omega^{\mathbf{Pł}(\mathbb{X})}(a_1, \ldots, a_n)$ (where $a_k \in A_{s_k}$, for $1 \leq k \leq n$) is computed as $N(\omega(x_1, \ldots, x_n))^{\mathbf{S}}(s_1, \ldots, s_n)$, exactly like in Agassiz sums. On the other hand, unlike in [25], the *arguments* to which $\omega^{\mathbf{A}_s}$ is applied are the images of $a_1, \ldots, a_n$ under functions that vary depending on $N(\omega(x_1, \ldots, x_n))$. In other words, we associate different operation symbols in the type of the indexing algebra to different relations on its universe, and correspondingly, to different classes of functions that link the fibres to one another.

While we consider the most general case in the work in progress [20], in the present paper we confine ourselves to a more restricted framework, which, however, subsumes all the motivating examples mentioned in the introduction. In all such examples we have sum representations where the indexing algebra is always a semilattice (possibly with zero) with an additional unary operation. Moreover, in such representations, all operations are computed according to Definition 2.2, like in Płonka sums, and therefore by recourse to a single relation (the semilattice order) and a single family of homomorphisms (the Płonka homomorphisms), except for a unique unary operation that requires a different relation and a different class of functions. We now list a number of definitions aimed at encompassing all these examples under a common umbrella.

Definition 3.1. (*E-type*) Let $\tau_1 \colon \Omega_1 \to \mathbb{N}$ and $\tau_2 \colon \Omega_2 \to \mathbb{N}$ be two similarity types such that τ_1 is a plural type, $\tau_2 = \langle 1 \rangle$, and $\Omega_1 \cap \Omega_2 = \emptyset$. By an *E-type* we mean a similarity type $\tau \colon \Omega_1 \cup \Omega_2 \to \mathbb{N}$ such that, for $i \in \{1, 2\}$, $\tau_i = \tau \restriction_{\Omega_i}$. An *E-type with constants* differs from an E-type for the condition $0 \in \tau_1[\Omega_1]$.

Let $\mathbf{A}$ be a τ_1-algebra and let $\tau \colon \Omega_1 \cup \Omega_2 \to \mathbb{N}$ be an E-type expanding τ_1. Assuming a fixed interpretation of the unary operation symbol in Ω_2, we denote by $\mathbf{A}^+$ the expansion $\left\langle A, \{\omega^{\mathbf{A}^+}\}_{\omega \in \Omega_1 \cup \Omega_2} \right\rangle$. Where appropriate, we shall refer to $\mathbf{A}^+$ as the τ-*expansion* of $\mathbf{A}$. Similarly, by the τ-expansion of an indexed family $\{\mathbf{A}_k\}_{k \in K}$ of τ_1-algebras, we shall mean the family $\{\mathbf{A}_k^+\}_{k \in K}$.

Definition 3.2. ($\circledast$-*semilattice*) A $\circledast$-*semilattice* is an algebra $\mathbf{S}^+ = \langle S, \cdot, \circledast \rangle$ of type $\langle 2, 1 \rangle$ where $\langle S, \cdot \rangle$ is a semilattice. A *unital* $\circledast$-*semilattice* is an algebra $\mathbf{S}^+ = \langle S, \cdot, \circledast, e \rangle$ of type $\langle 2, 1, 0 \rangle$ where $\langle S, \cdot, e \rangle$ is a unital semilattice.

For $\mathbf{S}^+$ a (unital) $\circledast$-semilattice, we denote by $\mathbf{S}$ its $\{\circledast\}$-free reduct. Examples of $\circledast$-semilattices include:

(1) (Unital) involutive semilattices, i.e. (unital) commutative idempotent *-semigroups [9];
(2) Boolean algebras, defined with join (or meet), complementation, and the bottom (or top) element as primitive operations;
(3) Pseudocomplemented semilattices [18];
(4) $\{\vee, \Diamond\}$- and $\{\wedge, \Box\}$-subreducts of positive modal algebras [12];
(5) $\{\vee, ^{-1}\}$- and $\{\wedge, ^{-1}\}$-subreducts of ℓ-groups [19].

As is well known, meet-semilattices are join-semilattices in the dual orderings, and *vice versa*. Throughout this paper, unless stated otherwise, by a $\circledast$-semilattice we shall mean a join-semilattice expanded with the unary operation $\circledast$.

Definition 3.3. (*Enriched semilattice direct system*) Let $\tau_1 \colon \Omega_1 \to \mathbb{N}$ be an arbitrary similarity type and let $\mathbb{S} = \left\langle \mathbf{S}, \{\mathbf{A}_s\}_{s\in S}, \mathrm{Hom}_{\leq}(\mathbb{S}) \right\rangle$ be a semilattice direct system of τ_1-algebras. For $\tau \colon \Omega_1 \cup \Omega_2 \to \mathbb{N}$ an E-type (possibly with constants) expanding τ_1, an *enriched semilattice direct system* (*E-system*) over $\mathbb{S}$ is a quintuple

$$\mathbb{S}^+ = \left\langle \mathbf{S}^+, \{\mathbf{A}_s^+\}_{s\in S}, R, \mathrm{Fun}_{\leq}(\mathbb{S}^+), \mathrm{Fun}_R(\mathbb{S}^+) \right\rangle$$

where:

(e_1) $\mathbf{S}^+$ is a $\circledast$-semilattice over $\mathbf{S}$ if $\tau_1^{-1}[0] = \emptyset$, a $\circledast$-semilattice with zero otherwise;
(e_2) $\{\mathbf{A}_s^+\}_{s\in S}$ is the τ-expansion of $\{\mathbf{A}_s\}_{s\in S}$;
(e_3) $\mathrm{Fun}_{\leq}(\mathbb{S}^+) = \mathrm{Hom}_{\leq}(\mathbb{S})$;
(e_4) $R \subseteq S^2$ is such that $\langle s, \circledast s\rangle \in R$, for all $s \in S$;
(e_5) $\mathrm{Fun}_R(\mathbb{S}^+) = \{\varphi_{st} \in \mathrm{Fun}(A_s, A_t) \mid \langle s, t\rangle \in R\}$ is a family of functions.

Clearly, E-systems generalise semilattice direct systems of algebras. The semilattice (with zero) is replaced by an expanded semilattice (with zero) as an indexing algebra, while an additional relation and family of functions are introduced with an eye to accounting for operations that do not behave like in Płonka sums. Observe that, unlike Definition 2.2, this definition imposes no requirement on these additional components of E-systems. Furthermore, note that the objects in $\mathrm{Fun}_{\leq}(\mathbb{S}^+)$ are the Płonka homomorphism of the original semilattice direct system $\mathbb{S}$, so they are not necessarily required to be τ-homomorphisms, but just τ_1-homomorphisms.

In order to distinguish between elements of $\mathrm{Fun}_{\leq}(\mathbb{S}^+)$ and $\mathrm{Fun}_R(\mathbb{S}^+)$, we adopt the following notational convention: we write φ^1_{st} for a generic member of $\mathrm{Fun}_{\leq}(\mathbb{S}^+)$ and φ^2_{st} for a generic member of $\mathrm{Fun}_R(\mathbb{S}^+)$.

Definition 3.4. ($N_{\mathbb{S}^+}$) Let τ be an E-type (with constants), and let ρ denote the similarity type $\langle 2, 1\rangle$ ($\langle 2, 1, 0\rangle$). For an E-system of τ-algebras $\mathbb{S}^+$, we define the function $N_{\mathbb{S}^+} \colon T_\tau(X) \to T_\rho(X)$ as follows:

(f$_1$) $N_{\mathbb{S}^+}(x) = x$, for all $x \in X$;

(f$_2$) $N_{\mathbb{S}^+}(\omega(p_1, \ldots, p_n)) = N_{\mathbb{S}^+}(p_1) \vee \cdots \vee N_{\mathbb{S}^+}(p_n)$, for all n-ary $\omega \in \Omega_1$ (with $n > 1$) and all $p_1, \ldots, p_n \in T_\tau(X)$;

(f$_3$) $N_{\mathbb{S}^+}(\omega(p)) = N_{\mathbb{S}^+}(p)$, for all unary $\omega \in \Omega_1$ and all $p \in T_\tau(X)$;

(f$_4$) $N_{\mathbb{S}^+}(\omega) = 0$, for all constants $\omega \in \Omega_1$;

(f$_5$) $N_{\mathbb{S}^+}(*p) = \circledast N_{\mathbb{S}^+}(p)$, for $*$ the unary τ_2-operation and $p \in T_\tau(X)$.

Proposition 3.5. *Let $\mathbb{S}^+$ be an E-system of algebras. The function $N_{\mathbb{S}^+}$ is a naming function and the indexing $\circledast$-semilattice of $\mathbb{S}^+$ is an element of $\mathfrak{SC}(N_{\mathbb{S}^+})$.*

Proof. Left to the reader. □

Definition 3.6. (*Enriched Płonka sum*) Let $\tau\colon \Omega_1 \cup \Omega_2 \to \mathbb{N}$ be an E-type (with constants). The *enriched Płonka sum* over an E-system $\mathbb{S}^+$ of τ-algebras is the τ-algebra

$$\mathbf{Pł}(\mathbb{S}^+) = \left\langle \bigcup_{s \in S} A_s, \left\{\omega^{\mathbf{Pł}(\mathbb{S}^+)}\right\}_{\omega \in \Omega_1 \cup \Omega_2} \right\rangle$$

where:

(es$_1$) for every n-ary $\omega \in \Omega_i$ ($n \geq 1$, $i \in \{1, 2\}$) and $a_1, \ldots, a_n \in \bigcup_{s \in S} A_s$,

$$\omega^{\mathbf{Pł}(\mathbb{S}^+)}(a_1, \ldots, a_n) = \omega^{\mathbf{A}_s^+}(\varphi^i_{s_1 s}(a_1), \ldots, \varphi^i_{s_n s}(a_n)),$$

where $a_k \in A_k$, $s = N_{\mathbb{S}^+}(\omega(x_1, \ldots, x_n))^{\mathbf{S}^+}(s_1, \ldots, s_n)$;

(es$_2$) for every $\omega \in \tau_1^{-1}[0]$, $\omega^{\mathbf{Pł}(\mathbb{S}^+)} = \omega^{\mathbf{A}_c^+}$, where $c = N_{\mathbb{S}^+}(\omega)^{\mathbf{S}^+} = 0^{\mathbf{S}^+}$.

Again, enriched Płonka sums generalise Płonka sums by prescribing a separate treatment for the interpretation of the single unary operation symbol in Ω_2. In Płonka sums, the value of a unary operation always inhabits the same fibre as its argument; here, it inhabits the fibre whose index is obtained by applying $\circledast$ to the index of the fibre of its argument.

4. Examples

4.1. Płonka sums, revisited

An ordinary semilattice direct system $\mathbb{S}$ with indexing semilattice $\mathbf{S}$ and fibres $\{\mathbf{A}_s\}_{s \in S}$ of type $\tau_1\colon \Omega_1 \to \mathbb{N}$ may be viewed as an E-system

$$\mathbb{S}^+ = \left\langle \mathbf{S}^+, \{\mathbf{A}_s^+\}_{s \in S}, \Delta_{\mathbf{S}^+}, \mathrm{Hom}_{\leq}(\mathbb{S}), \{\mathrm{id}_{\mathbf{A}_s}\}_{s \in S} \right\rangle$$

where:

(rs$_1$) $\mathbf{S}^+ = \langle S, \vee, \mathrm{id}_{\mathbf{S}} \rangle$ if $\tau^{-1}[0] = \emptyset$, $\mathbf{S}^+ = \langle S, \vee, \mathrm{id}_{\mathbf{S}}, 0 \rangle$ otherwise;

(rs$_2$) For each $s \in S$, $\mathbf{A}_s^+$ is an algebra of E-type $\tau\colon \Omega_1 \cup \Omega_2 \to \mathbb{N}$ (possibly with constants) obtained by expanding $\mathbf{A}_s$ with the unary operation $*^{\mathbf{A}_s^+} = \mathrm{id}_{\mathbf{A}_s}$;

(rs$_3$) $\Delta_{\mathbf{S}^+}$ is the diagonal relation over $\mathbf{S}^+$ (clearly, the family of functions associated with $\Delta_{\mathbf{S}^+}$ consists of the identity mappings on fibres).

We call the above defined E-system a *trivial enrichment* of $\mathbb{S}$. The reader can easily verify that, for $N\colon T_{\tau_1}(X) \to T_{\langle 2,0\rangle}(X)$ the naming function defined in Example 2.9 and $p \in T_{\tau_1}(X)$, we have:

$$N_{\mathbb{S}^+}(p)^{\mathbf{S}^+}(\overline{s}) = N(p)^{\mathbf{S}}(\overline{s})$$
$$N_{\mathbb{S}^+}(*p)^{\mathbf{S}^+}(\overline{s}) = \mathrm{id}_{\mathbf{S}^+}\left(N_{\mathbb{S}^+}(p)^{\mathbf{S}^+}(\overline{s})\right) = \mathrm{id}_{\mathbf{S}}\left(N(p)^{\mathbf{S}}(\overline{s})\right) = N(p)^{\mathbf{S}}(\overline{s})$$

It follows that $\mathbf{P}\boldsymbol{\ell}(\mathbb{S}^+)$ is nothing but $\mathbf{P}\boldsymbol{\ell}(\mathbb{S})$ expanded with the identity function $\mathrm{id}_{\mathbf{P}\boldsymbol{\ell}(\mathbb{S})}$. Indeed, for $a_1 \in A_{s_1}$, it holds that:

$$\begin{aligned} *^{\mathbf{P}\boldsymbol{\ell}(\mathbb{S}^+)}(a_1) &= *^{\mathbf{A}_s^+}\left(\varphi^2_{s_1 s}(a_1)\right) && \text{where } s = N_{\mathbb{S}^+}(*x_1)^{\mathbf{S}^+}(s_1) \\ &= \mathrm{id}_{\mathbf{A}_{s_1}}\big(\mathrm{id}_{\mathbf{A}_{s_1}}(a_1)\big) && \text{since } N_{\mathbb{S}^+}(*x_1)^{\mathbf{S}^+}(s_1) = \mathrm{id}_{\mathbf{S}}(s_1) = s_1 \\ &= a_1 \\ &= \mathrm{id}_{\mathbf{P}\boldsymbol{\ell}(\mathbb{S})}(a_1) \end{aligned}$$

In light of this construction, we obtain the following counterpart of Theorem 2.3.

Theorem 4.1. *If* $\mathbb{S}^+$ *is a trivial enrichment of a semilattice direct system* $\mathbb{S}$ *containing at least two algebras, then* $\mathbf{P}\boldsymbol{\ell}(\mathbb{S}^+)$ *satisfies precisely all regular identities satisfied in the fibres of* $\mathbb{S}^+$, *whereas any other identity is not satisfied by* $\mathbf{P}\boldsymbol{\ell}(\mathbb{S}^+)$.

4.2. Polin's variety

The next example concerns a variety that has intrigued several universal algebraists over the past few decades: *Polin's variety.* In the 1970s it was observed that many lattice identities weaker than modularity were not, in fact, truly weaker in the context of congruence varieties. More precisely, while none of these identities alone implied modularity, it was proven that any variety of algebras all of whose congruence lattices satisfied any of them was, in fact, congruence-modular [15,17,34]. This led to a conjecture suggesting that any variety having nontrivial congruence identities must be congruence-modular [32]. The conjecture was disproved by Polin [35, Thm. 1], who introduced a non-congruence-modular variety (subsequently named after him) with congruence identities. Actually, an even stronger property holds [10, Thm. 6.1]: a variety $\mathfrak{V}$ is not congruence-modular if and only if the variety of lattices generated by all congruence lattices of algebras in $\mathfrak{V}$ includes the variety generated by all congruence lattices of Polin algebras.

Beside providing a counterexample to this conjecture, Polin's variety instantiates some other fairly rare phenomena in universal algebra. For example, a well-known result by Jónsson [27, Thm. 1.2] states that if a variety has 3-permutable congruences, then it is congruence-modular. Although Polin's variety is not congruence 3-permutable for this reason, it is congruence 4-permutable [10, Thm. 7.6]. To mention another interesting property, it is the non-modular join of two disjoint congruence-distributive varieties (see below).

Definition 4.2. (*Polin algebra*) A *Polin algebra* is an algebra

$$\mathbf{P} = \langle P, \vee, \neg, \sim, 0\rangle$$

of type $\tau = \langle 2, 1, 1, 0\rangle$ satisfying the following identities:

(s1) $(x \vee y) \vee z \approx x \vee (y \vee z)$;
(s2) $x \vee y \approx y \vee x$;
(s3) $x \vee x \approx x$;
(s4) $0 \vee x \approx x$;
(s5) $x \vee \sim 0 \approx \sim 0$;
(e1) $\sim\sim\sim x \approx \sim x$;
(e2) $\sim\sim x \vee \sim\sim y \approx \sim\sim(x \vee y)$;
(e3) $\sim(x \vee y) \vee \sim x \approx \sim x$;
(e4) $x \vee \sim x \approx \sim 0$;
(e5) $x \vee \sim(\sim y \vee \sim z) \approx \sim(\sim(x \vee y) \vee \sim(x \vee z))$;
(i1) $\sim\neg x \approx \sim x$;
(i2) $x \vee \neg x \approx \sim\sim x$.
(i3) $\neg x \vee \neg(x \vee y) \approx \neg x \vee \neg\sim\sim y$
(i4) $\neg\neg x \approx x$
(i5) $x \vee \neg(\neg y \vee \neg z) \approx \neg(\neg(x \vee y) \vee \neg(x \vee z))$
(i6) $\neg\sim\sim x \vee \neg\sim\sim y \approx \neg\sim\sim(x \vee y)$
(i7) $\sim\sim x \vee \neg\sim\sim y \approx \sim\sim(x \vee y)$

The variety of Polin algebras will be denoted by $\mathfrak{P}$. It is an irregular variety, since the axioms (s5),(e3), (e4) fail to be regular identities.

$\mathfrak{P}$ is generated by a 4-element algebra, which is the direct product of two 2-element algebras. Each factor algebra can be viewed as a copy of the 2-element Boolean algebra in an expanded language with an additional unary operation, but crucially, the unary operation that plays the role of a Boolean negation is different in each factor.

Lemma 4.3. (Freese et al. [16, Ex. 6.12]) *For* $k \in \{i, j\}$, *let*

$$\mathbf{A}_k = \left\langle \{0,1\}, \vee^{\mathbf{A}_k}, \neg^{\mathbf{A}_k}, \sim^{\mathbf{A}_k}, 0^{\mathbf{A}_k} \right\rangle$$

be an algebra such that $a \vee^{\mathbf{A}_k} b = \max\{a, b\}$ *and the two unary operations are defined by the following tables*:

$\mathbf{A}_i$	0	1
$\neg$	1	0
$\sim$	1	1

$\mathbf{A}_j$	0	1
$\neg$	0	1
$\sim$	1	0

Then $\mathfrak{P} = \mathsf{V}(\mathbf{A}_i \times \mathbf{A}_j)$.

$\mathfrak{V}_i = \mathsf{V}(\mathbf{A}_i)$ and $\mathfrak{V}_j = \mathsf{V}(\mathbf{A}_j)$ are disjoint varieties whose join is $\mathfrak{P}$. Observe that both $\mathfrak{V}_i$ and $\mathfrak{V}_j$ clearly have Jónsson terms witnessing congruence-distributivity, but since they use symbols in different subtypes, neither set works for $\mathfrak{P}$. Moreover, observe that $\mathfrak{V}_i$ and $\mathfrak{V}_j$ are not independent in the sense of [24], for there are Polin algebras that are not isomorphic to the product of a member of $\mathfrak{V}_i$ and a member of $\mathfrak{V}_j$ [30].

We now examine Polin's representation mentioned in the introduction. Let $\mathfrak{BA}_\rho$ be the variety of Boolean algebras defined in the type $\rho = \langle 2, 1, 0\rangle$. Upon fixing an "external" structure $\mathbf{B} \in \mathfrak{BA}_\rho$, let an algebra $\mathbf{S}_b \in \mathfrak{BA}_\rho$ be assigned to each $b \in B$. Let now

$$\left\{\varphi_{bc} \in \mathrm{Hom}(\mathbf{S}_b, \mathbf{S}_c) \mid b \leq^{\mathbf{B}} c\right\}$$

be a family of homomorphisms satisfying:

(i) for all $b \in B$, $\varphi_{bb} = \mathrm{id}_{\mathbf{S}_b}$;
(ii) for all $b, c, d \in B$, $b \leq^{\mathbf{B}} c \leq^{\mathbf{B}} d \Rightarrow \varphi_{cd} \circ \varphi_{bc} = \varphi_{bd}$.

Set $P = \bigcup_{b \in B}(\{b\} \times S_b)$. We can now form an algebra $\mathbf{P}$ of type τ with universe P and operations defined as follows:

$$\begin{aligned}
\langle b, s\rangle \vee^{\mathbf{P}} \langle c, t\rangle &= \left\langle b \vee^{\mathbf{B}} c,\ \varphi_{bd}(s) \vee^{\mathbf{S}_d} \varphi_{cd}(t)\right\rangle \quad (\text{where } d = b \vee^{\mathbf{B}} c)\\
\neg^{\mathbf{P}}\langle b, s\rangle &= \left\langle b, \neg^{\mathbf{S}_b} s\right\rangle\\
\sim^{\mathbf{P}} \langle b, s\rangle &= \left\langle \neg^{\mathbf{B}} b, \neg^{\mathbf{S}_{\neg b}} 0^{\mathbf{S}_{\neg b}}\right\rangle\\
0^{\mathbf{P}} &= \left\langle 0^{\mathbf{B}}, 0^{\mathbf{S}_{0^{\mathbf{B}}}}\right\rangle.
\end{aligned}$$

Polin [35] shows that every such $\mathbf{P}$ is a Polin algebra, and conversely, every Polin algebra is isomorphic to one that is obtained from members of $\mathfrak{BA}_\rho$ via the above construction.

While clearly reminiscent of Płonka-type decompositions, and although the Boolean algebras of indices used herein are indeed semilattice-based algebras, this representation is not a Płonka sum. If it were, the output of the unary operation $\sim$ should belong to the same fibre as its input, which is not the case. Also, Agassiz systems are insufficient to account for it−if a belongs to A_b, then $\sim a$ belongs to $A_{\neg b}$, and generally $b \nleq^{\mathbf{B}} \neg b$. Despite that, Polin algebras can be represented as enriched Płonka sums, as we presently show. To this aim, observe that Boolean algebras can be viewed as algebras of type τ, upon adding to one of the usual set of axioms of type ρ the single identity $\sim x \approx x \vee \neg x$. Indeed, $\sim x \approx x \vee \neg x$ holds in a Polin algebra $\mathbf{P}$ if and only if the corresponding external Boolean algebra $\mathbf{B}$ is trivial, and hence in this case $\mathbf{P}$ is simply the trivial expansion of $\mathbf{S}_b$ by $\sim$, where b is the unique element of $\mathbf{B}$.

Theorem 4.4. (Representation theorem for Polin algebras) *Let* $\mathbf{P} \in \mathfrak{P}$. *Then there exists an E-system* $\mathbb{S}^+$ *such that* $\mathbf{P}$ *is isomorphic to* $\mathbf{Pł}(\mathbb{S}^+)$.

Proof. First observe that the type $\langle 2, 1, 1, 0\rangle$ of Polin algebras can be seen as an E-type with constants $\tau : \Omega_1 \cup \Omega_2 \to \mathbb{N}$ where $\Omega_1 = \{\vee, \neg, 0\}$, $\Omega_2 = \{\sim\}$, and arities are assigned as usual. The type τ can be used for defining Boolean algebras upon introducing the identity $\sim x \approx x \vee \neg x$. We denote by $\mathfrak{BA}_\tau$ the variety of Boolean algebras of type τ. Let

$$\mathbb{S} = \left\langle \mathbf{B}^-, \{\mathbf{A}_b\}_{b \in B}, \mathrm{Hom}_{\leq}(\mathbb{S})\right\rangle$$

be a semilattice direct system where:

- $\mathbf{B}^-$ is the $\neg$-free reduct of a Boolean algebra $\mathbf{B} \in \mathfrak{BA}_\rho$;
- $\{\mathbf{A}_b\}_{b \in B}$ is a family of $\mathfrak{BA}_\rho$-algebras indexed over $\mathbf{B}^-$;
- $\mathrm{Hom}_{\leq}(\mathbb{S})$ is defined as usual.

We can now build the required E-system as a quintuple

$$\mathbb{S}^+ = \left\langle \mathbf{B}, \{\mathbf{A}_b^+\}_{b \in B}, R, \mathrm{Fun}_{\leq}(\mathbb{S}^+), \mathrm{Fun}_R(\mathbb{S}^+)\right\rangle$$

specified as follows:

- the indexing $\circledast$-semilattice is the $\mathfrak{BA}_\rho$-algebra $\mathbf{B} = (\mathbf{B}^-)^+$;

- $\{\mathbf{A}_b^+\}_{b\in B}$ is the τ-expansion of $\{\mathbf{A}_b\}_{b\in B}$;
- $\mathrm{Fun}_{\leq}(\mathbb{S}^+) = \mathrm{Hom}_{\leq}(\mathbb{S})$
- for all $b, c \in B$, $\langle b, c\rangle \in R$ iff $b = \neg^{\mathbf{B}} c$;
- $\mathrm{Fun}_R(\mathbb{S}^+) = \{\psi_{bc} \in \mathrm{Fun}(A_b, A_c) \mid b = \neg^{\mathbf{B}} c\}$ is a family of arbitrary functions.

It remains to be checked that the requirements of Definition 3.6 are met. For the operations $\vee, \neg, 0$, this is true since in the Polin representation they behave like in Płonka sums: for example, if $b = N_{\mathbb{S}^+}(x_1 \vee x_2)^{\mathbf{B}}(b_1, b_2)$ and $a_k \in A_{b_k} (k \in \{1, 2\})$, then $a_1 \vee^{\mathbf{P}\text{ł}(\mathbb{S}^+)} a_2 = \varphi_{b_1 b}(a_1) \vee^{\mathbf{A}_b^+} \varphi_{b_2 b}(a_2)$. We now check the case of $\sim$. Let $b = N_{\mathbb{S}^+}(\sim x_1)^{\mathbf{B}}(b_1) = \neg b_1$, and $a_1 \in A_{b_1}$. Upon recalling that in each Boolean fibre the identity $\sim x \approx x \vee \neg x$ is satisfied, we have that $\sim^{\mathbf{P}\text{ł}(\mathbb{S}^+)} a_1 = \sim^{\mathbf{A}_b^+} \psi_{b_1 b}(a_1) = \sim^{\mathbf{A}_b^+} 0^{\mathbf{A}_b^+}$. □

The reader will now have a clearer intuition as to why in Definition 3.3, unlike in Definition 2.2 but along the lines of [22], we use generic functions instead of homomorphisms to link the fibres to one another. In the representation of a Polin algebra, two R-related fibres might well be such that the former is a trivial algebra, while the latter is a non-trivial one. Thus, in general, no homomorphism might be available to do the required job. Although our garden-variety functions might perform poorly in preserving operations, this is not a major issue since their application is immediately followed by an application of the operation $\sim$, which "crushes" everything to the top element of the fibre.

Example 4.5. (Representation of $\mathbf{A}_i \times \mathbf{A}_j$) Let $\mathbf{A}_i$ and $\mathbf{A}_j$ be the 2-element algebras defined in Lemma 4.3. Let us denote by $\mathbf{A}_j^-$ the $\neg$-free reduct of $\mathbf{A}_j$. Clearly, this algebra is isomorphic to the 2-element $\mathfrak{BA}_\rho$-algebra $\mathbf{B}_2 = \langle B, \vee, \sim, 0\rangle$ which, in turn, may be seen as a $\circledast$-semilattice $\mathbf{S}_2^+$ expanding the 2-element chain $\mathbf{S}_2$ with the Boolean complementation. Moreover, for τ the E-type with constants defined in Theorem 4.4, the algebra $\mathbf{A}_i$ may be seen as a τ-expansion $\mathbf{B}_2^+$ of the 2-element $\mathfrak{BA}_\rho$-algebra $\mathbf{B}_2 = \langle B, \vee, \neg, 0\rangle$.

It can be easily verified that the direct product $\mathbf{A}_i \times \mathbf{A}_j$ generating $\mathfrak{P}$ is isomorphic to the enriched Płonka sum over the E-system

$$\mathbb{S}^+_{\mathbf{A}_i\times\mathbf{A}_j} = \left\langle \mathbf{S}_2^+, \{\mathbf{A}_0, \mathbf{A}_1\}, R, \left\{\varphi^1_{00}, \varphi^1_{01}, \varphi^1_{11}\right\}, \left\{\varphi^2_{01}, \varphi^2_{10}\right\}\right\rangle$$

where R is as in Theorem 4.4 and:

- $\mathbf{S}_2^+ \cong \mathbf{B}_2 \cong \mathbf{A}_j^-$;
- $\mathbf{A}_0 \cong \mathbf{A}_1 \cong \mathbf{B}_2^+ \cong \mathbf{A}_i$;
- $\varphi^1_{01} = \varphi^2_{01}$ is the unique isomorphism from $\mathbf{A}_0$ to $\mathbf{A}_1$;
- $\varphi^2_{10} = (\varphi^2_{01})^{-1}$.

4.3. Pseudocomplemented semilattices

Pseudocomplemented semilattices [18] are an important and well-studied variety with nice structure theorems that are similar to (but simpler than) analogous results holding of pseudocomplemented distributive lattices or Stone algebras [29,8], and several connections to classical algebra [13]; see [6] for a good survey.

Definition 4.6. (*Pseudocomplemented semilattice*) A *pseudocomplemented semilattice* is an algebra $\mathbf{L} = \langle L, \wedge, \sim, 1\rangle$ of type $\langle 2, 1, 0\rangle$, where $\langle L, \wedge, 1\rangle$ is a meet-semilattice with unit and least element $0 = {\sim}1$, and for every $a, x \in L$, $x \leq {\sim} a$ iff $x \wedge a = 0$.

The class of pseudocomplemented semilattices is a variety, denoted $\mathfrak{PCS}$. Like $\mathfrak{P}$, it fails to be a regular variety.

Let $\mathbf{L} \in \mathfrak{PCS}$. An element $a \in L$ is said to be *closed* iff $a = {\sim}{\sim}a$. The set of closed elements of any $\mathbf{L} \in \mathfrak{PCS}$ is the universe of a Boolean algebra with respect to the restrictions of the operations in $\mathbf{L}$ and the defined join

$$x \vee y = {\sim}({\sim}x \wedge {\sim}y).$$

Moreover, the relation $\theta_{GF} = \{\langle a, b\rangle \in L^2 \mid {\sim}a = {\sim}b\}$ is a congruence on $\mathbf{L}$, called the *Glivenko–Frink congruence*. Each of its congruence classes a/θ_{GF} is the universe of a pointed subsemilattice of $\mathbf{L}$, and contains exactly one closed element ${\sim}{\sim}a$, which is the top element in a/θ_{GF}.

Recently, Katriňák and Guričan [28] proposed a fascinating generalisation of Chen and Grätzer's celebrated triple construction [7], originally introduced in the representation theory of Stone algebras and then extended to pseudocomplemented semilattices [29], but with significant limitations. This more general construction, which is very much in the same ballpark as Płonka sums, allows the authors to overcome these restrictions.

Definition 4.7. (*Abstract full triple*) An *abstract full triple* is a system

$$\mathbb{T} = \Big\langle \mathbf{B}, \{\mathbf{A}_b\}_{b\in B}, \mathrm{Hom}_{\geq}(\mathbb{T})\Big\rangle$$

consisting of:

- (t_1) a Boolean algebra $\mathbf{B}$;
- (t_2) a B-indexed family $\{\mathbf{A}_b\}_{b\in B}$ of meet-semilattices with unit 1_b, with pairwise disjoint universes, such that for every $b \in B$, $\mathbf{A}_b \cap B = \{b\} = 1_b$, and such that $A_0 = \{0^{\mathbf{B}}\}$;
- (t_3) $\mathrm{Hom}_{\geq}(\mathbb{T}) = \{\varphi_{bc} \in \mathrm{Hom}(\mathbf{A}_b, \mathbf{A}_c) \mid b \geq c\}$ is a family of homomorphisms such that:
  - for all $b \in B$, $\varphi_{bb} = \mathrm{id}_{\mathbf{A}_b}$;
  - for all $b, c, d \in B$, $\varphi_{cd} \circ \varphi_{bc} = \varphi_{bd}$ whenever $b \geq c \geq d$.

Theorem 4.8. [28, Thm. 5.1] *If* $\mathbb{T}$ *is an abstract full triple, then*

$$\mathbf{L}(\mathbb{T}) = \left\langle \bigcup_{b\in B} A_b, \wedge, \sim, 1\right\rangle$$

where:

- *for* $a_k \in A_{b_k} (k \in \{1, 2\})$ *and* $c = b_1 \wedge^{\mathbf{B}} b_2$:

$$a_1 \wedge^{\mathbf{L}(\mathbb{T})} a_2 = \varphi_{b_1 c}(a_1) \wedge^{\mathbf{A}_c} \varphi_{b_2 c}(a_2);$$

- *for* $a \in A_b$, $\sim^{\mathbf{L}(\mathbb{T})} a = 1^{\mathbf{A}_{\neg b}}$;
- $1^{\mathbf{L}(\mathbb{T})} = 1^{\mathbf{A}_1}$

is a pseudocomplemented semilattice.

Theorem 4.9. [28, Thm. 3.6] *Every* $\mathbf{L} \in \mathfrak{PCS}$ *is isomorphic to a pseudocomplemented semilattice arising out of the abstract full triple*

$$\mathbb{T} = \left\langle \mathbf{B}, \{\mathbf{A}_b\}_{b \in B}, \mathrm{Hom}_{\geq}(\mathbb{T}) \right\rangle$$

where $\mathbf{B}$ *is the Boolean algebra of closed elements of* $\mathbf{L}$, *the universe of each* $\mathbf{A}_b$ *is a congruence class of its Glivenko–Frink congruence, and for* $b \geq c$, $\varphi_{bc}(a) = c \wedge^{\mathbf{L}} a$.

Upon noticing that meet-semilattices with 1 can be regarded as algebras of type $\langle 2, 1, 0 \rangle$ satisfying the identity $\sim x \approx 1$, it is not hard to extract from Theorem 4.9 a representation of pseudocomplemented semilattices in terms of enriched Płonka sums, along the lines of Theorem 4.4. We thus state without a proof the following:

Theorem 4.10. (Representation theorem for pseudocomplemented semilattices) *Let* $\mathbf{L} \in \mathfrak{PCS}$. *Then there exists an E-system* $\mathbb{S}^+$ *such that* $\mathbf{L}$ *is isomorphic to* ***Pł***$(\mathbb{S}^+)$.

4.4. Enriched Płonka sums over involutive semilattices

In the preceding examples, the ⊛-semilattices used as algebras of indices are Boolean algebras. In this subsection, we examine certain sums over classes of E-systems whose indexing algebras are *involutive semilattices*. The earliest examples of such constructions are Dolinka and Vinčić's *involutorial Płonka sums* [11]. These will not be considered in detail. Instead, we focus on a different suggestion, recently advanced by T. Randriamahazaka in an attempt to represent algebras with a De Morgan negation that can be built via a sort of a twist product construction out of algebras with a dualised type [40]. Applications include De Morgan bisemilattices, a regular variety (which, however, is not the regularisation of any strongly irregular variety) used in the semantics of Angell's logic of analytic containment [1].

Definition 4.11. (*Dualised type*) A similarity type $\tau \colon \Omega \to \mathbb{N}$ is said to be *dualised* with respect to a function $d \colon \Omega \to \Omega$ if, for all $\omega \in \Omega$, $d(d(\omega)) = \omega$ and $\tau(d(\omega)) = \tau(\omega)$.

If τ is a dualised type with respect to d, then every τ-algebra $\mathbf{A} = \langle A, \{\omega^{\mathbf{A}}\}_{\omega \in \Omega} \rangle$ induces a τ-algebra $\mathbf{A}^d = \langle A, \{d(\omega)^{\mathbf{A}}\}_{\omega \in \Omega} \rangle$.

Definition 4.12. (*Involutive semilattice direct system*) Let $\tau \colon \Omega \to \mathbb{N}$ be a dualised type (with respect to d) such that Ω contains a designated operation symbol $\sim$ with $\tau(\sim) = 1$ and $d(\sim) = \sim$. An *involutive semilattice direct system* of τ-algebras is a quadruple

$$\mathbb{S}_{\mathrm{inv}} = \left\langle \mathbf{S}_{\mathrm{inv}}, \{\mathbf{A}_s\}_{s \in S}, \mathrm{Hom}_{\leq}(\mathbb{S}_{\mathrm{inv}}), \mathrm{Iso}_{\sim}(\mathbb{S}_{\mathrm{inv}}) \right\rangle$$

where:

(is$_1$) $\mathbf{S}_{\text{inv}}$ is an involutive semilattice $\langle S, \vee, \neg \rangle$ if $\tau^{-1}(0) = \emptyset$, an involutive semilattice with zero $\langle S, \vee, \neg, 0 \rangle$ otherwise;

(is$_2$) $\{\mathbf{A}_s\}_{s \in S}$ is an S-indexed family of τ-algebras with pairwise disjoint universes, satisfying the identity ${\sim} x \approx x$;

(is$_3$) $\text{Hom}_{\leq}(\mathbb{S}_{\text{inv}}) = \{\varphi_{st} \in \text{Hom}(\mathbf{A}_s, \mathbf{A}_t) \mid s \leq t\}$ is such that:
- for all $s \in S$, $\varphi_{ss} = \text{id}_{\mathbf{A}_s}$;
- for all $s, t, u \in S$, $\varphi_{tu} \circ \varphi_{st} = \varphi_{su}$ whenever $s \leq t \leq u$;

(is$_4$) $\text{Iso}_{\sim}(\mathbb{S}_{\text{inv}}) = \left\{\psi_s \in \text{Hom}\left(\mathbf{A}_s, \mathbf{A}^d_{\neg s}\right)\right\}$ is a family of isomorphisms such that:
- for all $s \in S$, $\psi_{\neg s} \circ \psi_s = \text{id}_{\mathbf{A}_s}$;
- for all $s, t \in S$, $\psi_t \circ \varphi_{st} = \varphi_{\neg s \, \neg t} \circ \psi_s$ whenever $s \leq t$.

If $\mathbb{S}_{\text{inv}}$ is an involutive semilattice direct system of τ-algebras, Randriamahazaka defines as follows a new τ-algebra $\mathbf{DPł}(\mathbb{S}_{\text{inv}})$:

Definition 4.13. (*De Morgan–Płonka sum*) The *De Morgan–Płonka sum* over $\mathbb{S}_{\text{inv}}$ is the τ-algebra

$$\mathbf{DPł}(\mathbb{S}_{\text{inv}}) = \left\langle \bigcup_{s \in S} A_s, \left\{\omega^{\mathbf{DPł}(\mathbb{S}_{\text{inv}})}\right\}_{\omega \in \Omega} \right\rangle,$$

where

- for every n-ary $\omega \in \Omega \setminus \{\sim\}$ (with $n \geq 1$) and $a_1, \ldots, a_n \in \bigcup_{s \in S} A_s$,
$$\omega^{\mathbf{DPł}(\mathbb{S}_{\text{inv}})}(a_1, \ldots, a_n) = \omega^{\mathbf{A}_s}(\varphi_{s_1 s}(a_1), \ldots, \varphi_{s_n s}(a_n)),$$
where $a_1 \in A_{s_1}, \ldots, a_n \in A_{s_n}$ and $s = s_1 \vee \cdots \vee s_n$.
- for every 0-ary $\omega \in \Omega$, $\omega^{\mathbf{DPł}(\mathbb{S}_{\text{inv}})} = \omega^{\mathbf{A}_0}$.
- ${\sim}^{\mathbf{DPł}(\mathbb{S}_{\text{inv}})} a = {\sim}^{\mathbf{A}^d_{\neg s}} (\psi_s(a))$, where $a \in A_s$.

If τ is as in Definition 4.12, a regular τ-identity $p \approx q$ is *balanced* whenever a variable x occurs in p in the scope of an even (odd) number of occurrences of $\sim$ iff it occurs in q in the scope of an even (odd) number of occurrences of $\sim$. It turns out that De Morgan-Płonka sums preserve exactly the balanced regular identities satisfied in all algebras that can be obtained from the fibres via a general version of the *twist product* construction, for which see e.g. [33].

Definition 4.14. (*Bilateralisation of an algebra*) If τ is as in Definition 4.12 and $\mathbf{A} = \langle A, \{\omega^{\mathbf{A}}\}_{\omega \in \Omega} \rangle$ is a τ-algebra, the *bilateralisation* of $\mathbf{A}$ is the τ-algebra $\flat\mathbf{A}$ with universe A^2 and such that:

- $\omega^{\flat\mathbf{A}}(\langle a_1, b_1 \rangle, \ldots, \langle a_n, b_n \rangle) = \langle \omega^{\mathbf{A}}(a_1, \ldots, a_n), d(\omega)^{\mathbf{A}}(b_1, \ldots, b_n) \rangle$, for every n-ary $\omega \in \Omega \setminus \{\sim\}$ $(n \geq 1)$ and any $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$;
- $\omega^{\flat\mathbf{A}} = \langle \omega^{\mathbf{A}}, d(\omega)^{\mathbf{A}} \rangle$, for every 0-ary $\omega \in \Omega$;
- ${\sim}^{\flat\mathbf{A}} \langle a, b \rangle = \langle {\sim}^{\mathbf{A}} b, {\sim}^{\mathbf{A}} a \rangle$.

In the next theorem, $\mathbf{S}_2$ denotes the 2-element semilattice.

Theorem 4.15. [40] *If τ is as in Definition* 4.12 *and $\mathbb{S}_{\text{inv}}$ is an involutive semilattice direct system of τ-algebras containing at least two algebras, such that*

$\flat\mathbf{S}_2$ is a subalgebra of $\mathbf{S}_{\mathrm{inv}}$, then all balanced regular identities satisfied in the bilateralisations of all algebras of $\mathbb{S}$ are satisfied in $\mathbf{DP}\boldsymbol{\ell}(\mathbb{S}_{\mathrm{inv}})$, whereas any other identity is not satisfied in $\mathbf{DP}\boldsymbol{\ell}(\mathbb{S}_{\mathrm{inv}})$.

The paper [40] contains indeed stronger results, including a generalisation of Theorem 2.4, on which we do not dwell. Instead, we show that De Morgan–Płonka sums are an instance of enriched Płonka sums.

Theorem 4.16. *Let $\tau\colon \Omega \to \mathbb{N}$ be as in Definition 4.12, and let $\mathbf{DP}\boldsymbol{\ell}(\mathbb{S}_{\mathrm{inv}})$ be the De Morgan–Płonka sum over an involutive semilattice direct system of τ-algebras $\mathbb{S}_{\mathrm{inv}}$. Then there exists an E-system $\mathbb{S}^+$ such that $\mathbf{DP}\boldsymbol{\ell}(\mathbb{S}_{\mathrm{inv}}) = \mathbf{P}\boldsymbol{\ell}(\mathbb{S}^+)$.*

Proof. To begin with, note that the indexing involutive semilattice of $\mathbb{S}_{\mathrm{inv}}$ may be seen as a $\circledast$-semilattice $\mathbf{S}^+$ expanding a join-semilattice $\mathbf{S}$. Further, one can immediately check that τ is an E-type with $\Omega_1 = \Omega\backslash\{\sim\}$ and $\Omega_2 = \{\sim\}$. Therefore, $\{\mathbf{A}_s\}_{s\in S}$ may be seen as the τ-expansion of a family $\{\mathbf{A}_s^-\}_{s\in S}$ of algebras of type $\tau_1 : \Omega_1 \to \mathbb{N}$. We can thus construct an underlying semilattice direct system

$$\mathbb{S} = \left\langle \mathbf{S}, \{\mathbf{A}_s^-\}_{s\in S}, \mathrm{Hom}_{\leq}(\mathbb{S}_{\mathrm{inv}}) \right\rangle.$$

The required E-system is now obtained as a quintuple

$$\mathbb{S}^+ = \left\langle \mathbf{S}^+, \{\mathbf{A}_s\}_{s\in S}, R, \mathrm{Fun}_{\leq}(\mathbb{S}^+), \mathrm{Fun}_R(\mathbb{S}^+) \right\rangle$$

where $\mathbf{S}^+ = \mathbf{S}_{\mathrm{inv}}$, $R := \{\langle s,t\rangle \mid t = \neg^{\mathbf{S}^+} s\}$, and the families of functions are given by $\mathrm{Fun}_{\leq}(\mathbb{S}^+) = \mathrm{Hom}_{\leq}(\mathbb{S}_{\mathrm{inv}})$ and $\mathrm{Fun}_R(\mathbb{S}^+) = \mathrm{Iso}_{\sim}(\mathbb{S}_{\mathrm{inv}})$.

We fold with the requirements of Definition 3.6. For $\omega \in \tau_1^{-1}[n]$ $(n > 0)$, $N_{\mathbb{S}^+}(\omega(x_1,\ldots,x_n)) = x_1 \vee\cdots\vee x_n$, $b = b_1 \vee\cdots\vee b_n$ and $a_k \in A_{b_k}$ for $1 \leq k \leq n$, by Definition 4.13

$$\omega^{\mathbf{P}\boldsymbol{\ell}(\mathbb{S}^+)}(a_1,\ldots,a_n) = \omega^{\mathbf{A}_b}(\varphi^1_{b_1 b}(a_1),\ldots,\varphi^1_{b_n b}(a_n)).$$

On the other hand, the only operation symbol in Ω_2 is $\sim$, whence, if $a_1 \in A_{b_1}$ and $b = N_{\mathbb{S}^+}(\sim x)^{\mathbf{S}^+}(b_1) = \neg^{\mathbf{S}^+} b_1$, again by Definition 4.13

$$\sim^{\mathbf{P}\boldsymbol{\ell}(\mathbb{S}^+)} a_1 = \sim^{\mathbf{A}_b} \varphi^2_{b_1 b}(a_1).$$

□

5. Preservation of identities

In this section, we investigate the problem as to which identities are preserved by enriched Płonka sums. Let us warn the reader straight away that the findings below are nowhere near the strong preservation results available for Płonka sums. Indeed, not only does Płonka prove that all varieties whose members are Płonka sums of algebras in some $\mathfrak{W}$ preserve precisely the regular identities of $\mathfrak{W}$ (as a corollary to Theorem 2.3), but he also gives a partial converse to this result, holding of regularisations of strongly irregular varieties (Theorem 2.4). Moreover, all and only regular identities are preserved not only from the variety being regularised to its regularisation, but also from the fibres of a single semilattice direct system to its Płonka sum.

Our more modest goal below will be to identify sufficient conditions for a variety $\mathfrak{V}$, generated by enriched Płonka sums over E-systems $\mathbb{S}^+$ with fibres in $\mathfrak{W}$ and indexing $\circledast$-semilattices in $\mathfrak{B}$, to preserve precisely the identities of $\mathfrak{W}$ that are $N_{\mathbb{S}^+}$-regular in $\mathfrak{B}$. This theorem will cover at least some of the motivating examples mentioned in our paper.

5.1. The main result

We set off by spelling out a number of properties that make an E-system "well-behaved". We aim at a preservation result for varieties that are at least *generated* by sums over well-behaved systems, although not all of their members may have such good representations.

Definition 5.1. (*Confluent E-system*) An E-system of τ-algebras

$$\mathbb{S}^+ = \left\langle \mathbf{S}^+, \{\mathbf{A}_s^+\}_{s\in S}, R, \mathrm{Fun}_{\leq}(\mathbb{S}^+), \mathrm{Fun}_R(\mathbb{S}^+) \right\rangle$$

is *confluent* in $\mathbf{S}^+$ if it satisfies the following properties:

(1) $R = R^{-1}$ (symmetry);
(2) $R^{-1} \circ R \subseteq \Delta_{\mathbf{S}^+}, R \circ R^{-1} \subseteq \Delta_{\mathbf{S}^+}$ (right and left univocity);
(3) $(\leq \circ\, R \circ \leq) \cup (R \circ \leq \circ\, R) \subseteq\, \leq \cup R \cup (\leq \circ\, R) \cup (R \circ \leq)$;
(4) Each element of $\mathrm{Fun}_{\leq}(\mathbb{S}^+) \cup \mathrm{Fun}_R(\mathbb{S}^+)$ is a τ-homomorphism and, moreover, for any $s, t \in S$, $\varphi^2_{ts} \circ \varphi^2_{st} = \mathrm{id}_{\mathbf{A}_s^+}$ whenever $\langle s, t\rangle \in R$;
(5) for $i, j \in \{1, 2\}$ and $x \in \{i, j\}$, whenever the indicated functions exist, the following diagram commutes (Figure 1).

Definition 5.2. (*Confluent variety*) Let $\mathfrak{S}$ be a class of $\circledast$-semilattices, and $\mathfrak{V}$ be a variety of E-type τ. $\mathfrak{V} = \mathsf{V}(\mathfrak{G})$ is said to be *confluent* in $\mathfrak{S}$ with generating set $\mathfrak{G}$ if each member of $\mathfrak{G}$ is an enriched Płonka sum over an E-system that is confluent in some $\mathbf{S}^+ \in \mathfrak{S}$.

Definition 5.2 subsumes at least some of the examples in Section 4.

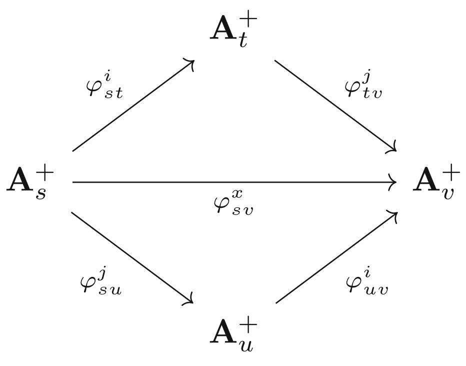


FIGURE 1. The *diamond property*

Theorem 5.3. *The following varieties are confluent*:

1. *The Polin variety* $\mathfrak{P}$.
2. *Every variety whose members are Płonka sums over a semilattice direct system of algebras.*
3. *Every variety whose members are De Morgan–Płonka sums over an involutive semilattice direct system of algebras.*

Proof. Since the proof of 2. is immediate, we focus on the remaining two items.

1. By Lemma 4.3, $\mathfrak{P}$ is generated by the 4-element algebra $\mathbf{A}_i \times \mathbf{A}_j$. We will refer to the representation of such an algebra provided by Example 4.5, in terms of a sum over the E-system $\mathbb{S}^+_{\mathbf{A}_i \times \mathbf{A}_j}$.

Clearly, the relation R is symmetric and (both left and right) univocal. This much accounts for (1) and (2) in Definition 5.1. As regards (3), it follows from the fact that any ordered pair consisting of two (not necessarily distinct) fibres in $\mathbb{S}^+_{\mathbf{A}_i \times \mathbf{A}_j}$ is in either $\leq$ or R. Condition (4) obtains because both the φ^1's and the φ^2's are isomorphisms. Finally, the Diamond property (5) holds because of the way the φ^1's and the φ^2's are defined.

3. In this case we have to prove that *every* involutive semilattice direct system of algebras has the properties in Definition 5.1. Again, it is easily seen that R is symmetric and (both left and right) univocal. We check (3). Suppose that $\langle a, b \rangle \in \leq \circ R \circ \leq$, i.e., there exist c, d such that $a \leq c, d \leq b, c = \neg d$. Since this is an involutive semilattice system, $d \leq b$ implies $\neg d \leq \neg b$ and hence $a \leq \neg b$. Therefore $\langle a, b \rangle \in \leq \circ R \subseteq \leq \cup R \cup (\leq \circ R) \cup (R \circ \leq)$. Suppose now that $\langle a, b \rangle \in R \circ \leq \circ R$, i.e., there exist c, d such that $a = \neg c, c \leq d, d = \neg b$. It follows that $\neg a \leq \neg b$ and hence $a \leq b$. So $\langle a, b \rangle \in \leq \subseteq \leq \cup R \cup (\leq \circ R) \cup (R \circ \leq)$.

Condition (5) follows from the definition of an involutive semilattice direct system, while (6) holds because of the Conditions (is_3) and (is_4) in Definition 4.12. □

Lemma 5.4. *Let* $\mathsf{V}(\mathfrak{G})$ *be a variety that is confluent in* $\mathfrak{S}$ *with generating set* $\mathfrak{G}$, *and let each* $\mathbf{A} \in \mathfrak{G}$ *be an enriched Płonka sum over a confluent E-system*

$$\mathbb{S} = \left\langle \mathbf{S}^+, \{\mathbf{A}^+_s\}_{s \in S}, R, \mathrm{Fun}_{\leq}(\mathbb{S}^+), \mathrm{Fun}_R(\mathbb{S}^+) \right\rangle$$

Let $\mathbf{A}^+_s, \mathbf{A}^+_t, \mathbf{A}^+_{s_1}, \ldots, \mathbf{A}^+_{s_m}, \mathbf{A}^+_{t_1}, \ldots, \mathbf{A}^+_{t_n}$ *be arranged in the following diagram (which, for convenience, we call* $\mathcal{D}$ *and where the* φ*'s and the* ψ*'s belong to* $\mathrm{Fun}_{\leq}(\mathbb{S}^+) \cup \mathrm{Fun}_R(\mathbb{S}^+)$) (*Figure* 2):
Let $\varphi, \psi \in \mathrm{Hom}(\mathbf{A}^+_s, \mathbf{A}^+_t)$ *be defined by*

$$\varphi = \varphi^{i_{m+1}}_{s_m t} \circ \cdots \circ \varphi^{i_1}_{s s_1}$$
$$\psi = \psi^{j_{n+1}}_{t_n t} \circ \cdots \circ \psi^{j_1}_{s t_1}.$$

Then $\varphi = \psi$.

Before proving the lemma, we need to make some preliminary considerations.

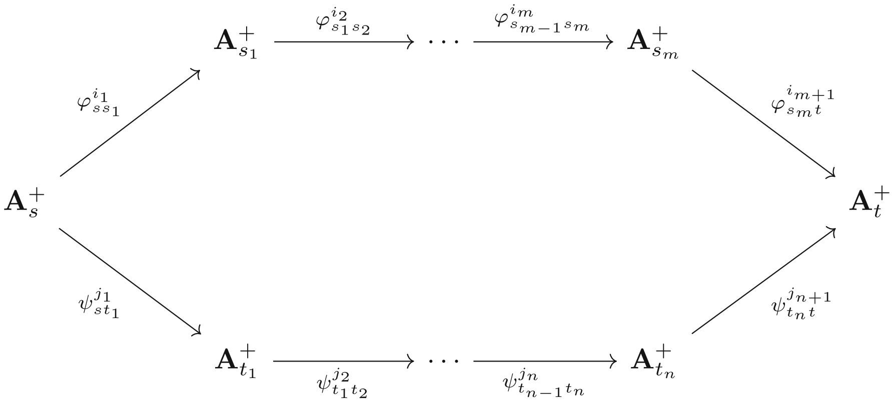


FIGURE 2. The diagram $\mathcal{D}$

Note. First, observe that the upper and lower edge sequences of $\mathcal{D}$ can be viewed, w.l.o.g., as alternations of φ^1- and φ^2-edges. Indeed, consider a fragment of the form

$$\cdots \longrightarrow \mathbf{A}_{x_{k-1}}^+ \xrightarrow{\varphi^1_{x_{k-1}x_k}} \mathbf{A}_{x_k}^+ \xrightarrow{\varphi^1_{x_k x_{k+1}}} \mathbf{A}_{x_{k+1}}^+ \longrightarrow \cdots$$

By Definition 3.3 (e$_3$), $x_{k-1} \leq x_{k+1}$ and $\varphi^1_{x_{k-1}x_{k+1}} = \varphi^1_{x_k x_{k+1}} \circ \varphi^1_{x_{k-1}x_k}$, so we can suppress $\mathbf{A}_{x_k}^+$ from the sequence of fibres, and so on until we have no adjoining φ^1-edges.

Similarly, in the case of a fragment of the form

$$\cdots \longrightarrow \mathbf{A}_{x_{k-1}}^+ \xrightarrow{\varphi^2_{x_{k-1}x_k}} \mathbf{A}_{x_k}^+ \xrightarrow{\varphi^2_{x_k x_{k+1}}} \mathbf{A}_{x_{k+1}}^+ \longrightarrow \cdots$$

by (1) and (2) in Definition 5.1, $R \circ R \subseteq \Delta_{\mathbf{S}^+}$ and so $x_{k-1} = x_{k+1}$, and by (4) $\varphi^2_{x_k x_{k+1}} \circ \varphi^2_{x_{k+1}x_k} = \mathrm{id}_{\mathbf{A}^+_{x_{k+1}}}$. Thus we can shorten the sequence until we have no adjoining φ^2-edges. From now on, we will always assume that both the upper and lower edge sequences of $\mathcal{D}$ contain no adjoining arrows representing homomorphisms of the same kind.

Second, it is not hard to see that the proof of the lemma requires some form of induction. We therefore introduce a measure of the complexity of $\mathcal{D}$. Let $\Pi(\mathcal{D})$ be the set of all (directed) paths in $\mathcal{D}$. We adopt the usual notation $|\mathcal{X}|$ for the length of a path $\mathcal{X} \in \Pi(\mathcal{D})$. Let now $\mathcal{U}$ and $\mathcal{L}$ be the upper and the lower edge sequences of $\mathcal{D}$, respectively. Clearly $\mathcal{U}, \mathcal{L} \in \Pi(\mathcal{D})$. We define the *weight* of $\mathcal{D}$ as the value $\mu(\mathcal{D}) = |\mathcal{U}| + |\mathcal{L}|$.

Proof. To prove the lemma, we proceed by induction on $\mu(\mathcal{D})$.

Base case ($\mu(\mathcal{D}) = 4$). We have one of the following situations (Figure 3):

(α_1) By (2), $s_1 = t_1$ whence $\varphi^1_{ss_1} = \varphi^1_{st_1}$ and $\varphi^2_{s_1 t} = \varphi^2_{t_1 t}$.

(β_1) By (2) again, $s_1 = t_1$ whence $\varphi^1_{s_1 t} = \varphi^1_{t_1 t}$ and $\varphi^2_{ss_1} = \varphi^2_{st_1}$.

(γ_1, δ_1) Straightforward consequence of (5).

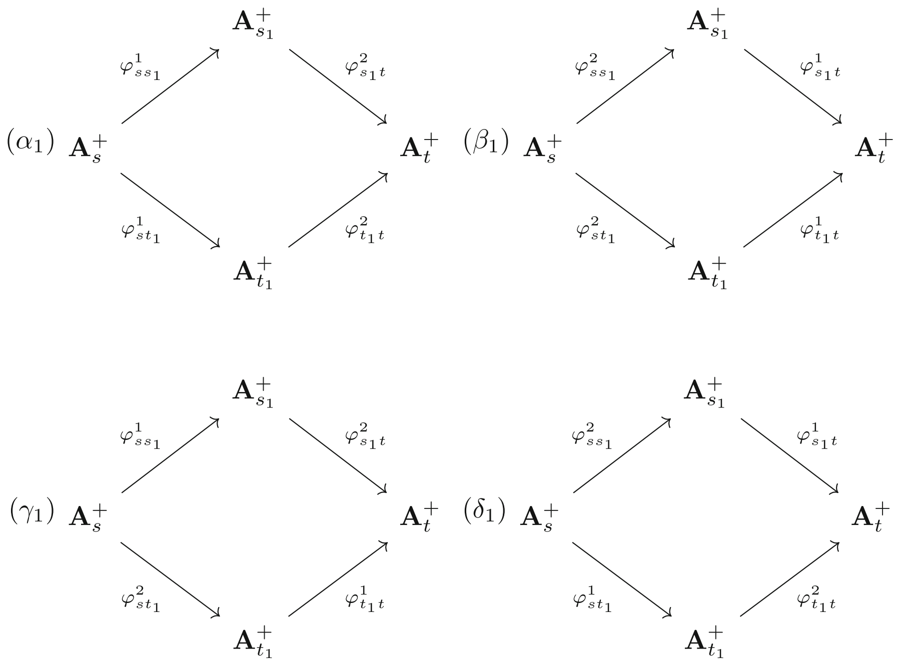


FIGURE 3. Base case of Lemma 5.4

Step. Let the lemma be true up to $\mu(\mathcal{D})-1$. For $\mathcal{X}\in\{\mathcal{U},\mathcal{L}\}$, we have one of the following situations:

(α_2) $\mathcal{X}$ is of the form

$$\mathbf{A}_s^+ \longrightarrow \cdots \longrightarrow \mathbf{A}_{x_{q-2}}^+ \xrightarrow{\varphi^1_{x_{q-2}x_{q-1}}} \mathbf{A}_{x_{q-1}}^+ \xrightarrow{\varphi^2_{x_{q-1}x_q}} \mathbf{A}_{x_q}^+ \xrightarrow{\varphi^1_{x_q t}} \mathbf{A}_t^+$$

(β_2) $\mathcal{X}$ is of the form

$$\mathbf{A}_s^+ \longrightarrow \cdots \longrightarrow \mathbf{A}_{x_{q-2}}^+ \xrightarrow{\varphi^2_{x_{q-2}x_{q-1}}} \mathbf{A}_{x_{q-1}}^+ \xrightarrow{\varphi^1_{x_{q-1}x_q}} \mathbf{A}_{x_q}^+ \xrightarrow{\varphi^2_{x_q t}} \mathbf{A}_t^+$$

Let us set

$$\mathcal{Y} = \mathbf{A}_{x_{q-2}}^+ \xrightarrow{\varphi^1_{x_{q-2}x_{q-1}}} \mathbf{A}_{x_{q-1}}^+ \xrightarrow{\varphi^2_{x_{q-1}x_q}} \mathbf{A}_{x_q}^+ \xrightarrow{\varphi^1_{x_q t}} \mathbf{A}_t^+$$

$$\mathcal{Z} = \mathbf{A}_{x_{q-2}}^+ \xrightarrow{\varphi^2_{x_{q-2}x_{q-1}}} \mathbf{A}_{x_{q-1}}^+ \xrightarrow{\varphi^1_{x_{q-1}x_q}} \mathbf{A}_{x_q}^+ \xrightarrow{\varphi^2_{x_q t}} \mathbf{A}_t^+$$

Observe that $|\mathcal{X}| = q+2$ and $|\mathcal{Y}| = |\mathcal{Z}| = 3$. By (3) and (5), we can always replace $\mathcal{Y}$ or $\mathcal{Z}$ with subpaths of length 2 or 1, thus obtaining a shorter sequence $\mathcal{X}'$ of overall length q or $q+1$, to which the IH applies. This concludes the proof. □

Theorem 5.5. *For $\mathfrak{S}$ a variety of $\circledast$-semilattices (with zero) and τ an E-type (with constants), let $\mathfrak{V} = \mathsf{V}(\mathfrak{G})$ and $\mathfrak{W}$ be nontrivial varieties of τ-algebras such that $\mathfrak{W} \leq \mathfrak{V}$ and all members of $\mathfrak{G}$ are isomorphic to enriched Płonka sums over E-systems $\mathbb{S}^+$ with fibres in $\mathfrak{W}$ and indexing algebras in $\mathfrak{S}$. If $\mathfrak{V}$ is confluent in $\mathfrak{S}$ with generating set $\mathfrak{G}$, then $\mathfrak{V}$ is the $N_{\mathbb{S}^+}$-regularisation of $\mathfrak{W}$ in $\mathfrak{S}$.*

Proof. We claim that, under our assumptions, $\mathrm{Id}(\mathfrak{V}) = \mathrm{Id}(\mathfrak{W}) \cap \mathrm{Id}_{N_{\mathbb{S}^+}}(\mathfrak{S})$.

The left-to-right inclusion is obvious. Indeed, each identity holding in $\mathfrak{V}$ is $N_{\mathbb{S}^+}$-regular by construction and, since $\mathfrak{W} \leq \mathfrak{V}$, we have $\mathrm{Id}(\mathfrak{V}) \subseteq \mathrm{Id}(\mathfrak{W})$. Therefore, $\mathrm{Id}(\mathfrak{V}) \subseteq \mathrm{Id}(\mathfrak{W}) \cap \mathrm{Id}_{N_{\mathbb{S}^+}}(\mathfrak{B})$.

As for the converse inclusion, it will suffice to check that the identities in $\mathrm{Id}(\mathfrak{W}) \cap \mathrm{Id}_{N_{\mathbb{S}^+}}(\mathfrak{S})$ hold in an arbitrary member $\mathbf{A}$ of $\mathfrak{G}$. Let $p \approx q \in \mathrm{Id}(\mathfrak{W}) \cap \mathrm{Id}_{N_{\mathbb{S}^+}}(\mathfrak{S})$. This identity is satisfiable by assumption, so $Var(p) \cap Var(q) \neq \emptyset$. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the syntactic trees of p and q, respectively. Choosing an initial variable $x_k \in Var(p) \cap Var(q)$, we compute the interpretations of p and q in $\mathbf{A}$ parameterised on x_k and obtain a number of pairs of tree diagrams as shown in the Appendix (Figures 5 and 6)

Let now $\mathcal{B}_1$ and $\mathcal{B}_2$ be, respectively, branches of $\mathcal{T}_1$ and $\mathcal{T}_2$ terminating with x_k. We thus have two sequences of the form

$$\mathbf{A}^+_{b_k} \xrightarrow{\varphi^i_{b_k b^{n_1}_{k\,d_1-1}}} \mathbf{A}^+_{b^{n_1}_{k\,d_1-1}} \xrightarrow{\varphi^i_{b^{n_1}_{k\,d_1-1} b^{n_1}_{k\,d_1-2}}} \mathbf{A}^+_{b^{n_1}_{k\,d_1-2}} \xrightarrow{\varphi^i_{b^{n_1}_{k\,d_1-2} b^{n_1}_{k\,d_1-3}}} \cdots \xrightarrow{\varphi^i_{b^{n_1}_{k1} b}} \mathbf{A}^+_{b}$$

$$\mathbf{A}^+_{c_k} \xrightarrow{\psi^i_{c_k c^{n_2}_{k\,d_2-1}}} \mathbf{A}^+_{c^{n_2}_{k\,d_2-1}} \xrightarrow{\psi^i_{c^{n_2}_{k\,d_2-1} c^{n_2}_{k\,d_2-2}}} \mathbf{A}^+_{c^{n_2}_{k\,d_2-2}} \xrightarrow{\psi^i_{c^{n_2}_{k\,d_2-2} c^{n_2}_{k\,d_2-3}}} \cdots \xrightarrow{\psi^i_{c^{n_2}_{k1} c}} \mathbf{A}^+_{c}$$

where d_1, d_2 and n_1, n_2 are, respectively, the depths and the indices of the branches, and $i \in \{1, 2\}$. Since $b_k = c_k$ by construction and $b = c$ by the $N_{\mathbb{S}^+}$-regularity of $p \approx q$, we can apply Lemma 5.4, obtaining $\varphi = \psi$, where

$$\varphi = \varphi^i_{b^{n_1}_{k1} b} \circ \cdots \circ \varphi^i_{b_k b^{n_1}_{k\,d_1-1}}$$
$$\psi = \psi^i_{c^{n_2}_{k1} c} \circ \cdots \circ \psi^i_{c_k c^{n_2}_{k\,d_2-1}}$$

The coincidence of the interpretations of p and q on $\mathbf{A}$ follows straightforwardly from the fact that $\mathfrak{W} \models p \approx q$ and $\mathfrak{W} \leq \mathfrak{V}$. Indeed, if $\mathbf{B}^+ \in \mathfrak{W}$, it holds that $\chi(p) = \chi(q)$ for all $\chi \in \mathrm{Hom}(\mathbf{T}_\tau(X), \mathbf{B}^+)$. In particular, for all $\chi \in \mathrm{Hom}(\mathbf{T}_\tau(X), \mathbf{A}^+_{b_k})$ such that $\chi(x_k) = a_k$, considering the homomorphism $\chi^* = \varphi \circ \chi = \psi \circ \chi$, we have that $\chi^*(p) = \chi^*(q)$. □

6. Applications to Polin's variety

Theorem 5.5, when specialised to Polin's variety, tells us that, for any E-system $\mathbb{S}^+$ defined as in Theorem 4.4, $\mathfrak{P}$ preserves precisely those Boolean identities of type τ that are $N_{\mathbb{S}^+}$-regular in the variety of Boolean algebras of type ρ. It is worth observing that such a result can also be obtained in a more ad-hoc way, using properties of the single finite generator of $\mathfrak{P}$. We detail below this shorter proof.

Theorem 6.1. $\mathrm{Id}(\mathfrak{P}) = \mathrm{Id}(\mathfrak{BA}_\tau) \cap \mathrm{Id}_{N_{\mathbb{S}^+}}(\mathfrak{BA}_\rho)$.

Proof. From left to right, suppose that $\mathfrak{P} \models p \approx q$. Then (keeping the notation of Lemma 4.3) $\mathbf{A}_i \times \mathbf{A}_j \models p \approx q$. Now, $\mathbf{A}_i$ is a quotient of $\mathbf{A}_i \times \mathbf{A}_j$ that is isomorphic to the 2-element Boolean algebra of type τ. Hence $\mathfrak{BA}_\tau \models p \approx q$. It remains to show that if $\mathfrak{P} \models p \approx q$, then $N_{\mathbb{S}^+}(p) \approx N_{\mathbb{S}^+}(q) \in \mathrm{Id}(\mathfrak{BA}_\rho)$, which we do contrapositively. If $N_{\mathbb{S}^+}(p) \approx N_{\mathbb{S}^+}(q) \notin \mathrm{Id}(\mathfrak{BA}^\rho)$, there exist a homomorphism φ from $\mathbf{T}_\rho(X)$ to the 2-element Boolean algebra of type ρ such that $\varphi(N_{\mathbb{S}^+}(p)) \neq \varphi(N_{\mathbb{S}^+}(q))$. Note that φ clearly induces a homomorphism of type τ which we denote with the same symbol, with a slight abuse. Let $\psi\colon \mathbf{T}_\tau(X) \to \mathbf{A}_i \times \mathbf{A}_j$ be such that $\psi(x) = \langle 0, \varphi(x)\rangle$ for any $x \in X$. We show by induction that for any $t \in T_\tau(X)$, $\pi \circ \psi(t) = \varphi(N_{\mathbb{S}^+}(t))$, where π is the projection map from $\mathbf{A}_i \times \mathbf{A}_j$ onto $\mathbf{A}_j$. We examine in detail the base case and the unary and binary operations.

- Suppose $t = x$. We have that $\pi \circ \psi(x) = \varphi(x) = \varphi(N_{\mathbb{S}^+}(x))$.
- Suppose $t = \neg s$. Then
$$\pi \circ \psi(\neg s) = \neg^{\mathbf{A}_j} \pi(\psi(s)) = \pi(\psi(s)) = \varphi(N_{\mathbb{S}^+}(s)) = \varphi(N_{\mathbb{S}^+}(\neg s)).$$
- Suppose $t = \sim s$. Then
$$\pi \circ \psi(\sim s) = \sim^{\mathbf{A}_j} \pi(\psi(s)) = \sim^{\mathbf{A}_j} \varphi(N_{\mathbb{S}^+}(s)) = \varphi(N_{\mathbb{S}^+}(\sim s)).$$
- Suppose $t = s_1 \vee s_2$. Then $\pi \circ \psi(s_1 \vee s_2) = \pi(\psi(s_1)) \vee^{\mathbf{A}_j} \pi(\psi(s_2)) = \varphi(N_{\mathbb{S}^+}(s_1)) \vee^{\mathbf{A}_j} \varphi(N_{\mathbb{S}^+}(s_2)) = \varphi(N_{\mathbb{S}^+}(s_1) \vee N_{\mathbb{S}^+}(s_2)) = \varphi(N_{\mathbb{S}^+}(s_1 \vee s_2))$.

Thus $\pi \circ \psi(p) = \varphi(N_{\mathbb{S}^+}(p)) \neq \varphi(N_{\mathbb{S}^+}(q)) = \pi \circ \psi(q)$, which means that $p \approx q$ fails in $\mathbf{A}_i \times \mathbf{A}_j$ and hence in $\mathfrak{P}$.

From right to left, suppose that $p \approx q$ fails in $\mathfrak{P}$, whence by Lemma 4.3 there exists $\psi\colon \mathbf{T}_\tau(X) \to \mathbf{A}_i \times \mathbf{A}_j$ such that $\psi(p) \neq \psi(q)$. If π_i (resp. π_j) is the projection map from $\mathbf{A}_i \times \mathbf{A}_j$ onto $\mathbf{A}_i$ (resp. onto $\mathbf{A}_j$), we have two homomorphisms $\psi_i := \pi_i \circ \psi\colon \mathbf{T}_\tau(X) \to \mathbf{A}_i$ and $\psi_j := \pi_j \circ \psi\colon \mathbf{T}_\tau(X) \to \mathbf{A}_j$. Thus, either $\psi_i(p) \neq \psi_i(q)$ or $\psi_j(p) \neq \psi_j(q)$. If the former, then $p \approx q \notin \mathrm{Id}(\mathfrak{BA}_\tau)$. If the latter, let $\mathbf{A}_j^*$ be the $\neg$-free reduct of $\mathbf{A}_j$ and define $\varphi\colon \mathbf{T}_\rho(X) \to \mathbf{A}_j^*$ by $\varphi(x) = \psi_j(x)$. We show by induction that for any τ-term t, $\varphi(N_{\mathbb{S}^+}(t)) = \psi_j(t)$. Again, we examine in detail the base case and the unary and binary operations.

- Suppose $t = x$. Then $\psi_j(x) = \varphi(x) = \varphi(N_{\mathbb{S}^+}(x))$.
- Suppose $t = \neg s$. Then $\psi_j(\neg s) = \neg^{\mathbf{A}_j} \psi_j(s) = \psi_j(s) = \varphi(N_{\mathbb{S}^+}(s)) = \varphi(N_{\mathbb{S}^+}(\neg s))$.
- Suppose $t = \sim s$. Then $\psi_j(\sim s) = \sim^{\mathbf{A}_j} \psi_j(s) = \sim^{\mathbf{A}_j^*} \varphi(N_{\mathbb{S}^+}(s)) = 1 - \varphi(N_{\mathbb{S}^+}(s)) = \varphi(N_{\mathbb{S}^+}(\sim s))$.
- Suppose $t = s_1 \vee s_2$. Then
$$\begin{aligned}\psi_j(s_1 \vee s_2) = \psi_j(s_1) \vee^{\mathbf{A}_j} \psi_j(s_2) &= \varphi(N_{\mathbb{S}^+}(s_1)) \vee^{\mathbf{A}_j^*} \varphi(N_{\mathbb{S}^+}(s_2)) \\ &= \varphi(N_{\mathbb{S}^+}(s_1) \vee N_{\mathbb{S}^+}(s_2)) = \varphi(N_{\mathbb{S}^+}(s_1 \vee s_2)).\end{aligned}$$

Thus $\varphi(N_{\mathbb{S}^+}(p)) = \psi_j(p) \neq \psi_j(q) = \varphi(N_{\mathbb{S}^+}(q))$, which shows that $p \approx q \notin \mathrm{Id}_{N_{\mathbb{S}^+}}(\mathfrak{BA}_\rho)$. □

Next, we see how the results in this paper bear on the relationships between Polin algebras and other varieties that have been addressed (directly or indirectly) in the previous sections.

Lemma 6.2. *Let* $\mathbf{P} \in \mathfrak{P}$. *The following conditions are equivalent*:

(1) $\mathbf{P}$ *satisfies the quasi-identity* $\sim x \approx 1 \Rightarrow x \approx 0$.
(2) $\mathbf{P}$ *satisfies the identity* $\sim 1 \approx 0$.
(3) *In the representation of Theorem 4.4, the bottom fibre is a trivial algebra.*
(4) *The* $\neg$*-free reduct of* $\mathbf{P}$ *is a pseudocomplemented semilattice.*

Proof. (1) implies (2). Clear, since easily $\sim\sim 1 = 1$.

(2) is equivalent to (3). Suppose (2) holds, and assume ex absurdo that the top element of the bottom fibre of (the representation of) P is $a \neq 0$. Then $\sim 1 = a$, contradicting (2). The converse follows from Theorem 4.4.

(2) implies (4). By [6, Thm. 3.1.7], pseudocomplemented semilattices can be axiomatised by semilattice identities plus the following four identities:

$$\text{(PS1) } x \vee 1 \approx 1 \qquad \text{(PS2) } x \vee \sim 1 \approx x$$
$$\text{(PS3) } x \vee \sim x \vee y \approx 1 \quad \text{(PS4) } x \vee \sim(x \vee y) \approx x \vee \sim y.$$

Of these, the semilattice identities, (PS1) and (PS3) are easily proved from the axioms of Definition 4.2, using axioms (s1)-(s3), (s5), and (e4). For $\mathbb{S}^+$ any E-system defined as in Theorem 4.4, (PS4) is a valid Boolean τ-identity that is $N_{\mathbb{S}^+}$-regular in $\mathfrak{BA}_\rho$, hence it holds in $\mathbf{P}$ by Theorem 6.1. Finally, if (2) holds, then $x \vee \sim 1 = x \vee 0 = x$, i.e., (PS2) holds.

(4) implies (1). This quasi-identity is clearly satisfied in pseudocomplemented semilattices. □

The variety $\mathfrak{IBSL}$ of *involutive bisemilattices* (for which see [2,38] or [3, Ch. 2.4]) is the regularisation of $\mathfrak{BA}_\rho$. We conclude this section by showing that involutive bisemilattices satisfy precisely the same identities as the $\sim$-free reducts of Polin algebras.

Theorem 6.3. *The equational theory of* $\mathfrak{P}$ *is a conservative extension of the equational theory of* $\mathfrak{IBSL}$.

Proof. Involutive bisemilattices satisfy precisely the regular identities of type ρ that are satisfied by Boolean algebras. By Theorem 6.1, Polin algebras satisfy the identities in $\mathrm{Id}(\mathfrak{BA}_\tau) \cap \mathrm{Id}_{N_{\mathbb{S}^+}}(\mathfrak{BA}_\rho)$. Let $p \approx q$ be a $\sim$-free identity of type τ, which is in fact an identity of type ρ. If it is in $\mathrm{Id}(\mathfrak{BA}_\tau) \cap \mathrm{Id}_{N_{\mathbb{S}^+}}(\mathfrak{BA}_\rho)$, then it holds in all Boolean algebras and, moreover, it is regular by Theorem 4.4, since its $N_{\mathbb{S}^+}$-translation contains no occurrences of $\neg$ and hence it is a semilattice identity with zero. Conversely, if it is a regular Boolean identity, then in particular it is $N_{\mathbb{S}^+}$-regular in $\mathfrak{BA}_\rho$ (because it contains no occurrences of $\neg$) and belongs to $\mathrm{Id}(\mathfrak{BA}_\tau)$. Hence the $\sim$-free reducts of Polin algebras satisfy the same identities as involutive bisemilattices. □

7. Conclusions and open problems

The list of examples of enriched Płonka sums discussed in this paper is by no means exhaustive. For instance, *Bochvar algebras* [14] are the equivalent algebraic semantics of Bochvar's external logic, a time-honoured three-valued logic with applications e.g. in the theory of computer malfunctions. Recently, Bonzio and Pra Baldi [4] gave a representation of Bochvar algebras that can indeed be subsumed under our generalisation. On the other hand, it may be more difficult to harness into this framework other kindred constructions [21, 26, 31].

It would be interesting to extend this concept to encompass the case of arbitrary indexing algebras (as in Agassiz sums), while allowing for a finite but otherwise unrestricted set of relations and corresponding classes of transition maps. Beyond the greater technical difficulties associated with this more ambitious goal—an endeavour already initiated in the ongoing work [20])—it would also be necessary to identify relevant examples that involve more than two relations.

Theorem 5.5 has certain limitations, primarily due to the somewhat ad hoc nature of the confluence condition. Ideally, one would seek more natural sufficient conditions to ensure the preservation of $N_{\mathbb{S}^+}$-regular identities. Moreover, in favourable cases, such as in Płonka sums or De Morgan–Płonka sums, $N_{\mathbb{S}^+}$-regular identities admit an independent, purely syntactical description (in terms of regular identities and balanced regular identities, respectively). In general, however, this does not appear to hold. A worthwhile direction for further research would be to explore whether similar independent characterisations exist for other examples examined in this paper.

Finally, a partial converse to Theorem 5.5 is available for both Płonka sums and De Morgan–Płonka sums. For example (Theorem 2.4), whenever $\mathfrak{V}$ is a strongly irregular variety of type τ and $\mathbf{A}$ is a τ-algebra in $R(\mathfrak{V})$, we have that $\mathbf{A}$ is decomposable as a Płonka sum over a semilattice direct system of algebras in $\mathfrak{V}$. A potential avenue for generalisation would be to refine the notion of a partition function appropriately. We intend to pursue this goal in future work.

Appendix

In order to see Lemma 5.4 in action, consider identity (e3) in the equational base for $\mathfrak{P}$ written above. Let us draw the syntactic trees $\mathcal{T}_1$ (left) and $\mathcal{T}_2$ (right) of the terms in the identity (Figure 4).

We first show that (e3) is $N_{\mathbb{S}^+}$-regular. We have:

$$\begin{aligned} N_{\mathbb{S}^+}({\sim}(x_1 \vee x_2) \vee {\sim} x_1) &= N_{\mathbb{S}^+}({\sim}(x_1 \vee x_2)) \vee N_{\mathbb{S}^+}({\sim} x_1) \\ &= \neg N_{\mathbb{S}^+}(x_1 \vee x_2) \vee \neg N_{\mathbb{S}^+}(x_1) \\ &= \neg(N_{\mathbb{S}^+}(x_1) \vee N_{\mathbb{S}^+}(x_2)) \vee \neg N_{\mathbb{S}^+}(x_1) \\ &= \neg(x_1 \vee x_2) \vee \neg x_1 \end{aligned}$$

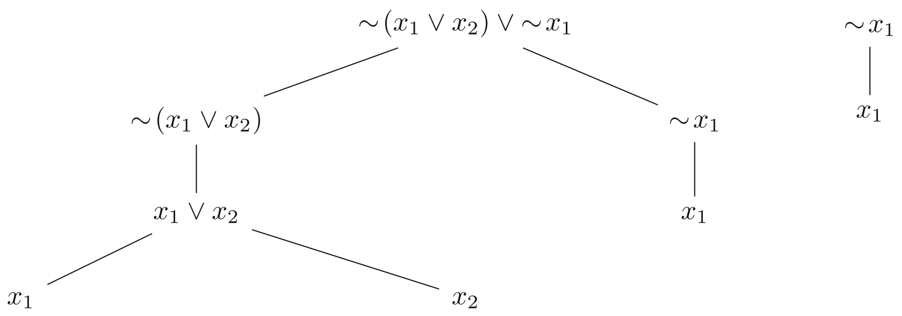


FIGURE 4. Syntactic trees $\mathcal{T}_1$ of $\sim(x_1 \vee x_2) \vee \sim x_1$ (left) and $\mathcal{T}_2$ of $\sim x_1$ (right)

$$
\begin{aligned}
N_{\mathbb{S}^+}(\sim x_1) &= \neg N_{\mathbb{S}^+}(x_1)\\
&= \neg x_1
\end{aligned}
$$

The identity $\neg(x_1 \vee x_2) \vee \neg x_1 \approx \neg x_1$ is a version of the absorption identity, clearly satisfied in $\mathfrak{BA}_\rho$. Moving along the paths of the two trees in the upward direction, we now compute the interpretations in $\mathbf{P\l}(\mathbb{S}^+_{\mathbf{A}_i \times \mathbf{A}_j})$ of the terms $\sim(x_1 \vee x_2) \vee \sim x_1$ and $\sim x_1$ (see Example 4.5).

To streamline our notation, we set $N = N_{\mathbb{S}^+_{\mathbf{A}_i \times \mathbf{A}_j}}$ and $\mathbf{A} = \mathbf{P\l}(\mathbb{S}^+_{\mathbf{A}_i \times \mathbf{A}_j})$. Starting from the left branch of $\mathcal{T}_1$ (in which we obviously include the terminal bifurcation), for $b_1 = x_1^{\mathbf{B}_2}$ and $b_2 = x_2^{\mathbf{B}_2}$, we define

$$
\begin{aligned}
b^1_{12} &= N(x_1 \vee x_2)^{\mathbf{B}_2}(b_1, b_2)\\
&= N(x_1)^{\mathbf{B}_2}(\bar{b}) \vee^{\mathbf{B}_2} N(x_2)^{\mathbf{B}_2}(\bar{b})\\
&= b_1 \vee^{\mathbf{B}_2} b_2.
\end{aligned}
$$

Thus we have $(x_1 \vee x_2)^{\mathbf{A}}(a_1, a_2) = \varphi^1_{b_1 b^1_{12}}(a_1) \vee^{\mathbf{A}_{b^1_{12}}} \varphi^1_{b_2 b^1_{12}}(a_2)$, for $a_1 \in A_{b_1}$ and $a_2 \in A_{b_2}$.

Note. Let us spend a few lines to justify our notation. Let $\mathcal{T}$ be the syntactic tree of an arbitrary τ-term p. We select $x_k \in Var(p)$, calling it, for reasons that will become clear later, the *initial variable*. The interpretation of $N(x_k) = x_k$ in $\mathbf{B}_2$ will be an element $b_k \in \{0, 1\}$. Let now p^n_d be the subterm of p occurring in the n-th branch of $\mathcal{T}$ at depth d. We write b^n_{kd} to denote the element $N(p^n_d(\overline{x}))^{\mathbf{B}_2}(\bar{b})$, *emphasising the fact that we have reached the node labelled by p^n_d starting from the initial variable x_k*. In $\mathcal{T}_1$, we wrote b^1_{12} to denote $N(x_1 \vee x_2)^{\mathbf{B}_2}(b_1, b_2)$. This is because we chose x_1 as the initial variable. If we had chosen x_2, we would clearly have written b^1_{22}.

Having clarified the notation, let us continue by defining

$$
\begin{aligned}
b^1_{11} &= N(\sim(x_1 \vee x_2))^{\mathbf{B}_2}(b_1, b_2)\\
&= \neg^{\mathbf{B}_2} N(x_1 \vee x_2)^{\mathbf{B}_2}(b_1, b_2)\\
&= \neg^{\mathbf{B}_2}\left[N(x_1)^{\mathbf{B}_2}(\bar{b}) \vee^{\mathbf{B}_2} N(x_2)^{\mathbf{B}_2}(\bar{b})\right]
\end{aligned}
$$

$$= \neg^{\mathbf{B}_2} \left(b_1 \vee^{\mathbf{B}_2} b_2\right)$$
$$= \neg^{\mathbf{B}_2} b^1_{12}.$$

So we obtain

$$\begin{aligned} &\sim(x_1 \vee x_2)^{\mathbf{A}}(a_1, a_2) \\ &= \neg^{\mathbf{A}_{b^1_{11}}} \left(\varphi^2_{b^1_{12}b^1_{11}} \left((x_1 \vee x_2)^{\mathbf{A}}(a_1, a_2)\right)\right) \\ &= \neg^{\mathbf{A}_{b^1_{11}}} \left(\varphi^2_{b^1_{12}b^1_{11}} \left(\varphi^1_{b_1 b^1_{12}}(a_1) \vee^{\mathbf{A}_{b^1_{12}}} \varphi^1_{b_2 b^1_{12}}(a_2)\right)\right) \\ &= \neg^{\mathbf{A}_{b^1_{11}}} \left[\left(\varphi^2_{b^1_{12}b^1_{11}} \circ \varphi^1_{b_1 b^1_{12}}\right)(a_1) \vee^{\mathbf{A}_{b^1_{11}}} \left(\varphi^2_{b^1_{12}b^1_{11}} \circ \varphi^1_{b_2 b^1_{12}}\right)(a_2)\right] \end{aligned}$$

Consider now the right branch of $\mathcal{T}_1$. For $b_1 = x_1^{\mathbf{B}_2}$ and $a_1 \in A_{b_1}$, let

$$\begin{aligned} b^2_{11} &= N(\sim x_1)^{\mathbf{B}_2}(b_1) \\ &= \neg^{\mathbf{B}_2} \left[N(x_1)^{\mathbf{B}_2}(\bar{b})\right] \\ &= \neg^{\mathbf{B}_2} b_1. \end{aligned}$$

Therefore, $(\sim x_1)^{\mathbf{A}}(a_1) = \neg^{\mathbf{A}_{b^2_{11}}} \left(\varphi^2_{b_1 b^2_{11}}(a_1)\right)$. In light of our computations, it is clear that the value

$$b = N(\sim(x_1 \vee x_2) \vee \sim x_1)^{\mathbf{B}_2}(b_1, b_2)$$

can be expressed as

$$\begin{aligned} b &= b^1_{11} \vee^{\mathbf{B}_2} b^2_{11} \\ &= \neg^{\mathbf{B}_2} b^1_{12} \vee^{\mathbf{B}_2} \neg^{\mathbf{B}_2} b_1 \\ &= \neg^{\mathbf{B}_2} \left(b_1 \vee^{\mathbf{B}_2} b_2\right) \vee^{\mathbf{B}_2} \neg^{\mathbf{B}_2} b_1. \end{aligned}$$

Hence we have

$$\begin{aligned} &\sim(x_1 \vee x_2) \vee \sim x_1{}^{\mathbf{A}}(a_1, a_2) \\ &\quad= \varphi^1_{b^1_{11} b} \left(\sim(x_1 \vee x_2)^{\mathbf{A}}(a_1, a_2)\right) \vee^{\mathbf{A}_b} \varphi^1_{b^2_{11} b} \left((\sim x_1)^{\mathbf{A}}(a_1)\right) \\ &\quad= \varphi^1_{b^1_{11} b} \left[\neg^{\mathbf{A}_{b^1_{11}}} \left[\left(\varphi^2_{b^1_{12}b^1_{11}} \circ \varphi^1_{b_1 b^1_{12}}\right)(a_1) \vee^{\mathbf{A}_{b^1_{11}}} \left(\varphi^2_{b^1_{12}b^1_{11}} \circ \varphi^1_{b_2 b^1_{12}}\right)(a_2)\right]\right] \\ &\qquad \vee^{\mathbf{A}_b} \varphi^1_{b^2_{11} b} \left[\neg^{\mathbf{A}_{b^2_{11}}} \left(\varphi^2_{b_1 b^2_{11}}(a_1)\right)\right] \\ &\quad= \neg^{\mathbf{A}_b} \left[\left(\varphi^1_{b^1_{11} b} \circ \varphi^2_{b^1_{12}b^1_{11}} \circ \varphi^1_{b_1 b^1_{12}}\right)(a_1) \vee^{\mathbf{A}_b} \left(\varphi^1_{b^1_{11} b} \circ \varphi^2_{b^1_{12}b^1_{11}} \circ \varphi^1_{b_2 b^1_{12}}\right)(a_2)\right] \\ &\qquad \vee^{\mathbf{A}_b} \neg^{\mathbf{A}_b} \left[\left(\varphi^1_{b^2_{11} b} \circ \varphi^2_{b_1 b^2_{11}}(a_1)\right)\right]. \end{aligned}$$

As for $\mathcal{T}_2$, which is a single-branch tree of depth 1, the procedure is immediate. Indeed, for $c_1 = x_1^{\mathbf{B}_2}$ and $a_1 \in A_{c_1}$, we obtain

$$(\sim x_1)^{\mathbf{A}}(a_1) = \neg^{\mathbf{A}_c} \left(\varphi^2_{c_1 c}(a_1)\right),$$

where $c = N(\sim x_1)^{\mathbf{B}_2}(c_1)$.

And here comes a crucial point. The fact that $\sim(x_1 \vee x_2) \vee \sim x_1 \approx \sim x_1$ is N-regular brings about that $b = c$. However, in principle, nothing guarantees that the two terms occurring in the identity are interpreted on the same object

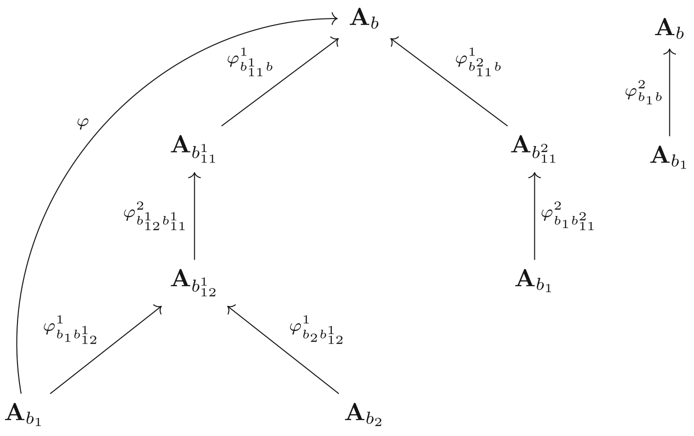


FIGURE 5. $\varphi = \varphi^1_{b^1_{11}b} \circ \varphi^2_{b^1_{12}b^1_{11}} \circ \varphi^1_{b_1b^1_{12}}$ and $\psi = \varphi^2_{b_1b}$

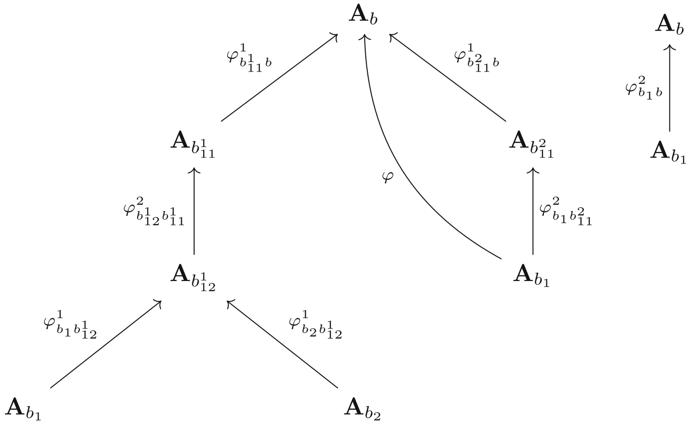


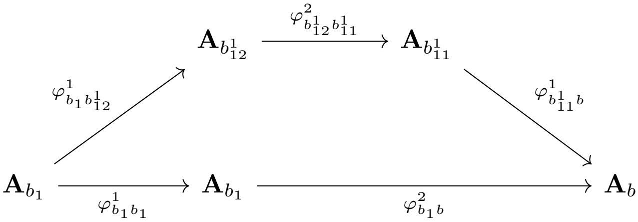
FIGURE 6. $\varphi = \varphi^1_{b^2_{11}b} \circ \varphi^2_{b_1b^2_{11}}$ and $\psi = \varphi^2_{b_1b}$

in $\mathbf{A}$. Let us first observe that any satisfiable identity $p \approx q$ is such that $Var(p) \cap Var(q) \neq \emptyset$. In particular,

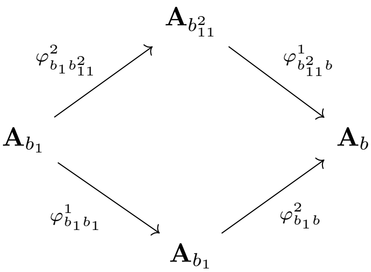
$$\mathit{Var}\Big(\sim(x_1 \vee x_2) \vee \sim x_1\Big) \cap \mathit{Var}\Big(\sim x_1\Big) = \{x_1\}.$$

Since $b_1 = c_1 = x_1^{\mathbf{B}_2}$ and $a_1 = x_1^{\mathbf{A}}$, to establish that $\mathbf{A} \models \sim(x_1 \vee x_2) \vee \sim x_1 \approx \sim x_1$, it is sufficient to exhibit a pair of homomorphisms $\varphi, \psi \in \mathrm{Hom}(\mathbf{A}_{b_1}, \mathbf{A}_b)$ which agree on the value assigned to a_1, *on the basis of the transformations successively applied in the interpretation of the subterms of* $\sim(x_1 \vee x_2) \vee \sim x_1$ *and* $\sim x_1$. Indeed φ and ψ can be chosen as shown in Figures 5 and 6.

This is where Lemma 5.4 comes in. Although these two pairs of diagrams are displayed in such a way as to mirror the structure of the syntactic trees in Figure 4, the reader can immediately verify that in both cases the initial conditions of Lemma 5.4 are satisfied. Indeed, in relation to Figure 5, we have



and therefore $\varphi = \psi$. Instead, the diagram we obtain from Figure 6 is



and also in this case $\varphi = \psi$. We conclude that $\mathbf{A} \models {\sim}(x_1 \vee x_2) \vee {\sim} x_1 \approx {\sim} x_1$.

In general, with reference to Theorem 5.5, the fibres over which an algebra $\mathbf{A} \in \mathfrak{G}$ is constructed are not necessarily isomorphic copies of one another—the functions connecting them might be generic homomorphisms. However, the checking method is essentially the same as above.

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Declarations

Conflict of interest The authors declare that they have no competing interests.

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