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Isometry-invariant solutions to a critical problem on non-compact Riemannian manifolds

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Abstract

We analyse an elliptic equation with critical growth set on a d-dimensional ($d \geq 3$) Hadamard manifold $(\mathcal{M}, g)$. By adopting a variational perspective, we prove the existence of non-zero non-negative solutions invariant under the action of a specific family of isometries. Our result remains valid when the original nonlinearity is singularly perturbed. Preserving the same variational approach, but considering other groups of isometries, we finally show that when $\mathcal{M} = \mathbb{R}^d$, $d > 3$, and the nonlinearity is odd, there exist at least $(-1)^d + \left[\frac{d-3}{2}\right]$ pairs of sign-changing solutions.

MSC: 58J05; 53C21; 35A15; 35J60

Keywords: Non-compact Riemannian manifold; Critical equation; Isometry; Existence; Multiplicity

1. Introduction

In the present paper we address the existence and multiplicity of solutions to the problem

$$-\Delta_g w + w = |w|^{\frac{4}{d-2}} w + \lambda\alpha(\sigma) f(w), \quad \sigma \in \mathcal{M},\ w \in H^1_g(\mathcal{M}), \tag{P_λ}$$

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where $(\mathcal{M}, g)$ is a d-dimensional, $d \geq 3$, Hadamard manifold, Δ_g is the usual Laplace-Beltrami operator on $(\mathcal{M}, g)$, expressed in local coordinates $(x_1, \dots, x_d)$ by

$$\Delta_g w := g^{ij} \left(\frac{\partial^2 w}{\partial x_i \partial x_j} - \Gamma^k_{ij} \frac{\partial w}{\partial x_k} \right),$$

(Γ^k_{ij} denotes as usual the Christoffel's symbols), the function $\alpha : \mathcal{M} \to \mathbb{R}$ satisfies suitable integrability and symmetry conditions, λ is a positive real parameter and $f : \mathbb{R} \to \mathbb{R}$ plays the role of a subcritical perturbation.

It is immediate to recognise in (P_λ) a Yamabe-type equation, though set in a non-compact framework and subcritically perturbed. As largely known, the Yamabe problem on a compact d-dimensional $(d > 2)$ boundaryless Riemannian manifold $(\mathcal{M}, g)$, consists in finding a metric $\tilde{g}$ on $\mathcal{M}$ with constant scalar curvature $\tilde{k}$ and pointwise conformal to g. If one sets $\tilde{g} := w^{4/(d-2)} g$, where $w > 0$ is the conformal factor, the existence of such a metric is equivalent to the existence of a positive solution to the semilinear equation

$$-\Delta_g w + \frac{d-2}{4(d-1)} k(\sigma) w = \tilde{k} w^{\frac{d+2}{d-2}}. \tag{1}$$

The resolution of (1), obtained by techniques spanning differential geometry, functional analysis and partial differential equations, was obtained by the efforts, in chronological order, of Yamabe [1], Trudinger [2], Aubin [3] and Schoen [4].

Passing from the compact to the non-compact framework, the situation, as expected, gets much more delicate and, as shown by Jin [5], it is possible to exhibit examples of complete non-compact manifolds for which the Yamabe problem does not have any solution. The literature on the subject contains only partial results, mainly related to the case of non-positive scalar curvature, and the issue of the existence is not settled in full generality. Among the papers which require the non-positivity of the scalar curvature, it is worth mentioning [6] by Aviles and McOwen. In [7] and [8], Kim introduced the so-called Yamabe constant at infinity to deal with the non-compact Yamabe problem on manifolds of positive scalar curvature and his existence result was later improved in [9]. Other interesting contributions in which no explicit assumptions on the curvature are required, are instead [10–12]. In particular, in the recent [11], instead of assuming the positivity of the scalar curvature, the author requires the infimum of the L^2-spectrum of the conformal Laplacian with respect to the complete metric g to be positive. We also point out that recent results for parametric equations and systems related to (1) have been established in [13] and [14], where, by Liapunov-Schmidt reduction procedures, concentration phenomena of positive solutions are investigated as the parameter approaches zero.

In this paper, still working in the non-compact setting, we basically perturb equation (1) by a subcritical term which, as shown later, may very well include singularities, and focus on the existence and multiplicity of solutions to (P_λ) by variational techniques. The main obstruction in this direction is evidently represented by the loss of compactness which we try to regain by investigating the presence of group symmetries. If G is a compact subgroup of isometries of a non-compact homogeneous Riemannian manifold $(\mathcal{M}, g)$, Skrzypczak and Tintarev [15] have recently proved that, under very general conditions, the G-invariant subspace of a normed vector space embeds compactly into $L^p(\mathcal{M})$, $p \in (1, \infty)$, if and only if G is coercive, i.e. has no orbits with a uniformly bounded diameter in a neighbourhood of infinity (see [15, Theorem 1.3]). As a plain consequence of this result, if G fixes only a point $\sigma_0 \in \mathcal{M}$, the subspace of the functions radially symmetric with respect to σ_0 is still compactly embedded in $L^p(\mathcal{M})$.

As said at the beginning, we restrict our attention to a class of manifolds for which the group G meeting the last condition can be made explicit in several cases, i.e. to Hadamard manifolds (complete, simply connected and with everywhere non-positive sectional curvature). By virtue of Cartan-Hadamard's theorem, every Hadamard manifold $(\mathcal{M}, g)$ is diffeomorphic to $\mathbb{R}^d$, $d = \dim \mathcal{M}$, but there exist other interesting geometric objects having this structure, for instance, the hyperbolic space and the cone of symmetric positive-definite matrices endowed with a suitable metric, as shown in the next section; see also [16] and the references therein.

Settled the compactness issue by means of this group-theoretical approach, we are going to apply variational techniques to establish existence and multiplicity of solutions to (P_λ). The main ingredients are a local lower semicontinuity property of the energy, which opens the way for the direct minimization, and some estimates for the functional

$$w \mapsto \frac{\sup_{\|z\| \le \varrho} \widetilde{\mathcal{E}}_\lambda(z) - \widetilde{\mathcal{E}}_\lambda(w)}{\varrho^2 - \|w\|^2}, \quad w \in H^1_g(\mathcal{M}), \tag{2}$$

where $\varrho > 0$ is small enough, $\|\cdot\|$ is the standard norm on $H^1_g(\mathcal{M})$ and

$$\widetilde{\mathcal{E}}_\lambda(w) := \frac{d-2}{2d} \int_{\mathcal{M}} |w|^{\frac{2d}{d-2}} d\sigma_g + \lambda \int_{\mathcal{M}} \alpha(\sigma) \left(\int_0^w f(s) ds \right) d\sigma_g,$$

for all $w \in H^1_g(\mathcal{M})$. By assuming that the nonlinearity f is merely subcritical and with an appropriate asymptotic behaviour at 0, and that λ falls within the positive range of a suitable rational function, we prove that the energy associated with (P_λ) and restricted to the subspace of $H^1_g(\mathcal{M})$ of the G-symmetric functions, admits a local minimum. Palais' principle of symmetric criticality allows us finally to get our existence result. This approach can be traced, for instance, to the recent papers [17–19], where it has been applied in different contexts and, in particular, to [20] where the compact counterpart of problem (P_λ) has been discussed.

In what follows, just to give a concrete example of our main result, we formulate it in a quite common framework, i.e. the hyperbolic space $\mathbb{H}^d$, after briefly recalling the Poincaré model of this space.

Denoting by $B(0,1) := \left\{x \in \mathbb{R}^d : |x| < 1\right\}$ the open unit ball of $\mathbb{R}^d$, the Poincaré ball model of $\mathbb{H}^d$ is simply $B(0,1)$ endowed with the Riemannian metric

$$g_{ij}(x) = \varrho^2(x)\delta_{ij}, \tag{3}$$

for all $x \in B(0,1)$ and for all $i, j = 1, \dots, d$, where $\varrho : B(0,1) \to \mathbb{R}$ is the function defined by

$$\varrho(x) = \frac{2}{1 - |x|^2} \quad \text{for all } x \in B(0,1).$$

The Riemannian volume element $d\sigma_g$ on this manifold is given by

$$d\sigma_g = \sqrt{\det g}\, dx = (\varrho(x))^d dx,$$

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where dx is the Lebesgue volume element of $\mathbb{R}^d$. It is customary to set

$$\partial B(0,1) = \partial \mathbb{H}^d = \{\infty\} \quad \text{and} \quad \overline{\mathbb{H}^d} = \mathbb{H}^d \cup \partial \mathbb{H}^d = \overline{B}(0,1).$$

The geodesic distance $d_g(x, y)$ between $x, y \in \mathbb{H}^d$ is expressed by

$$d_g(x,y) = \cosh^{-1}\left(1 + \frac{2|x-y|^2}{(1-|x|^2)(1-|y|^2)}\right),$$

from which one deduces that

$$d_g(x,0) = \ln\left(\frac{1+|x|}{1-|x|}\right)$$

and that

$$B(0,r) = B_g\left(0, \ln\left(\frac{1+r}{1-r}\right)\right),$$

for any $r \in (0,1)$, being $B_g(x,r) := \{y \in \mathbb{H}^d : d_g(x,y) < r\}$ the open geodesic ball with centre x and radius $r > 0$. Taking account of the expression of the Laplace-Beltrami operator on a local chart of coordinates $(x_1, \ldots, x_d)$ and of the geodesic distance in $\mathbb{H}^d$, the hyperbolic Laplacian Δ_H takes the form

$$\Delta_H = \frac{1}{4}(1-|x|^2)^2 \sum_{i=1}^{d} \frac{\partial^2}{\partial {x_i}^2} + \frac{d-2}{2}(1-|x|^2) \sum_{i=1}^{d} x_i \frac{\partial}{\partial x_i} = \frac{\Delta}{\varrho^2} + \frac{d-2}{\varrho}\langle x, \nabla \rangle .$$

In this framework, our existence result reads as follows:

Theorem 1. *Let $G := SO(d_1) \times \ldots \times SO(d_k)$, with $d_j \geq 2$, $j = 1, \ldots, k$ and $d_1 + \ldots + d_k = d$, where $SO(j)$ is the special orthogonal group in dimension j, let $\alpha \in L^1(\mathbb{H}^d) \cap L^\infty(\mathbb{H}^d) \setminus \{0\}$ be non-negative and radially symmetric with respect to the origin and let $f : \mathbb{R} \to \mathbb{R}$ satisfy*

(f_1) *there exist $\kappa > 0$, $q \in (1,2)$ and $p \in [2, 2d/(d-2))$, such that*

$$|f(t)| \leq \kappa(|t|^{q-1} + |t|^{p-1}), \quad \text{for all } t \in \mathbb{R},$$

(f_2) $$\liminf_{t \to 0^+} \frac{\int_0^t f(s)ds}{t^2} = +\infty.$$

Furthermore, let $r \in [2, 2d/(d-2))$ and $l : (0, +\infty) \to \mathbb{R}$ be the function

$$l(t) := \frac{t - c_{2^*}^{2^*} t^{2^*-1}}{\kappa\left(c_r^q \, \|\alpha\|_{L^{\frac{r}{r-q}}(\mathbb{H}^d)} t^{q-1} + c_p^p \, \|\alpha\|_{L^\infty(\mathbb{H}^d)} t^{p-1}\right)} \quad \text{for all } t > 0, \tag{4}$$

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where

$$c_\nu := \sup_{w\in H^1_g(\mathbb{H}^d)\setminus\{0\}} \frac{\|w\|_{L^\nu(\mathbb{H}^d)}}{\|w\|_{H^1_g(\mathbb{H}^d)}}, \quad \nu \in \left[2, \frac{2d}{d-2}\right].$$

Then, there exists an open interval $\Lambda \subseteq \big(0, \max_{t\in[0,+\infty)} l(t)\big)$ *such that, for every* $\lambda \in \Lambda$, *the problem*

$$-\Delta_H w + w = |w|^{\frac{4}{d-2}} w + \lambda\alpha(x) f(w), \quad x \in \mathbb{H}^d,\ w \in H^1_g(\mathbb{H}^d), \qquad (\tilde{P}_\lambda)$$

admits a non-zero and non-negative G*-invariant solution* $w_{0,\lambda} \in H^1_g(\mathbb{H}^d)$.

The strategy adopted to tackle problem (P_λ) continues to be valid if we add a singular term to f, namely

$$\begin{cases} -\Delta_g w + w = w^{\frac{d+2}{d-2}} + \lambda\alpha(\sigma)\big(f(w) + w^{r-1}\big), & \sigma \in \mathcal{M} \\ w \in H^1_g(\mathcal{M}), \quad w > 0 \text{ in } \mathcal{M}, \end{cases} \qquad (P^\star_\lambda)$$

where $r \in (0,1)$ and $f : [0,+\infty) \to [0,+\infty)$ is again continuous and subcritical. In this case the rational functional which naturally arises with this approach takes a slightly more complicated form and more attention has to be paid to the definition (and verification) of a weak solution to ($P^\star_\lambda$).

In the final section we address the issue of the multiplicity of solutions to (P_λ) and we restrict ourselves to $\mathbb{R}^d$, making the additional assumption that f is odd. It is clear that in this case, being the energy functional even, the solutions to (P_λ) appear in pairs. Borrowing an idea originally proposed by Bartsch and Willem [21] and later developed by Kristály, Moroşanu and O'Regan [22], we repeatedly apply a similar methodology to specific subspaces of $H^1(\mathbb{R}^d)$ which preserve the compact embedding in $L^p(\mathbb{R}^d)$ and whose pairwise intersections contain only the null function. Thanks to these facts we are able to prove, for $\lambda > 0$ sufficiently small, a dimension-depending multiplicity result: if $d > 3$ and $d \neq 5$ there exist at least $1 + (-1)^d + \left[\frac{d-3}{2}\right]$ pair of solutions, one of which is radial, and the remaining ones are non-radial and sign-changing; when $d = 5$ there exists at least a pair of non-trivial radial solutions.

The paper is structured as follows. In Section 2 we fix notations and introduce the functional framework of our problem. In Sections 3 and 4 we prove the existence results for Problems (P_λ) and ($P^\star_\lambda$), respectively. The final Section 5 is devoted to the multiplicity of solutions in the Euclidean setting with symmetric nonlinearity.

2. Preliminaries

In this section we briefly recall some notions from Riemannian geometry needed in the sequel and then illustrate the functional framework we will move in. We refer the reader to the following sources [23–25] for detailed derivations of the geometric quantities, their motivation and further applications.

Let $(\mathcal{M}, g)$ be a d-dimensional Riemannian manifold, $d \geq 3$, and let g_{ij} be the components of the metric g. Denote by $T_\sigma M$ the tangent space at $\sigma \in \mathcal{M}$ and by $TM = \bigcup_{\sigma\in\mathcal{M}} T_\sigma M$ the tangent bundle. Let $d_g : \mathcal{M} \times \mathcal{M} \to [0,+\infty)$ be the usual distance function associated with g,

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$B_g(\sigma, r) := \{x \in \mathcal{M} : d_g(\sigma, x) < r\}$ and $\overline{B}_g(\sigma, r) := \{x \in \mathcal{M} : d_g(\sigma, x) \leq r\}$ be the open and closed geodesic balls centred at $\sigma \in \mathcal{M}$ and of radius $r > 0$.

If $C_0^\infty(\mathcal{M})$ denotes, as customary, the space of real-valued compactly supported smooth functions on $\mathcal{M}$, we set

$$\|w\| := \left(\int_{\mathcal{M}} |\nabla_g w|^2 d\sigma_g + \int_{\mathcal{M}} |w|^2 d\sigma_g \right)^{1/2}, \tag{5}$$

for every $w \in C_0^\infty(\mathcal{M})$, where $\nabla_g w$ is the covariant derivative of w and $d\sigma_g$ is the Riemannian measure on $\mathcal{M}$, related to the Lebesgue measure dx in $\mathbb{R}^d$ by the formula $d\sigma_g = \sqrt{\det g}\, dx$. We set

$$\mathrm{Vol}_g(\Omega) := \int_{\Omega} d\sigma_g$$

for every bounded open set $\Omega \subset \mathcal{M}$. Fixed a system of local coordinates $(x_1, \dots, x_d)$, $\nabla_g w$ can be represented by

$$(\nabla_g^2 w)_{ij} = \frac{\partial^2 w}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial w}{\partial x_k},$$

where

$$\Gamma_{ij}^k := \frac{1}{2} \left(\frac{\partial g_{lj}}{\partial x_i} + \frac{\partial g_{li}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k} \right) g^{lk}$$

are the usual Christoffel's symbols and g^{lk} are the elements of the inverse matrix of g (here and anywhere else, Einstein's summation convention is tacitly adopted). It is useful to remind that, for every fixed $\sigma_0 \in \mathcal{M}$, the eikonal equation

$$|\nabla_g d_g(\sigma_0, \cdot)| = 1 \tag{6}$$

is satisfied almost everywhere in $\mathcal{M} \setminus \{\sigma_0\}$. The Laplace-Beltrami operator is the differential operator $\Delta_g w = \mathrm{div}(\nabla_g w)$ and its local expression is

$$\Delta_g w := g^{ij} \left(\frac{\partial^2 w}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial w}{\partial x_k} \right) = -\frac{1}{\sqrt{\det(g_{ij})}} \frac{\partial}{\partial x_m} \left(\sqrt{\det(g_{ij})} g^{km} \frac{\partial w}{\partial x_k} \right).$$

The space $H_g^1(\mathcal{M})$ is defined to be the completion of $C_0^\infty(\mathcal{M})$ with respect to the norm (5) and it turns out to be a Hilbert space equipped with the inner product

$$\langle w_1, w_2 \rangle := \int_{\mathcal{M}} \left(\langle \nabla_g w_1, \nabla_g w_2 \rangle_g + w_1 w_2 \right) d\sigma_g, \tag{7}$$

for every $w_1, w_2 \in H^1_g(\mathcal{M})$. The open (respectively, closed) ball centred at $w \in H^1_g(\mathcal{M})$ of radius $r > 0$ will be indicated by $B(w, r)$ (respectively, $\overline{B}(w, r)$), and the sphere $\{z \in H^1_g(\mathcal{M}) : \|w - z\| = r\}$ by $\partial B(w, r)$.

The notion of curvature on $(\mathcal{M}, g)$ is described by means of the Riemann tensor that assigns, to each point $\sigma \in \mathcal{M}$, a multilinear function $R_{(\mathcal{M},g)} : T_\sigma\mathcal{M} \times T_\sigma\mathcal{M} \times T_\sigma\mathcal{M} \times T_\sigma\mathcal{M} \to \mathbb{R}$ satisfying the pair of conditions

$$R_{(\mathcal{M},g)}(X_1, X_2, X_3, X_4) = -R_{(\mathcal{M},g)}(X_2, X_1, X_4, X_3) = R_{(\mathcal{M},g)}(X_4, X_3, X_2, X_1),$$
$$R_{(\mathcal{M},g)}(X_1, X_2, X_3, X_4) + R_{(\mathcal{M},g)}(X_2, X_3, X_1, X_4) + R_{(\mathcal{M},g)}(X_3, X_1, X_2, X_4) = 0,$$

for any $X_1, X_2, X_3, X_4 \in T_\sigma\mathcal{M}$. It locally takes the form

$$R^h_{ijk} = \frac{\partial \Gamma^h_{jk}}{\partial x_i} - \frac{\partial \Gamma^h_{ik}}{\partial x_j} + \Gamma^r_{jk}\Gamma^h_{ir} - \Gamma^r_{ik}\Gamma^h_{jr}.$$

The Ricci tensor $\mathrm{Ric}_{(\mathcal{M},g)}$ on $(\mathcal{M}, g)$ is defined to be the trace of $R_{(\mathcal{M},g)}$ and in local coordinates is represented by

$$R_{ik} = R^h_{ihk} = \frac{\partial \Gamma^h_{hk}}{\partial x_i} - \frac{\partial \Gamma^h_{ik}}{\partial x_h} + \Gamma^r_{hk}\Gamma^h_{ir} - \Gamma^r_{ik}\Gamma^h_{hr}.$$

Given a point $\sigma \in \mathcal{M}$ and a two-dimensional plane $\pi \subset T_\sigma\mathcal{M}$ with basis $\{X_1, X_2\}$, the sectional curvature of π is defined by

$$k_\pi = k(X_1, X_2) := \frac{R_{(\mathcal{M},g)}(X_1, X_2, X_1, X_2)}{|X_1|^2_g |X_2|^2_g - \langle X_1, X_2\rangle^2_g}.$$

Since the above definition is independent of the choice of the vectors $\{X_1, X_2\}$, one can compute k_π by working with an orthonormal basis of π. A Riemannian manifold $(\mathcal{M}, g)$ is said to be a Hadamard manifold if it is complete, simply connected and with everywhere non-positive sectional curvature, i.e. such that $k(X_1, X_2) \leq 0$ for every $X_1, X_2 \in T_\sigma\mathcal{M}$ linearly independent tangent vectors. A Hadamard manifold $(\mathcal{M}, g)$ is termed homogeneous if the group of all isometries of $\mathcal{M}$ acts transitively on $\mathcal{M}$. Cartan-Hadamard's theorem guarantees that every Hadamard manifold $(\mathcal{M}, g)$ is diffeomorphic to $\mathbb{R}^d$, $d = \dim \mathcal{M}$, in striking contrast to Meyer's theorem, which states that any complete Riemannian manifold $(\mathcal{M}, g)$ of strictly positive Ricci curvature is compact. Besides the Euclidean space, there exist other interesting geometric objects having the structure of a Hadamard manifold, as shown shortly afterwards.

For the rest of this section and for the whole next one, without further mentioning, we will always assume that $(\mathcal{M}, g)$ is a d-dimensional, $d \geq 3$, homogeneous Hadamard manifold.

Referring to Hoffman and Spruck [26], the Sobolev embedding $H^1_g(\mathcal{M}) \hookrightarrow L^\nu(\mathcal{M})$ is continuous (but not compact) for every $\nu \in [2, 2^*]$, where $2^* = 2d/(d-2)$. In the light of this result, we indicate by c_ν the positive constant

$$c_\nu := \sup_{w \in H^1_g(\mathcal{M}) \setminus \{0\}} \frac{\|w\|_\nu}{\|w\|},$$

$\|\cdot\|_\nu$ denoting as usual the L^ν-norm on $\mathcal{M}$.

Since problem (P_λ) is set in a non-compact framework, we will adopt a group-theoretical approach to identify suitable symmetric subspaces of $H^1_g(\mathcal{M})$ for which the compactness of the embedding in $L^\nu(\mathcal{M})$ can be regained. Let G be a subgroup of $\mathrm{Isom}_g(\mathcal{M})$, the group of the isometries of $(\mathcal{M}, g)$. We say that $w : \mathcal{M} \to \mathbb{R}$ is G-invariant if $w(\tau(\sigma)) = w(\sigma)$ for every $\sigma \in \mathcal{M}$ and $\tau \in G$, and set

$$\mathrm{Fix}_G(\mathcal{M}) := \{\sigma \in \mathcal{M} : \tau(\sigma) = \sigma, \text{ for all } \tau \in G\}.$$

We focus now on a specific family of subgroups of $\mathrm{Isom}_g(\mathcal{M})$: fixed $\sigma_0 \in \mathcal{M}$, denote by

$$\mathcal{G}_{\sigma_0} = \left\{G \subset \mathrm{Isom}_g(\mathcal{M}) : G \text{ is compact, connected and } \mathrm{Fix}_G(\mathcal{M}) = \{\sigma_0\}\right\}. \tag{8}$$

Some noteworthy prototypes of manifolds $(\mathcal{M}, g)$ and related groups of isometries $G \in \mathcal{G}_{\sigma_0}$ are the following ones; cf. [16] and the references therein.

If $(\mathcal{M}, g) = (\mathbb{R}^d, \mathrm{g_{euc}})$ where $\mathrm{g_{euc}}$ is the canonical Euclidean metric, one can choose $\sigma_0 = 0$ and $G := SO(d_1) \times \ldots \times SO(d_k)$ for a splitting of $d = d_1 + \ldots + d_k$, with $d_j \geq 2$, $j = 1, \ldots, k$, where $SO(l)$ is the special orthogonal group in dimension l.

As already anticipated in the Introduction, the hyperbolic space $(\mathbb{H}^d, \mathrm{g_{hyp}})$, where $\mathrm{g_{hyp}}$ is defined by (3), turns out to be a homogeneous Hadamard manifold with $k_\pi \equiv -1$. Then, picking σ_0 and G like in the Euclidean setting above, one can prove that $G \in \mathcal{G}_{\sigma_0}$.

As a third example, we turn our attention to the symmetric positive-definite matrices. Denote by $\mathrm{Sym}(d, \mathbb{R})$ the set of all symmetric $d \times d$ real matrices and by $P(d, \mathbb{R}) \subset \mathrm{Sym}(d, \mathbb{R})$ the open cone of symmetric positive-definite matrices. As $P(d, \mathbb{R})$ is an open set of $\mathrm{Sym}(d, \mathbb{R})$, it turns out to be a differential manifold of dimension $d(d+1)/2$. The tangent space $T_X P(d, \mathbb{R})$ at $X \in P(d, \mathbb{R})$ is naturally isomorphic via translation to $\mathrm{Sym}(d, \mathbb{R})$ and it is possible to define a scalar product on it by putting

$$\langle A, B\rangle_X = \mathrm{Tr}(X^{-1} B X^{-1} A) \tag{9}$$

for all $X \in P(d, \mathbb{R})$, $A, B \in T_X(P(d, \mathbb{R})) \simeq \mathrm{Sym}(d, \mathbb{R})$, where $\mathrm{Tr}(A)$ denotes the trace of $A \in \mathrm{Sym}(d, \mathbb{R})$. The formula (9) defines a Riemannian metric on $P(d, \mathbb{R})$. If $\widetilde{P}(d, \mathbb{R}) \subset P(d, \mathbb{R})$ is the subspace of matrices with determinant 1, then $(\widetilde{P}(d, \mathbb{R}), \langle\cdot, \cdot\rangle)$ is a homogeneous Hadamard manifold with non-constant sectional curvature and the special linear group $SL(d, \mathbb{R})$ leaves $\widetilde{P}(d, \mathbb{R})$ invariant and acts transitively on it. Moreover, for every $\tau \in SL(d, \mathbb{R})$, the map $[\tau] : \widetilde{P}(d, \mathbb{R}) \to \widetilde{P}(d, \mathbb{R})$ defined by

$$[\tau](X) = \tau X \tau^t,$$

where τ^t is the transpose of τ, is an isometry. If $G = SO(d, \mathbb{R})$, one can show that $\mathrm{Fix}_G(\widetilde{P}(d, \mathbb{R})) = \{I_d\}$, with I_d identity matrix. So, on $(\widetilde{P}(d, \mathbb{R}), \langle\cdot, \cdot\rangle)$, the choices $\sigma_0 = I_d$ and $G = SO(d, \mathbb{R})$ fulfil the requirements.

We say that a function $w \in H^1_g(\mathcal{M})$ is a weak solution to (P_λ) if

$$\langle w, z\rangle = \int_{\mathcal{M}} |w|^{\frac{4}{d-2}} w z \, d\sigma_g + \lambda \int_{\mathcal{M}} \alpha(\sigma) f(w) z \, d\sigma_g, \quad \text{for all } z \in H^1_g(\mathcal{M}).$$

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It is easily seen that such a solution is a critical point of the energy naturally associated with (P_λ), i.e. the functional

$$\mathcal{E}_\lambda(w) := \frac{1}{2}\|w\|^2 - \frac{1}{2^*}\int_{\mathcal{M}} |w|^{2^*} d\sigma_g - \lambda J(w), \qquad \text{for all } w \in H^1_g(\mathcal{M}), \tag{10}$$

where

$$J(w) := \int_{\mathcal{M}} \alpha(\sigma)F(w)d\sigma_g,$$

and

$$F(t) := \int_0^t f(s)ds, \qquad \text{for all } t \in \mathbb{R}.$$

In what follows, we fix $\sigma_0 \in \mathcal{M}$, $G \in \mathcal{G}_{\sigma_0}$ and assume that the weight α satisfies

(α_1) $\alpha \in L^1(\mathcal{M}) \cap L^\infty(\mathcal{M}) \setminus \{0\}$, is non-negative and radially symmetric w.r.t. $\sigma_0 \in \mathcal{M}$, i.e., there exists $\psi : [0, +\infty) \to \mathbb{R}$ such that $\alpha(\sigma) = \psi(d_g(\sigma_0, \sigma))$ for all $\sigma \in \mathcal{M}$.

The action of G on $H^1_g(\mathcal{M})$ is defined as usual by

$$(\tau w)(\sigma) = w(\tau^{-1}(\sigma)), \qquad \text{for all } \tau \in G,\ w \in H^1_g(\mathcal{M}),\ \sigma \in \mathcal{M}. \tag{11}$$

The assumptions on α and the structure of G allow us to deduce the following result.

Proposition 2. *If α satisfy (α_1) and $G \in \mathcal{G}_{\sigma_0}$, then $\mathcal{E}_\lambda$ is G-invariant*

Proof. First, notice that the action (11) is isometric. Indeed, if $w \in H^1_g(\mathcal{M})$ and $\tau \in G$, one has

$$\begin{aligned}\|\tau w\|^2 &= \int_{\mathcal{M}} \left(|\nabla_g(\tau w)(\sigma)|^2_\sigma + |\tau w(\sigma)|^2\right) d\sigma_g \\ &= \int_{\mathcal{M}} \left(|\nabla_g w(\tau^{-1}(\sigma))|^2_{\tau^{-1}(\sigma)} + |w(\tau^{-1}(\sigma))|^2\right) d\sigma_g \\ &= \|w\|^2,\end{aligned}$$

where, among the other things, we have made use of the chain rule

$$\nabla_g(\tau w)(\sigma) = D\tau_{\tau^{-1}(\sigma)}\nabla_g w(\tau^{-1}(\sigma)), \qquad \text{for every } \sigma \in \mathcal{M},$$

$D\tau_{\tau^{-1}(\sigma)} : T_{\tau^{-1}(\sigma)}\mathcal{M} \to T_\sigma\mathcal{M}$ being the differential of τ at the point $\tau^{-1}(\sigma)$, and the fact that $D\tau_{\tau^{-1}(\sigma)}$ is inner product-preserving. Moreover, for every $\tau \in G$ and $\sigma \in \mathcal{M}$, one has

$$\alpha(\tau(\sigma)) = \psi(d_g(\sigma_0, \tau(\sigma))) = \psi(d_g(\tau(\sigma_0), \tau(\sigma))) = \psi(d_g(\sigma_0, \sigma)) = \alpha(\sigma)$$

and hence

$$J(\tau w) = \int_{\mathcal{M}} \alpha(\sigma) F((\tau w)(\sigma))\, d\sigma_g = \int_{\mathcal{M}} \alpha(\sigma) F\left(w(\tau^{-1}(\sigma))\right) d\sigma_g = J(w),$$

yielding the G-invariance of $\mathcal{E}_\lambda$. □

As a next step, denote by

$$H^1_{g,G}(\mathcal{M}) := \left\{ w \in H^1_g(\mathcal{M}) : \tau w = w \text{ for all } \tau \in G \right\}$$

the subspace of G-invariant functions of $H^1_g(\mathcal{M})$ and by $\mathcal{E}_{\lambda,G}$ the restriction of $\mathcal{E}_\lambda$ to $H^1_{g,G}(\mathcal{M})$. A recent embedding result (à la Lions) due to Skrzypczak and Tintarev, already mentioned in the Introduction, will be decisive in our next arguments. We state it below in a convenient form (see also [16]).

Theorem 3 *([15]). Let $(\mathcal{M}, g)$ be a d-dimensional, $d \geq 3$, homogeneous Hadamard manifold and G be a compact connected subgroup of $\mathrm{Isom}_g(\mathcal{M})$ such that $\mathrm{Fix}_G(\mathcal{M})$ is a singleton. Then $H^1_{g,G}(\mathcal{M}) \hookrightarrow L^\nu(\mathcal{M})$ compactly for any $\nu \in (2, 2^*)$.*

Noticing that any $G \in \mathcal{G}_{\sigma_0}$ acts continuously on $H^1_g(\mathcal{M})$ by (11), and that $\mathcal{E}_\lambda$ is G-invariant, on account of Palais' principle of symmetric criticality (recalled below for the sake of completeness) we are legitimized to look for critical points of the energy functional constrained on $H^1_{g,G}(\mathcal{M})$.

Theorem 4 *([27]). Let X be a real Banach space, G be a compact topological group acting continuously on X by a map $[\tau, u] \mapsto \tau u$ from $G \times X \to X$, and $\Phi : X \to \mathbb{R}$ be a G-invariant C^1-function. If u is a critical point of $\Phi_{|\mathrm{Fix}_G(X)}$, then u is also a critical point of Φ.*

3. Existence of isometry-invariant solutions

We start by proving that, when f is in the subcritical regime, the functional $\mathcal{E}_{\lambda,G}$ is locally sequentially weakly lower semicontinuous.

Proposition 5. *Assume (α_1) and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying*

(f_1) *there exist $\kappa > 0$, $q \in (1, 2)$ and $p \in [2, 2^*)$, such that*

$$|f(t)| \leq \kappa(|t|^{q-1} + |t|^{p-1}), \quad \textit{for all } t \in \mathbb{R}.$$

Then, there exists $\varrho_0 > 0$ such that $\mathcal{E}_{\lambda,G}$ is sequentially weakly lower semicontinuous in $\overline{B}(0, \varrho_0)$.

Proof. Let $\varrho > 0$, $\{w_j\} \subset \overline{B}(0, \varrho)$ be such that $w_j \rightharpoonup w_\infty \in \overline{B}(0, \varrho)$, and let $\widehat{\mathcal{E}}_G : H^1_{g,G}(\mathcal{M}) \to \mathbb{R}$ be the functional defined by

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$$\widehat{\mathcal{E}}_G(w) := \frac{1}{2}\|w\|^2 - \frac{1}{2^*}\int\limits_{\mathcal{M}} |w|^{2^*} d\sigma_g,$$

for every $w \in H^1_{g,G}(\mathcal{M})$. With the aid of the well-known geometric property

$$\|w_2\|^2 - \|w_1\|^2 - 2\langle w_1, w_2 - w_1\rangle = \|w_1 - w_2\|^2, \quad \text{for all } w_1, w_2 \in H^1_{g,G}(\mathcal{M}),$$

and of Brézis-Lieb's Lemma, we deduce that

$$\begin{aligned}\liminf_{j\to\infty}(\widehat{\mathcal{E}}_G(w_j) - \widehat{\mathcal{E}}_G(w_\infty)) &= \liminf_{j\to\infty}\left(\frac{1}{2}\left(\|w_j\|^2 - \|w_\infty\|^2\right)\right.\\ &\qquad \left. - \frac{1}{2^*}\int\limits_{\mathcal{M}}\left(|w_j|^{2^*} - |w_\infty|^{2^*}\right)d\sigma_g\right)\\ &\geq \liminf_{j\to\infty}\left(\frac{1}{2}\|w_j - w_\infty\|^2 - \frac{1}{2^*}\int\limits_{\mathcal{M}}|w_j - w_\infty|^{2^*}d\sigma_g\right) \qquad (12)\\ &\geq \liminf_{j\to\infty}\left(\frac{1}{2} - \frac{c_{2^*}^{2^*}}{2^*}\|w_j - w_\infty\|^{2^*-2}\right)\|w_j - w_\infty\|^2\\ &\geq \liminf_{j\to\infty}\left(\frac{1}{2} - \frac{c_{2^*}^{2^*}}{2^*}\varrho^{2^*-2}\right)\|w_j - w_\infty\|^2.\end{aligned}$$

So, for

$$0 < \varrho \leq \bar{\varrho} := \left(\frac{d}{(d-2)c_{2^*}^{2^*}}\right)^{\frac{d-2}{4}}, \tag{13}$$

we get

$$\liminf_{j\to\infty}\widehat{\mathcal{E}}_G(w_j) \geq \widehat{\mathcal{E}}_G(w_\infty). \tag{14}$$

Now, let us prove that the functional $J_G := J_{|H^1_{g,G}(\mathcal{M})}$ is sequentially weakly continuous on the whole $H^1_{g,G}(\mathcal{M})$. Arguing by contradiction, let $\{w_j\} \subset H^1_{g,G}(\mathcal{M})$ be such that $w_j \rightharpoonup \bar{w} \in H^1_{g,G}(\mathcal{M})$ and suppose that there exists $\varepsilon > 0$ such that $|J(w_j) - J(\bar{w})| \geq \varepsilon$ for all $j \in \mathbb{N}$. Let $r_1, r_2, r_3 \in \mathbb{R}$ satisfy

$$2 \leq r_1 \leq 2^*, \quad \frac{2^* r_1}{(2^*-1)r_1 - 2^*q + 2^*} < r_2 < \frac{2r_1}{r_1 - 2q + 2},$$
$$r_3 := \frac{r_1 r_2}{(r_1 - q + 1)r_2 - r_1}.$$

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Due to (f_1), for a suitable real sequence $\{\eta_j\} \subset (0, 1)$, one has

$$
\begin{aligned}
\varepsilon \le |J_G(w_j) - J_G(\bar{w})| &\le \int_{\mathcal{M}} \alpha(\sigma)|F(w_j) - F(\bar{w})| d\sigma_g \\
&\le \int_{\mathcal{M}} \alpha(\sigma)|(w_j - \bar{w}) f(\bar{w} + \eta_j(w_j - \bar{w}))| d\sigma_g \\
&\le \kappa \int_{\mathcal{M}} \alpha(\sigma)|w_j - \bar{w}| \left(|\bar{w} + \eta_j(w_j - \bar{w})|^{q-1} + |\bar{w} + \eta_j(w_j - \bar{w})|^{p-1}\right) d\sigma_g \\
&\le \kappa \int_{\mathcal{M}} \alpha(\sigma)|w_j - \bar{w}|(|w_j|^{q-1} + |w_j|^{p-1}) d\sigma_g \\
&\le \kappa \left(\|\alpha\|_{r_2} \|w_j - \bar{w}\|_{r_3} \|w_j\|_{r_1}^{q-1} + \|\alpha\|_\infty \|w_j - \bar{w}\|_p \|w_j\|_p^{p-1}\right)
\end{aligned}
\tag{15}
$$

and in the light of the compact embeddings $H^1_{g,G}(\mathcal{M}) \hookrightarrow L^{r_3}(\mathcal{M})$, $H^1_{g,G}(\mathcal{M}) \hookrightarrow L^p(\mathcal{M})$, the last expression tends to 0 as $j \to \infty$, a contradiction.

As a result, taking (13), (14) and (15) into account, $\mathcal{E}_{\lambda,G}$ turns out to be sequentially weakly lower semicontinuous in $\overline{B}(0, \varrho_0)$, provided that $\varrho_0 \in (0, \bar{\varrho})$. □

The next proposition provides some estimates for the functional consisting of the critical term and the potential J.

Proposition 6. *Let $\lambda > 0$, $f : \mathbb{R} \to \mathbb{R}$ satisfy (f_1), and let $\widetilde{\mathcal{E}}_{\lambda,G} : H^1_{g,G}(\mathcal{M}) \to \mathbb{R}$ be the functional defined by*

$$\widetilde{\mathcal{E}}_{\lambda,G}(w) := \frac{1}{2^*} \int_{\mathcal{M}} |w|^{2^*} d\sigma_g + \lambda J_G(w)$$

for any $w \in H^1_{g,G}(\mathcal{M})$. Then the following facts hold:

(i) if

$$\limsup_{\varepsilon \to 0^+} \frac{\sup_{w \in \overline{B}(0,\varrho_0)} \widetilde{\mathcal{E}}_{\lambda,G}(w) - \sup_{\overline{B}(0,\varrho_0 - \epsilon)} \widetilde{\mathcal{E}}_{\lambda,G}(w)}{\varepsilon} < \varrho_0 \tag{16}$$

for some $\varrho_0 > 0$, then

$$\inf_{\eta < \varrho_0} \frac{\sup_{w \in \overline{B}(0,\varrho_0)} \widetilde{\mathcal{E}}_{\lambda,G}(w) - \sup_{w \in \overline{B}(0,\eta)} \widetilde{\mathcal{E}}_{\lambda,G}(w)}{\varrho_0^2 - \eta^2} < \frac{1}{2}; \tag{17}$$

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(ii) if (17) *is satisfied for some* $\varrho_0 > 0$*, then*

$$\inf_{w\in B(0,\varrho_0)} \frac{\sup\limits_{z\in \overline{B}(0,\varrho_0)} \widetilde{\mathcal{E}}_{\lambda,G}(z) - \widetilde{\mathcal{E}}_{\lambda,G}(w)}{\varrho_0^2 - \|w\|^2} < \frac{1}{2}. \tag{18}$$

Proof. (i) From the identity

$$\begin{aligned} &\frac{\sup\limits_{w\in \overline{B}(0,\varrho_0)} \widetilde{\mathcal{E}}_{\lambda,G}(w) - \sup\limits_{w\in \overline{B}(0,\varrho_0-\epsilon)} \widetilde{\mathcal{E}}_{\lambda,G}(w)}{\varrho_0^2 - (\varrho_0-\varepsilon)^2} \\ &= \frac{\sup\limits_{w\in \overline{B}(0,\varrho_0)} \widetilde{\mathcal{E}}_{\lambda,G}(w) - \sup\limits_{w\in \overline{B}(0,\varrho_0-\epsilon)} \widetilde{\mathcal{E}}_{\lambda,G}(w)}{\varepsilon} \cdot \frac{1}{2\varrho_0 - \varepsilon}, \end{aligned}$$

it follows that

$$\limsup_{\varepsilon\to 0^+} \frac{\sup\limits_{w\in \overline{B}(0,\varrho_0)} \widetilde{\mathcal{E}}_{\lambda,G}(w) - \sup\limits_{w\in \overline{B}(0,\varrho_0-\epsilon)} \widetilde{\mathcal{E}}_{\lambda,G}(w)}{\varrho_0^2 - (\varrho_0-\varepsilon)^2} < \frac{1}{2}. \tag{19}$$

Now, by (19) there exists $\bar{\varepsilon}_0 > 0$ such that

$$\frac{\sup\limits_{w\in \overline{B}(0,\varrho_0)} \widetilde{\mathcal{E}}_{\lambda,G}(w) - \sup\limits_{w\in \overline{B}(0,\varrho_0-\varepsilon)} \widetilde{\mathcal{E}}_{\lambda,G}(w)}{\varrho_0^2 - (\varrho_0-\varepsilon)^2} < \frac{1}{2}$$

for every $\varepsilon \in (0, \bar{\varepsilon}_0)$; setting $\eta_0 := \varrho_0 - \varepsilon_0$, with $\varepsilon_0 \in (0, \bar{\varepsilon}_0)$, it follows that

$$\frac{\sup\limits_{w\in \overline{B}(0,\varrho_0)} \widetilde{\mathcal{E}}_{\lambda,G}(w) - \sup\limits_{w\in \overline{B}(0,\eta_0)} \widetilde{\mathcal{E}}_{\lambda,G}(w)}{\varrho_0^2 - \eta_0^2} < \frac{1}{2}$$

and thus the conclusion follows.

(ii) Thanks to inequality (17) one has

$$\sup_{w\in \overline{B}(0,\eta_0)} \widetilde{\mathcal{E}}_{\lambda,G}(w) > \sup_{w\in \overline{B}(0,\varrho_0)} \widetilde{\mathcal{E}}_{\lambda,G}(w) - \frac{1}{2}(\varrho_0^2 - \eta_0^2) \tag{20}$$

for some $0 < \eta_0 < \varrho_0$. Arguing as in Proposition 5, $\widetilde{\mathcal{E}}_{\lambda,G}$ is weakly lower semicontinuous in $\overline{B}(0, \eta_0)$ and therefore

$$\sup_{w\in \partial B(0,\eta_0)} \widetilde{\mathcal{E}}_{\lambda,G}(w) = \sup_{w\in \overline{\partial B(0,\eta_0)}^*} \widetilde{\mathcal{E}}_{\lambda,G}(w) = \sup_{w\in \overline{B}(0,\eta_0)} \widetilde{\mathcal{E}}_{\lambda,G}(w),$$

where $\overline{\partial B(0,\eta_0)}^*$ is the weak closure of $\partial B(0,\eta_0)$ in $H^1_{g,G}(\mathcal{M})$. By (20) there exists therefore $w_0 \in H^1_{g,G}(\mathcal{M})$ with $\|w_0\| = \eta_0$ such that

$$\widetilde{\mathcal{E}}_{\lambda,G}(w_0) > \sup_{w\in \overline{B}(0,\varrho_0)} \widetilde{\mathcal{E}}_{\lambda,G}(w) - \frac{1}{2}(\varrho_0^2 - \eta_0^2),$$

and the second claim is proved as well. □

We are now in a position to prove our existence result.

Theorem 7. *Let $\sigma_0 \in \mathcal{M}$, $G \in \mathcal{G}_{\sigma_0}$, α satisfy (α_1), f satisfy (f_1) in addition to*

(f_2) $\liminf_{t\to 0^+} \dfrac{F(t)}{t^2} = +\infty.$

Moreover, for any $r \in [2, 2^*)$, let $l : (0, +\infty) \to \mathbb{R}$ be the function defined by

$$l(t) := \frac{t - c_{2^*}^{2^*} t^{2^*-1}}{\kappa \left(c_r^q \|\alpha\|_{\frac{r}{r-q}} t^{q-1} + c_p^p \|\alpha\|_\infty t^{p-1}\right)} \quad \textit{for all } t > 0. \tag{21}$$

Then, there exists an open interval $\Lambda \subseteq \left(0, \max_{t\in[0,+\infty)} l(t)\right)$ such that, for every $\lambda \in \Lambda$, (P_λ) admits a non-zero G-invariant solution $w_{0,\lambda} \in H^1_g(\mathcal{M})$.

Remark 8. It is worth observing that, for any function $f : \mathbb{R} \to \mathbb{R}$ satisfying the pair of conditions $(f_1) - (f_2)$, the function

$$f^+(t) := \begin{cases} f(t) & \text{if } t > 0 \\ 0 & \text{if } t \le 0, \end{cases}$$

fulfils the same hypotheses. As a result, Theorem 7 provides the existence of a critical point of the functional

$$\mathcal{E}_\lambda^+(w) := \frac{1}{2}\|w\|^2 - \frac{1}{2^*}\int_{\mathcal{M}} (w^+)^{2^*} d\sigma_g - \lambda \int_{\mathcal{M}} \alpha(\sigma) F^+(w) d\sigma_g, \tag{22}$$

for all $w \in H^1_g(\mathcal{M})$, with

$$F^+(t) := \int_0^t f^+(s)ds, \quad \text{for all } t \in \mathbb{R}.$$

But any critical point $w_{0,\lambda}$ of $\mathcal{E}_\lambda^+$ has $w_{0,\lambda}^- := \max\{0, -w_{0,\lambda}\} = 0$. Indeed, taking also account of the relation

$$\left\langle w_{0,\lambda}, w_{0,\lambda}^{-}\right\rangle = \int_{\mathcal{M}} \left\langle \nabla_g w_{0,\lambda}, \nabla_g w_{0,\lambda}^{-}\right\rangle_g d\sigma_g + \int_{\mathcal{M}} w_{0,\lambda} w_{0,\lambda}^{-} d\sigma_g$$
$$= -\int_{\mathcal{M}} \left(|\nabla_g w_{0,\lambda}^{-}|^2 + (w_{0,\lambda}^{-})^2\right) d\sigma_g,$$

where $w_{0,\lambda}^{+} := \max\{w_{0,\lambda}, 0\}$, we get

$$\begin{aligned} 0 &= \mathcal{E}_\lambda^{+'}(w_{0,\lambda})(w_{0,\lambda}^{-}) \\ &= \left\langle w_{0,\lambda}, w_{0,\lambda}^{-}\right\rangle - \int_{\mathcal{M}} (w_{0,\lambda}^{+})^{2^*-1} w_{0,\lambda}^{-} d\sigma_g - \lambda \int_{\mathcal{M}} \alpha(\sigma) f^{+}(w_{0,\lambda}) w_{0,\lambda}^{-} d\sigma_g \\ &= -\left\| w_{0,\lambda}^{-}\right\|^2 \\ &\le 0. \end{aligned}$$

This means that, under the assumptions of Theorem 7, any solution to (P_λ) is non-negative in $\mathcal{M}$. Analogously, if $f : \mathbb{R} \to \mathbb{R}$ satisfies (f_1) and

(f_2') $\liminf_{t\to 0^-} \frac{F(t)}{t^2} = +\infty,$

one can prove the existence of a non-positive solution to (P_λ) by considering the functional

$$\mathcal{E}_\lambda^{-}(w) := \frac{1}{2}\|w\|^2 - \frac{1}{2^*}\int_{\mathcal{M}} (w^{-})^{2^*} d\sigma_g - \lambda \int_{\mathcal{M}} \alpha(\sigma) F^{-}(w) d\sigma_g, \tag{23}$$

where

$$F^{-}(t) := \int_0^t f^{-}(s)ds$$

and

$$f^{-}(t) := \begin{cases} f(t) & \text{if } t < 0 \\ 0 & \text{if } t \ge 0. \end{cases}$$

Proof of Theorem 7. Since $\lim_{t\to 0^+} l(t) = 0$, $\lim_{t\to+\infty} l(t) = -\infty$ and $l > 0$ in $(0, \delta)$, δ small enough, there exists $\varrho_{\max} > 0$ such that $l(\varrho_{\max}) = \max_{t\in[0,+\infty)} l(t)$. Set $\varrho_0 := \min\{\bar\varrho, \varrho_{\max}\}$, $\bar\varrho$ being defined by (13), and $\Lambda := (0, l(\varrho_0))$.

Taking $\lambda \in \Lambda$, there exists $\varrho_{0,\lambda} \in (0, \varrho_0)$ such that

$$0 < \lambda < \frac{\varrho_{0,\lambda} - c_{2^*}^{2^*}\varrho_{0,\lambda}^{2^*-1}}{\kappa\left(c_r^q \|\alpha\|_{\frac{r}{r-q}} \varrho_{0,\lambda}^{q-1} + c_p^p \|\alpha\|_\infty \varrho_{0,\lambda}^{p-1}\right)}. \tag{24}$$

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Since $\varrho_{0,\lambda} < \bar{\varrho}$, by Proposition 5, the functional $\mathcal{E}_{\lambda,G}$ is sequentially weakly lower semicontinuous in $\overline{B}(0, \varrho_{0,\lambda})$ and then there exists $w_{0,\lambda} \in \overline{B}(0, \varrho_{0,\lambda})$ such that

$$\mathcal{E}_{\lambda,G}(w_{0,\lambda}) = \min_{w\in\overline{B}(0,\varrho_{0,\lambda})} \mathcal{E}_{\lambda,G}(w).$$

Suppose by contradiction that $\|w_{0,\lambda}\| = \varrho_{0,\lambda}$. Fix $\varepsilon \in (0, \varrho_{0,\lambda})$ and define

$$\varphi(\varepsilon, \varrho_{0,\lambda}) := \varepsilon^{-1} \left(\sup_{w\in\overline{B}(0,\varrho_{0,\lambda})} \widetilde{\mathcal{E}}_{\lambda,G}(w) - \sup_{w\in\overline{B}(0,\varrho_{0,\lambda}-\varepsilon)} \widetilde{\mathcal{E}}_{\lambda,G}(w) \right).$$

By (f_1), Hölder and Sobolev inequalities, we get

$$\begin{aligned}
\varphi(\varepsilon, \varrho_{0,\lambda}) &\leq \varepsilon^{-1} \sup_{w\in\overline{B}(0,1)} \int_{\mathcal{M}} \left| \int_{(\varrho_{0,\lambda}-\varepsilon)w(\sigma)}^{\varrho_{0,\lambda}w(\sigma)} \left(|t|^{2^*-1} + \lambda\alpha(\sigma)|f(t)|\right) dt \right| d\sigma_g \\
&\leq \varepsilon^{-1} \sup_{w\in\overline{B}(0,1)} \int_{\mathcal{M}} \left| \int_{(\varrho_{0,\lambda}-\varepsilon)w(\sigma)}^{\varrho_{0,\lambda}w(\sigma)} \left(|t|^{2^*-1} + \lambda\kappa\alpha(\sigma)\left(|t|^{q-1} + |t|^{p-1}\right)\right) dt \right| d\sigma_g \\
&\leq \varepsilon^{-1} \sup_{w\in\overline{B}(0,1)} \Bigg(\frac{\varrho_{0,\lambda}^{2^*} - (\varrho_{0,\lambda}-\varepsilon)^{2^*}}{2^*} \|w\|_{2^*}^{2^*} + \lambda\kappa \frac{\varrho_{0,\lambda}^{q} - (\varrho_{0,\lambda}-\varepsilon)^{q}}{q} \\
&\qquad \cdot \int_{\mathcal{M}} \alpha(\sigma)|w|^q d\sigma_g + \lambda\kappa \frac{\varrho_{0,\lambda}^{p} - (\varrho_{0,\lambda}-\varepsilon)^{p}}{p} \int_{\mathcal{M}} \alpha(\sigma)|w|^p d\sigma_g \Bigg) \\
&\leq \varepsilon^{-1} \sup_{w\in\overline{B}(0,1)} \Bigg(\frac{\varrho_{0,\lambda}^{2^*} - (\varrho_{0,\lambda}-\varepsilon)^{2^*}}{2^*} \|w\|_{2^*}^{2^*} + \lambda\kappa \frac{\varrho_{0,\lambda}^{q} - (\varrho_{0,\lambda}-\varepsilon)^{q}}{q} \|\alpha\|_{\frac{r}{r-q}} \|w\|_r^q \\
&\qquad + \lambda\kappa \frac{\varrho_{0,\lambda}^{p} - (\varrho_{0,\lambda}-\varepsilon)^{p}}{p} \|\alpha\|_\infty \|w\|_p^p \Bigg) \\
&\leq \frac{c_{2^*}^{2^*}}{2^*} \left(\frac{\varrho_{0,\lambda}^{2^*} - (\varrho_{0,\lambda}-\varepsilon)^{2^*}}{\varepsilon} \right) + \frac{\lambda\kappa c_r^q \|\alpha\|_{\frac{r}{r-q}}}{q} \left(\frac{\varrho_{0,\lambda}^{q} - (\varrho_{0,\lambda}-\varepsilon)^{q}}{\varepsilon} \right) \\
&\qquad + \frac{\lambda\kappa c_p^p \|\alpha\|_\infty}{p} \left(\frac{\varrho_{0,\lambda}^{p} - (\varrho_{0,\lambda}-\varepsilon)^{p}}{\varepsilon} \right),
\end{aligned}$$

and taking the limsup for $\varepsilon \to 0^+$ we get

$$\limsup_{\varepsilon\to0^+} \varphi(\varepsilon, \varrho_{0,\lambda}) \leq c_{2^*}^{2^*} \varrho_{0,\lambda}^{2^*-1} + \lambda\kappa c_r^q \|\alpha\|_{\frac{r}{r-q}} \varrho_{0,\lambda}^{q-1} + \lambda\kappa c_p^p \|\alpha\|_\infty \varrho_{0,\lambda}^{p-1}, \tag{25}$$

which, due to (24), forces

$$\limsup_{\varepsilon\to 0^+}\varphi(\varepsilon,\varrho_{0,\lambda})<\varrho_{0,\lambda}.$$

Therefore, invoking Proposition 6, one has

$$\inf_{w\in B(0,\varrho_{0,\lambda})}\frac{\sup\limits_{z\in\overline{B}(0,\varrho_{0,\lambda})}\widetilde{\mathcal{E}}_{\lambda,G}(z)-\widetilde{\mathcal{E}}_{\lambda,G}(w)}{\varrho_{0,\lambda}^2-\|w\|_g^2}<\frac{1}{2}$$

and there exists $\bar{w}_\lambda\in B(0,\varrho_{0,\lambda})$ such that, for every $w\in\overline{B}(0,\varrho_{0,\lambda})$,

$$\widetilde{\mathcal{E}}_{\lambda,G}(w)\le\sup_{\overline{B}(0,\varrho_{0,\lambda})}\widetilde{\mathcal{E}}_{\lambda,G}(w)<\widetilde{\mathcal{E}}_{\lambda,G}(\bar{w}_\lambda)+\frac{1}{2}(\varrho_{0,\lambda}^2-\|\bar{w}_\lambda\|^2),$$

which we can rewrite as

$$\mathcal{E}_{\lambda,G}(\bar{w}_\lambda):=\frac{1}{2}\|\bar{w}_\lambda\|^2-\widetilde{\mathcal{E}}_{\lambda,G}(\bar{w}_\lambda)<\frac{\varrho_{0,\lambda}^2}{2}-\widetilde{\mathcal{E}}_{\lambda,G}(w).\tag{26}$$

Evaluating the previous inequality at $w=w_{0,\lambda}$ we deduce

$$\mathcal{E}_{\lambda,G}(\bar{w}_\lambda)<\frac{1}{2}\varrho_{0,\lambda}^2-\widetilde{\mathcal{E}}_{\lambda,G}(w_{0,\lambda})=\mathcal{E}_{\lambda,G}(w_{0,\lambda}),$$

against the minimality of $w_{0,\lambda}$. In conclusion $w_{0,\lambda}\in B(0,\varrho_{0,\lambda})$ and is therefore a local minimum for $\mathcal{E}_{\lambda,G}$ in the H_g^1-topology and a solution to (P_λ).

We now show that $w_{0,\lambda}$ is not identically zero on $\mathcal{M}$. Let $a<b$ be two positive constants, and define the following annular domain

$$A_a^b(\sigma_0):=\left\{\sigma\in\mathcal{M}:b-a<d_g(\sigma_0,\sigma)<a+b\right\}.$$

By (α_1) one can find two real numbers $\gamma>\delta>0$ and $\alpha_0>0$ such that

$$\operatorname*{essinf}_{\sigma\in A_\delta^\gamma(\sigma_0)}\alpha(\sigma)\ge\alpha_0>0.\tag{27}$$

Now, set

$$w_{\gamma,\delta}(\sigma):=\begin{cases}0 & \text{if } \sigma\in\mathcal{M}\setminus A_\delta^\gamma(\sigma_0)\\ 1 & \text{if } \sigma\in A_{\delta/2}^\gamma(\sigma_0)\\ \dfrac{2(\delta-|d_g(\sigma_0,\sigma)-\gamma|)}{\delta} & \text{if } \sigma\in A_\delta^\gamma(\sigma_0)\setminus A_{\delta/2}^\gamma(\sigma_0),\end{cases}\tag{28}$$

for every $\sigma\in\mathcal{M}$. It is clear that $\operatorname{supp} w_{\gamma,\delta}\subseteq A_\delta^\gamma(\sigma_0)$ and $\|w_{\gamma,\delta}\|_\infty=1$. By the definition of $w_{\gamma,\delta}$ and exploiting also (6), we have

$$\begin{aligned}
\|w_{\gamma,\delta}\|^2 &= \int\limits_{A_\delta^\gamma(\sigma_0)} \left(|\nabla_g w_{\gamma,\delta}|^2 + |w_{\gamma,\delta}|^2\right) d\sigma_g \\
&= \int\limits_{A_{\delta/2}^\gamma(\sigma_0)} |\nabla_g w_{\gamma,\delta}|^2 d\sigma_g + \int\limits_{A_{\delta/2}^\gamma(\sigma_0)} |w_{\gamma,\delta}|^2 d\sigma_g \\
&\quad + \int\limits_{A_\delta^\gamma(\sigma_0)\backslash A_{\delta/2}^\gamma(\sigma_0)} |\nabla_g w_{\gamma,\delta}|^2 d\sigma_g + \int\limits_{A_\delta^\gamma(\sigma_0)\backslash A_{\delta/2}^\gamma(\sigma_0)} |w_{\gamma,\delta}|^2 d\sigma_g \\
&\leq \mathrm{Vol}_g(A_\delta^\gamma(\sigma_0)) + \frac{4}{\delta^2} \int\limits_{A_\delta^\gamma(\sigma_0)\backslash A_{\delta/2}^\gamma(\sigma_0)} \left|\nabla_g(\delta - |d_g(\sigma_0,\sigma) - \gamma|)\right|^2 d\sigma_g \\
&= \mathrm{Vol}_g(A_\delta^\gamma(\sigma_0)) + \frac{4}{\delta^2} \int\limits_{A_\delta^\gamma(\sigma_0)\backslash A_{\delta/2}^\gamma(\sigma_0)} |\nabla_g|d_g(\sigma_0,\sigma) - \gamma||^2 d\sigma_g \\
&= \mathrm{Vol}_g(A_\delta^\gamma(\sigma_0)) + \frac{4}{\delta^2} \mathrm{Vol}_g\left(A_\delta^\gamma(\sigma_0) \setminus A_{\delta/2}^\gamma(\sigma_0)\right) \\
&\leq \left(1 + \frac{4}{\delta^2}\right) \mathrm{Vol}_g(A_\delta^\gamma(\sigma_0)).
\end{aligned}$$

In the same wake, we can deduce that

$$\int\limits_{\mathcal{M}} |w_{\gamma,\delta}|^{2^*} d\sigma_g \geq \int\limits_{A_{\delta/2}^\gamma(\sigma_0)} |w_{\gamma,\delta}|^{2^*} d\sigma_g = \mathrm{Vol}_g(A_{\delta/2}^\gamma(\sigma_0)).$$

Thanks to (f_2), there exists a sequence $\{t_j\} \subset \mathbb{R}^+$, with $t_j \to 0$ as $j \to +\infty$, such that

$$F(t_j) \geq c t_j^2,$$

for all $c > 0$ and for sufficiently large j. Defining now the sequence $w_j := t_j w_{\gamma,\delta}$, $j \in \mathbb{N}$, on the basis of the previous estimates we get, for j large enough,

$$\begin{aligned}
\mathcal{E}_{\lambda,G}(w_j) &\leq \left(\frac{1}{2} + \frac{2}{\delta^2}\right) \mathrm{Vol}_g(A_\delta^\gamma(\sigma_0)) t_j^2 - \frac{1}{2^*} \mathrm{Vol}_g(A_{\delta/2}^\gamma(\sigma_0)) t_j^{2^*} \\
&\quad - \lambda c \alpha_0 \mathrm{Vol}_g(A_{\delta/2}^\gamma(\sigma_0)) t_j^2 \\
&= \left(\frac{1}{2} \mathrm{Vol}_g(A_\delta^\gamma(\sigma_0)) + \frac{2}{\delta^2} \mathrm{Vol}_g(A_\delta^\gamma(\sigma_0)) - \lambda c \alpha_0 \mathrm{Vol}_g(A_{\delta/2}^\gamma(\sigma_0))\right) t_j^2 \\
&\quad - \frac{1}{2^*} \mathrm{Vol}_g(A_{\delta/2}^\gamma(\sigma_0)) t_j^{2^*}
\end{aligned}$$

By choosing $c > 0$ large enough, we can easily deduce that 0 is not a local minimizer of $\mathcal{E}_{\lambda,G}$ and hence cannot coincide with $w_{0,\lambda}$. This concludes the proof. □

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4. Nonlinearity with singular terms

The underlying idea of the proof of Theorem 7 remains valid when adding a term singular at zero, i.e. to treat the following singular variant of problem (P_λ):

$$\begin{cases} -\Delta_g w + w = w^{\frac{d+2}{d-2}} + \lambda\alpha(\sigma)\left(f(w) + w^{r-1}\right), & \sigma \in \mathcal{M} \\ w \in H^1_g(\mathcal{M}), \quad w > 0 \text{ in } \mathcal{M}, \end{cases} \tag{$P^\star_\lambda$}$$

where $r \in (0,1)$ and $f : [0,+\infty) \to [0,+\infty)$ is continuous and subcritical. In this context a weak solution to ($P^\star_\lambda$) is meant to be any $w \in H^1_g(\mathcal{M})$ such that $w > 0$ a.e. in $\mathcal{M}$ and

$$\langle w, z\rangle - \int_{\mathcal{M}} w^{\frac{d+2}{d-2}} z\, d\sigma_g - \lambda \int_{\mathcal{M}} \alpha(\sigma)\left(f(w) + w^{r-1}\right) z\, d\sigma_g = 0$$

for each $z \in H^1_g(\mathcal{M})$. The energy naturally associated with ($P^\star_\lambda$) is

$$\begin{aligned} \mathcal{E}_\lambda(w) := {} & \frac{1}{2}\|w\|^2 - \frac{1}{2^*}\int_{\mathcal{M}} (w^+)^{2^*} d\sigma_g \\ & - \lambda \int_{\mathcal{M}} \alpha(\sigma) F(w^+) d\sigma_g - \frac{\lambda}{r}\int_{\mathcal{M}} \alpha(\sigma)(w^+)^r d\sigma_g, \end{aligned} \tag{29}$$

for all $w \in H^1_g(\mathcal{M})$, and its restriction to $H^1_{g,G}(\mathcal{M})$ is denoted as before by $\mathcal{E}_{\lambda,G}$.

Our existence result for problem ($P^\star_\lambda$) reads as follows:

Theorem 9. *Let $\sigma_0 \in \mathcal{M}$, $G \in \mathcal{G}_{\sigma_0}$, α satisfy (α_1) and let $f : [0,+\infty) \to [0,+\infty)$ be a continuous function for which*

(f_1') *there exist $\kappa > 0$, $q \in (1,2)$ and $p \in [2, 2^*)$, such that*

$$f(t) \le \kappa(|t|^{q-1} + |t|^{p-1}), \quad \text{for all } t \ge 0.$$

Let $r_1 \in [2, 2^*)$ and $m : [0,+\infty) \to \mathbb{R}$ be the function defined by

$$m(t) := \frac{t^{2-r} - c_{2^*}^{2^*} t^{2^*-r}}{c_{r_1}^r \|\alpha\|_{\frac{r_1}{r_1-r}} + \kappa c_{r_1}^q \|\alpha\|_{\frac{r_1}{r_1-q}} t^{q-r} + \kappa c_p^p \|\alpha\|_\infty t^{p-r}}.$$

Then, there exists an open interval $\Lambda \subseteq \left(0, \max_{t\in[0,+\infty)} m(t)\right)$ such that, for every $\lambda \in \Lambda$, ($P^\star_\lambda$) admits a non-trivial G-invariant weak solution $w_{1,\lambda} \in H^1_g(\mathcal{M})$.

Proof. In order to prove that $\mathcal{E}_{\lambda,G}$ satisfies Proposition 5, it suffices to show that the functional $J_r : H^1_{g,G}(\mathcal{M}) \to \mathbb{R}$ defined by

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$$J_r(w) := \frac{1}{r}\int_{\mathcal{M}} \alpha(\sigma)(w^+)^r d\sigma_g, \qquad \text{for all } w \in H^1_{g,G}(\mathcal{M}),$$

is sequentially weakly lower semicontinuous in $H^1_{g,G}(\mathcal{M})$. To this end, let $\{w_j\} \subset H^1_{g,G}(\mathcal{M})$ such that $w_j \rightharpoonup w_\infty \in H^1_{g,G}(\mathcal{M})$. The sequence $\{w_j\}$ is bounded in $H^1_{g,G}(\mathcal{M})$ and, by the compactness of the embedding $H^1_{g,G}(\mathcal{M}) \hookrightarrow L^{r_1}(\mathcal{M})$, one has $w_j \to w_\infty$ in $L^{r_1}(\mathcal{M})$. Moreover, up to a subsequence,

$$|w_j(\sigma)| \le W(\sigma) \quad \text{and} \quad w_j \to w_\infty \text{ a.e. in } \mathcal{M},$$

for some $W \in L^{r_1}(\mathcal{M})$, and therefore

$$\int_{\mathcal{M}} \alpha(\sigma)|w|^r d\sigma_g \le \int_{\mathcal{M}} \alpha(\sigma) W^r d\sigma_g \le \|\alpha\|_{\frac{r_1}{r_1-r}} \|W\|^r_{r_1}.$$

The dominated convergence theorem then yields $J_r(w_j) \to J_r(w_\infty)$, as desired.

It is clear that the functional

$$H^1_{g,G}(\mathcal{M}) \ni w \mapsto \frac{1}{2^*}\int_{\mathcal{M}} (w^+)^{2^*} d\sigma_g + \lambda \int_{\mathcal{M}} \alpha(\sigma)\left(F(w^+) + \frac{(w^+)^r}{r}\right) d\sigma_g$$

fulfils Proposition 6. So, arguing exactly as in Theorem 7 we deduce that the functional $\mathcal{E}_{\lambda,G}$ admits a minimizer $w_{1,\lambda}$ on a sufficiently small ball $\overline{B}(0, \varrho_{0,\lambda}) \subset H^1_{g,G}(\mathcal{M})$ and such a minimizer is not identically zero. Indeed, fixing $w \in H^1_{g,G}(\mathcal{M})$, $w > 0$ on $\mathcal{M}$, if $t > 0$ one has

$$\begin{aligned}\mathcal{E}_{\lambda,G}(tw) \le{}& \frac{1}{2}\|w\|^2 t^2 - \frac{1}{2^*}\|w\|^{2^*}_{2^*} t^{2^*} + \frac{\lambda\kappa}{q}\int_{\mathcal{M}} \alpha(\sigma) w^q d\sigma_g\, t^q \\ &+ \frac{\lambda\kappa}{p}\int_{\mathcal{M}} \alpha(\sigma) w^p d\sigma_g\, t^p - \frac{\lambda}{r}\int_{\mathcal{M}} \alpha(\sigma) w^r d\sigma_g\, t^r\end{aligned}$$

and hence $\mathcal{E}_{\lambda,G}(tw)$ is negative for t small enough. Finally, we show that $w_{1,\lambda}$ weakly solves ($P^\star_\lambda$). The proof develops along the same line as [17,28]; we illustrate it below for the sake of completeness. Let us start by proving that $w_{1,\lambda} > 0$ a.e. in $\mathcal{M}$. One has

$$\begin{aligned}0 &\le \mathcal{E}_{\lambda,G}(w_{1,\lambda} + tw^-_{1,\lambda}) - \mathcal{E}_{\lambda,G}(w_{1,\lambda}) \\ &= \frac{1}{2}\left(\left\|w_{1,\lambda} + tw^-_{1,\lambda}\right\|^2 - \|w_{1,\lambda}\|^2\right) - \frac{1}{2^*}\int_{\mathcal{M}}\left(\left(\left(w_{1,\lambda} + tw^-_{1,\lambda}\right)^+\right)^{2^*} - \left(w^+_{1,\lambda}\right)^{2^*}\right) d\sigma_g \\ &\quad - \frac{\lambda}{r}\int_{\mathcal{M}} \alpha(\sigma)\left(\left(\left(w_{1,\lambda} + tw^-_{1,\lambda}\right)^+\right)^r - \left(w^+_{1,\lambda}\right)^r\right) d\sigma_g\end{aligned}$$

$$-\lambda \int_{\mathcal{M}} \alpha(\sigma)\left(F\left(\left(w_{1,\lambda}+tw_{1,\lambda}^-\right)^+\right)-F\left(w_{1,\lambda}^+\right)\right)d\sigma_g,$$

where we exploited that, for t small enough, $w_{1,\lambda}+tw_{1,\lambda}^- \in \overline{B}(0,\varrho_{0,\lambda})$ and $w_{1,\lambda}+tw_{1,\lambda}^- = w_{1,\lambda}^+$. As a result, we get

$$\begin{aligned}\lim_{t\to 0}\frac{\mathcal{E}_{\lambda,G}(w_{1,\lambda}+tw_{1,\lambda}^-)-\mathcal{E}_{\lambda,G}(w_{1,\lambda})}{t} &= \int_{\mathcal{M}}\left(\left\langle \nabla_g w_{1,\lambda}, \nabla_g w_{1,\lambda}^-\right\rangle_g + w_{1,\lambda}w_{1,\lambda}^-\right)d\sigma_g \\ &= -\left\|w_{1,\lambda}\right\|^2\end{aligned}$$

and thus $w_{1,\lambda}\geq 0$ a.e. in $\mathcal{M}$. If $\mathcal{M}_1\subset\mathcal{M}$ has positive Riemannian measure and $w_{1,\lambda}=0$ in $\mathcal{M}_1$, taking $z\in H^1_{g,G}(\mathcal{M})$, $z>0$, and $t>0$ small enough, we get

$$\begin{aligned} 0 &\leq \mathcal{E}_{\lambda,G}(w_{1,\lambda}+tz)-\mathcal{E}_{\lambda,G}(w_{1,\lambda}) \\ &\leq \frac{1}{2}\left(\left\|w_{1,\lambda}+tz\right\|^2-\left\|w_{1,\lambda}\right\|^2\right)-\frac{1}{2^*}\int_{\mathcal{M}}\left(\left(w_{1,\lambda}+tz\right)^{2^*}-\left(w_{1,\lambda}\right)^{2^*}\right)d\sigma_g \\ &\quad -\frac{\lambda t^r}{r}\int_{\mathcal{M}_1}\alpha(\sigma)z^r d\sigma_g-\lambda\int_{\mathcal{M}}\alpha(\sigma)\left(F\left(w_{1,\lambda}+tz\right)-F\left(w_{1,\lambda}\right)\right)d\sigma_g. \end{aligned}$$

Dividing by t and passing to the limit we obtain

$$0\leq \lim_{t\to 0}\frac{\mathcal{E}_{\lambda,G}(w_{1,\lambda}+tz)-\mathcal{E}_{\lambda,G}(w_{1,\lambda})}{t}\leq -\infty,$$

and hence such a set $\mathcal{M}_1$ cannot exist and $w_{1,\lambda}>0$ all over $\mathcal{M}$.

Now let us prove that

$$\left\langle w_{1,\lambda}, z\right\rangle-\int_{\mathcal{M}} w_{1,\lambda}^{\frac{d+2}{d-2}} z d\sigma_g-\lambda\int_{\mathcal{M}}\alpha(\sigma)\left(f(w_{1,\lambda})+w_{1,\lambda}^{r-1}\right)zd\sigma_g\geq 0 \tag{30}$$

for every non-negative test function $z\in H^1_{g,G}(\mathcal{M})$. Fix such a function z, take a decreasing sequence $\{t_j\}\subset(0,1)$ such that $t_j\to 0$ as $j\to+\infty$ and set

$$z_j := \frac{(w_{1,\lambda}+t_j z)^r-w_{1,\lambda}^r}{t_j}, \quad \text{for all } j\in\mathbb{N}.$$

Arguing similarly to before and considering that, by Fatou's lemma,

$$\int_{\mathcal{M}}\alpha(\sigma)w_{1,\lambda}^{r-1}zd\sigma_g\leq\frac{1}{r}\liminf_{j\to+\infty}\int_{\mathcal{M}}\alpha(\sigma)z_j d\sigma_g,$$

we deduce

$$0 \le \liminf_{j\to\infty} \frac{\mathcal{E}_{\lambda,G}(w_{1,\lambda}+t_j z)-\mathcal{E}_{\lambda,G}(w_{1,\lambda})}{t_j}$$
$$\le \langle w_{1,\lambda}, z\rangle - \int_{\mathcal{M}} w_{1,\lambda}^{\frac{d+2}{d-2}} z d\sigma_g - \lambda \int_{\mathcal{M}} \alpha(\sigma) f(w_{1,\lambda}) z d\sigma_g - \lambda \int_{\mathcal{M}} \alpha(\sigma) w_{1,\lambda}^{r-1} z d\sigma_g,$$

as claimed. Now, choosing $\varepsilon \in (0,1)$ so that $w_{1,\lambda} + t w_{1,\lambda} \in \overline{B}(0, \varrho_{0,\lambda})$ for $|t| \le \varepsilon$, define $\psi : [\varepsilon, \varepsilon] \to \mathbb{R}$ by $\psi(t) = \mathcal{E}_{\lambda,G}((1+t)w_{1,\lambda})$. Since ψ has a minimum at $t=0$ we easily deduce that

$$\|w_{1,\lambda}\|^2 - \int_{\mathcal{M}} w_{1,\lambda}^{2^*} d\sigma_g - \lambda \int_{\mathcal{M}} \alpha(\sigma)\left(f(w_{1,\lambda})w_{1,\lambda} + w_{1,\lambda}^{r}\right) d\sigma_g = 0. \tag{31}$$

Now suppose $h \in H^1_{g,G}(\mathcal{M})$ and define

$$z := (w_{1,\lambda}+\varepsilon h)^+, \quad \mathcal{M}_0^+ := \{\sigma \in \mathcal{M} : w_{1,\lambda}(\sigma) + \varepsilon h(\sigma) \ge 0\},$$
$$\mathcal{M}^- := \{\sigma \in \mathcal{M} : w_{1,\lambda}(\sigma) + \varepsilon h(\sigma) < 0\}.$$

Plugging z into (30) and taking (31) into account, we get

$$\begin{aligned}
0 \le& \int_{\mathcal{M}_0^+} \left\langle \nabla_g w_{1,\lambda} \nabla_g (w_{1,\lambda}+\varepsilon h)\right\rangle_g d\sigma_g + \int_{\mathcal{M}_0^+} w_{1,\lambda}(w_{1,\lambda}+\varepsilon h) d\sigma_g \\
&- \int_{\mathcal{M}_0^+} w_{1,\lambda}^{2^*-1}(w_{1,\lambda}+\varepsilon h) d\sigma_g - \lambda \int_{\mathcal{M}_0^+} \alpha(\sigma) f(w_{1,\lambda})(w_{1,\lambda}+\varepsilon h) d\sigma_g \\
&- \lambda \int_{\mathcal{M}_0^+} \alpha(\sigma) w_{1,\lambda}^{r-1}(w_{1,\lambda}+\varepsilon h) d\sigma_g \\
=& \left(\int_{\mathcal{M}} - \int_{\mathcal{M}^-}\right)\Big(\left\langle \nabla_g w_{1,\lambda} \nabla_g (w_{1,\lambda}+\varepsilon h)\right\rangle_g + w_{1,\lambda}(w_{1,\lambda}+\varepsilon h) \\
&- w_{1,\lambda}^{2^*-1}(w_{1,\lambda}+\varepsilon h) - \lambda\alpha(\sigma) f(w_{1,\lambda})(w_{1,\lambda}+\varepsilon h) \\
&- \lambda\alpha(\sigma) w_{1,\lambda}^{r-1}(w_{1,\lambda}+\varepsilon h)\Big) d\sigma_g \\
=& \varepsilon \int_{\mathcal{M}} \Big(\left\langle \nabla_g w_{1,\lambda} \nabla_g h\right\rangle_g + w_{1,\lambda} h - w_{1,\lambda}^{2^*-1} h - \lambda\alpha(\sigma) f(w_{1,\lambda}) h - \lambda\alpha(\sigma) w_{1,\lambda}^{r-1} h\Big) d\sigma_g \\
&- \int_{\mathcal{M}^-} \Big(\left\langle \nabla_g w_{1,\lambda} \nabla_g (w_{1,\lambda}+\varepsilon h)\right\rangle_g + w_{1,\lambda}(w_{1,\lambda}+\varepsilon h) - w_{1,\lambda}^{2^*-1}(w_{1,\lambda}+\varepsilon h) \\
&- \lambda\alpha(\sigma) f(w_{1,\lambda})(w_{1,\lambda}+\varepsilon h) - \lambda\alpha(\sigma) w_{1,\lambda}^{r-1}(w_{1,\lambda}+\varepsilon h)\Big) d\sigma_g
\end{aligned}$$

$$\le \varepsilon \int\limits_{\mathcal{M}} \left(\langle \nabla_g w_{1,\lambda} \nabla_g h \rangle_g + w_{1,\lambda} h - w_{1,\lambda}^{2^*-1} h - \lambda \alpha(\sigma) f(w_{1,\lambda}) h - \lambda \alpha(\sigma) w_{1,\lambda}^{r-1} h \right) d\sigma_g$$

$$- \varepsilon \int\limits_{\mathcal{M}^-} \left(\langle \nabla_g w_{1,\lambda} \nabla_g h \rangle_g + w_{1,\lambda} h \right) d\sigma_g.$$

Considering that the Riemannian measure of $\mathcal{M}^-$ goes to 0 as $\varepsilon \to 0$, dividing by ε and taking the limit as $\varepsilon \to 0$ we get

$$\langle w_{1,\lambda}, h \rangle - \int\limits_{\mathcal{M}} w_{1,\lambda}^{\frac{d+2}{d-2}} h d\sigma_g - \lambda \int\limits_{\mathcal{M}} \alpha(\sigma) \left(f(w_{1,\lambda}) + w_{1,\lambda}^{r-1} \right) h d\sigma_g \ge 0.$$

The arbitrariness of h implies that the above inequality holds for $-h$ as well and so $w_{1,\lambda}$ is a weak solution to ($P_\lambda^\star$). This concludes the proof. □

5. The case of $\mathbb{R}^d$: multiple solutions in the presence of symmetries

In this final section we consider the case when $(\mathcal{M}, g) = (\mathbb{R}^d, g_{\text{euc}})$ is the canonical Euclidean space and study the existence of multiple solutions (radial and not) for (P_λ) in the presence of a symmetric nonlinear term f.

We will denote a point in $\mathbb{R}^d$ by x (instead of σ) and $\|\cdot\|$, $\|\cdot\|_\nu$ will represent the usual norms on $H^1(\mathbb{R}^d)$ and $L^\nu(\mathbb{R}^d)$, respectively.

Let either $d = 4$ or $d \ge 6$ and consider the subgroup $H_{d,i} \subset O(d)$ given by

$$H_{d,i} := \begin{cases} O(d/2) \times O(d/2) & \text{if } i = \dfrac{d-2}{2} \\ O(i+1) \times O(d-2i-2) \times O(i+1) & \text{if } i \neq \dfrac{d-2}{2}, \end{cases}$$

for every $i \in I := \{1, ..., \tau_d\}$, where

$$\tau_d := (-1)^d + \left[\frac{d-3}{2} \right].$$

For every $i \in I$, define the map $\eta_{H_{d,i}} : \mathbb{R}^d \to \mathbb{R}^d$ as follows

$$\eta_{H_{d,i}}(x) := \begin{cases} (x_3, x_1) & \text{if } i = \dfrac{d-2}{2} \text{ and } x := (x_1, x_3) \in \mathbb{R}^{d/2} \times \mathbb{R}^{d/2} \\ (x_3, x_2, x_1) & \text{if } i \neq \dfrac{d-2}{2} \text{ and } x := (x_1, x_2, x_3) \in \mathbb{R}^{i+1} \times \mathbb{R}^{d-2i-2} \times \mathbb{R}^{i+1}. \end{cases}$$

By definition, $\eta_{H_{d,i}} \notin H_{d,i}$ and one has, in addition, $\eta_{H_{d,i}} H_{d,i} \eta_{H_{d,i}}^{-1} = H_{d,i}$ (i.e. $\eta_{H_{d,i}}$ belongs to the normalizer of $H_{d,i}$ in $O(d)$) and $\eta_{H_{d,i}}^2 = \text{id}_{\mathbb{R}^d}$ for every $i \in I$. Now, let us consider the compact group

$$H_{d,\eta_i} := \langle H_{d,i}, \eta_{H_{d,i}} \rangle = H_{d,i} \cup \eta_{H_{d,i}} H_{d,i}$$

and the action $\circledast_i : H_{d,\eta_i} \times H^1(\mathbb{R}^d) \to H^1(\mathbb{R}^d)$ of H_{d,η_i} on $H^1(\mathbb{R}^d)$ given by

$$(h \circledast_i w)(x) := \begin{cases} w(h^{-1}x) & \text{if } h \in H_{d,i} \\ -w\left(\tau^{-1}\eta_{H_{d,i}}^{-1} x\right) & \text{if } h = \eta_{H_{d,i}}\tau \in H_{d,\eta_i} \setminus H_{d,i},\ \tau \in H_{d,i} \end{cases} \tag{32}$$

for every $w \in H^1(\mathbb{R}^d)$, $x \in \mathbb{R}^d$. We notice that $\circledast_i$ is defined for every element of H_{d,η_i}. Indeed, if $h \in H_{d,\eta_i}$, then either $h \in H_{d,i}$ or $h = \eta_{H_{d,i}}\tau \in H_{d,\eta_i} \setminus H_{d,i}$, with $\tau \in H_{d,i}$.

Setting

$$\text{Fix}_{H_{d,\eta_i}}(H^1(\mathbb{R}^d)) := \{w \in H^1(\mathbb{R}^d) : h \circledast_i w = w \ \text{ for all } h \in H_{d,\eta_i}\},$$

for every $i \in I$, following [21], the embedding

$$\text{Fix}_{H_{d,\eta_i}}(H^1(\mathbb{R}^d)) \hookrightarrow L^\nu(\mathbb{R}^d) \tag{33}$$

is compact for every $\nu \in (2, 2^*)$. The next important result illustrates the relation, in terms of inclusion, between H_{d,η_i}-invariant and radial symmetric functions, and between H_{d,η_i}-invariant functions as i ranges in I (cf. [22, Theorem 2.2]).

Proposition 10. *The following facts hold:*

(i) *if* $d = 4$ *or* $d \geq 6$, *then* $\text{Fix}_{H_{d,\eta_i}}(H^1(\mathbb{R}^d)) \cap \text{Fix}_{O(d)}(H^1(\mathbb{R}^d)) = \{0\}$, *for every* $i \in I$;

(ii) *if* $d = 6$ *or* $d \geq 8$, *then* $\text{Fix}_{H_{d,\eta_i}}(H^1(\mathbb{R}^d)) \cap \text{Fix}_{H_{d,\eta_j}}(H^1(\mathbb{R}^d)) = \{0\}$, *for every* $i, j \in I$ *and* $i \neq j$.

By adopting the same approach as the previous section we can prove the following result.

Theorem 11. *Let* $d \geq 3$, α *satisfy* (α_1) *with* $\sigma_0 = 0$, $f : \mathbb{R} \to \mathbb{R}$ *satisfy* $(f_1) - (f_2)$, $r \in [2, 2^*)$ *and* $l : (0, +\infty) \to \mathbb{R}$ *be defined by* (21).

Then, setting $\lambda^* := \max_{t \in [0,+\infty)} l(t)$, *there exists an open interval* $\Lambda \subseteq (0, \lambda^*)$ *such that, for every* $\lambda \in \Lambda$, *the problem*

$$-\Delta w + w = |w|^{\frac{4}{d-2}} w + \lambda\alpha(x) f(w), \quad x \in \mathbb{R}^d,\ w \in H^1(\mathbb{R}^d), \tag{$\widetilde{P}_\lambda$}$$

admits a non-zero non-negative radial solution $w_{0,\lambda} \in H^1(\mathbb{R}^d)$.

Proof. We consider the group $G = O(d)$ acting on $H^1(\mathbb{R}^d)$ by the standard linear and isometric action (11). By (α_1), $\mathcal{E}_\lambda$ is $O(d)$-invariant and $\mathcal{E}_{\lambda,O(d)}$, $\widetilde{\mathcal{E}}_{\lambda,O(d)}$ are easily seen to satisfy Propositions 5 and 6, respectively. So, arguing as in Theorem 7, we deduce the existence of an open interval Λ such that, for $\lambda \in \Lambda \subseteq (0, \lambda^\star)$, $\mathcal{E}_{\lambda,O(d)}$ possesses a local minimizer $w_{0,\lambda} \in H^1_{O(d)}(\mathbb{R}^d)$. The conclusion then is a consequence of Theorem 4. □

Remark 12. We notice that, if in Theorem 11 we keep all the assumptions and only replace (α_1) by

(α_1') $\alpha \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \setminus \{0\}$, is non-negative and H_{d,η_i}-invariant for some $i \in I$, then, we get the existence of at least one non-trivial solution, invariant under the action of H_{d,η_i} on $\mathbb{R}^d$.

Set

$$c_{i,\nu} := \sup\left\{\frac{\|u\|_\nu}{\|u\|} : u \in \mathrm{Fix}_{H_{d,\eta_i}}(H^1(\mathbb{R}^d)) \setminus \{0\}\right\},$$

for every $\nu \in (2, 2^*)$ and $i \in I$, and

$$\lambda_i^\star := \max_{t\geq 0} \frac{t - c_{i,2^*}^{2^*} t^{2^*-1}}{\kappa\left(c_{i,r}^q \|\alpha\|_{\frac{r}{r-q}} t^{q-1} + c_{i,p}^p \|\alpha\|_\infty t^{p-1}\right)}. \tag{34}$$

The main result of this section reads as follows.

Theorem 13. *Assume $d > 3$ and let f and α be as in Theorem 7. In addition, suppose that the nonlinearity f is odd.*

Then, there exists an open interval $\Lambda \subseteq (0, \lambda_\star)$, where

$$\lambda_\star := \begin{cases} \lambda^\star & \text{if } d = 5 \\ \min\{\lambda^\star, \lambda_i^\star : i \in I\} & \text{if } d \neq 5, \end{cases}$$

such that, for every $\lambda \in \Lambda$, problem $(\widetilde{P}_\lambda)$ admits at least

$$t_d := 1 + (-1)^d + \left[\frac{d-3}{2}\right]$$

pairs of non-trivial weak solutions $\{\pm w_{\lambda,i}\} \subset H^1(\mathbb{R}^d)$.

Moreover, if $d \neq 5$ problem $(\widetilde{P}_\lambda)$ admits at least

$$\tau_d := (-1)^d + \left[\frac{d-3}{2}\right]$$

pairs of sign-changing weak solutions $\{\pm z_{\lambda,i}\}_{i\in I} \subset H^1(\mathbb{R}^d)$.

Proof. We start by considering the case $d = 5$. The oddity of f implies the evenness of $\mathcal{E}_{\lambda,O(d)}$. So Theorem 11 guarantees that, for every $\lambda \in \Lambda \subseteq (0, \lambda^\star)$, problem $(\widetilde{P}_\lambda)$ admits at least $t_5 = 1$ non-trivial pair of radial weak solutions $\{\pm w_{0,\lambda}\} \subset H^1(\mathbb{R}^d)$.

Assume now that $d > 3$ and $d \neq 5$. For every $\lambda > 0$ and $i = 1, 2, ..., \tau_d$, consider the restrictions

$$\mathcal{E}_{\lambda,i} := \mathcal{E}_{\lambda|\mathrm{Fix}_{H_{d,\eta_i}}(H^1(\mathbb{R}^d))}.$$

We notice that the range of d forces $\tau_d \geq 1$ and thus $|I| \geq 1$.

First, let us show that $\mathcal{E}_{\lambda,i}$ satisfies the properties required by our approach. The functional $\mathcal{E}_\lambda$ is H_{d,η_i}-invariant, i.e. $\mathcal{E}_\lambda(h \circledast_i w) = \mathcal{E}_\lambda(w)$ for all $h \in H_{d,\eta_i}$ and $w \in H^1(\mathbb{R}^d)$. Indeed, H_{d,η_i} acts isometrically on $H^1(\mathbb{R}^d)$ and, thanks to (α_1), one has

$$\int_{\mathbb{R}^d} \alpha(x)F((h \circledast_i w)(x))dx = \int_{\mathbb{R}^d} \alpha(x)F(w(h^{-1}x))dx = \int_{\mathbb{R}^d} \alpha(y)F(w(y))dy$$

if $h \in H_{d,i}$, and

$$\int_{\mathbb{R}^d} \alpha(x)F((h \circledast_i w)(x))dx = \int_{\mathbb{R}^d} \alpha(x)F(w(\tau^{-1}\eta_{H_{d,i}}^{-1}x))dx = \int_{\mathbb{R}^d} \alpha(y)F(w(y))dy$$

if $h = \eta_{H_{d,i}}\tau \in H_{d,\eta_i} \setminus H_{d,i}$. It is also clear that $\mathcal{E}_{\lambda,i}$ is sequentially weakly lower semicontinuous on a suitable ball $\overline{B}(0, \varrho_{0,i}) \subset H^1(\mathbb{R}^d)$ and that $\widetilde{\mathcal{E}}_{\lambda,i}$ fulfils Proposition 6. Then, retracing the same steps as Theorem 7, for $\lambda \in (0, \lambda_i^*)$ there exists $z_{\lambda,i}$ local minimizer of $\mathcal{E}_{\lambda,i}$ and due to Palais' principle of symmetric criticality, the pairs of critical points $\{\pm z_{\lambda,i}\}$ of $\mathcal{E}_{\lambda,i}$ are also critical points of $\mathcal{E}_\lambda$.

We have to show now that $z_{\lambda,i} \not\equiv 0$ in $\mathrm{Fix}_{H_{d,\eta_i}}(H^1(\mathbb{R}^d))$. We use in a suitable way certain test functions introduced in [22]. Let $0 < \delta < \gamma$ such that condition (27) holds and $\delta \geq \dfrac{\gamma}{5 + 4\sqrt{2}}$. Set $\vartheta \in (0, 1)$ and define $v_{\vartheta,i} \in H^1(\mathbb{R}^d)$ as follows: if $i \neq (d-2)/2$ and $x = (x_1, x_2, x_3) \in \mathbb{R}^{d/2} \times \mathbb{R}^{d-2i-2} \times \mathbb{R}^{d/2}$,

$$\begin{aligned} v_{\vartheta,i}(x) := \Bigg[& \left(\frac{\gamma - \delta}{4} - \max\left\{ \sqrt{\left(|x_1|^2 - \frac{\gamma + 3\delta}{4}\right)^2 + |x_3|^2}, \vartheta\frac{\gamma - \delta}{4} \right\} \right)^+ \\ & - \left(\frac{\gamma - \delta}{4} - \max\left\{ \sqrt{\left(|x_3|^2 - \frac{\gamma + 3\delta}{4}\right)^2 + |x_1|^2}, \vartheta\frac{\gamma - \delta}{4} \right\} \right)^+ \Bigg] \\ & \cdot \left(\frac{\gamma - \delta}{4} - \max\left\{ |x_2|, \vartheta\frac{\gamma - \delta}{4} \right\} \right)^+ \frac{16}{(\gamma - \delta)^2(1 - \vartheta)^2}, \end{aligned}$$

while, for every $x = (x_1, x_3) \in \mathbb{R}^{d/2} \times \mathbb{R}^{d/2}$,

$$\begin{aligned} v_{\vartheta,(d-2)/2}(x) := \Bigg[& \left(\frac{\gamma - \delta}{4} - \max\left\{ \sqrt{\left(|x_1|^2 - \frac{\gamma + 3\delta}{4}\right)^2 + |x_3|^2}, \vartheta\frac{\gamma - \delta}{4} \right\} \right)^+ \\ & - \left(\frac{\gamma - \delta}{4} - \max\left\{ \sqrt{\left(|x_1|^2 - \frac{\gamma + 3\delta}{4}\right)^2 + |x_3|^2}, \vartheta\frac{\gamma - \delta}{4} \right\} \right)^+ \Bigg] \\ & \cdot \frac{4}{(\gamma - \delta)(1 - \vartheta)}. \end{aligned}$$

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Next, for every $\mu \in (0, 1]$, let us consider the disjoint sets

$$Q_{\mu,1} := \left\{ (x_1, x_3) \in \mathbb{R}^{i+1} \times \mathbb{R}^{i+1} : \sqrt{\left(|x_1|^2 - \frac{\gamma + 3\delta}{4} \right)^2 + |x_3|^2} \le \mu \frac{\gamma - \delta}{4} \right\},$$

$$Q_{\mu,2} := \left\{ (x_1, x_3) \in \mathbb{R}^{i+1} \times \mathbb{R}^{i+1} : \sqrt{\left(|x_3|^2 - \frac{\gamma + 3\delta}{4} \right)^2 + |x_1|^2} \le \mu \frac{\gamma - \delta}{4} \right\},$$

and define

$$D_{\mu,(d-2)/2} := \left\{ (x_1, x_3) \in \mathbb{R}^{d/2} \times \mathbb{R}^{d/2} : (x_1, x_3) \in Q_{\mu,1} \cup Q_{\mu,2} \right\},$$

$$D_{\mu,i} := \Big\{ (x_1, x_2, x_3) \in \mathbb{R}^{i+1} \times \mathbb{R}^{d-2i-2} \times \mathbb{R}^{i+1} : (x_1, x_3) \in Q_{\mu,1} \cup Q_{\mu,2}, \quad |x_2| \le \mu \frac{\gamma - \delta}{4} \Big\}, \quad \forall i \neq \frac{d-2}{2}.$$

The sets $D_{\mu,i}$ have positive Lebesgue measure and are H_{d,η_i}-invariant. Moreover, for every $\vartheta \in (0, 1)$, the special shape of $v_{\vartheta,i}$ guarantees that $v_{\vartheta,i} \in \mathrm{Fix}_{H_{d,\eta_i}}(H^1(\mathbb{R}^d))$ and that

(j_1) $\mathrm{supp}(v_{\vartheta,i}) = D_{1,i} \subseteq A_\delta^\gamma(0)$;
(j_2) $\|v_{\vartheta,i}\|_\infty = 1$;
(j_3) $|v_{\vartheta,i}(x)| = 1$ for every $x \in D_{\vartheta,i}$.

Thanks to (f_2), there exists a sequence $\{t_j\} \subset \mathbb{R}^+$, with $t_j \to 0$ as $j \to +\infty$, such that

$$F(t_j) \ge c t_j^2,$$

for all $c > 0$ and for sufficiently large j. Thus, let $w_{j,i} := t_j v_{\vartheta,i}$ for any $j \in \mathbb{N}$. Of course $w_{j,i} \in \mathrm{Fix}_{H_{d,\eta_i}}(H^1(\mathbb{R}^d))$ for any $j \in \mathbb{N}$. Furthermore, taking the properties (j_1)–(j_3) into account, since F is even (this implies that $F(w_{j,i}(x)) = F(|w_{j,i}(x)|) = F(t_j)$ for every $x \in D_{\vartheta,i}$) we get, for j large enough,

$$\begin{aligned}
\mathcal{E}_{\lambda,i}(w_{j,i}) &= \frac{1}{2} \left\| v_{\vartheta,i} \right\|^2 t_j^2 - \frac{1}{2^*} \int_{D_{1,i}} (v_{\vartheta,i})^{2^*} dx \, t_j^{2^*} - \lambda \int_{D_{\vartheta,i}} \alpha(x) F(w_{j,i}) dx \\
&\quad - \lambda \int_{A_\delta^\gamma(0) \setminus D_{\vartheta,i}} \alpha(x) F(w_{j,i}) dx \\
&\le \frac{1}{2} \left\| v_{\vartheta,i} \right\|^2 t_j^2 - \frac{1}{2^*} \int_{D_{1,i}} (v_{\vartheta,i})^{2^*} dx \, t_j^{2^*} \\
&\quad - \lambda c \alpha_0 \left(|D_{\vartheta,i}| + \int_{A_\delta^\gamma(0) \setminus D_{\vartheta,i}} (v_{\vartheta,i})^2 dx \right) t_j^2 .
\end{aligned}$$

By choosing $c > 0$ large enough, we can easily deduce that 0 is not a local minimizer of $\mathcal{E}_{\lambda,i}$ and hence cannot coincide with $z_{\lambda,i}$.

In conclusion, if $\lambda < \lambda_\star$, taking Proposition 10 into account, problem $(\widetilde{P}_\lambda)$ admits at least $t_d := \tau_d + 1$ pairs of non-trivial weak solutions $\{\pm w_{\lambda,i}\}$. By construction, it is clear that

$$\tau_d := (-1)^d + \left[\frac{d-3}{2}\right]$$

pairs of the attained solutions are sign-changing. The proof is then completed. □

Remark 14. We observe that Theorem 13 is not relevant in dimension three. However, also in this case, it provides the existence of one pair of non-trivial and radially symmetric solutions to $(\widetilde{P}_\lambda)$ whenever λ is sufficiently small.

Remark 15. For the sake of completeness, we point out that the approach presented in this paper could also be followed to investigate a larger class of elliptic equations governed by several differential operators, like the ones considered in [29–31]. However, for this wider class of energies, some different technical tools are needed to get analogous existence results. We intend to address these interesting cases in future investigations.

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