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Research article

Harnack inequality and Liouville-type theorems for Ornstein–Uhlenbeck and Kolmogorov operators

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Abstract: We prove, with a purely analytic technique, a one-side Liouville theorem for a class of Ornstein–Uhlenbeck operators $\mathcal{L}_0$ in $\mathbb{R}^N$, as a consequence of a Liouville theorem at "$t = -\infty$" for the corresponding Kolmogorov operators $\mathcal{L}_0 - \partial_t$ in $\mathbb{R}^{N+1}$. In turn, this last result is proved as a corollary of a global Harnack inequality for non-negative solutions to $(\mathcal{L}_0 - \partial_t)u = 0$ which seems to have an independent interest in its own right. We stress that our Liouville theorem for $\mathcal{L}_0$ cannot be obtained by a probabilistic approach based on recurrence if $N > 2$. We provide a self-contained proof of a Liouville theorem involving recurrent Ornstein–Uhlenbeck stochastic processes in the Appendix.

Keywords: Liouville theorems; Harnack inequalities; Kolmogorov operators; Ornstein–Uhlenbeck operators; mean value formulae

Dedicato a Sandro Salsa con stima ed amicizia.

1. Introduction and main results

The main "motivation" of this paper is to provide a purely analytical proof of a *one-side* Liouville theorem for the following Ornstein–Uhlenbeck operator in $\mathbb{R}^N$:

$$\mathscr{L}_0 := \varDelta + \langle Bx, \nabla \rangle, \tag{1.1}$$

where $\varDelta$ is the Laplace operator, while $\langle\ ,\ \rangle$ and ∇ denote, respectively, the inner product and the gradient in $\mathbb{R}^N$. Moreover B is a $N \times N$ real matrix which we suppose to satisfy the following condition: letting

$$E(t) := \exp(-tB), \tag{1.2}$$

then,

$$b := \sup_{t \in \mathbb{R}} \|E(t)\| < \infty. \tag{H}$$

It is not difficult to show that condition (H) is equivalent to the following one:

B is diagonalizable over the complex field
with all the eigenvalues on the imaginary axis.

This condition is satisfied in particular if $B = -B^T$ and if $B^2 = -\mathbb{I}_N$, where $\mathbb{I}_N$ is the $N \times N$ identity matrix.

Our positive (one-side) Liouville Theorem for (1.1) is the following one.

Theorem 1.1. *Let v be a smooth* solution to*

$$\mathscr{L}_0 v = 0 \text{ in } \mathbb{R}^N.$$

If $\inf_{\mathbb{R}^N} v > -\infty$, *then v is constant.*

If we assume the solution v to be bounded both from below and from above then the conclusion of Theorem 1.1 immediately follows from a theorem due to Priola and Zabczyk [14, Theorem 3.1], which, for the operator $\mathscr{L}_0$ in (1.1), takes this form:
Consider the Ornstein–Uhlenbeck operator

$$\mathscr{L}_0 = \varDelta + \langle Bx, \nabla \rangle,$$

where B is any $N \times N$ constant matrix. Then the following statements are equivalent:

(i) *$\mathscr{L}_0$ has the simple Liouville property, i.e.,*

$$\mathscr{L}_0 v = 0 \text{ in } \mathbb{R}^N, \ \sup_{\mathbb{R}^N} |v| < \infty \implies v \equiv \text{constant};$$

(ii) *the real part of every eigenvalue of the matrix B is non-positive.*

If the matrix B satisfies (H), its eigenvalues have real part equal to zero. Then, the aforementioned Priola and Zabczyk theorem implies that the *bounded* solutions to $\mathscr{L}_0 v = 0$ in $\mathbb{R}^N$ are constant.

Theorem 1.1 is a Corollary of the following Liouville theorem "at $t = -\infty$" for the *evolution counterpart* of $\mathscr{L}_0$, i.e., for the Kolmogorov operator in $\mathbb{R}^{N+1} = \mathbb{R}^N_x \times \mathbb{R}_t$

$$\mathcal{L} := \varDelta + \langle Bx, \nabla \rangle - \partial_t. \tag{1.3}$$

*$\mathscr{L}_0$ is hypoelliptic, so that every distributional solution to $\mathscr{L}_0 v = 0$ actually is of class C^∞.

Theorem 1.2. *Let u be a smooth solution to*

$$\mathscr{L}u = 0 \textit{ in } \mathbb{R}^{N+1}.$$

If $\inf_{\mathbb{R}^N} u > -\infty$*, then*

$$\lim_{t\to-\infty} u(x,t) = \inf_{\mathbb{R}^{N+1}} u \quad \textit{for every } x \in \mathbb{R}^N.$$

It easy to show that this theorem implies Theorem 1.1. Indeed, let $v : \mathbb{R}^N \longrightarrow \mathbb{R}$ be a smooth and bounded below solution to $\mathscr{L}_0 v = 0$ in $\mathbb{R}^N$. Then, letting

$$u(x,t) = v(x), \quad x \in \mathbb{R}^N, \quad t \in \mathbb{R},$$

we have

$$\mathscr{L}u = 0 \text{ in } \mathbb{R}^{N+1} \quad \text{and} \quad \inf_{\mathbb{R}^{N+1}} u = \inf_{\mathbb{R}^N} v > -\infty.$$

Then, by Theorem 1.2,

$$\inf_{\mathbb{R}^N} v = \inf_{\mathbb{R}^{N+1}} u = \lim_{t\to-\infty} u(x,t) = v(x) \quad \text{for every } x \in \mathbb{R}^N.$$

Hence, v is constant.

From Theorem 1.2 it also follows a Liouville theorem for *bounded* solutions to $\mathscr{L}u = 0$ (for a related result see Theorem 3.6 in [13]).

Theorem 1.3. *Let u be a bounded smooth solution to*

$$\mathscr{L}u = 0 \textit{ in } \mathbb{R}^{N+1}.$$

Then, u is constant.

Proof. Let

$$m = \inf_{\mathbb{R}^{N+1}} u \quad \text{and} \quad M = \sup_{\mathbb{R}^{N+1}} u.$$

Applying Theorem 1.2 to $M - u$ and $u - m$, we obtain for every $x \in \mathbb{R}^N$ that:

$$0 = \inf_{\mathbb{R}^{N+1}} (M - u) = \lim_{t\to-\infty} (M - u(x,t))$$

and

$$0 = \inf_{\mathbb{R}^{N+1}} (u - m) = \lim_{t\to-\infty} (u(x,t) - m).$$

Hence, $M = m$ and u is constant. □

Theorem 1.2 is, in turn, a consequence of a "global" Harnack inequality for non-negative solutions to $\mathscr{L}u = 0$ in $\mathbb{R}^{N+1}$. To state this inequality we need to recall that $\mathscr{L}$ is left translation invariant on the Lie group $\mathbb{K} = (\mathbb{R}^{N+1}, \circ)$ with composition law

$$(x,t) \circ (y,\tau) = (y + E(\tau)x, t + \tau),$$

see [12]. For every z_0 in $\mathbb{R}^{N+1}$ we define the "paraboloid"

$$P(z_0) = z_0 \circ P,$$

where

$$P = \left\{ (x,t) \in \mathbb{R}^{N+1} \ : \ t < -\frac{|x|^2}{4} \right\}.$$

Then, inspired by an idea used in [8] for classical parabolic operators, and exploiting Mean Value formulas for solutions to $\mathscr{L}u = 0$, we establish the following Harnack inequality.

Theorem 1.4. *Let $z_0 \in \mathbb{R}^{N+1}$ and let u be a non-negative smooth solution to*

$$\mathscr{L}u = 0 \ \textit{in}\ \mathbb{R}^{N+1}.$$

Then, there exists a positive constant C, independent of u and z_0, such that

$$u(z) \leq C u(z_0),$$

for every $z \in P(z_0)$.

We will prove this theorem in Section 5. Here we show how it implies Theorem 1.2 by using the following lemma (for the reader's convenience we postpone its proof to Section 3).

Lemma 1.5. *For every $x \in \mathbb{R}^N$ and for every $z_0 \in \mathbb{R}^{N+1}$ there exists a real number $T = T(x, z_0)$ such that*

$$(x,t) \in P(z_0) \qquad \forall t < T.$$

Proof of Theorem 1.2. Let u be a smooth bounded below solution to $\mathscr{L}u = 0$ in $\mathbb{R}^{N+1}$. Define

$$m = \inf_{\mathbb{R}^{N+1}} u.$$

Then, for every $\varepsilon > 0$, there exists $z_\varepsilon \in \mathbb{R}^{N+1}$ such that

$$u(z_\varepsilon) - m < \varepsilon.$$

Theorem 1.4 applies to function $u - m$, so that

$$u(z) - m < C(u(z_\varepsilon) - m) < C\varepsilon, \tag{1.4}$$

for every $z \in P(z_\varepsilon)$, where $C > 0$ does not depend on z and on ε. Let us now fix $x \in \mathbb{R}^N$. By Lemma 1.5, there exists $T = T(z_\varepsilon, x) \in \mathbb{R}$ such that $(x,t) \in P(z_\varepsilon)$ for every $t < T$. Then, from (1.4), we get

$$0 \leq u(x,t) - m \leq C\varepsilon \qquad \forall t < T.$$

This means

$$\lim_{t \to -\infty} u(x,t) = m.$$

□

We conclude the introduction with the following remark.

Remark 1.6. One-side Liouville theorems for a class of Ornstein–Uhlenbeck operators can be proved by a probabilistic approach based on recurrence of the corresponding Ornstein–Uhlenbeck process. We present this approach in Appendix, showing how it leads to one-side Liouville theorems also for degenerate Ornstein–Uhlenbeck operators. However, the results obtained with this probabilistic approach contain Theorem 1.1 only in the case $N = 2$. We mention that, in this last case, Theorem 1.1 is contained in [3], where a full description of the Martin boundary for a non-degenerate two-dimensional Ornstein–Uhlenbeck operator is given.

We also mention that under particular assumptions on the matrix B that make the operator $\mathscr{L}$ homogenous with respect to a group of dilations, asymptotic Liouville theorems at $t = -\infty$ for the solutions to $\mathscr{L}u = 0$ in $\mathbb{R}^{N+1}$ are known (see [10] and the references therein); as a consequence, in such cases, one-side Liouville theorems for the solutions to $\mathscr{L}_0 v = 0$ hold.

2. Some preliminaries

The matrix
$$E(\tau) = \exp(-\tau B), \quad \tau \in \mathbb{R},$$
introduced in (1.2), plays a crucial rôle for the operator $\mathscr{L}$. First of all, as already recalled in the Introduction, defining the composition law $\circ$ in $\mathbb{R}^{N+1}$ as follows:
$$(x, t) \circ (y, \tau) = (y + E(\tau)x, t + \tau), \tag{2.1}$$
we obtain a Lie group
$$\mathbb{K} = (\mathbb{R}^{N+1}, \circ),$$
on which $\mathscr{L}$ is left translation invariant (see [12]; see also [1], Section 4.1.4).

As already observed, assumption (H) implies
$$\sigma(B) := \{\text{eigenvalues of } B\} \subseteq i\mathbb{R}.$$

Then, since B has real entries, $-\lambda \in \sigma(B)$ if $\lambda \in \sigma(B)$. As a consequence,
$$\operatorname{trace}(B) = 0.$$
A fundamental solution for $\mathscr{L}$ is given by
$$\Gamma(z, \zeta) = \gamma(\zeta^{-1} \circ z), \tag{2.2}$$
where,
$$\gamma(z) = \gamma(x, t) = \begin{cases} 0 \text{ if } t \le 0, \\ \\ \dfrac{(4\pi)^{-\frac{N}{2}}}{\sqrt{\det C(t)}} \exp\left(-\dfrac{1}{4}\langle C^{-1}(t)x, x\rangle\right) \text{ if } t > 0, \end{cases}$$
and

$$C(t) = \int_0^t E(s)E(s)^T \, ds,$$

(see [12, (1.7)], and keep in mind that trace $(B) = 0$ since B satisfies (H)).

It is noteworthy to stress that

$$C(t) \text{ is symmetric and } C(t) > 0$$

for every $t > 0$.

The solutions to $\mathscr{L}u = 0$ in $\mathbb{R}^{N+1}$ satisfy the following Mean Value formula: for every $z_0 \in \mathbb{R}^{N+1}$, $r > 0$ and $p \in \mathbb{N}$,

$$u(z_0) = \frac{1}{r} \int_{\Omega_r^{(p)}(z_0)} u(z) W_r^{(p)}(z_0^{-1} \circ z) \, dz, \tag{2.3}$$

where

$$\Omega_r^{(p)}(z_0) = \left\{ z \ : \ \phi_p(z_0, z) > \frac{1}{r} \right\},$$

with

$$\phi_p(z_0, z) := \frac{\Gamma(z_0, z)}{(4\pi(t_0 - t))^{\frac{p}{2}}},$$

if $z = (x, t)$ and $z_0 = (x_0, t_0)$.

Remark 2.1. If $z \in \Omega_r^{(p)}(z_0)$, then $\Gamma(z_0, z) > 0$, hence $t_0 - t > 0$.

Moreover,

$$W_r^{(p)}(z) = \omega_p R_r^p(0, z) \left\{ W(z) + \frac{p}{4(p+2)} \left(\frac{R_r(0, z)}{t} \right)^2 \right\}, \tag{2.4}$$

where ω_p denotes the Lebesgue measure of the unit ball of $\mathbb{R}^p$,

$$W(z) = W(x, t) = \frac{1}{4} \left| C^{-1}(t) x \right|^2, \tag{2.5}$$

and

$$R_r(0, z) = \sqrt{4(-t) \log(r \phi_p(0, z))}. \tag{2.6}$$

A complete proof of the Mean Value formula (2.3) can be found in Section 5 of [2].

3. Proof of Lemma 1.5

Let $z_0 = (x_0, t_0)$ and $z = (x, t)$. Then,

$$z \in P(z_0) = z_0 \circ P \iff z_0^{-1} \circ z \in P \iff (x - E(t - t_0)x_0, t - t_0) \in P.$$

Hence, keeping in mind the definition of P,

$$z = (x, t) \in P(z_0) \iff \frac{|x - E(t - t_0)x_0|^2}{4(t_0 - t)} < 1. \tag{3.1}$$

On the other hand, from (H), we have

$$\frac{|x - E(t-t_0)x_0|^2}{4(t_0-t)} \le \frac{(|x| + b|x_0|)^2}{4(t_0-t)} \longrightarrow 0, \quad \text{as } t \longrightarrow -\infty.$$

Therefore: for every fixed $z_0 \in \mathbb{R}^{N+1}$ and $x \in \mathbb{R}$, there exists $T = T(z_0, x)$ s.t.

$$z = (x,t) \in P(z_0) \qquad \forall\, t < T.$$

4. A two "onions" lemma

The aim of this section is to prove a geometrical lemma on the level sets $\Omega_r^{(p)}$ (which we call $\mathscr{L}$-*"onions"*), that will play a crucial rôle in the proof of the Harnack inequality in Theorem 1.4.
First of all we resume that hypothesis (H) implies:

$$\frac{1}{b^2}|x|^2 \le t\langle C^{-1}(t)x, x\rangle \le b^2|x|^2, \tag{4.1}$$

for every $t \in \mathbb{R}$ and for every $x \in \mathbb{R}^N$.
Indeed, from (H), we obtain

$$b := \sup_{t\in\mathbb{R}} \|E(t)^T\| < \infty.$$

Since we are considering the operator norm, we have

$$|E(s)^T y| \le b|y| = b|E(-s)^T E(s)^T y| \le b^2 |E(s)^T y|,$$

so that

$$\frac{1}{b}|y| \le |E(s)^T y| \le b|y|$$

for every $t \in \mathbb{R}$ and every $y \in \mathbb{R}^N$.
Then, since

$$\langle C(t)y, y\rangle = \int_0^t |E(s)^T y|^2\, ds,$$

we get

$$\frac{1}{b^2}|y|^2 \le \frac{1}{t}\langle C(t)y, y\rangle \le b^2|y|^2$$

for every $y \in \mathbb{R}^N$ and $t \in \mathbb{R} \setminus \{0\}$.
If in these inequalities we choose

$$y = (C(t))^{-\frac{1}{2}}x \qquad \text{if } t > 0$$

and

$$y = (-C(t))^{-\frac{1}{2}}x \qquad \text{if } t < 0,$$

we immediately obtain (4.1).
Now, for every $r > 0$, define

$$\Sigma_r = \left\{ z = (x,t) \;:\; t = -r^{\frac{2}{N+p}},\ |x|^2 < -4t \right\}.$$

Then, the following lemma holds

Lemma 4.1. *For every $p \in \mathbb{N}$, there exists a constant $\theta = \theta(p) > 1$ such that,*

$$\Omega^{(p)}_{\theta r}(0) \supseteq \Omega^{(p)}_{r}(z) \quad \forall\, z \in \Sigma_r, \quad \forall\, r > 0.$$

Proof. Let $r > 0$ and $z \in \Sigma_r$. Then $z = (x, t)$, with

$$t = -r^{\frac{2}{N+p}} \quad \text{and} \quad |x|^2 < 4r^{\frac{2}{N+p}}.$$

Let us now take $\zeta = (\xi, \tau) \in \Omega^{(p)}_r(z)$. This means

$$\begin{gathered} \phi_p(z,\zeta) > \frac{1}{r} \\ \Longleftrightarrow \\ \langle C^{-1}(t-\tau)(x - E(t-\tau)\xi), x - E(t-\tau)\xi\rangle < \log \frac{r}{(4\pi(t-\tau))^{\frac{N+p}{2}}}. \end{gathered} \tag{4.2}$$

Analogously,

$$\begin{gathered} \zeta \in \Omega^{(p)}_{\theta r}(0) \\ \Longleftrightarrow \\ \langle C^{-1}(-\tau)E(-\tau)\xi, E(-\tau)\xi\rangle < \log \frac{\theta r}{(4\pi(-\tau))^{\frac{N+p}{2}}}. \end{gathered}$$

On the other hand, by (4.1) and (H),

$$\langle C^{-1}(-\tau)E(-\tau)\xi, E(-\tau)\xi\rangle \le b^4 \frac{|\xi|^2}{|\tau|},$$

so that, $\zeta = (\xi, \tau) \in \Omega^{(p)}_{\theta r}(0)$ if $\tau < 0$ and

$$|\xi|^2 < \frac{1}{b^4}|\tau| \log \frac{\theta r}{(4\pi|\tau|)^{\frac{N+p}{2}}}. \tag{4.3}$$

Then, to prove our lemma, it is enough to show that inequality (4.2) implies (4.3). Now, from (4.2), using (H), (4.1) and the inclusion $z = (x, t) \in \Sigma_r$, we obtain (we assume $b \ge 1$ so that $b^2 \le b^4$)

$$\begin{aligned} |\xi|^2 &\le b^2|E(t-\tau)\xi|^2 \\ &\le 2b^2(|E(t-\tau)\xi - x|^2 + |x|^2) \\ &\le 2b^4\Big((t-\tau)\langle C^{-1}(t-\tau)(E(t-\tau)\xi - x), E(t-\tau)\xi - x\rangle + 4|t|\Big) \\ &< 2b^4\left((t-\tau)\log\frac{r}{(4\pi(t-\tau))^{\frac{N+p}{2}}} + 4|t|\right). \end{aligned}$$

Therefore, we will obtain (4.3), and hence the lemma, if for a suitable $\theta > 1$ independent of z and ζ, the following inequality holds

$$2b^4\left((t-\tau)\log\frac{r}{(4\pi(t-\tau))^{\frac{N+p}{2}}} + 4|t|\right) \le \frac{1}{b^4}|\tau|\log\frac{\theta r}{(4\pi|\tau|)^{\frac{N+p}{2}}}. \tag{4.4}$$

To simplify the notation we put

$$\frac{r}{(4\pi)^{\frac{N+p}{2}}} = \rho^{\frac{N+p}{2}} \iff \rho = \frac{r^{\frac{2}{N+p}}}{4\pi}.$$

Hence, since $z \in \Sigma_r$,

$$|t| = 4\pi\rho,$$

and inequality (4.4) can be written as follows:

$$A_0(t-\tau)\log\frac{\rho}{t-\tau} + A_1\rho \le A_2|\tau|\log\frac{\theta\rho}{|\tau|}, \tag{4.5}$$

and the A_i's are strictly positive constants independent of z and ζ.
Since $\zeta \in \Omega_r^{(p)}(z)$, we have

$$\frac{1}{r} < \phi_\rho(z,\zeta) \le \left(\frac{1}{4\pi(t-\tau)}\right)^{\frac{N+p}{2}},$$

then,

$$0 < t-\tau < \rho.$$

As a consequence, since

$$4\pi\rho = |t| < |\tau| \le |\tau - t| + |t| < \rho + 4\pi\rho,$$

we get

$$\frac{1}{4\pi+1} \le \frac{\rho}{|\tau|} \le \frac{1}{4\pi}.$$

Thus, the left hand side of (4.5) can be estimated from above as follows:

$$A_0(t-\tau)\log\frac{\rho}{t-\tau} + A_1\rho = \rho\left(A_o\frac{t-\tau}{\rho}\log\frac{\rho}{t-\tau} + A_1\right) \le \rho(A_0 S + A_1),$$

where

$$S = \sup\left\{ s\log\frac{1}{s} \;:\; 0 < s < 1\right\}.$$

Moreover, the right hand side of (4.5) can be estimated from below as follows:

$$A_2|\tau|\log\frac{\theta\rho}{|\tau|} \ge \rho 4\pi A_2\log\frac{\theta}{4\pi+1}.$$

Therefore, if we choose $\theta > 0$ such that

$$A_0 S + A_1 \le 4\pi A_2 \log\frac{\theta}{4\pi+1}$$

inequality (4.5) is satisfied. This completes the proof.

□

5. Proof of Theorem 1.4

Since $\mathscr{L}$ is left translation invariant on the Lie group $(\mathbb{K}, \circ)$, it is enough to prove Theorem 1.4 in the case $z_0 = 0 \in \mathbb{R}^{N+1}$. In particular, it is enough to prove the inequality

$$u(z) \le Cu(z_0), \qquad \text{with } z_0 = 0, \tag{5.1}$$

for every non-negative smooth solution u to

$$\mathscr{L}u = 0 \text{ in } \mathbb{R}^{N+1},$$

and for every $z = (x, t) \in P = \{(x, t) \ : \ |x|^2 < -4t\}$.
The constant C in (5.1) has to be independent of u. To this end, taken a non-negative global solution u to $\mathscr{L}u = 0$, we start with the Mean Value formula for u on the $\mathscr{L}$-level set $\Omega^{(p)}_{2\theta r}(z_0)$, with $p > 4$ and with θ given by Lemma 4.1:

$$u(z_0) = \frac{1}{2\theta r} \int_{\Omega^{(p)}_{2\theta r}(z_0)} u(\zeta) W^{(p)}_{2\theta r}(z_0^{-1} \circ \zeta)\, d\zeta. \tag{5.2}$$

Let us arbitrarily fix $z = (x, t) \in P$. Then $t < 0$ and $|x|^2 < 4|t|$. In (5.2) we choose $r > 0$ such that

$$t = -r^{\frac{2}{N+p}}.$$

By Lemma 4.1 we have the inclusion

$$\Omega^{(p)}_{2\theta r}(z_0) \supseteq \Omega^{(p)}_{r}(z),$$

so that, since $u \ge 0$, from (5.2) we get

$$u(z_0) \ge \frac{1}{2\theta r} \int_{\Omega^{(p)}_{r}(z)} u(\zeta) W^{(p)}_{2\theta r}(z_0^{-1} \circ \zeta)\, d\zeta. \tag{5.3}$$

Let us now prove that, for a suitable positive constant C independent of u and of z, we have ($z_0^{-1} = z_0 = 0$):

$$\frac{W^{(p)}_{2\theta r}(z_0^{-1} \circ \zeta)}{W^{(p)}_{r}(z^{-1} \circ \zeta)} \ge \frac{2\theta}{C} \qquad \forall \zeta \in \Omega^{(p)}_{r}(z). \tag{5.4}$$

It will follow, from (5.3),

$$\begin{aligned} u(z_0) \quad &\ge \quad \frac{1}{rC} \int_{\Omega^{(p)}_{r}(z)} u(\zeta) W^{(p)}_{r}(z^{-1} \circ \zeta)\, d\zeta \\ &\qquad \text{(again by the Mean Value formula (2.3))} \\ &= \quad \frac{1}{C} u(z), \end{aligned}$$

i.e., $u(z) \le Cu(z_0)$, which is (5.1).

To prove (5.4) we first estimate from below $W^{(p)}_{2\theta r}(z_0^{-1} \circ \zeta)$. From the very definition of this kernel, by keeping in mind that $z_0 = 0$, and letting $\zeta = (\xi, \tau)$, we obtain:

$$\begin{aligned} W^{(p)}_{2\theta r}(z_0^{-1} \circ \zeta) &\geq \frac{p\omega_p}{4(p+2)} \frac{(R_{2\theta r}(z_0, \zeta))^{p+2}}{|\tau|^2} \\ &= c'_p |\tau|^{\frac{p+2}{2}-2} (\log(2\theta r \phi_p(z_0, \zeta)))^{\frac{p}{2}+1} \\ &\quad (\phi_p(z_0, \zeta) \geq \tfrac{1}{\theta r} \text{ since } \zeta \in \Omega^{(p)}_r(\zeta) \subseteq \Omega^{(p)}_{\theta r}(z_0)) \\ &\geq c'_p (\log(2\theta))^{\frac{p}{2}+1} |\tau|^{\frac{p}{2}-1} \\ &\quad (\text{if } p > 2) \\ &\geq c_p |t|^{\frac{p}{2}-1} \\ &= c_p r^{\frac{p-2}{p+N}}. \end{aligned}$$

Here, and in what follows, $c'_p, c''_p, \ldots, c_p$ denote strictly positive constants only depending on p. So, we have proved the following inequality

$$W^{(p)}_{2\theta r}(z_0^{-1} \circ \zeta) \geq c_p r^{\frac{p-2}{p+N}} \qquad \forall \zeta \in \Omega^{(p)}_r(z). \tag{5.5}$$

Now we estimate $W^{(p)}_r(z^{-1} \circ \zeta)$ from above, estimating, separately

$$K_1(z, \zeta) = R^p_r(0, z^{-1} \circ \zeta) W(z^{-1} \circ \zeta) \tag{5.6}$$

and

$$K_2(z, \zeta) = \frac{R^{p+2}_r(z_0, z^{-1} \circ \zeta)}{(t-\tau)^2}. \tag{5.7}$$

We have

$$\begin{aligned} K_1(z, \zeta) &= \left(4(t-\tau) \log\left(r \frac{\Gamma(z, \zeta)}{(4\pi(t-\tau))^{\frac{N+p}{2}}}\right)\right)^{\frac{p}{2}} W(z^{-1} \circ \zeta)) \\ &\leq 2^p \left((t-\tau) \log \frac{r}{(t-\tau)^{\frac{N+p}{2}}}\right)^{\frac{p}{2}} W(z^{-1} \circ \zeta). \end{aligned} \tag{5.8}$$

Moreover, from (2.5) and (4.1), we obtain

$$\begin{aligned} W(z^{-1} \circ \zeta) &= \frac{1}{4} \left| C^{-1}(\tau - t)(\xi - E(\tau - t)x) \right|^2 \\ &\leq \frac{b^4}{4} \frac{|\xi - E(\tau - t)x|^2}{(\tau - t)^2}. \end{aligned} \tag{5.9}$$

To estimate the right hand side of this inequality we use the inclusion $\zeta \in \Omega^{(p)}_r(z)$ which implies:

$$\phi_p(z, \zeta) > \frac{1}{r}$$

$$\Longleftrightarrow$$

$$\left(\frac{1}{(4\pi(t-\tau))}\right)^{\frac{N+p}{2}} \exp\left(-\frac{1}{4}\langle C^{-1}(t-\tau)(x-E(t-\tau)\xi), x-E(t-\tau)\xi\rangle\right) > \frac{1}{r}$$

$$\Longleftrightarrow$$

$$\langle C^{-1}(t-\tau)(x-E(t-\tau)\xi), x-E(t-\tau)\xi\rangle < \log \frac{r}{(4\pi(t-\tau))^{\frac{N+p}{2}}}.$$

This inequality, keeping in mind (4.1), implies

$$|x-E(t-\tau)\xi|^2 \leq b^2(t-\tau)\log\frac{r}{(4\pi(t-\tau))^{\frac{N+p}{2}}}.$$

Then

$$\begin{aligned} |\xi - E(\tau-t)x|^2 &\leq \|E(\tau-t)\|^2\,|E(t-\tau)\xi - x|^2 \\ &\leq b^4(t-\tau)\log\frac{r}{(4\pi(t-\tau))^{\frac{N+p}{2}}} \\ &\leq c_p' \frac{r^{\frac{2}{N+p}}}{4\pi}, \end{aligned}$$

where

$$c_p' = b^4 \sup\left\{ s\log\frac{1}{s} : 0 < s < 1\right\}.$$

Using this estimate in (5.9) and (5.8) we obtain:

$$K_1(z,\zeta) \leq c_p'' r^{\frac{2}{N+p}} (t-\tau)^{\frac{p}{2}-1}\left(\log\frac{r}{(4\pi(t-\tau))^{\frac{N+p}{2}}}\right)^{\frac{p}{2}} \leq c_p r^{\frac{p-2}{N+p}}, \qquad (5.10)$$

where, $c_p = c_p''' S_p$, with

$$S_p = \sup\left\{ s^{\frac{p}{2}-2}\left(\log\frac{1}{s}\right)^{\frac{p}{2}} : 0 < s < 1\right\}.$$

We stress that $S_p < \infty$ since $p > 4$.
The same estimate holds for K_2. Indeed:

$$\begin{aligned} K_2(z,\zeta) &\leq c_p'(t-\tau)^{\frac{p}{2}-1}\left(\log\frac{r}{(4\pi(t-\tau))^{\frac{N+p}{2}}}\right)^{\frac{p+2}{2}} \\ &\leq c^p r^{\frac{2}{N+p}(\frac{p}{2}-1)} = c_p r^{\frac{p-2}{N+p}}, \end{aligned} \qquad (5.11)$$

where,

$$c_p = c_p' \sup\left\{ s^{\frac{p}{2}-1}\left(\log\frac{1}{s}\right)^{\frac{p+2}{2}} : 0 < s < 1\right\} < \infty.$$

Keeping in mind (5.6) and (5.7), and the very definition of $W_r^{(p)}(z,\zeta)$, from inequalities (5.10) and (5.11) we obtain

$$W_r^{(p)}(z^{-1}\circ\zeta) \le c_p r^{\frac{p-2}{p+N}} \qquad \forall\zeta\in\Omega_r^{(p)}(z). \tag{5.12}$$

This inequality, together with (5.5), implies (5.4), and completes the proof of Theorem 1.4.

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Conflict of interest

The authors declare no conflict of interest.

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6. Appendix: A one-side Liouville theorem for Ornstein–Uhlenbeck operators by recurrence

Here we show a one-side Liouville theorem for some Ornstein–Uhlenbeck (OU) operators based on recurrence of the corresponding OU stochastic processes.

It is a general fact from probabilistic potential theory (see in particular [9]) that recurrence of a Markov process is equivalent to the fact that all excessive functions are constants (we also mention that the equivalence between excessive functions and super harmonic functions has been established in a general setting; see [6] and the references therein). On the other hand, a characterization of recurrent OU processes is known (see [7] which extends the seminal paper [5]; see also [15] for connections between recurrence and stochastic controllability).

We present the main steps to prove a one-side Liouville theorem in a self-contained way. Comparing with [5,7,9], we simplify some proofs; see in particular the proof of Theorem 6.6 in which we also use a result in [14]. We do not appeal to the general theory of Markov processes but we use some basic stochastic calculus. It seems to be an open problem to find a purely analytic approach to proving such result.

Let Q be a non-negative symmetric $N \times N$ matrix and let B be a real $N \times N$ matrix. The OU operator we consider is

$$\mathcal{K}_0 = \frac{1}{2}\,\mathrm{tr}(QD^2) + \langle Bx, \nabla\rangle = \frac{1}{2}\,\mathrm{div}(Q\nabla) + \langle Bx, \nabla\rangle. \tag{6.1}$$

We will always assume the well-known Kalman controllability condition:

$$\mathrm{rank}[Q, BQ, \dots, B^{N-1}Q] = N, \tag{6.2}$$

see [4,7,11,12,14,15] and the references therein. Under this assumption $\mathcal{K}_0$ is hypoelliptic, see [12]. Before stating the Liouville theorem we recall that a matrix C is stable if all its eigenvalues have negative real part.

Theorem 6.1. *Assume* (6.2). *Let* $v : \mathbb{R}^N \to \mathbb{R}$ *be a non-negative* C^2*-function such that* $\mathcal{K}_0 v \leq 0$ *on* $\mathbb{R}^N$. *Then* v *is constant if the following condition holds:*

(HR) *The real Jordan representation of* B *is*

$$\begin{pmatrix} B_0 & 0 \\ 0 & B_1 \end{pmatrix} \tag{6.3}$$

where B_0 *is stable and* B_1 *is at most of dimension 2 and of the form* $B_1 = [0]$ *or* $B_1 = \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix}$ *for some* $\alpha \in \mathbb{R}$ *(in this case we need* $N \geq 2$*).*

The proof of Theorem 6.1 will immediately follow by Lemma 6.4 and Theorem 6.6 below.

Remark 6.2. Note that when $N = 2$ the matrix $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ does not satisfy (HR). On the other hand $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ verifies (HR) with $\alpha = 0$. Moreover, an example of possibly degenerate two-dimensional OU operator for which the one-side Liouville theorem holds is

$$\mathcal{K}_0 = \partial^2_{xx} + a\partial^2_{yy} + x\partial_y - y\partial_x, \qquad a \geq 0.$$

Remark 6.3. It is well-known, that condition (6.2) is equivalent to the fact that

$$Q_t = \int_0^t \exp(sB)\, Q \exp(sB^T) ds \text{ is positive definite for all } t > 0 \tag{6.4}$$

(cf. [4, 7, 12]). Note that $C(t) = \exp(-tB)Q_t \exp(-tB^T)$ is used in [12] and in Section 5 of [2] with Q replaced by A.

Let us introduce the OU stochastic process starting at $x \in \mathbb{R}^N$. It is the solution to the following linear SDE

$$X_t^x(\omega) = x + \int_0^t BX_s^x(\omega) ds \;+\; \sqrt{Q}\, W_t(\omega), \quad t \geq 0, \quad x \in \mathbb{R}^N,\; \omega \in \Omega, \tag{6.5}$$

see, for instance, [7, 14]. Here $W = (W_t)$ is a standard N-dimensional Wiener process defined a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ (the expectation with respect to $\mathbb{P}$ is denoted by $\mathbb{E}$; as usual in the sequel we often do not indicate the dependence on $\omega \in \Omega$).

For any non-empty open set $O \subset \mathbb{R}^N$, we consider the *hitting time* $\tau_O^x = \inf\{t \geq 0 \,:\, X_t^x \in O\}$ (if $\{\cdot\}$ is empty we write $\tau_O^x = \infty$).

Now we recall the notion of recurrence. The OU process $(X_t^x)_{t\geq 0} = X^x$ is *recurrent* if for any $x \in \mathbb{R}^N$, for any non-empty open set $O \subset \mathbb{R}^N$, one has

$$\phi_O(x) = \mathbb{P}(\tau_O^x < \infty) = 1. \tag{6.6}$$

Thus recurrence means that with probability one, the OU process reaches in finite time any open set starting from any initial position x.

Lemma 6.4. *Suppose that the OU process is recurrent. Let $v \in C^2(\mathbb{R}^N)$ be a non-negative function such that $\mathcal{K}_0 v \leq 0$ on $\mathbb{R}^N$. Then v is constant.*

Proof. We will adapt an argument used in the proof of Lemma 3.2 of [9] to show that excessive functions are constant for recurrent Markov processes.

Let us fix $x \in \mathbb{R}^N$. Applying the Itô formula and using the fact that $\mathcal{K}_0 v \leq 0$ we get, $\mathbb{P}$-a.s.,

$$v(X_t^x) = v(x) + \int_0^t \mathcal{K}_0 v(X_s^x) ds + M_t \leq v(x) + M_t, \quad t \geq 0,$$

where we are considering the martigale $M = (M_t)$, $M_t = \int_0^t \nabla v(X_s^x) \cdot \sqrt{Q} dW_s$.

Let $O \subset \mathbb{R}^N$ be a non-empty open set and consider the hitting time τ_O^x. We have $0 \leq v(X^x_{t\wedge\tau_O^x}) \leq v(x) + M_{t\wedge\tau_O^x}$, $t \geq 0$. By the Doob optional stopping theorem we obtain

$$\mathbb{E}[v(X^x_{t\wedge\tau_O^x})] \leq v(x), \quad t \geq 0.$$

Hence

$$v(x) \geq \mathbb{E}[v(X^x_{n\wedge\tau^x_O})] \geq \mathbb{E}[v(X^x_{n\wedge\tau^x_O})\, 1_{\{\tau^x_O<\infty\}}], \quad x \in \mathbb{R}^N, \quad n \geq 1. \tag{6.7}$$

Recall that $\mathbb{P}(\tau^x_O < \infty) = 1$, for any $x \in \mathbb{R}^N$. By the Fatou lemma (using also the continuity of the paths of the OU process) we infer

$$\mathbb{E}[v(X^x_{\tau^x_O})] = \mathbb{E}[\liminf_{n\to\infty} v(X^x_{n\wedge\tau^x_O})] \leq v(x). \tag{6.8}$$

Now we argue by contradiction. Suppose that v is not constant. Then there exists $0 < a < b$, $z \in \mathbb{R}^N$ such that $v(z) < a$ and $U = \{v > b\} = \{x \in \mathbb{R}^N : v(x) > b\}$ which is a non-empty open set. By (6.8) with $x = z$ we obtain

$$a > v(z) \geq \mathbb{E}\big[v(X^z_{\tau^z_U})\big] \geq b$$

because on the event $\{\tau^z_U < \infty\}$ we know that $X^z_{\tau^z_U} \in \{v \geq b\}$. We have found the contradiction $a > b$. Thus v is constant. □

Recall the *OU Markov semigroup* $(P_t) = (P_t)_{t\geq 0}$,

$$P_t f(x) = (P_t f)(x) = \mathbb{E}[f(X^x_t)] = \int_{\mathbb{R}^N} f(y)\, p_t(x,y)dy, \quad t > 0, \tag{6.9}$$

where $x \in \mathbb{R}^N$, $f : \mathbb{R}^N \to \mathbb{R}$ Borel and bounded and $p_t(x,y) = \dfrac{e^{-\frac{|Q_t^{-1/2}(e^{tB}x-y)|^2}{2}}}{\sqrt{(2\pi)^N \det(Q_t)}}$. We set $P_0 f = f$. The associated *potential* of a non-negative Borel function $g : \mathbb{R}^N \to \mathbb{R}$ is

$$Ug(x) = \int_0^\infty P_t g(x)dt, \quad x \in \mathbb{R}^N. \tag{6.10}$$

Clearly, in general it can also assume the value ∞ (cf. [9]).

Remark 6.5. Let A be an empty open set and let 1_A be the indicator function of A. The probabilistic interpretation of $U1_A$ is as follows. First one defines the sojourn time or occupation time of A (by the OU process starting at x) as

$$J^x_A(\omega) = \int_0^\infty 1_A(X^x_t(\omega))dt, \quad \omega \in \Omega;$$

it is the total amount of time that the sample path $t \mapsto X^x_t(\omega)$ spends in A. Then $\mathbb{E}[J^x_A] = \int_0^\infty \mathbb{E}[1_A(X^x_t)]dt = U1_A(x)$ is the average sojourn time or the expected occupation time of A.

The next result is a reformulation of a theorem in [7] at page 822 (see also the comments before such theorem and [5]). Erickson proves some parts of the theorem and refers to [5] for the proof of the remaining parts.

Theorem 6.6. *Assume* (6.2). *The next conditions for the OU process are equivalent.*

(i) Condition (HR) holds.

(ii) $\int_1^\infty \frac{1}{\sqrt{\det(Q_t)}}dt = \infty$.

(iii) For any $x, y \in \mathbb{R}^N$,

$$\int_1^\infty p_t(x,y)dt = \infty. \tag{6.11}$$

(iv) The OU process (X^x_t) *is recurrent.*

We will only deal with the proofs of *(i)* ⇒ *(ii)* ⇒ *(iii)* and *(i)* ⇒ *(iv)*; the last implication is needed to prove the one-side Liouville theorem in Lemma 6.4.

The proof of the recurrence *(i)* ⇒ *(iv)* is different and simpler than the proof given in [5] which also [7] mentions (see the remark below for more details).

Remark 6.7. In [5] it is proved that *(iii)* ⇒ *(iv)* by showing first that (iii) implies that, for any non-empty open set O, one has $U1_O \equiv \infty$, and then using a quite involved Khasminskii argument (see pages 142–143 in [5]) which uses the strong Markov property, the irreducibility and strong Feller property of the OU process. Alternatively, the fact that $U1_O \equiv \infty$, for any non-empty open set O, is equivalent to recurrence can be obtained using a potential theoretical approach involving excessive functions as in [9] (see in particular the proof that (ii) implies (iv) in Proposition 2.4 and Lemma 3.1 in [9]).

Proof. (**i**) ⇒ (**ii**). This can be proved as in the proof of Lemma 6.1 in [5] by using the Jordan decomposition of the matrix B (see also the remarks in [7]).

(**ii**) ⇒ (**iii**) Note that $Q_t \le Q_T$ (in the sense of positive symmetric matrices) if $0 < t \le T$. Hence by the Courant-Fischer min-max principle, we have $\lambda(t) \le \lambda(T)$ (where $\lambda(s)$ is the minimal eigenvalue of Q_s). Hence, there exists $M > 0$ such that, for $t \ge 1$,

$$\langle Q_t^{-1}(e^{tB}x - y), e^{tB}x - y\rangle \le \frac{1}{\lambda(t)}|e^{tB}x - y|^2 \le \frac{M}{\lambda(1)}(|x|^2 + |y|^2).$$

Then $p_t(x,y) \ge \exp(-\frac{M}{2\lambda(1)}(|x|^2 + |y|^2))\, \frac{1}{\sqrt{(2\pi)^N \det(Q_t)}}$, $t \ge 1$, and (6.11) holds if (ii) is satisfied.

(**i**) ⇒ (**iv**) The proof of this assertion is inspired by [9] and uses also the Liouville-type theorem for bounded harmonic function proved in [14].

Let us fix a non-empty open set $O \subset \mathbb{R}^N$ and consider the function $\phi_O = \phi : \mathbb{R}^N \to [0,1]$ (cf. (6.6)), $\phi(x) = \mathbb{P}(\tau_O^x < \infty)$, $x \in \mathbb{R}^N$. *We have to prove that ϕ is identically 1.*

Using the OU semigroup (P_t) we first check that

$$P_r\phi(x) \le \phi(x), \quad r \ge 0, \quad x \in \mathbb{R}^N. \tag{6.12}$$

This is a known fact. We briefly recall the proof for the sake of completeness. Let us fix $x \in \mathbb{R}^N$ and $r > 0$ and note that ϕ is a Borel and bounded function. Since $\mathbb{P}(X_{t+r}^x \in O, \ \text{for some } t \ge 0) \le \mathbb{P}(X_t^x \in O, \ \text{for some } t \ge 0) = \phi(x)$, we get (6.12) by the Markov property:

$$\begin{aligned}\mathbb{P}(X_{t+r}^x \in O, \ \text{for some } t \ge 0) &= \mathbb{E}[\mathbb{E}[1_{\{X_{t+r}^x \in O, \text{ for some } t \ge 0\}} \setminus \mathcal{F}_r]] \\ &= \mathbb{E}[\phi(X_r^x)] = P_r\phi(x).\end{aligned}$$

Now take any decreasing sequence (r_n) of positive numbers converging to 0, i.e., $r_n \downarrow 0$. We have $\{X_t^x \in O, \ \text{for some } t \ge 0\} = \cup_{n\ge1}\{X_{t+r_n}^x \in O, \ \text{for some } t \ge 0\}$ (increasing union) and so $\mathbb{P}(X_{t+r_n}^x \in O, \ \text{for some } t \ge 0) = P_{r_n}\phi(x) \uparrow \phi(x)$. Hence

$$P_s\phi(x) \uparrow \phi(x), \quad \text{as } s \to 0^+, x \in \mathbb{R}^N. \tag{6.13}$$

Since $\phi \ge 0$, properties (6.12) and (6.13) say that ϕ is an excessive function.

Let us fix $s > 0$ and introduce the non-negative function $f_s = \frac{(f - P_s \phi)}{s}$. We have

$$0 \leq U f_s(x) = \frac{1}{s} \int_0^s P_t \phi(x) dt < \infty, \quad x \in \mathbb{R}^N. \tag{6.14}$$

Indeed, for any $T > s$,

$$\begin{aligned} 0 \leq \frac{1}{s} \int_0^T P_t(\phi - P_s \phi)(x) dt &= \frac{1}{s} \int_0^T P_t \phi(x) dt - \frac{1}{s} \int_0^T P_{t+s} \phi(x) dt \\ = \frac{1}{s} \int_0^T P_t \phi(x) dt - \frac{1}{s} \int_s^{T+s} P_t \phi(x) dt &= \frac{1}{s} \int_0^s P_t \phi(x) dt - \frac{1}{s} \int_T^{T+s} P_t \phi(x) dt \\ &\leq \frac{1}{s} \int_0^s P_t \phi(x) dt \end{aligned}$$

(in the last passage we have used that $\phi \geq 0$). Passing to the limit as $T \to \infty$ we get (6.14). Now by the Fubini theorem, for any $s > 0$,

$$\infty > U f_s(x) = \int_0^\infty dt \int_{\mathbb{R}^N} f_s(y) p_t(x, y) dy \geq \int_{\mathbb{R}^N} f_s(y) \Big(\int_1^\infty p_t(x, y) dt \Big) dy.$$

Since we know (6.11) we deduce that $f_s = 0$, a.e. on $\mathbb{R}^N$. This means that, for any $s \geq 0$,

$$\phi(x) = P_s \phi(x), \quad \text{for any } x \in \mathbb{R}^N \quad \text{a.e..} \tag{6.15}$$

It follows that, for any $t > 0$,

$$P_t \phi(x) = P_t(P_s \phi)(x) = P_s(P_t \phi)(x), \quad s \geq 0, \tag{6.16}$$

holds, for any $x \in \mathbb{R}^N$ (not only *a.e.*). Thus, for any $t > 0$, $P_t \phi$ is a bounded harmonic function for (P_t). By hypothesis (HR) and Theorem 3.1 in [14] we deduce that $P_t \phi \equiv c_t$ for some constant c_t.

Since ϕ is excessive we know that $P_t \phi(x) \uparrow \phi(x)$ as $t \to 0^+$, $x \in \mathbb{R}^N$. It follows that $c_t \uparrow c_0$ and $\phi \equiv c_0$. Take $z \in O$. We have $\phi(z) = 1$. Hence ϕ is identically 1 and the proof is complete. □