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## Value function computation in fuzzy models by differential evolution

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| **Corresponding Author:** | Maria Letizia Guerra<br>Univeristy of Bologna<br>Bologna, ITALY |
| **Corresponding Author Secondary Information:** | |
| **Corresponding Author's Institution:** | Univeristy of Bologna |
| **Corresponding Author's Secondary Institution:** | |
| **First Author:** | Maria Letizia Guerra |
| **First Author Secondary Information:** | |
| **Order of Authors:** | Maria Letizia Guerra |
| | Laerte Sorini |
| | Luciano Stefanini |
| **Order of Authors Secondary Information:** | |
| **Author Comments:** | Dear Editor,<br>here within enclosed is our paper<br><br>Value function computation in fuzzy models by differential evolution<br><br>for consideration to be published.<br><br>The manuscript is the authors' original work and has not been published nor has it been submitted simultaneously elsewhere.<br><br>The main contribution of the paper is the evaluation of the importance of a correct computation of the extension principle when working with applications involving fuzzy numbers.<br><br>Best regards,<br><br>Prof. Maria Letizia Guerra, corresponding author |

# Value function computation in fuzzy models by differential evolution

Maria Letizia Guerra*, Laerte Sorini and Luciano Stefanini†

**Abstract**

In this paper we show that the possibilistic mean values produce computation results that may differ in a non trivial may from those obtained with the fuzzy extension principle. The evidence is carried out by comparing some examples derived from several models in finance and economics.

## 1 Introduction

Many models in social sciences are obtained with a strong probabilistic theoretical basis but they can not solve all the uncertainty sources. We believe that an efficient way to treat uncertainty is the use of fuzzy numbers because they are a family of graduated intervals of possibilities and they can so represent the imprecision about some parameters or variables of the model. The correct way to obtain the fuzzy version of a model, preserving its probabilistic nature, is the extension principle.

In particular, in many economic applications, the computation of the possibilistic mean value is a central issue of the problem and its congruent use is based on the fuzzy extension principle that avoids problems of lack of congruence and feasibility of the solutions.

In order to show the primary importance of a correct use of the extension principle we consider some examples as the research of the value function of an option; due to the fluctuations of the financial market, input data cannot be considered to be precise and fuzzy numbers represent a good way to model variables such as risk free interest rate or volatility also when the option is not a plain vanilla but an Asian one (see [20] for details).

We compare the results with those obtained in [2] where a formula for fuzzy option values that involves the possibilistic mean value and variance of fuzzy numbers (introduced in [1]) is applied. A second analysis is carried out with the models presented in [14], [15] and [18].

*Department of Mathematics, University of Bologna, Italy; mletizia.guerra@unibo.it
†Department of Economics, Society and Politics, University of Urbino, Italy; laerte.sorini@uniurb.it, luciano.stefanini@uniurb.it

The paper is organized as follows: in second section we approach the critical aspects connected with the application of the extension principle and we describe the differential evolution optimization method. Before the last section devoted to conclusions, in the third section we study some computational experiments in order to prove the relevance of the correct application of the extension principle in some economic models.

## 2 Differential Evolution algorithms for fuzzy arithmetic

In many applications the computation of fuzzy-valued functions is the central issue. A rigorous methodology is required in order to avoid problems of lack of congruence and feasibility of the solutions.

If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and we are interested in its fuzzy extension $f : \mathcal{F} \rightarrow \mathcal{F}$, it can be obtained through the extension principle:

$$[f(u)]_\alpha = [\min \{f(x) \mid x \in [u]_\alpha\}, \max \{f(x) \mid x \in [u]_\alpha\}] \tag{1}$$

and in general it follows that:

$$E(f(u)) \neq f(E(u)) \qquad \sigma(f(u)) \neq f(\sigma(u))$$

and

$$\overline{M}(f(u)) \neq f\left(\overline{M}(u)\right)$$

where the equalities hold only when the function $f$ is affine or linear but in most part of applications based on the introduction of the extension principle it is not true.

The error can be not negligible on the computed values but also on the core interval.

Some simplifications can avoid the massive computation of min and max but the critical aspect is about the shape of the fuzzy number. In fact, when the information about the shape of the fuzzy numbers is lost before the application of the extension principle, the consequence is the lack of one of the most relevant factor in the uncertainty propagation from the parameters to the value function.

If $u_k = (u_{k,i}^-, \delta u_{k,i}^-, u_{k,i}^+, \delta u_{k,i}^+)_{i=0,1,\dots,N}$ are the LU-fuzzy representations of the $n$ input quantities and $v = (v_i^-, \delta v_i^-, v_i^+, \delta v_i^+)_{i=0,1,\dots,N}$, the $\alpha-cuts$ of $v$ are obtained by solving (1).

For each $\alpha = \alpha_i$, $i = 0, 1, ..., N$ the min{} and the max{} can occur either at a point whose components $x_{k,i}$ are internal to the corresponding intervals $[u_{k,i}^-, u_{k,i}^+]$ or are coincident with one of the extremal values; denote by $\widehat{x}_i^- = (\widehat{x}_{1,i}^-, ..., \widehat{x}_{n,i}^-)$ and $\widehat{x}_i^+ = (\widehat{x}_{1,i}^+, ..., \widehat{x}_{n,i}^+)$ the points where the min and the max take place; then $v_i^- = f(\widehat{x}_{1,i}^-, ..., \widehat{x}_{n,i}^-)$ and $v_i^+ = f(\widehat{x}_{1,i}^+, ..., \widehat{x}_{n,i}^+)$ and the slopes

$\delta v_i^-$, $\delta v_i^+$ are computed (as $f$ is differentiable) by

$$\begin{aligned}
\delta v_i^- &= \sum_{\substack{k=1 \\ \widehat{x}_{k,i}^- = u_{k,i}^-}}^{n} \frac{\partial f(\widehat{x}_{1,i}^-, ..., \widehat{x}_{n,i}^-)}{\partial x_k} \delta u_{k,i}^- \\
&\quad + \sum_{\substack{k=1 \\ \widehat{x}_{k,i}^- = u_{k,i}^+}}^{n} \frac{\partial f(\widehat{x}_{1,i}^-, ..., \widehat{x}_{n,i}^-)}{\partial x_k} \delta u_{k,i}^+ \\
\delta v_i^+ &= \sum_{\substack{k=1 \\ \widehat{x}_{k,i}^+ = u_{k,i}^-}}^{n} \frac{\partial f(\widehat{x}_{1,i}^+, ..., \widehat{x}_{n,i}^+)}{\partial x_k} \delta u_{k,i}^- \\
&\quad + \sum_{\substack{k=1 \\ \widehat{x}_{k,i}^+ = u_{k,i}^+}}^{n} \frac{\partial f(\widehat{x}_{1,i}^+, ..., \widehat{x}_{n,i}^+)}{\partial x_k} \delta u_{k,i}^+ .
\end{aligned} \tag{2}$$

We adopt an algorithmic approach to describe the application of differential evolution methods to calculate the fuzzy extension of multivariable function, associated to the LU representation.

Let $v = f(u_1, u_2, ..., u_n)$ denote the fuzzy extension of a continuous function $f$ in $n$ variables; it is well known that the fuzzy extension of $f$ to normal upper semicontinuous fuzzy intervals (with compact support) has the level-cutting commutative property, i.e. the $\alpha-cuts$ $[v_\alpha^-, v_\alpha^+]$ of $v$ are the images of the $\alpha-cuts$ of $(u_1, u_2, ..., u_n)$ and are obtained by solving the box-constrained optimization problems

$$(EP)_\alpha : \begin{cases} v_\alpha^- = \min \left\{ f(x_1, x_2, ..., x_n) | x_k \in [u_{k,\alpha}^-, u_{k,\alpha}^+], \ k = 1, 2, ..., n \right\} \\ v_\alpha^+ = \max \left\{ f(x_1, x_2, ..., x_n) | x_k \in [u_{k,\alpha}^-, u_{k,\alpha}^+], \ k = 1, 2, ..., n \right\} . \end{cases} \tag{3}$$

Except for simple elementary cases for which the optimization problems above can be solved analytically, the direct application of *(EP)* is difficult and computationally expensive.

The main and possibly critical steps in the algorithm above is the solution of the optimization problems (1), depending on the dimension $n$ of the solution space and on the possibility of many local optimal points (if the min and the max points are not located with sufficient precision, an underestimation of the fuzziness may be produced and the propagation of the errors may grow without control).

A careful exploitation of the min and max problems can produce efficient solution methods, all existing general methods (in cases where the structure of the min and max subproblems do not suggest specific efficient procedures) try to take advantage of the nested structure of the box-constraints for different values of $\alpha$.

We suggest here a relatively simple procedure, based on the differential evolution ($DE$) method of Storn and Price (detailed in [12]) adapted in order to take into account both the nested property of $\alpha - cuts$ and the min and max problems over the same domains.

The general idea of $DE$ to find $min$ or $max$ of $\{f(x_1,...,x_n)|(x_1,...,x_n)\in\mathbb{A}\subset\mathbb{R}^n\}$ is simple. Start with an initial "population" $(x_1,...,x_n)^{(1)},...,(x_1,...,x_n)^{(p)}\in\mathbb{A}$ of $p$ feasible points; at each iteration obtain a new set of points by recombining randomly the individuals of the current population and by selecting the best generated elements to continue in the next generation.

If the extension algorithm is used in combinations with the LU-fuzzy representation for differentiable membership functions (and differentiable extended functions), then the number $N+1$ of $\alpha-cuts$ (and correspondingly of min/max optimizations) can be sufficiently small. Many experiments suggest that $N=10$ is in general quite sufficient to obtain good approximations.

The idea of $DE$ to find $min$ or $max$ of $\{f(x_1,...,x_n)|(x_1,...,x_n)\in\mathbb{A}\subset\mathbb{R}^n\}$ is simple: start with an initial "population" $x^{(1)}=(x_1,...,x_n)^{(1)}$, ..., $x^{(p)}=(x_1,...,x_n)^{(p)}\in\mathbb{A}$ of $p$ feasible points for each generation (i.e. for each iteration) to obtain a new set of points by recombining randomly the individuals of the current population and by selecting the best generated elements to continue in the next generation. The initial population is chosen randomly and should try to cover uniformly the entire parameter space.

Denote by $x^{(k,g)}$ the $k-$th vector of the population at iteration (generation) $g$ and by $x_j^{(k,g)}$ its $j-$th component ($j=1,...,n$).

At each iteration, the method generates a set of candidate points $y^{(k,g)}$ to substitute the elements $x^{(k,g)}$ of the current population, if $y^{(k,g)}$ is better.

To generate $y^{(k,g)}$ two operations are applied: recombination and crossover.

A typical recombination operates on a single component $j\in\{1,...,n\}$ and generates a new perturbed vector of the form $v_j^{(k,g)}=x_j^{(r,g)}+\gamma[x_j^{(s,g)}-x_j^{(t,g)}]$, where $r,s,t\in\{1,2,...,p\}$ are chosen randomly and $\gamma\in]0,2]$ is a constant (eventually chosen randomly for the current iteration) that controls the amplification of the variation.

The potential diversity of the population is controlled by a crossover operator, that construct the candidate $y^{(k,g)}$ by crossing randomly the components of the perturbed vector $v_j^{(k,g)}$ and the old vector $x_j^{(k,g)}$:

$$y_j^{(k,g)}=\begin{cases} v_j^{(k,g)} & \text{if } \ j\in\{j_1,j_2,...,j_h\} \\ x_j^{(k,g)} & \text{if } \ j\notin\{j_1,j_2,...,j_h\}\end{cases}$$

where $h$ is a random integer between 0 and $n$ (it is 0 with probability $q$) and $j_1,j_2,...,j_h$ are random components if $h$ is not 0; so, the components of each individual of the current population are modified to $y_j^{(k,g)}$ by a given probability $q$. Typical values are $\gamma\in[0.2,0.95]$, $q\in[0.7,1.0]$ and $p\geq 5n$ (the higher $p$, the lower $\gamma$).

The candidate $y^{(k,g)}$ is then compared to the existing $x^{(k,g)}$ by evaluating the objective function at $y^{(k,g)}$ : if $f(y^{(k,g)})$ is better than $f(x^{(k,g)})$ then $y^{(k,g)}$

substitutes $x^{(k,g)}$ in the new generation $g+1$, otherwise $x^{(k,g)}$ is retained.

To take into account the particular nature of our problem, we modify the basic procedure and examine two different strategies.

Let $[u_{k,i}^{-}, u_{k,i}^{+}]$, $k = 1, 2, ..., n$ and $f : \mathbb{R}^n \to \mathbb{R}$ be given; we have to find $v_i^{-}$ and $v_i^{+}$ according to (1) for $i = 0, 1, ..., N$. The slope parameters $\delta v_i^{-}$, $\delta v_i^{+}$ are computed in (2).

The first strategy is implemented in algorithm 1. Function $ran(0,1)$ generates a random uniform number in [0,1].

SPDE (Single Population DE procedure): start with the $(\alpha = 1) - cut$ back to the $(\alpha = 0) - cut$ so that the optimal solutions at a given level can be inserted into the "starting" populations of lower levels; use two distinct populations and perform the recombinations such that, during generations, one of the populations specializes to find the minimum and the other to find the maximum.

**Algorithm 1:** (Frame of SPDE).

Choose $p \approx 10n$, $g_{\max} \approx 500$, $q$ and $\gamma$.<br>
Select $(x_1^{(l)}, ..., x_n^{(l)})$, $x_k^{(l)} \in [u_{k,N}^{-}, u_{k,N}^{+}]$<br>
$\forall k$, $l = 1, ..., 2p$ evaluate $y^{(l)} = f(x_1^{(l)}, ..., x_n^{(l)})$<br>
for $i = N, N-1, ..., 0$<br>
  for $g = 1, 2, ..., g_{\max}$<br>
  (up to $g_{\max}$ generations or other stopping rule)<br>
    for $l = 1, 2, ..., 2p$<br>
      select (randomly) $r, s, t \in \{1, 2, ..., 2p\}$<br>
      and $j^* \in \{1, 2, ..., n\}$<br>
      for $j = 1, 2, ..., n$<br>
        if ($j = j^*$ or $ran(0,1) < q$)<br>
          then $x_j' = x_j^{(r)} + \gamma[x_j^{(s)} - x_j^{(t)}]$<br>
          else $x_j' = x_j^{(l)}$<br>
        ensure that $u_{j,i}^{-} \le x_j' \le u_{j,i}^{+}$<br>
      end<br>
      evaluate $y = f(x_1', ..., x_n')$<br>
      if $l \le p$ and $y < y^{(l)}$ then<br>
         substitute $(x_1, ..., x_n)^{(l)}$ with $(x_1', ..., x_n')$<br>
      if $l > p$ and $y > y^{(l)}$ then<br>
         substitute $(x_1, ..., x_n)^{(l)}$ with $(x_1', ..., x_n')$<br>
    end<br>
  end<br>
  $v_i^{-} = y^{(l^*)} = \min\left\{y^{(l)} | l = 1, 2, ..., p\right\}$<br>
  $(\widehat{x}_{1,i}^{-}, ..., \widehat{x}_{n,i}^{-}) = (x_1, ..., x_n)^{(l^*)}$<br>
  $v_i^{+} = y^{(l^{**})} = \max\left\{y^{(p+l)} | l = 1, 2, ..., p\right\}$<br>
  $(\widehat{x}_{1,i}^{+}, ..., \widehat{x}_{n,i}^{+}) = (x_1, ..., x_n)^{(l^{**})}$<br>
  if $i < N$<br>
    select $(x_1^{(l)}, ..., x_n^{(l)})$, $x_k^{(l)} \in [u_{k,i-1}^{-}, u_{k,i-1}^{+}]$<br>
    $\forall k$, $l = 1, ..., 2p$

including $(\widehat{x}_{1,i}^{-},...,\widehat{x}_{n,i}^{-})$ and $(\widehat{x}_{1,i}^{+},...,\widehat{x}_{n,i}^{+})$
endif
end

The second strategy is implemented in algorithm 2.

MPDE (Multi Populations DE procedure): use $2(N+1)$ populations to solve simultaneously all the box-constrained problems (1); $N+1$ populations specialize for the min and the others for the max and the current best solution for level $\alpha_i$ is valid also for levels $\alpha_0, ..., \alpha_{i-1}$.

**Algorithm 2:** (Frame of MPDE).

Choose $p \approx 5n$, $g_{\max} \approx 500$, $q$ and $\gamma$.
Select $(x_1^{(l,i)}, ..., x_n^{(l,i)})$, $x_k^{(l,i)} \in [u_{k,i}^{-}, u_{k,i}^{+}]$
$\forall k$, $l = 1, ..., 2p$, $i = 0, 1, ..., N$
let $y^{(l,i)} = f(x_1^{(l,i)}, ..., x_n^{(l,i)})$
let $v_i^{-} = \min\left\{y^{(l,j)} | j = 0, ..., i, \forall l\right\}$
let $v_i^{+} = \max\left\{y^{(l,j)} | j = 0, ..., i, \forall l\right\}$
let $\widehat{x}_i^{-}, \widehat{x}_i^{+} \in R^n$ the points where $v_i^{-}$, $v_i^{+}$ are taken
for $g = 1, 2, ..., g_{\max}$
  (up to $g_{\max}$ generations or other stopping rule)
  for $i = N, N-1, ..., 0$
    for $l = 1, 2, ..., p$
      select (randomly) $r, s, t \in \{1, 2, ..., p\}$
      and $k^* \in \{1, 2, ..., n\}$
      for $k = 1, 2, ..., n$
        if ($k = k^*$ or $ran(0,1) < q$) then
          $x_k' = x_k^{(r,i)} + \gamma[x_k^{(s,i)} - x_k^{(t,i)}]$
          $x_k'' = x_k^{(p+r,i)} + \gamma[x_k^{(p+s,i)} - x_k^{(p+t,i)}]$
          ensure $u_{k,i}^{-} \leq x_k', x_k'' \leq u_{k,i}^{+}$
        else
          $x_k' = x_k^{(l,i)}$, $x_k'' = x_k^{(p+l,i)}$
        endif
      end
      let $y' = f(x_1', ..., x_n')$ and $y'' = f(x_1'', ..., x_n'')$;
      if $y' < y^{(l,i)}$ (population for min)
        substitute $(x_1, ..., x_n)^{(l,i)}$ with $(x_1', ..., x_n')$
      if $y'' > y^{(p+l,i)}$ (population for max)
        substitute $(x_1, ..., x_n)^{(p+l,i)}$ with $(x_1'', ..., x_n'')$
      if $y'$ or $y''$ are better
        update values $\{v_j^{-}, v_j^{+}, \widehat{x}_j^{-}, \widehat{x}_j^{+} | j = 0, ..., i\}$
      endif
    end
  end
end

In our case, as we have simple box-constraints, it is easy to produce feasible starting populations, as we have to generate random numbers $x_j^{(k,0)}$ between the lower $u_{j,i}^-$ and the upper $u_{j,i}^+$ values.

During the iterations, we use a variant of the method above, where the $y^{(k,g)}$ are progressively forced to be feasible or with small infeasibilities and a penalty is assigned to infeasible values:

*(i)* modify $y_j^{(k,g)}$ to fit $[u_{j,i}^- - \frac{\varepsilon}{g^2}, u_{j,i}^+ + \frac{\varepsilon}{g^2}]$, $j = 1, 2, ..., n$ with small $\varepsilon \sim 10^{-2}(u_{j,i}^+ - u_{j,i}^-)$, so that the eventual infeasibilities decrease rapidly during the generation process;

*(ii)* if the candidate point $y^{(k,g)}$ is infeasible and has a value $f(y^{(k,g)})$ better than the current best feasible value $f(x^{(best,g)})$ then a penalty is added and the value of $y^{(k,g)}$ is elevated to $f(x^{(best,g)})+\varepsilon'$ (for the min problems) or reduced to $f(x^{(best,g)})-\varepsilon'$ (for the max problem), being $\varepsilon' \sim 10^{-3}$ a small positive number.

To decide that a solution is found, we use the following simple rule: choose a fixed tolerance *tol* $\sim 10^{-3}, 10^{-4}$ and a number $\widehat{g} \sim 20, 30$ of generations; if for $\widehat{g}$ subsequent iterations all the values $v_i^-$ and $v_i^+$ are changed less than *tol*, then the procedure stops and the found solution is assumed to be optimal. In any case, no more than 500 iterations are performed (but this limit was never reached during the computations). More details can be found in [12].

## 3 Evidence from fuzzy valued functions

The Black and Scholes formula for a call option $C_0(S, r, \sigma, K, T) = S_0 N(d_1) - Ke^{-rT} N(d_2)$ is expressed as a function of the underlying stock price process $\{S_t\}_{t\geq 0}$ satisfying

$$dS_t = rS_t dt + \sigma S_t dW_t$$

and of the constant risk-free interest rate $r$, the constant volatility $\sigma$, the constant strike price $K$ and the constant time to maturity $T$, and where

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \text{ and } d_2 = d_1 - \sigma\sqrt{T}$$

and $N(x)$ is the cumulated normal distribution function

$$N(x) = \int_{-\infty}^{x} \Phi(t)\, dt \text{ with } \Phi(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}.$$

The Black-Scholes option pricing formula was extended by Merton to the case of dividends-paying stocks as:

$$C_0(S, r, \sigma, K, T) = S_0 e^{-\delta T} N(d_1) - Ke^{-rT} N(d_2) \quad (4)$$

where

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \text{ and } d_2 = d_1 - \sigma\sqrt{T}$$

We now turn our attention to the determination of the fuzzy version of the Black and Scholes formula that can be obtained by the direct application of the extension principle.

We search for a fuzzy version of the call option price by modelling the underlying stock price, the volatility and the risk-free interest rate as fuzzy numbers, incorporating uncertainty in the Black and Scholes model. The fuzzy variables become:

$$S = \left(S_i^-, \delta S_i^-, S_i^+, \delta S_i^+\right)_{i=0,1,\ldots,N}$$

$$\sigma = \left(\sigma_i^-, \delta\sigma_i^-, \sigma_i^+, \delta\sigma_i^+\right)_{i=0,1,\ldots,N}$$

$$r = \left(r_i^-, \delta r_i^-, r_i^+, \delta r_i^+\right)_{i=0,1,\ldots,N}$$

and the fuzzy call option price takes the form:

$$C = \left(C_i^-, \delta C_i^-, C_i^+, \delta C_i^+\right)_{i=0,1,\ldots,N}$$

It then follows that

$$C_i^- = C\left(S_i^-, \sigma_i^-, r_i^-, K, T\right) \quad (5)$$

$$C_i^+ = C\left(S_i^+, \sigma_i^+, r_i^+, K, T\right) \quad (6)$$

The fact that the function $f(x_1, x_2, ..., x_n)$ in (1) has the sufficiently simpler form, implies that the analytical expressions for $v_i^-$, $\delta v_i^-$, $v_i^+$ and $\delta v_i^+$ can be explicitly obtained. The preliminary studies about the fuzzy option pricing are in [**?**] and [**?**], a deeper investigation can be found in [8].

By comparing our proposed methodology with some important results we are able to focus how relevant is the correct numerical application of the extension principle.

Yoshida in [17] introduces fuzzy logic for the Black and Scholes stochastic model by deriving the expected prices of European options for triangle-type shape function. He calls fuzzy factor of the model, the fuzziness $c$ of the volatility $\sigma$ because it is recognized as the most difficult variable to estimate.

Yoshida computes the expected price (equal to 0.774283) for a European call option with time to maturity $T = 0.5$, strike price $K = 35$, interest rate $r = 5\%$, volatility $\sigma = 25\%$ with $c = 0.05$ and underlying stock price $S = 30$. The LU methodology in the same case produces a crisp call option price equal to 0.7694 with $\alpha = 0$ support $[0.4459, 1.1298]$ (Yoshida does not report his value). When we introduce uncertainty also in the interest rate and in the underlying we obtain a crisp value equal to 0.7655 and a fuzzy call price that has a nonlinear and asymmetric shape that can be a way to obtain more information about the call option price for example about the possibility to have higher or smaller values, as it is shown in [8].

In fuzzy calculus, as it is well known, the linear shape of fuzzy numbers looses when also simple arithmetic operations are applied.

Zmeskal in [21] formulates a fuzzy-stochastic model that can not be solved analytically but as a non-linear programming problem with the input data that are linear fuzzy numbers. The empirical application in his work is devoted to the finding of the fuzzy firm value.

Thiagarajah, Appadoo, Thavaneswaran (TAT) in [14] assume that the expiry date and exercise price are always known and are nonfuzzy. They model the uncertainty of interest rate, volatility, and stock price using adaptive fuzzy numbers. They replace the fuzzy interest rate, the fuzzy stock price and the fuzzy volatility by possibilistic mean value in the fuzzy Black–Scholes formula. The price $S$ is an adaptive fuzzy number of the form $\widetilde{S}_0 = (S_1, S_2, S_3, S_4)_n$ where $n > 0$ and for $n = 1$ the fuzzy number is a trapezoidal one, for $n > 1$ or $n < 1$ the fuzzy number is nomore linear and it can be a concentration or a dilation of the trapezoidal fuzzy number; in a similar way they model the interest rate $\widetilde{r} = (r_1, r_2, r_3, r_4)_n$ and the volatility $\widetilde{\sigma} = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)_n$. They consider the usual Black–Scholes formula with exercise price $K$ and with the variables as adaptive fuzzy numbers also in the expressions of $d_1$ and $d_2$:

$$C_0(S, r, \sigma, K, T) = \widetilde{S}_0 e^{-\delta T} N(d_1) - K e^{-\widetilde{r}T} N(d_2) \tag{7}$$

Given the variables $\widetilde{S}_t = [158, 160, 162, 164]_n$, $\widetilde{r} = [0.03, 0.04, 0.05, 0.06]_n$, $\sigma = [0.29, 0.31, 0.33, 0.35]_n$ and $K = 140$. they obtain the following support for the call option price: $[34.343 + 2.8275\alpha^{\frac{1}{n}}, 42.74 - 2.7703\alpha^{\frac{1}{n}}]$ for various level of $n$ and $\alpha$. In the following table we compare our methodology in the computation of the call option price:

| | TAT | Lower Upper |
|---|---|---|
| $\alpha = 0.8$ | $[36.60, 40.52]$ | $[37.31, 39.23]$ |
| $\alpha = 0.9$ | $[36.89, 40.25]$ | $[38.12, 39.11]$ |

As it is clear with a low level of uncertainty the support stays too large in tha TAT approach while it is reduced when our methodology is applied.

Wu in [15] and [16] fully justifies the use of fuzzy numbers to model uncertainty in option pricing and he applies the Black and Scholes formula to find the fuzzy call option price when three key variables are triangular fuzzy numbers.

At first, we test the LU-representation in the same simulated example as in [15]: the valuation of a call option with a three months time to maturity $T = 0.25$, a strike price $K = 30$ and the interest rate $r$, the underlying stock price $S$ and the volatility $\sigma$ are modelled as triangular fuzzy numbers having respectively the following supports $[4.8\%, 5.2\%]$, $[32, 34]$, $[8\%, 12\%]$.

Consequently, we price a call option that cannot be *out of the money* (so that will be always exercised) because the crisp strike price is always smaller than the smallest value of the fuzzy underlying stock price.

The first consideration attains the fact that the LU approach is computationally simpler and most of all it does not overestimate the fuzziness as it is shown in the next table:

| | WU | Lower Upper |
|---|---|---|
| $\alpha = 0.98$ | $[3.3092, 3.4534]$ | $[3.3611, 3.4016]$ |
| $\alpha = 0.99$ | $[3.3453, 3.4174]$ | $[3.3712, 3.3914]$ |

where the level cut intervals 0.98 and 0.99 are significantly smaller than intervals estimated in [15]. The same behavior is even more evident for level cut intervals with a higher degree of uncertainty as it is shown in the next table:

| | WU | Lower Upper |
|---|---|---|
| $\alpha = 0.70$ | $[3.2231, 3.5754]$ | $[3.2012, 3.5564]$ |
| $\alpha = 0.71$ | $[3.2164, 3.5501]$ | $[3.1987, 3.5321]$ |

## 4 Conclusions

The paper shows that the congruent use of possibilistic mean values with the fuzzy extension principle must be pursued in order to obtain results that can well highlight the positive aspects of uncertainty modeling through fuzzy numbers and most of all the correct use of the principle avoids the overestimation effect producing more readable results also for practitioners.

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