

A Brezis–Nirenberg splitting approach for nonlocal fractional equations

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To access the original version see: <https://doi.org/10.1016/j.na.2014.10.025>

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A BREZIS-NIRENBERG SPLITTING APPROACH FOR NONLOCAL FRACTIONAL EQUATIONS

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ABSTRACT. In this paper we consider problems modeled by the following nonlocal fractional equation

$$\begin{cases} (-\Delta)^s u + a(x)u = \mu f(u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $s \in (0, 1)$ is fixed, Ω is an open bounded subset of \mathbb{R}^n , $n > 2s$, with Lipschitz boundary, $(-\Delta)^s$ is the fractional Laplace operator and μ is a real parameter.

Under two different types of conditions on the functions a and f , by using a famous critical point theorem in the presence of splitting established by Brezis and Nirenberg, we obtain the existence of at least two nontrivial weak solutions for our problem.

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1. INTRODUCTION

1.1. Setting of the problem. In the last years an always increasing interest has been shown towards nonlocal fractional problems, both for their intriguing structure, which motivates academic research, and for their presence in many models coming from real-world applications.

In this paper we are interested in nonlocal problems depending on parameters. This kind of equations models a wide class of problems arising in applications and, of course, in these cases the parameters have a physical interpretation. The interest in considering problems with parameters is, at least, twofold: on one hand, finding solutions, and, on the other hand, studying how these solutions depend on the parameters.

Key words and phrases. Fractional Laplacian, nonlocal problems, variational methods, critical point theory, integrodifferential operators.

2010 AMS Subject Classification: Primary: 49J35, 35A15, 35S15; Secondary: 47G20, 45G05.

Both the authors were supported by the INDAM-GNAMPA Project 2014 *Proprietà geometriche e analitiche per problemi non-locali*. The second author was supported by the MIUR National Research Project *Variational and Topological Methods in the Study of Nonlinear Phenomena* and by the ERC grant ϵ (*Elliptic Pde's and Symmetry of Interfaces and Layers for Odd Nonlinearities*).

Precisely, here we study the following nonlocal equation

$$(1.1) \quad \begin{cases} -\mathcal{L}_K u + a(x)u = \mu f(u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $s \in (0, 1)$, Ω is an open bounded subset of \mathbb{R}^n , $n > 2s$, with smooth boundary, μ is a real parameter, $a : \Omega \rightarrow \mathbb{R}$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ are two functions verifying the conditions stated in the sequel and \mathcal{L}_K is the integrodifferential operator defined as follows

$$(1.2) \quad \mathcal{L}_K u(x) := \int_{\mathbb{R}^n} \left(u(x+y) + u(x-y) - 2u(x) \right) K(y) dy, \quad x \in \mathbb{R}^n,$$

with the kernel $K : \mathbb{R}^n \setminus \{0\} \rightarrow (0, +\infty)$ such that

$$(1.3) \quad mK \in L^1(\mathbb{R}^n), \text{ where } m(x) = \min\{|x|^{-2s}, 1\};$$

$$(1.4) \quad \text{there exists } \theta > 0 \text{ such that } K(x) \geq \theta|x|^{-(n+2s)} \text{ for any } x \in \mathbb{R}^n \setminus \{0\}.$$

A model for K is given by the singular kernel $K(x) = |x|^{-(n+2s)}$ which gives rise to the fractional Laplace operator $-(-\Delta)^s$, defined, up to normalization factors, as

$$(1.5) \quad -(-\Delta)^s u(x) := \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} dy, \quad x \in \mathbb{R}^n.$$

The operator \mathcal{L}_K has a nonlocal nature: this is the reason why the Dirichlet datum in (1.1) is given in $\mathbb{R}^n \setminus \Omega$ and not simply on the boundary $\partial\Omega$, as it happens in the classical case of Laplacian equations.

Aim of this paper is to get the existence of multiple weak solutions for problem (1.1). By a weak solution for (1.1), we mean a function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$(1.6) \quad \begin{cases} \int_{\mathbb{R}^n \times \mathbb{R}^n} (u(x) - u(y))(\varphi(x) - \varphi(y))K(x-y)dx dy + \int_{\Omega} a(x)u(x)\varphi(x)dx \\ \quad \quad \quad = \mu \int_{\Omega} f(u(x))\varphi(x)dx \quad \forall \varphi \in X_0 \\ u \in X_0. \end{cases}$$

Here and in the sequel we set

$$X_0 := \{g \in X : g = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\},$$

where the functional space X denotes the linear space of Lebesgue measurable functions from \mathbb{R}^n to \mathbb{R} such that the restriction to Ω of any function g in X belongs to $L^2(\Omega)$ and

the map $(x, y) \mapsto (g(x) - g(y))\sqrt{K(x-y)}$ is in $L^2((\mathbb{R}^n \times \mathbb{R}^n) \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega), dxdy)$, with $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$.

1.2. Two multiplicity results. In this paper we prove two multiplicity results for problem (1.1). Recently, in the literature appeared some results on the existence of multiple solutions for nonlocal equations, see for instance, [3, 15, 16, 21] and the references therein.

Here we consider different kinds of conditions on the data. First of all, we assume that the function $a : \Omega \rightarrow \mathbb{R}$ is such that

$$(1.7) \quad a \in L^\infty(\Omega);$$

$$(1.8) \quad a \geq 0 \text{ a.e. } x \in \Omega;$$

while $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the following assumptions

$$(1.9) \quad f \in C^1(\mathbb{R}) \text{ with } f(0) = f'(0) = 0;$$

$$(1.10) \quad -\infty < \liminf_{|t| \rightarrow +\infty} \frac{f(t)}{t} \leq \limsup_{|t| \rightarrow +\infty} \frac{f(t)}{t} < 0;$$

$$(1.11) \quad \text{there exists } \bar{t} \in \mathbb{R} \text{ such that } F(\bar{t}) > 0,$$

where

$$(1.12) \quad F(t) = \int_0^t f(\tau) d\tau, \quad t \in \mathbb{R}.$$

A model for f is given by the function

$$f(t) = \begin{cases} t^2 - t^3 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0, \end{cases}$$

or, more generally, by

$$f(t) = \begin{cases} t^\alpha - t^\beta & \text{if } t \geq 0 \\ 0 & \text{if } t < 0, \end{cases}$$

with $1 < \alpha < \beta$.

Since $f(0) = 0$ by assumption, of course $u \equiv 0$ is a solution of problem (1.1). In the following result we show that problem (1.1) admits, at least, two nontrivial solutions:

Theorem 1. *Let $s \in (0, 1)$, $n > 2s$, Ω be an open bounded subset of \mathbb{R}^n with continuous boundary. Let $K : \mathbb{R}^n \setminus \{0\} \rightarrow (0, +\infty)$ be a function satisfying (1.3) and (1.4) and let a be a function satisfying (1.7) and (1.8), and f verifying (1.9)–(1.11).*

Then, problem (1.1) admits at least two nontrivial weak solutions, provided $\mu > 0$ is large enough.

If we consider the model case when $-\mathcal{L}_K = (-\Delta)^s$, in the limit case when $s = 1$ we get the Laplace operator $-\Delta$. Hence, the classical counterpart of problem (1.1) is given by

$$(1.13) \quad \begin{cases} -\Delta u + a(x)u = \mu f(u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Problem (1.13) was firstly studied in [6] (see also [11, Theorem 6]), where the authors got the analogous of Theorem 1 for (1.13), thanks to a critical point theorem in the presence of splitting proved along the same paper. Adapting, in a suitable way, this type of arguments to the nonlocal setting, here we prove Theorem 1.

Under the assumptions of Theorem 1 the function a is bounded and non-negative. A natural question is whether or not the result stated in Theorem 1 holds true removing the sign condition on a . A partial answer will be given in the next theorem, where we assume that a is constant and negative in Ω and a multiplicity result for the following problem

$$(1.14) \quad \begin{cases} -\mathcal{L}_K u - \gamma u = f(u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

is provided. Before stating this theorem, we need to consider the eigenvalue problem related to the operator $-\mathcal{L}_K$, that is the following problem

$$(1.15) \quad \begin{cases} -\mathcal{L}_K u = \lambda u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

There exists a non-decreasing sequence of positive eigenvalues λ_k for (1.15), as proved in [26, Proposition 9 and Appendix A], where a spectral theory for general integrodifferential nonlocal operators was developed (see also [22, 28, 29] for further properties of the spectrum of $-\mathcal{L}_K$ and of its eigenfunctions).

Now, we can introduce the assumptions on γ and f , given by:

$$(1.16) \quad 0 < \gamma < \lambda_1,$$

where λ_1 is the first eigenvalue of $-\mathcal{L}_K$ with homogeneous Dirichlet boundary data, while $f : \mathbb{R} \rightarrow \mathbb{R}$ is a function such that

$$(1.17) \quad f \in C(\mathbb{R});$$

$$(1.18) \quad -\infty < \liminf_{|t| \rightarrow \infty} \frac{f(t)}{t} \leq \limsup_{|t| \rightarrow \infty} \frac{f(t)}{t} < \lambda_1 - \gamma;$$

there exist an integer $k \geq 1$ and a constant $\delta > 0$ such that

$$(1.19) \quad \frac{f(t)}{t} \geq \lambda_k - \gamma \text{ for any } 0 < |t| < \delta \text{ and } \limsup_{t \rightarrow 0} \frac{f(t)}{t} < \lambda_{k+1} - \gamma.$$

Conditions (1.18) and (1.19) are, somehow, resonance conditions for the problem. In the literature different types of resonance assumptions have been considered: see, e.g., [4, 7, 10, 13, 14, 17, 18, 19, 33, 34, 35] and references therein, when dealing with elliptic problems.

In this setting our result can be stated as follows:

Theorem 2. *Let $s \in (0, 1)$, $n > 2s$, Ω be an open bounded subset of \mathbb{R}^n with continuous boundary. Let $K : \mathbb{R}^n \setminus \{0\} \rightarrow (0, +\infty)$ be a function satisfying (1.3) and (1.4), let γ be such that (1.16) holds true and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function verifying (1.17)–(1.19).*

Then, problem (1.14) admits at least two nontrivial weak solutions.

As the one of Theorem 1, also the proof of Theorem 2 is based on the critical point theorem in the presence of splitting due to Brezis and Nirenberg in [6]. In both cases we find critical points of the Euler-Lagrange functionals associated with problems (1.1) and (1.14) respectively, showing that their geometric structure and their compactness properties fit with the requirements of [6, Theorem 4].

As for the geometry, the condition required in [6, Theorem 4] is that the functional has a local linking at 0 (see Appendix A). Also, in [6] a crucial assumption is that the functional is bounded from below. More general situations, in the presence of a local linking, were considered in [11], where also the cases of superquadratic or asymptotically quadratic functionals are discussed. Some results obtained in [11] were generalized in [20].

The present paper is organized as follows. In Section 2 we recall some basic definitions and notations useful along this work. Section 3 deals with the multiplicity results for problem (1.1) in the case when $a \geq 0$, while Section 4 is devoted to the case when a is constant and negative. Finally, in Appendix A we recall the abstract critical point theorem we use along this paper in order to get our multiplicity results.

2. BASIC DEFINITIONS AND NOTATIONS

This section is devoted to the notations used along the paper. First of all, we briefly recall some basic definitions related to the functional space X_0 . The reader familiar with this topic may skip this section and go directly to the next one.

The space X_0 is endowed with the norm

$$(2.1) \quad X_0 \ni v \mapsto \|v\|_{X_0} := \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} |v(x) - v(y)|^2 K(x - y) dx dy \right)^{1/2}.$$

Also $(X_0, \|\cdot\|_{X_0})$ is a Hilbert space (for this see [25, Lemma 7]), with scalar product

$$(2.2) \quad \langle u, v \rangle_{X_0} := \int_{\mathbb{R}^n \times \mathbb{R}^n} (u(x) - u(y))(v(x) - v(y)) K(x - y) dx dy.$$

We would remark that the definition of the space X_0 is inspired, but not equivalent, to the one of the fractional Sobolev spaces. Indeed, the usual fractional Sobolev space $H^s(\Omega)$ is endowed with the so-called *Gagliardo norm* (see, for instance [1, 8]) given by

$$(2.3) \quad \|g\|_{H^s(\Omega)} := \|g\|_{L^2(\Omega)} + \left(\int_{\Omega \times \Omega} \frac{|g(x) - g(y)|^2}{|x - y|^{n+2s}} dx dy \right)^{1/2}.$$

It is easy to see that, even in the model case in which $K(x) = |x|^{-(n+2s)}$, the norms in (2.1) and (2.3) are not the same: this makes the space X_0 not equivalent to the usual fractional Sobolev spaces and the classical fractional Sobolev space approach not sufficient for studying our problem from a variational point of view.

For further details on the fractional Sobolev spaces we refer to [8] and to the references therein, while for other details on X and X_0 we refer to [24], where these functional spaces

were introduced, and also to [25, 26, 27], where various properties of these spaces were proved.

With respect to the eigenvalue problem (1.15), we recall that it possesses a divergent sequence of eigenvalues

$$0 < \lambda_1 < \lambda_2 \leq \dots \leq \lambda_k \leq \lambda_{k+1} \leq \dots$$

In the sequel, we will denote by e_k the eigenfunction related to the eigenvalue λ_k , $k \in \mathbb{N}$. From [26, Proposition 9], we know that we can choose $\{e_k\}_k$ normalized in such a way that this sequence provides an orthonormal basis in $L^2(\Omega)$ and an orthogonal basis in X_0 , so that for any $k, i \in \mathbb{N}$ with $k \neq i$

$$(2.4) \quad \langle e_k, e_i \rangle_{X_0} = 0 = \int_{\Omega} e_k(x) e_i(x) dx$$

and

$$(2.5) \quad \|e_k\|_{X_0}^2 = \lambda_k \|e_k\|_{L^2(\Omega)}^2 = \lambda_k.$$

For a complete study of the spectrum of the integrodifferential operator $-\mathcal{L}_K$ we refer to [22, Proposition 2.3], [26, Proposition 9 and Appendix A] and [28, Proposition 4].

Along this paper we look for solutions of the problem (1.6), which represents the Euler-Lagrange equation of the functional $\mathcal{J}_{K,a} : X_0 \rightarrow \mathbb{R}$ defined as

$$(2.6) \quad \begin{aligned} \mathcal{J}_{K,a}(u) := & \frac{1}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(y)|^2 K(x - y) dx dy + \frac{1}{2} \int_{\Omega} a(x) |u(x)|^2 dx \\ & - \mu \int_{\Omega} F(u(x)) dx, \end{aligned}$$

where F is the function defined in (1.12). Note that $\mathcal{J}_{K,a} \in C^1(X_0)$ thanks to the assumptions on a and f and also due to the embedding properties of X_0 into the classical Lebesgue spaces (see [25, Lemmas 6 and 8] and [26, Lemma 9]).

3. A MULTIPLICITY RESULT: THE CASE WHEN $a(x) \geq 0$

In this section we prove the multiplicity result stated in Theorem 1. The proof of this result relies on an abstract critical point theorem in the presence of splitting, due to Brezis and Nirenberg (see [6, Theorem 4]), that we recall in Appendix A for reader's convenience.

Here we consider the case when the function a satisfies conditions (1.7) and (1.8), while f verifies (1.9)–(1.11).

3.1. Some preliminary lemmas. First of all, we need some preliminary results.

Lemma 3. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying conditions (1.9) and (1.11). Then, there exists $\bar{u} \in X_0$ such that*

$$\int_{\Omega} F(\bar{u}(x)) dx > 0.$$

Proof. Fix a point $x_0 \in \Omega$ and choose $\tau > 0$ in such a way that

$$\bar{B}(x_0, \tau) := \{x \in \mathbb{R}^n : |x - x_0| \leq \tau\} \subseteq \Omega,$$

where $|\cdot|$ denotes the usual Euclidean norm in \mathbb{R}^n . Furthermore, let $\bar{t} \in \mathbb{R}$ be as in condition (1.11) and fix $\sigma_0 \in (0, 1)$ for which

$$(3.1) \quad F(\bar{t})\sigma_0^n - (1 - \sigma_0^n) \max_{|t| \leq |\bar{t}|} |F(t)| > 0.$$

Note that this choice is admissible thanks to assumption (1.11).

Let $\bar{u} \in C_0^1(\Omega) \subset X_0$ (see [24, Lemma 5.1]) be such that

$$\bar{u}(x) := \begin{cases} 0 & \text{if } x \in \mathbb{R}^n \setminus B(x_0, \tau) \\ \bar{t} & \text{if } x \in B(x_0, \sigma_0 \tau), \end{cases}$$

and

$$|\bar{u}(x)| \leq |\bar{t}|$$

if $x \in B(x_0, \tau) \setminus B(x_0, \sigma_0 \tau)$.

We claim that

$$(3.2) \quad \int_{\Omega} F(\bar{u}(x)) dx \geq \left[F(\bar{t})\sigma_0^n - (1 - \sigma_0^n) \max_{|t| \leq |\bar{t}|} |F(t)| \right] \omega_n \tau^n,$$

where ω_n denotes the volume of the unit ball in \mathbb{R}^n . At this purpose, first of all, note that

$$(3.3) \quad |\bar{u}(x)| \leq |\bar{t}| \quad \text{in } \Omega.$$

Moreover, by the construction of \bar{u} , (3.3) and the fact that $F(0) = 0$, it follows that

$$(3.4) \quad \begin{aligned} \int_{B(x_0, \tau) \setminus B(x_0, \sigma_0 \tau)} F(\bar{u}(x)) dx &\geq - \int_{B(x_0, \tau) \setminus B(x_0, \sigma_0 \tau)} |F(\bar{u}(x))| dx \\ &\geq - \max_{|t| \leq |\bar{t}|} |F(t)| \int_{B(x_0, \tau) \setminus B(x_0, \sigma_0 \tau)} dx \\ &= - \max_{|t| \leq |\bar{t}|} |F(t)| (1 - \sigma_0^n) \tau^n \omega_n \end{aligned}$$

and

$$(3.5) \quad \int_{\mathbb{R}^n \setminus B(x_0, \tau)} F(\bar{u}(x)) dx = 0.$$

Consequently, relations (3.4) and (3.5) and again the definition of \bar{u} yield

$$(3.6) \quad \begin{aligned} \int_{\Omega} F(\bar{u}(x)) dx &= \int_{B(x_0, \sigma_0 \tau)} F(\bar{u}(x)) dx + \int_{B(x_0, \tau) \setminus B(x_0, \sigma_0 \tau)} F(\bar{u}(x)) dx \\ &= \int_{B(x_0, \sigma_0 \tau)} F(\bar{t}) dx + \int_{B(x_0, \tau) \setminus B(x_0, \sigma_0 \tau)} F(\bar{u}(x)) dx \\ &\geq F(\bar{t})\sigma_0^n \tau^n \omega_n - \max_{|t| \leq |\bar{t}|} |F(t)| (1 - \sigma_0^n) \tau^n \omega_n \\ &= \left[F(\bar{t})\sigma_0^n - (1 - \sigma_0^n) \max_{|t| \leq |\bar{t}|} |F(t)| \right] \omega_n \tau^n > 0, \end{aligned}$$

thanks to (3.1). Clearly, this ends the proof of Lemma 3. \square

Lemma 4. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying conditions (1.9) and (1.10). Then, for any $\varepsilon > 0$ there exists a positive constant M_ε , depending on ε , such that*

$$F(t) \leq \varepsilon |t|^2 + M_\varepsilon |t|$$

and

$$F(t) \leq \varepsilon |t|^2 + M_\varepsilon |t|^{2^*}$$

for any $t \in \mathbb{R}$, where $2^* := 2n/(n - 2s)$.

Proof. By (1.10) we get that for any $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that for any $t \in \mathbb{R}$ with $|t| > t_\varepsilon$

$$\frac{f(t)}{t} < 2\varepsilon,$$

so that

$$(3.7) \quad f(t) < 2\varepsilon t \quad \text{for } t > t_\varepsilon \quad \text{and} \quad f(t) > 2\varepsilon t \quad \text{for } t < -t_\varepsilon.$$

Also, since f is a continuous function in \mathbb{R} , by Weierstrass's theorem we get that

$$(3.8) \quad |f(t)| \leq M_\varepsilon \quad \text{for any } t \in [-t_\varepsilon, t_\varepsilon]$$

for a suitable positive constant M_ε , depending on ε .

Taking into account (1.12), (3.7) and (3.8) and integrating, we get

$$F(t) \leq \varepsilon |t|^2 + M_\varepsilon |t|$$

for any $t \in \mathbb{R}$.

By (3.7) and taking into account that $2^* > 2$, we also get that for any $\varepsilon > 0$

$$(3.9) \quad f(t) < 2\varepsilon|t|^{2^*-1} \quad \text{for } t > t_\varepsilon \quad \text{and} \quad f(t) > -2\varepsilon|t|^{2^*-1} \quad \text{for } t < -t_\varepsilon.$$

As above, by (3.8) and (3.9), we have that

$$F(t) \leq \varepsilon|t|^2 + M_\varepsilon|t|^{2^*}$$

for any $t \in \mathbb{R}$. This ends the proof of Lemma 4. \square

Lemma 5. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying conditions (1.9) and (1.10). Then, there exists a positive constant C such that*

$$|f(t)| \leq C(1 + |t|)$$

for any $t \in \mathbb{R}$.

Proof. By (1.10) there exist $\tilde{t} > 0$, $\alpha_1 \in \mathbb{R}$ and $\alpha_2 > 0$ such that

$$(3.10) \quad \alpha < \frac{f(t)}{t} < \alpha_2 \quad \text{for any } t \in \mathbb{R}, |t| > \tilde{t}.$$

Moreover, by (1.9) we easily get that

$$(3.11) \quad |f(t)| \leq \tilde{C} \quad \text{for any } t \in [-\tilde{t}, \tilde{t}].$$

Hence, the assertion of Lemma 5 comes from (3.10) and (3.11). \square

3.2. Proof of Theorem 1. The strategy for proving Theorem 1 will be showing that the functional $\mathcal{J}_{K,a}$ satisfies the assumptions of Theorem 8 in Appendix A. Of course, $\mathcal{J}_{K,a} \in C^1(X_0)$, thanks to our hypotheses on K , a and f , and $\mathcal{J}_{K,a}(0) = 0$, since $F(0) = 0$ by (1.12). Moreover, thanks to Lemma 3, we have that

$$\begin{aligned} \mathcal{J}_{K,a}(\bar{u}) &:= \frac{1}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} |\bar{u}(x) - \bar{u}(y)|^2 K(x-y) dx dy + \frac{1}{2} \int_{\Omega} a(x) |\bar{u}(x)|^2 dx \\ &\quad - \mu \int_{\Omega} F(\bar{u}(x)) dx < 0, \end{aligned}$$

for $\mu > 0$ sufficiently large, so that

$$\inf_{u \in X_0} \mathcal{J}_{K,a}(u) < 0.$$

Now, let us prove that

$$(3.12) \quad \inf_{u \in X_0} \mathcal{J}_{K,a}(u) > -\infty.$$

Indeed, by Lemma 4, (1.8) and the fact that $\mu > 0$, we get that, for any $\varepsilon > 0$ one has

$$\begin{aligned} \mathcal{J}_{K,a}(u) &:= \frac{1}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(y)|^2 K(x-y) dx dy + \frac{1}{2} \int_{\Omega} a(x) |u(x)|^2 dx \\ &\quad - \mu \int_{\Omega} F(u(x)) dx \\ (3.13) \quad &\geq \frac{1}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(y)|^2 K(x-y) dx dy - \mu \int_{\Omega} F(u(x)) dx \\ &\geq \frac{1}{2} \|u\|_{X_0}^2 - \mu\varepsilon \|u\|_{L^2(\Omega)}^2 - \mu M_\varepsilon \|u\|_{L^1(\Omega)} \\ &\geq \frac{1}{2} \|u\|_{X_0}^2 - \frac{\mu\varepsilon}{\lambda_1} \|u\|_{X_0}^2 - \mu \widetilde{M}_\varepsilon \|u\|_{X_0}, \end{aligned}$$

for some positive constant $\widetilde{M}_\varepsilon$ (here we used the Sobolev embedding theorem). Suppose that $\inf_{u \in X_0} \mathcal{J}_{K,a}(u) = -\infty$. Then, by (3.13), we also have

$$(3.14) \quad \inf_{u \in X_0} \left(\frac{1}{2} \|u\|_{X_0}^2 - \frac{\mu\varepsilon}{\lambda_1} \|u\|_{X_0}^2 - \mu\widetilde{M}_\varepsilon \|u\|_{X_0} \right) = -\infty.$$

Now, choose $\varepsilon > 0$ such that

$$(3.15) \quad \frac{1}{2} - \frac{\mu\varepsilon}{\lambda_1} > 0.$$

It is easily seen that, with this choice, (3.14) is a contradiction. This proves (3.12).

Finally, note that by (3.13) and (3.15) we also get that

$$(3.16) \quad \mathcal{J}_{K,a} \text{ is a coercive functional on } X_0$$

and this will be used in the following.

Now, let us show that $\mathcal{J}_{K,a}$ satisfies the Palais-Smale condition. For this, let $\{u_j\}_j$ be a sequence in X_0 such that

$$(3.17) \quad \{\mathcal{J}_{K,a}(u_j)\}_j \text{ is bounded in } X_0$$

and

$$(3.18) \quad \sup \left\{ |\langle \mathcal{J}'_{K,a}(u_j), \varphi \rangle| : \varphi \in X_0, \|\varphi\|_{X_0} = 1 \right\} \rightarrow 0, \text{ as } j \rightarrow +\infty.$$

Since $\mathcal{J}_{K,a}$ is coercive by (3.16) and (3.17) holds true, it is easy to see that

$$(3.19) \quad \{u_j\}_j \text{ is bounded in } X_0.$$

As a consequence of this and of the fact that X_0 is a reflexive space (being a Hilbert space, by [25, Lemma 7]), there exists $u_\infty \in X_0$ such that, up to a subsequence, $\{u_j\}_j$ converges to u_∞ weakly in X_0 , that is

$$(3.20) \quad \int_{\mathbb{R}^n \times \mathbb{R}^n} (u_j(x) - u_j(y))(\varphi(x) - \varphi(y))K(x-y) dx dy \rightarrow \int_{\mathbb{R}^n \times \mathbb{R}^n} (u_\infty(x) - u_\infty(y))(\varphi(x) - \varphi(y))K(x-y) dx dy,$$

for any $\varphi \in X_0$, as $j \rightarrow +\infty$. Moreover, by [25, Lemma 8] and [5, Theorem IV.9], we get that

$$(3.21) \quad \begin{aligned} u_j &\rightarrow u_\infty && \text{in } L^q(\mathbb{R}^n) \text{ for any } q \in [1, 2^*) \\ u_j &\rightarrow u_\infty && \text{a.e. in } \mathbb{R}^n \end{aligned}$$

as $j \rightarrow +\infty$.

By (3.18) and the fact that $\{u_j\}_j$ is bounded in X_0 (see (3.19)) we have that

$$(3.22) \quad \begin{aligned} 0 \leftarrow \langle \mathcal{J}'_{K,a}(u_j), u_j - u_\infty \rangle &= \int_{\mathbb{R}^n \times \mathbb{R}^n} |u_j(x) - u_j(y)|^2 K(x-y) dx dy \\ &\quad - \int_{\mathbb{R}^n \times \mathbb{R}^n} (u_j(x) - u_j(y))(u_\infty(x) - u_\infty(y))K(x-y) dx dy \\ &\quad - \int_{\Omega} a(x)u_j(x)(u_j(x) - u_\infty(x))dx \\ &\quad - \mu \int_{\Omega} f(u_j(x))(u_j(x) - u_\infty(x))dx \end{aligned}$$

as $j \rightarrow +\infty$. Note that, by (1.7) and (3.21) we have that

$$(3.23) \quad \int_{\Omega} a(x)u_j(x)(u_j(x) - u_\infty(x))dx \rightarrow 0$$

as $j \rightarrow +\infty$. Moreover, by Lemma 5, (1.9) and again (3.21) and using the Dominated Convergence Theorem, we have that

$$(3.24) \quad \int_{\Omega} f(u_j(x))(u_j(x) - u_{\infty}(x))dx \rightarrow 0$$

as $j \rightarrow +\infty$.

All in all, by (3.20) with $\varphi = u_{\infty}$ and (3.22)–(3.24) we deduce that

$$(3.25) \quad \int_{\mathbb{R}^n \times \mathbb{R}^n} |u_j(x) - u_j(y)|^2 K(x-y) dx dy \rightarrow \int_{\mathbb{R}^n \times \mathbb{R}^n} |u_{\infty}(x) - u_{\infty}(y)|^2 K(x-y) dx dy$$

as $j \rightarrow +\infty$.

Finally, we have that

$$\begin{aligned} \|u_j - u_{\infty}\|_{X_0}^2 &= \|u_j\|_{X_0}^2 + \|u_{\infty}\|_{X_0}^2 \\ &\quad - 2 \int_{\mathbb{R}^n \times \mathbb{R}^n} (u_j(x) - u_j(y))(u_{\infty}(x) - u_{\infty}(y))K(x-y) dx dy \\ &\rightarrow 2\|u_{\infty}\|_{X_0}^2 - 2\|u_{\infty}\|_{X_0}^2 = 0, \end{aligned}$$

as $j \rightarrow +\infty$, thanks to (3.20) and (3.25). Hence, $u_j \rightarrow u_{\infty}$ strongly in X_0 as $j \rightarrow +\infty$, concluding the proof of the Palais-Smale condition.

Finally, we need to analyze the geometry of the functional $\mathcal{J}_{K,a}$. At this purpose, with the notations of Theorem 8, we put

$$E_1 := X_0 \setminus \{0\}$$

and

$$E_2 := \{0\}.$$

To show that our functional has a local linking at 0, we just have to show that

$$(3.26) \quad \mathcal{J}_{K,a}(u) \geq 0 \quad \text{for any } u \in X_0 \setminus \{0\} \text{ with } \|u\|_{X_0} \leq R$$

for some positive constant R . For this it is enough to use Lemma 4. Indeed, by this, (1.8) and the fact that $\mu > 0$, we get that for any $\varepsilon > 0$

$$\begin{aligned} \mathcal{J}_{K,a}(u) &\geq \frac{1}{2}\|u\|_{X_0}^2 - \mu \int_{\Omega} F(u(x)) dx \\ &\geq \frac{1}{2}\|u\|_{X_0}^2 - \mu\varepsilon\|u\|_{L^2(\Omega)}^2 - \mu M_{\varepsilon}\|u\|_{L^{2^*}(\Omega)}^{2^*} \\ (3.27) \quad &\geq \frac{1}{2}\|u\|_{X_0}^2 - \frac{\mu\varepsilon}{\lambda_1}\|u\|_{X_0}^2 - \frac{\mu M_{\varepsilon}}{S_K}\|u\|_{X_0}^{2^*} \\ &= \left(\frac{1}{2} - \frac{\mu\varepsilon}{\lambda_1}\right)\|u\|_{X_0}^2 - \frac{\mu M_{\varepsilon}}{S_K}\|u\|_{X_0}^{2^*}, \end{aligned}$$

where S_K is the critical fractional Sobolev constant in the continuous embedding $X_0 \hookrightarrow L^{2^*}(\Omega)$ (see [25, Lemma 6] and [27, Lemma 9]). Choosing $\varepsilon > 0$ such that $\frac{1}{2} - \frac{\mu\varepsilon}{\lambda_1} > 0$ (note that this choice is admissible due to the fact that $\mu > 0$) and taking into account that $2 < 2^*$, by (3.27) we deduce that

$$\mathcal{J}_{K,a}(u) \geq 0,$$

provided $\|u\|_{X_0}$ is small enough, say $\|u\|_{X_0} \leq R$ for a suitable small $R > 0$. Hence, (3.26) is proved.

Since all the assumptions of Theorem 8 are satisfied, provided $\mu > 0$ is sufficiently large, we get that the functional $\mathcal{J}_{K,a}$ has at least two nontrivial critical points and this concludes the proof of Theorem 1.

4. A MULTIPLICITY RESULT: THE CASE WHEN a IS CONSTANT AND NEGATIVE

This section is devoted to the proof of Theorem 2, that is to the proof of a multiplicity result for problem (1.14). In this setting, we weaken the regularity assumptions on f : indeed, here we just need f to be a continuous function, and not a C^1 -function, as in the previous case. In addition, on f we consider the resonance conditions (1.18) and (1.19).

4.1. Some preliminary results. Also in this case, first of all, we need to prove some lemmas which will be useful in the sequel.

Lemma 6. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying conditions (1.17) and (1.18). Then, there exists a constant $\tilde{t} > 0$ such that*

$$F(t) \leq M\tilde{t} + \frac{\lambda_1 - \gamma - \varepsilon}{2} |t|^2$$

for any $t \in \mathbb{R}$, where $M := \max_{t \in [-\tilde{t}, \tilde{t}]} |f(t)|$.

Proof. By (1.18) there exists $\tilde{t} > 0$, $\alpha \in \mathbb{R}$ and $\varepsilon \in (0, \lambda_1 - \gamma)$ such that

$$(4.1) \quad \alpha < \frac{f(t)}{t} < \lambda_1 - \gamma - \varepsilon \quad \text{for any } t \in \mathbb{R}, |t| > \tilde{t}.$$

Moreover, by (1.17) we have

$$(4.2) \quad |f(t)| \leq M \quad \text{for any } t \in [-\tilde{t}, \tilde{t}].$$

Hence, by (4.1) and (4.2) and integrating, we obtain

$$\begin{aligned} F(t) &= \int_0^{\tilde{t}} f(\tau) d\tau + \int_{\tilde{t}}^t \frac{f(\tau)}{\tau} \tau d\tau \\ &\leq M\tilde{t} + (\lambda_1 - \gamma - \varepsilon) \int_{\tilde{t}}^t \tau d\tau \\ &= M\tilde{t} + \frac{\lambda_1 - \gamma - \varepsilon}{2} (|t|^2 - \tilde{t}^2) \\ &\leq M\tilde{t} + \frac{\lambda_1 - \gamma - \varepsilon}{2} |t|^2, \end{aligned}$$

for any $t \in \mathbb{R}$, with $t > \tilde{t}$. For $t < -\tilde{t}$ we can argue in the same way. Thus, we get

$$(4.3) \quad F(t) \leq M\tilde{t} + \frac{\lambda_1 - \gamma - \varepsilon}{2} |t|^2,$$

for any $t \in \mathbb{R}$ with $|t| > \tilde{t}$. In order to conclude the proof of Lemma 6, it is enough to observe that

$$F(t) := \int_0^t f(\tau) d\tau \leq M|t| \leq M\tilde{t},$$

for any $t \in [-\tilde{t}, \tilde{t}]$. This inequality and (4.3) give the assertion. \square

Lemma 7. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying conditions (1.17) and (1.18). Then, there exists a positive constant C such that*

$$|f(t)| \leq C(1 + |t|)$$

for any $t \in \mathbb{R}$.

Proof. We can argue exactly as in the proof of Lemma 5. \square

4.2. Proof of Theorem 2. Here the idea consists in applying Theorem 8 to the Euler-Lagrange functional associated with problem (1.14), that is $\mathcal{J}_{K,\gamma} : X_0 \rightarrow \mathbb{R}$ defined as

$$(4.4) \quad \mathcal{J}_{K,\gamma}(u) := \frac{1}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(y)|^2 K(x-y) dx dy - \frac{\gamma}{2} \int_{\Omega} |u(x)|^2 dx - \int_{\Omega} F(u(x)) dx,$$

where F is again the function defined in (1.12).

As we already noted, the functional $\mathcal{J}_{K,\gamma}$ turns out to be well-defined and smooth on X_0 , thanks to the assumptions on K and f . Also $\mathcal{J}_{K,\gamma}(0) = 0$ by definition of F .

Now, let us prove that $\mathcal{J}_{K,\gamma}$ is bounded from below. Indeed, by Lemma 6, it follows that

$$(4.5) \quad \begin{aligned} \mathcal{J}_{K,\gamma}(u) &:= \frac{1}{2} \|u\|_{X_0}^2 - \frac{\gamma}{2} \int_{\Omega} |u(x)|^2 dx - \int_{\Omega} F(t) dx \\ &\geq \frac{1}{2} \left(1 - \frac{\gamma}{\lambda_1}\right) \|u\|_{X_0}^2 - \int_{\Omega} \left(M\tilde{t} + \frac{\lambda_1 - \gamma - \varepsilon}{2} |u(x)|^2\right) dx \\ &= \frac{1}{2} \left(1 - \frac{\gamma}{\lambda_1}\right) \|u\|_{X_0}^2 - \frac{\lambda_1 - \gamma - \varepsilon}{2} \|u\|_{L^2(\Omega)}^2 - M\tilde{t}|\Omega|. \end{aligned}$$

Hence, we get

$$\mathcal{J}_{K,\gamma}(u) \geq -M\tilde{t}|\Omega|,$$

for every $u \in X_0$, that is $\mathcal{J}_{K,\gamma}$ is bounded from below in X_0 .

Now we claim that

$$(4.6) \quad \mathcal{J}_{K,\gamma} \text{ is coercive in } X_0.$$

Indeed, by the fact that

$$\|u\|_{L^2(\Omega)}^2 \leq \frac{1}{\lambda_1} \|u\|_{X_0}^2 \quad \text{for any } u \in X_0$$

(for this see the variational formulation of λ_1 given in [26, Proposition 9]), and (4.5), one has

$$\mathcal{J}_{K,\gamma}(u) \geq \frac{1}{2} \left(1 - \frac{\gamma}{\lambda_1}\right) \|u\|_{X_0}^2 - \left(\frac{\lambda_1 - \gamma - \varepsilon}{2\lambda_1}\right) \|u\|_{X_0}^2 - M\tilde{t}|\Omega|.$$

Thus,

$$(4.7) \quad \mathcal{J}_{K,\gamma}(u) \geq \frac{\varepsilon}{2\lambda_1} \|u\|_{X_0}^2 - M\tilde{t}|\Omega| \quad \text{for any } u \in X_0,$$

which shows the claim.

Now, let us prove that the functional $\mathcal{J}_{K,\gamma}$ satisfies the Palais-Smale compactness condition. At this purpose, let $\{u_j\}_j$ be a sequence in X_0 such that $\{\mathcal{J}_{K,\gamma}(u_j)\}_j$ is bounded and

$$(4.8) \quad \sup \left\{ |\langle \mathcal{J}'_{K,\gamma}(u_j), \varphi \rangle| : \varphi \in X_0, \|\varphi\|_{X_0} = 1 \right\} \rightarrow 0, \quad \text{as } j \rightarrow +\infty.$$

Since $\mathcal{J}_{K,\gamma}$ is coercive and $\{\mathcal{J}_{K,\gamma}(u_j)\}_j$ is bounded, the sequence $\{u_j\}_j$ turns out to be bounded in X_0 . Hence, being X_0 a reflexive space (it is a Hilbert space, by [25, Lemma 7]), there exists $u_\infty \in X_0$ such that, up to a subsequence, $\{u_j\}_j$ converges to u_∞ weakly in X_0 , that is (3.20) holds true. As a consequence, by applying [25, Lemma 8] and [5, Theorem IV.9], we know also that (3.21) is satisfied.

By (4.8) and the fact that $\{u_j\}_j$ is bounded in X_0 , we have

$$\begin{aligned}
(4.9) \quad 0 \leftarrow \langle \mathcal{J}'_{K,\gamma}(u_j), u_j - u_\infty \rangle &= \int_{\mathbb{R}^n \times \mathbb{R}^n} |u_j(x) - u_j(y)|^2 K(x-y) dx dy \\
&\quad - \int_{\mathbb{R}^n \times \mathbb{R}^n} (u_j(x) - u_j(y))(u_\infty(x) - u_\infty(y)) K(x-y) dx dy \\
&\quad - \gamma \int_{\Omega} u_j(x)(u_j(x) - u_\infty(x)) dx \\
&\quad - \int_{\Omega} f(u_j(x))(u_j(x) - u_\infty(x)) dx
\end{aligned}$$

as $j \rightarrow +\infty$.

Now, observe that, by Lemma 7, the Hölder inequality, (1.7) and (3.21), we get

$$\begin{aligned}
(4.10) \quad &\left| \gamma \int_{\Omega} u_j(x) (u_j(x) - u_\infty(x)) dx + \int_{\Omega} f(u_j(x))(u_j(x) - u_\infty(x)) dx \right| \\
&\leq \gamma \|u_j\|_{L^2(\Omega)} \|u_j - u_\infty\|_{L^2(\Omega)} \\
&\quad + C \int_{\Omega} (1 + |u_j(x)|) |u_j(x) - u_\infty(x)| dx \\
&\leq \gamma \|u_j\|_{L^2(\Omega)} \|u_j - u_\infty\|_{L^2(\Omega)} \\
&\quad + C \|u_j - u_\infty\|_{L^1(\Omega)} + C \|u_j\|_{L^2(\Omega)} \|u_j - u_\infty\|_{L^2(\Omega)} \rightarrow 0,
\end{aligned}$$

as $j \rightarrow +\infty$. Hence, taking into account (3.20) with $\varphi = u_\infty$ and (4.10), relation (4.9) gives that

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |u_j(x) - u_j(y)|^2 K(x-y) dx dy \rightarrow \int_{\mathbb{R}^n \times \mathbb{R}^n} |u_\infty(x) - u_\infty(y)|^2 K(x-y) dx dy,$$

that is

$$(4.11) \quad \|u_j\|_{X_0} \rightarrow \|u_\infty\|_{X_0},$$

as $j \rightarrow +\infty$.

Arguing as in the final part of the proof of the validity of the Palais-Smale condition in Subsection 3.2, we obtain that $u_j \rightarrow u_\infty$ strongly in X_0 as $j \rightarrow +\infty$, concluding the proof of the Palais-Smale condition.

Finally, let us study the geometry of the energy functional $\mathcal{J}_{K,\gamma}$. At this purpose, let $k \in \mathbb{N}$ be as in assumption (1.19). With the notations of Theorem 8 in Appendix A, we put

$$E_1 := \left\{ u \in X_0 : \langle u, e_j \rangle_{X_0} = 0 \text{ for any } j = 1, \dots, k \right\}$$

and

$$E_2 := \text{span}\{e_1, \dots, e_k\}.$$

One clearly has $X_0 = E_1 \oplus E_2$ and $\dim E_2 < +\infty$. The next step consists in verifying that the functional $\mathcal{J}_{K,\gamma}$ has a local linking at 0, that is

$$(4.12) \quad \mathcal{J}_{K,\gamma}(u) \geq 0, \quad \forall u \in E_1 \text{ with } \|u\|_{X_0} \leq R$$

and

$$(4.13) \quad \mathcal{J}_{K,\gamma}(u) \leq 0, \quad \forall u \in E_2 \text{ with } \|u\|_{X_0} \leq R$$

for some $R > 0$.

First of all, let us prove (4.12). At this purpose, note that, by (1.19) there exist $\beta \in (0, \lambda_{k+1} - \gamma)$ and $\varrho \in (0, \tilde{t})$ such that

$$\frac{f(t)}{t} < \beta,$$

for any t such that $|t| \in (0, \varrho)$. Hence,

$$\begin{aligned}
 \int_{|u(x)| < \varrho} F(u(x)) \, dx &= \int_{|u(x)| < \varrho} \left(\int_0^{u(x)} f(t) \, dt \right) \, dx \\
 &= \int_{|u(x)| < \varrho} \left(\int_0^{u(x)} \frac{f(t)}{t} \, t \, dt \right) \, dx \\
 (4.14) \qquad &\leq \frac{\beta}{2} \int_{|u(x)| < \varrho} |u(x)|^2 \, dx \\
 &\leq \frac{\beta}{2} \int_{\Omega} |u(x)|^2 \, dx.
 \end{aligned}$$

Now, let us fix $p \in (2, 2^*)$. Due to Lemma 6, one has

$$(4.15) \qquad F(t) \leq M\tilde{t} + \frac{1}{2}(\lambda_1 - \gamma - \varepsilon)|t|^2 \leq \left(\frac{M\tilde{t}}{\varrho^p} + \frac{\lambda_1 - \gamma - \varepsilon}{\varrho^{p-2}} \right) |t|^p,$$

provided $|t| \geq \varrho$. Thus, the Sobolev embedding theorem gives

$$(4.16) \qquad \int_{|u(x)| \geq \varrho} F(u(x)) \, dx \leq \left(\frac{M\tilde{t}}{\varrho^p} + \frac{\lambda_1 - \gamma - \varepsilon}{\varrho^{p-2}} \right) \|u\|_{L^p(\Omega)}^p \leq c^* \|u\|_{X_0}^p,$$

where

$$c^* := \left(\frac{M\tilde{t}}{\varrho^p} + \frac{\lambda_1 - \gamma - \varepsilon}{\varrho^{p-2}} \right) c_p^p$$

and c_p is the constant of the embedding $X_0 \hookrightarrow L^p(\Omega)$.

Now, if $u \in E_1$, then $u = \sum_{i=k+1}^{+\infty} \beta_i e_i$, for suitable $\beta_i \in \mathbb{R}$, where $i \in \mathbb{N}$ and $i \geq k+1$.

Owing to (2.4) and (2.5), one has

$$\|u\|_{L^2(\Omega)}^2 = \sum_{i=k+1}^{+\infty} \beta_i^2 \int_{\Omega} e_i(x)^2 \, dx = \sum_{i=k+1}^{+\infty} \frac{\beta_i^2}{\lambda_i} \langle e_i, e_i \rangle_{X_0} \leq \frac{1}{\lambda_{k+1}} \|u\|_{X_0}^2,$$

i.e.,

$$(4.17) \qquad \|u\|_{L^2(\Omega)}^2 \leq \frac{1}{\lambda_{k+1}} \|u\|_{X_0}^2,$$

for any $u \in E_1$. Then, by (4.14), (4.16) and (4.17) we get

$$\begin{aligned}
 \mathcal{J}_{K, \lambda}(u) &\geq \frac{1}{2} \left(1 - \frac{\gamma}{\lambda_{k+1}} \right) \|u\|_{X_0}^2 - \int_{|u(x)| < \varrho} F(u(x)) \, dx - \int_{|u(x)| \geq \varrho} F(u(x)) \, dx \\
 &\geq \frac{1}{2} \left(1 - \frac{\gamma}{\lambda_{k+1}} \right) \|u\|_{X_0}^2 - \frac{\beta}{2} \|u\|_{L^2(\Omega)}^2 - c^* \|u\|_{X_0}^p \\
 &\geq \frac{1}{2} \left(1 - \frac{\gamma}{\lambda_{k+1}} - \frac{\beta}{\lambda_{k+1}} \right) \|u\|_{X_0}^2 - c^* \|u\|_{X_0}^p.
 \end{aligned}$$

Since $p > 2$, if $\|u\|_{X_0}$ is small enough, say $\|u\|_{X_0} \leq R_1$, with $R_1 > 0$, by the above relation it follows that (4.12) holds true.

Now, let us prove (4.13). For this, again due to (1.19) it follows that

$$(4.18) \qquad F(t) = \int_0^t \frac{f(\tau)}{\tau} \, \tau \, d\tau \geq \frac{\lambda_k - \gamma}{2} |t|^2,$$

provided $0 < |t| < \delta$. Since E_2 is finite dimensional, we can find a positive constant R_2 such that $\|u\|_{\infty} < \delta$, if $u \in E_2$ and $\|u\|_{X_0} \leq R_2$. Consequently, thanks to (4.18), for any $u \in E_2$ with $\|u\|_{X_0} \leq R_2$, we get

$$(4.19) \qquad F(u(x)) \geq \frac{\lambda_k - \gamma}{2} |u(x)|^2,$$

for a.e. $x \in \mathbb{R}^n$.

Moreover, if $u \in E_2$, then

$$u = \sum_{i=1}^k \alpha_i e_i$$

for suitable $\alpha_i \in \mathbb{R}$, where $i = 1, \dots, k$. Owing to (2.4) and (2.5) one has

$$\|u\|_{X_0}^2 = \sum_{i=1}^k \alpha_i^2 \langle e_i, e_i \rangle_{X_0} = \sum_{i=1}^k \alpha_i^2 \lambda_i \int_{\Omega} |e_i(x)|^2 dx \leq \lambda_k \int_{\Omega} |u(x)|^2 dx.$$

This fact and (4.19) imply that for any $u \in E_2$ with $\|u\|_{X_0} \leq R_2$ we have

$$\begin{aligned} \mathcal{J}_{K,\gamma}(u) &\leq \frac{1}{2} \|u\|_{X_0}^2 - \frac{\gamma}{2} \|u\|_{L^2(\Omega)}^2 - \frac{\lambda_k - \gamma}{2} \|u\|_{L^2(\Omega)}^2 \\ &\leq \frac{\lambda_k}{2} \|u\|_{L^2(\Omega)}^2 - \frac{\gamma}{2} \|u\|_{L^2(\Omega)}^2 - \frac{\lambda_k - \gamma}{2} \|u\|_{L^2(\Omega)}^2 = 0, \end{aligned}$$

which clearly means that (4.13) holds true.

Choosing $R := \min\{R_1, R_2\}$, we get that (4.12) and (4.13) are satisfied. Hence the geometric features required by Theorem 8 are respected by the functional $\mathcal{J}_{K,\gamma}$.

By (4.12) and (4.13) it is easily seen that

$$\inf_{u \in X_0} \mathcal{J}_{K,\gamma}(u) \leq 0.$$

If $\inf_{u \in X_0} \mathcal{J}_{K,\gamma}(u) = 0$, then, by (4.12) we get that $\mathcal{J}_{K,\gamma}(u) = 0$ for every $u \in E_2$ with $\|u\|_{X_0} \leq R$. This fact implies that all the functions $u \in E_2$ with $\|u\|_{X_0} \leq R$ are weak solutions of problem (1.1).

On the other hand, if $\inf_{u \in X_0} \mathcal{J}_{K,\gamma}(u) < 0$, Theorem 8 ensures the existence of at least two nontrivial critical points for the energy functional $\mathcal{J}_{K,\gamma}$. Anyway, we have the existence of at least two nontrivial weak solutions for problem (1.1). The proof of Theorem 2 is now complete.

5. SOME FINAL COMMENTS

The results stated in Theorem 2 still holds true if the function $f = f(t)$ is replaced with $f = f(x, t)$, that is, more precisely, if $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that

$$(5.1) \quad -\infty < \liminf_{|t| \rightarrow \infty} \frac{f(x, t)}{t} \leq \limsup_{|t| \rightarrow \infty} \frac{f(x, t)}{t} < \lambda_1 - \gamma \quad \text{uniformly a.e. } x \in \Omega$$

there exist an integer $k \geq 1$ and a constant $\delta > 0$ such that

$$(5.2) \quad \begin{aligned} \frac{f(x, t)}{t} &\geq \lambda_k - \gamma \quad \text{for every } 0 < |t| < \delta \quad \text{uniformly a.e. } x \in \Omega \\ \text{and } \limsup_{t \rightarrow 0} \frac{f(x, t)}{t} &< \lambda_{k+1} - \gamma \quad \text{uniformly a.e. } x \in \Omega. \end{aligned}$$

In this case we just need more care in proving the estimates on f and its primitive F we used along the proofs of our results.

Finally, we just would like to recall that very recently in [32] Teng studied the existence of two nontrivial solutions for a parametric nonlocal hemivariational inequalities with Dirichlet boundary condition, by using a non-smooth critical point theorem due to Arcoya and Carmona [2]. For completeness we just point out that, by using a non-smooth version of the Brezis-Nirenberg result obtained by Wu in [33, Theorem 2.3], our approach can be exploited

for proving the existence of two nontrivial solutions for nonlocal differential inclusions of the form

$$(5.3) \quad \begin{cases} -\mathcal{L}_K u + a(x)u \in \partial j(x, u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where j is a suitable measurable function such that $j(x, \cdot)$ is locally Lipschitz continuous for a.e. $x \in \Omega$. Here $\partial j(x, \cdot)$ denotes the generalized subdifferential in the sense of Clarke.

APPENDIX A. THE BREZIS-NIRENBERG THEOREM IN THE PRESENCE OF SPLITTING

In order to prove the multiplicity results stated in Theorem 1 and Theorem 2 our main tool is given by the celebrated critical point theorem in the presence of splitting established by Brezis and Nirenberg in [6, Theorem 4]. For reader's convenience and for making this paper self-contained, we recall it here below:

Theorem 8 ([6, Theorem 4]). *Let $(E, \|\cdot\|)$ be a Banach space such that $E = E_1 \oplus E_2$ with $\dim E_2 < \infty$. Let $\mathcal{I} \in C^1(E)$ with $\mathcal{I}(0) = 0$, satisfying the Palais-Smale condition and assume that, for some $R > 0$*

$$(A.1) \quad \begin{aligned} \mathcal{I}(u) &\geq 0 \text{ for } u \in E_1 \text{ with } \|u\| \leq R \\ \mathcal{I}(u) &\leq 0 \text{ for } u \in E_2 \text{ with } \|u\| \leq R. \end{aligned}$$

Assume also that \mathcal{I} is bounded from below and $\inf_{u \in E} \mathcal{I}(u) < 0$.

Then, \mathcal{I} has at least two nontrivial critical points.

Condition (A.1), which means that the functional \mathcal{I} has a *local linking* at 0, was introduced by Liu and Li in [12]. Together with the Palais-Smale condition, the local linking property and the boundedness from below of the functional \mathcal{I} are the main assumptions in Theorem 8, whose proof is based on Ekeland's variational principle and on a general deformation lemma.

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