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Calculus of Variations

On critical Kirchhoff problems driven by the fractional Laplacian

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Abstract

We study a nonlocal parametric problem driven by the fractional Laplacian operator combined with a Kirchhoff-type coefficient and involving a critical nonlinearity term in the Sobolev embedding sense. Our approach is of variational and topological nature. The obtained results can be viewed as a nontrivial extension to the nonlocal setting of some recent contributions already present in the literature.

Mathematics Subject Classification 35S15 · 35J20

1 Introduction

The equation that goes under the name of *Kirchhoff equation* was proposed in [17] as a model for the transverse oscillation of a stretched string in the form

$$\rho h\, \partial_{tt}^2 u - \left(p_0 + \frac{\mathcal{E} h}{2L} \int_0^L |\partial_x u|^2 \, dx \right) \partial_{xx}^2 u + \delta\, \partial_t u + f(x, u) = 0 \tag{1}$$

for $t \geq 0$ and $0 < x < L$, where $u = u(t, x)$ is the lateral displacement at time t and at position x, $\mathcal{E}$ is the Young modulus, ρ is the mass density, h is the cross section area, L the length of the string, p_0 is the initial stress tension, δ the resistance modulus and g the external force. Kirchhoff actually considered only the particular case of (1) with $\delta = f = 0$.

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Through the years, this model was generalized in several ways that can be collected in the form

$$\partial_{tt}^2 u - M(\|u\|^2)\Delta u = f(t, x, u), \quad x \in \Omega \tag{2}$$

for a suitable function $M\colon [0,\infty) \to \mathbb{R}$, called *Kirchhoff function*. The set Ω is a bounded domain of $\mathbb{R}^N$, and $\|u\|^2 = \|\nabla u\|_2^2$ denotes the Dirichlet norm of u. The basic case corresponds to the choice

$$M(t) = a + bt^{\gamma-1}, \quad a \geq 0,\ b \geq 0,\ \gamma \geq 1.$$

When $M(0) = 0$, i.e. $a = 0$, the equation is called *degenerate*. *Stationary solutions* to (3) solve the equation

$$\begin{cases} -M(\|u\|^2)\Delta u = f(x, u), & x \in \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

We refer to [26] for a recent survey of the results connected to this model.

The existence and multiplicity of solutions to Kirchhoff problems under the effect of a critical nonlinearity f have received considerable attention. The term critical refers here to the rough assumption that $f(u) \sim |u|^{2^*-2}u$ with $2^* = 2N/(N-2)$. The natural setting of the corresponding equation in $H_0^1(\Omega)$ yields a *lack of compactness*, since the embedding of $H_0^1(\Omega)$ into $L^{2^*}(\Omega)$ is only continuous. Straightforward techniques of Calculus of Variations fail, and more advanced results from Critical Point Theory must be used. In particular, P.-L. Lions' Concentration-Compactness appears as a natural tool for the analysis of the loss of compactness.

The relevant outcome is that the Kirchhoff function M interacts with the critical growth of the nonlinearity g: the validity of the Palais–Smale compactness condition holds only under a condition like

$$a^{\frac{N-4}{2}} b \geq C_2(N),$$

and a similar inequality ensures that the associated Euler functional is weakly lower semicontinuous. For some very recent results on Kirchhoff-type problems, see [15,18] as well as [27] for related topics.

In a very recent paper, Faraci and Silva (see [13]) obtained several quantitative results for the problem

$$\begin{cases} -\left(a + b\displaystyle\int_\Omega |\nabla u|^2\, dx\right)\Delta u = |u|^{2^*-2}u + \lambda g(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, $N > 4$, a and b a repositive fixed numbers, λ is a parameter and g is a Carathéodory function that satisfies suitable growth conditions. By using a fibering-type approach, the authors of [13] investigate existence, non-existence and multiplicity of solutions to (4). In a previous paper, see [14], Faraci, Farkas and Kristály studied Eq. (4) with $g(x, u) = 0$ and under suitable assumptions on the parameters a and b they proved that the functional associated to the problem is sequentially weakly lower semicontinuous, satisfies the Palais–Smale condition and is convex.

The purpose of the present paper is to extend part of these results to the *fractional* counterpart of the Kirchhoff problem

$$\begin{cases} \left(a+b\int_{\mathcal{Q}}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\right)(-\Delta)^s u=|u|^{2_s^*-2}u+\lambda g(x,u) & \text{in } \Omega \\ u=0 & \text{in } \mathbb{R}^N\setminus\Omega \end{cases} \qquad (P_{a,b}^{\lambda})$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $\mathcal{Q}=\mathbb{R}^{2N}\setminus\mathcal{O}$ and $\mathcal{O}=\Omega^c\times\Omega^c$, a and b are strictly positive real numbers, $s\in(0,1)$, $N>4s$ and $2_s^*:=2N/(N-2s)$ denotes the critical exponent for the Sobolev embedding of $H^s(\mathbb{R}^N)$ into Lebesgue spaces. g is a function that satisfies hypothesis similar to the one in (4) adapted to the non local case. The fractional Laplacian in ($P_{a,b}^{\lambda}$) is defined as

$$(-\Delta)^s u(x)=K_{N,s}\lim_{\epsilon\to 0^+}\int_{\mathbb{R}^N\setminus B_\epsilon(0)}\frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy$$

where

$$\frac{1}{K_{N,s}}:=\int_{\mathbb{R}^N}\frac{1-\cos\zeta_1}{|\zeta|^{N+2s}}\,d\zeta.$$

Since the parameter s is given, we will work with a rescaled version of the operator and this enables us to assume that $K_{N,s}=1$. For references about the fractional Laplacian we refer to [11], [1] and to the monograph [23]. We define the space X as the set of functions $u\colon\mathbb{R}^N\to\mathbb{R}$ such that $u|_\Omega\in L^2(\Omega)$ and

$$\left\{(x,y)\mapsto\frac{u(x)-u(y)}{|x-y|^{N/2+s}}\right\}\in L^2(\mathcal{Q}),$$

endowed with the norm

$$\|u\|_X=\|u\|_{L^2(\Omega)}+\left(\int_{\mathcal{Q}}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\right)^{\frac{1}{2}}. \tag{5}$$

We also set

$$X_0^s(\Omega):=\left\{u\in X:\ u=0 \text{ a.e. in } \mathbb{R}^N\setminus\Omega\right\}.$$

We introduce the best Sobolev constant as

$$S_{N,s}:=\inf_{u\in X_0^s(\Omega)}\frac{\|u\|^2}{\|u\|_{2_s^*}^2} \tag{6}$$

where

$$\|u\|^2:=\int_{\mathcal{Q}}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy.$$

The norm introduced in the previous equation is induced by the scalar product

$$\langle u,v\rangle_{X_0^s(\Omega)}:=\int_{\mathcal{Q}}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy \quad \text{for all } u,v\in X_0^s(\Omega)$$

and we recall that in $X_0^s(\Omega)$ it is equivalent to (5). For further details we refer the reader to [28, Lemma 6]. As it is easy to check, looking for solution of ($P_{a,b}^{\lambda}$) is equivalent to finding the critical points of the functional $\mathcal{I}_{a,b}^{\lambda}\colon X_0^s(\Omega)\to\mathbb{R}$ associated to the problem:

$$\mathcal{I}^{\lambda}_{a,b}(u) := \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{2^*_s}\|u\|^{2^*_s}_{2^*_s} - \lambda \int_{\Omega} G(x,u)\,dx$$

where we denote with $G(x,t) = \int_0^t g(x,\omega)d\omega$. Arguing as in [30, Proposition 1.12], we get

$$\left(\mathcal{I}^{\lambda}_{a,b}\right)'(u)\,[v] = \left(a + b\|u\|^2\right)\langle u, v\rangle_{X^s_0(\Omega)} - \int_{\Omega} |u|^{2^*_s-2}uv\,dx - \lambda \int_{\Omega} g(x,u)v\,dx \quad (7)$$

for all $u, v \in X^s_0(\Omega)$. When we have $g(x,u) = 0$ we will use the notation

$$\mathcal{I}_{a,b}(u) := \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{2^*_s}\|u\|^{2^*_s}_{2^*_s}$$

and we point out that $\mathcal{I}_{a,b}$ is a C^2-functional.

The interest in generalizing to the fractional case the model introduced by Kirchhoff does not arise only for mathematical purposes. In fact, following the ideas of [6] and the concept of fractional perimeter, Fiscella and Valdinoci proposed in [16] an equation describing the behaviour of a string constrained at the extrema in which appears the fractional length of the rope. The interested reader can also consult [7–9] and the references therein for further motivations and applications of operators similar to the one proposed in $(P^{\lambda}_{a,b})$.

Recently, problem similar to (5) has been extensively investigated by many authors using different techniques and producing several relevant results. In [16] Fiscella and Valdinoci showed the existence of a non-negative solution of mountain pass type for an equation with a critical term perturbed with a subcritical nonlinearity. With the same spirit of the previous one, in their seminal paper [4], Autuori, Fiscella and Pucci generalize these results to the degenerate case, i.e $M(0) = 0$, without requiring monotonicity assumption on the function M. We stress that in these two articles the operator taken into account is more general to the one we consider here, but the two coincide making a particular choice on the kernel; see also the paper [24] due to Molica Bisci and Vilasi. In the recent [21], Liu, Squassina and Zhang studied ground state solutions for the Kirchhoff equation plus a potential with a non linear term asymptotic to a power with critical growth in low dimension. It is also worth mentioning [22] where Mingqi, Rădulescu and Zhang proved the existence of nontrivial radial solutions in the non-degenerate and degenerate cases for the non local Kirchhoff problem in which the fractional Laplacian is replaced by the fractional magnetic operator.

Despite all the results cited above, to the best of our knowledge, in literature there are still no articles summarizing the situation of different kind of solutions at different level of energy for the fractional Kirchhoff problem. Furthermore, even if some of the results we are going to prove are known, we present a proof based on a adaptation to the fractional case due to Palatucci and Pisante ([25]) of the Lions second concentration-compactness principle; for the original version of the lemma we refer to [20], as well as [19].

For the reader's convenience, we collect here our main results.

Theorem 1 *Define*

$$L_{N,s} := \frac{4s(N-4s)^{\frac{N-4s}{2s}}}{N^{\frac{N-2s}{2s}} S^{\frac{N}{2s}}_{N,s}}, \quad PS_{N,s} := \frac{2s(N-4s)^{\frac{N-4s}{2s}}}{(N-2s)^{\frac{N-2s}{2s}} S^{\frac{N}{2s}}_{N,s}},$$

and

$$C_{N,s} := \frac{2s(N-4s)^{\frac{N-4s}{2s}}(N+2s)^{\frac{N-2s}{2s}}}{(N-2s)^{\frac{N-2s}{s}} S^{\frac{N}{2s}}_{N,s}}.$$

The following assertions holds true:

- (i) *the energy functional* $\mathcal{I}_{a,b}$ *is sequentially weakly lower semicontinuous on* $X_0^s(\Omega)$ *if and only if* $a^{\frac{N-4s}{2s}} b \geq L_{N,s}$.
- (ii) *If* $a^{(N-4s)/2s} b \geq \mathsf{PS}_{N,s}$, *the functional* $\mathcal{I}_{a,b}$ *satisfies the compactness Palais–Smale condition at level* $c \in \mathbb{R}$.
- (iii) *If* $a^{(N-4s)/2s} b \geq C_{N,s}$, *then the functional* $\mathcal{I}_{a,b}$ *is convex on* $X_0^s(\Omega)$.

Theorem 1 guarantees the validity of some crucial properties such as the sequentially weakly lower semicontinuity and the Palais–Smale condition. As we are going to see in the next statement, these facts enable us to use traditional variational methods to completely describe the situation for problem ($P_{a,b}^{\lambda}$). We begin providing two results about the existence of global minimizers at different level of energy.

Theorem 2 *Let* $a, b \in \mathbb{R}^+$ *such that* $a^{(N-4s)/2s} b \geq L_{N,s}$ *and set*

$$\iota_\lambda^s := \inf \left\{ \mathcal{I}_{a,b}^{\lambda}(u) \mid u \in X_0^s(\Omega) \setminus \{0\} \right\} \quad \textit{for any } \lambda > 0.$$

There exists $\overline{\lambda}_0^s \geq 0$ *such that for any* $\lambda > \overline{\lambda}_0^s$ *it is possible to find* $u_\lambda^s \in X_0^s(\Omega) \setminus \{0\}$ *such that* $\mathcal{I}_{a,b}^{\lambda}(u_\lambda^s) = \iota_\lambda^s < 0$.

Theorem 3 *Let* $\lambda = \overline{\lambda}_0^s$. *The following statements hold:*

- (i) *if* $a^{(N-2s)/2s} b > L_{N,s}$ *then there exists* $u_\lambda^s \in X_0^s(\Omega) \setminus \{0\}$ *such that* $\iota_{\overline{\lambda}_0^s}^s = \mathcal{I}_{a,b}^{\overline{\lambda}_0^s} = 0$;
- (ii) *if* $a^{(N-2s)/2s} b = L_{N,s}$, *then* $u = 0$ *in the only minimizer for* $\iota_{\overline{\lambda}_0^s}^s$.

In the next Theorem we give some information on what happens when we do not keep fixed the parameters a, and b. It asserts we have some kind of stability when the product $a^{(N-4s)/2s} b$ becomes close to $L_{N,s}$.

Theorem 4 *Let* $(a_k)_k$, $(b_k)_k$ *be a sequence of real positive numbers such that* $a_k \to a$, $b_k \to b$ *and* $a_k^{(N-4s)/2s} b_k \searrow L_{N,s}$. *Setting* $\lambda_k := \overline{\lambda}_0^s(a_k, b_k)$ *we have that* $\lambda_k \to 0$ *as* $k \to \infty$. *Furthermore, if* $(u_k)_k \subset X_0^s(\Omega) \setminus \{0\}$ *such that* $\lambda_k = \lambda_0^s(u_k)$ *then* $u_k \rightharpoonup 0$ *and*

$$\frac{\|u_k\|_{2_s^*}^2}{\|u_k\|^2} \to S_{N,s}.$$

Next statement shows the existence of solution of mountain pass type when $\lambda \geq \overline{\lambda}_0^s$.

Theorem 5 *If* $\lambda \geq \overline{\lambda}_0^s$, *then there exists a* $v_\lambda^s \in X_0^s \Omega \setminus \{0\}$ *such that* $\mathcal{I}_{a,b}^{\lambda}(v_\lambda^s) = c_\lambda^s$ *and* $\left(\mathcal{I}_{a,b}^{\lambda}\right)' (v_\lambda^s) = 0$ *where*

$$c_\lambda^s := \inf_{h \in \Gamma_\lambda^s} \max_{\zeta \in [0,1]} \mathcal{I}_{a,b}^{\lambda}(h(\zeta))$$

and

$$\Gamma_\lambda^s := \left\{ h \in C\left([0,1], X_0^s(\Omega)\right) \mid h(0) = 0, \ h(1) = u_{\overline{\lambda}_0^s}^s \right\}.$$

Finally we focus on the case $\lambda \in (\overline{\lambda}_0^s - \delta, \overline{\lambda}_0^s)$ for some $\delta > 0$ small.

Theorem 6 *Set*

$$\hat{\iota}^s_\lambda := \inf\{\mathcal{I}^\lambda_{a,b}(u) \mid u \in X^s_0(\Omega),\ \|u\| \geq r\}$$

for some $r > 0$. *There exist* $\delta, r > 0$ *such that for any* $\lambda \in (\overline{\lambda}^s_0 - \delta, \overline{\lambda}^s_0)$ *the value* $\hat{\iota}^s_\lambda$ *is attained at a function* $w^s_\lambda \in X^s_0(\Omega)$ *satisfying* $\|w^s_\lambda\| > r$.

Theorem 7 *For any* $\lambda \in (\overline{\lambda}^s_0 - \delta, \overline{\lambda}^s_0)$ *there is* $v^s_\lambda \in X^s_0(\Omega) \setminus \{0\}$ *such that* $\mathcal{I}^\lambda_{a,b}(v^s_\lambda) = c^s_\lambda$ *and* $\left(\mathcal{I}^\lambda_{a,b}\right)'(v^s_\lambda) = 0$ *where*

$$c^s_\lambda := \inf_{h \in \Gamma^s_\lambda} \max_{\zeta \in [0,1]} \mathcal{I}^\lambda_{a,b}(h(\zeta))$$

and

$$\Gamma^s_\lambda := \{h \in C\left([0,1], X^s_0(\Omega)\right) : h(0) = 0,\ h(1) = w^s_\lambda\}$$

Our paper is organized as follows: in Sect. 1 we present the classic Kirchhoff model, its generalization to the non local case and we collect in a synthetic way our main results. In Sect. 2 we prove for the functional associated to the problem with $g(x, u) = 0$ the weak lower semicontinuity, the validity of the Palais–Smale condition and the convexity under suitable assumption on the parameters a and b. Since the perturbation g will have a subcritical growth, we decided to prove these conditions for the problem with the pure power in order to ease notation. We stress that the functional associated to the perturbed problem still verifies these properties and proofs need only minor adjustments. In Sect. 3 we prove the existence of global minimizers, local minimizers and mountain pass type solutions with different energy level at varying of the parameter λ. At the end of Sect. 3, strengthening the hypothesis on the non linear term g, we are able to give also a non existence result for problem ($P^\lambda_{a,b}$).

2 Semicontinuity and the validity of the Palais–Smale condition

In this section we completely describe the range of parameters a and b for which the functional $\mathcal{I}_{a,b}$ associated to the problem

$$\begin{cases} \left(a + b\displaystyle\int_{\mathcal{Q}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\right)(-\Delta)^s u = |u|^{2^*_s-2}u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{$P_{a,b}$}$$

is (sequentially) weakly lower semicontinuous.

Proof of Theorem 1 (i) We assume that $a^{\frac{N-4s}{2s}}b \geq L_{N,s}$, and we choose a sequence $(u_n)_n \subset X^s_0(\Omega)$ such that $u_n \rightharpoonup u$. Since the embedding $X^s_0(\Omega) \hookrightarrow L^p(\Omega)$ is compact (see for instance [29, Lemma 9]), u_n converges to u strongly in $L^p(\Omega)$ for any $p \in \left[1, 2^*_s\right)$. We notice that

$$\begin{aligned} \|u_n - u\|^2 + 2\langle u_n - u, u\rangle_{X^s_0(\Omega)} &= \langle u_n, u_n\rangle_{X^s_0(\Omega)} + \langle u, u\rangle_{X^s_0(\Omega)} - 2\langle u_n, u\rangle_{X^s_0(\Omega)} \\ &+ 2\langle u_n, u\rangle_{X^s_0(\Omega)} - 2\langle u, u\rangle_{X^s_0(\Omega)} = \|u_n\|^2 - \|u\|^2. \end{aligned} \tag{8}$$

Hence

$$\|u_n\|^2 - \|u\|^2 = \|u_n - u\|^2 + 2\langle u_n - u, u\rangle_{X^s_0(\Omega)} = \|u_n - u\|^2 + o(1)$$

as $n \to \infty$. After that, we compute

$$\|u_n\|^4 - \|u\|^4 = \left(\|u_n\|^2 - \|u\|^2\right)\left(\|u_n\|^2 + \|u\|^2\right)$$
$$= \left(\|u_n - u\|^2 + o(1)\right)\left(\|u_n - u\|^2 + 2\|u\|^2 + o(1)\right). \tag{9}$$

Finally, using the Brezis-Lieb Lemma (see [5, Theorem 1]), we have

$$\|u_n - u\|_{2_s^*}^{2_s^*} = \|u_n\|_{2_s^*}^{2_s^*} - \|u\|_{2_s^*}^{2_s^*} + o(1) \tag{10}$$

as $n \to \infty$. Putting together (8), (9), (10) and the Sobolev inequality (6) we obtain

$$\begin{aligned}
\mathcal{I}_{a,b}(u_n) - \mathcal{I}_{a,b}(u) &= \frac{a}{2}\left(\|u_n\|^2 - \|u\|^2\right) + \frac{b}{4}\left(\|u_n\|^4 - \|u\|^4\right) - \frac{1}{2_s^*}\left(\|u_n\|_{2_s^*}^{2_s^*} - \|u\|_{2_s^*}^{2_s^*}\right) \\
&= \frac{a}{2}\|u_n - u\|^2 + \frac{b}{4}\left(\|u_n - u\|^4 + 2\|u\|^2\|u_n - u\|^2\right) \\
&\quad - \frac{1}{2_s^*}\|u_n - u\|_{2_s^*}^{2_s^*} + o(1) \\
&\geq \frac{a}{2}\|u_n - u\|^2 + \frac{b}{4}\|u_n - u\|^4 - \frac{S_{N,s}^{-\frac{2_s^*}{2}}}{2_s^*}\|u_n - u\|^{2_s^*} + o(1) \\
&= \|u_n - u\|^2\left[\frac{a}{2} + \frac{b}{4}\|u_n - u\|^2 - \frac{S_{N,s}}{2_s^*}\|u_n - u\|^{2_s^*-2}\right] + o(1)
\end{aligned} \tag{11}$$

as $n \to \infty$. At this point, we introduce the auxiliary function

$$f_{N,s}(\zeta) = \frac{a}{2} + \frac{b}{4}\zeta^2 - \frac{S_{N,s}^{-\frac{2_s^*}{2}}}{2_s^*}\zeta^{2_s^*-2}, \quad \zeta \geq 0.$$

It is easy to verify that the function $f_{N,s}$ attains its minimum at the point

$$m_{N,s} = \left(\frac{b}{2}\frac{2_s^*}{2_s^* - 2}S_{N,s}^{\frac{2_s^*}{2}}\right)^{\frac{1}{2_s^*-4}},$$

and that

$$a^{\frac{N-4s}{2s}}b \geq L_{N,s} \Leftrightarrow f_{N,s}(m_{N,s}) = \frac{1}{2}\left(a - b^{-\frac{2s}{N-4s}}L_{N,s}^{\frac{2s}{N-4s}}\right) \geq 0 \tag{12}$$

From (11) and (12) it follows that

$$\liminf_{n\to\infty}\left(\mathcal{I}_{a,b}(u_n) - \mathcal{I}_{a,b}(u)\right) \geq \liminf_{n\to\infty}\|u_n - u\|^2 f_{N,s}(\|u_n - u\|) \geq 0,$$

which concludes this part of the proof.

Conversely, we proceed by contradiction, assuming that the functional $\mathcal{I}_{a,b}$ is sequentially weakly lower semicontinuous but

$$a^{\frac{N-4s}{2s}}b < L_{N,s} \tag{13}$$

Let $\{u_n\}_n \subset X_0^s(\Omega)$ be a minimizing sequence for $S_{N,s}$. By homogeneity we may assume furthermore that $\|u_n\|_{2_s^*} = 1$ for every n, so that we deduce that the sequence $\{u_n\}_n$ must be bounded. Up to a subsequence, we have that $u_n \rightharpoonup u$ in $X_0^s(\Omega)$ for some $u \in X_0^s(\Omega) \setminus \{0\}$. Besides, exploiting the weak lower semicontinuity of the norm, we have that $\|u\| \leq$

$\liminf_{n\to\infty} \|u_n\| =: L$ and there exists a subsequence $\{u_{n_k}\}_k$ such that $L = \lim_{k\to\infty} \|u_{n_k}\|$. We point out that $L > 0$, since $u \neq 0$. Now $N > 4s$ implies that $0 < 2_s^* - 2 < 2$, and $\lim_{x\to+\infty} f_{N,s}(x) = +\infty$. As we have already seen, the function $f_{N,s}$ attains its minimum at the point $m_{N,s}$ and this, together with (13), implies $f_{N,s}(m_{N,s}) < 0$. Set $c = m_{N,s}/L > 0$. We notice that

$$\begin{aligned}\liminf_{n\to\infty} \mathcal{I}_{a,b}(cu_n) &\leq \liminf_{k\to\infty} \mathcal{I}_{a,b}(cu_{n_k})\\ &= \liminf_{k\to\infty} \|cu_{n_k}\|^2 f_{N,s}(\|cu_{n_k}\|) = (cL)^2 f_{N,s}(cL)\\ &= (cL)^2 f_{N,s}(m_{N,s}) \leq \|cu\|^2 f_{N,s}(m_{N,s}) \leq \|cu\|^2 f_{N,s}(\|cu\|). \qquad (14)\end{aligned}$$

We also have that

$$\begin{aligned}\|cu\|^2 f_{N,s}(\|cu\|) &= \frac{a}{2}\|cu\|^2 + \frac{b}{4}\|cu\|^4 - \frac{S_{N,s}^{2_s^*/2}}{2_s^*}\|cu\|\\ &\leq \frac{a}{2}\|cu\|^2 + \frac{b}{4}\|cu\|^4 - \frac{1}{2_s^*}\int_\Omega |cu|^{2_s^*}\,dx = \mathcal{I}_{a,b}(cu). \qquad (15)\end{aligned}$$

Comparing (13) with (14) we get

$$\liminf_{n\to\infty} \mathcal{I}_{a,b}(cu_n) \leq \mathcal{I}_{a,b}(cu). \qquad (16)$$

We claim that a strict inequality holds in (16). Indeed, if we had equality, the function cu would attain the minimum in (6). This is impossible, since $\Omega \neq \mathbb{R}^N$ (see [10, Theorem 1.1]). The proof is complete. □

Proof of Theorem 1 (ii) Let $\{u_n\}_n \subset X_0^s(\Omega)$ be a $(PS)_c$ sequence, i.e. $\mathcal{I}_{a,b}(u_n) \to c$ and $\mathcal{I}'_{a,b}(u_n) \to 0$ as $n \to \infty$. Recalling (6), we observe that

$$\mathcal{I}_{a,b}(u) = a\|u\|^2 + b\|u\|^4 - \int_\Omega |u|^{2_s^*}\,dx \geq a\|u\|^2 + b\|u\|^4 - S_{N,s}^{-\frac{2_s^*}{2}}\|u\|^{2_s^*}.$$

Since $2_s^* < 4$ we have that $\mathcal{I}_{a,b}$ is coercive, and from that we can deduce the boundedness of the sequence $\{u_n\}_n$. From [29, Lemma 9], up to a subsequence, we have

$$\begin{cases} u_n \rightharpoonup u & \text{in } X_0^s(\Omega)\\ u_n \to u & \text{in } L^p(\Omega) \text{ for all } p \in \left[1, 2_s^*\right)\\ u_n \to u & \text{a.e in } \mathbb{R}^N. \end{cases}$$

Using the Hölder inequality, it is straightforward to see that the sequence $\{u_n\}_n$ is also bounded in the space $\mathcal{M}(\Omega)$, thus there exists two finite measures μ and ν such that

$$(-\Delta)^s u_n \rightharpoonup^* \mu \quad \text{and} \quad |u_n|^{2_s^*} \rightharpoonup^* \nu \quad \text{in } \mathcal{M}(\Omega)$$

From [25, Theorem 1.5], it follows that either $u_n \to u$ in $L^{2_s^*}(\Omega)$ or there exist a set J at most countable, two real sequences $\{\mu_j\}_{j\in J}$, $\{\nu_j\}_{j\in J}$ and distinct points $\{x_j\}_{j\in J} \subset \mathbb{R}^N$ such that

$$\nu = |u|^{2_s^*} + \sum_{j\in J} \nu_j \delta_{x_j} \qquad (17)$$

and

$$\mu = (-\Delta)^s u + \tilde{\mu} + \sum_{j\in J} \mu_j \delta_{x_j} \qquad (18)$$

for some positive finite measure $\tilde{\mu}$, where

$$\nu_j \leq S_{N,s} \mu_j^{2_s^*}. \tag{19}$$

Claim: the set J is empty.

If not, there exists an index j_0 such that $\nu_{j_0} \neq 0$ at x_{j_0}. Fix $\varepsilon > 0$ and consider a cut-off function ϑ_ε such that

$$\begin{cases} 0 \leq \vartheta_\varepsilon \leq 1 & \text{in } \Omega \\ \vartheta_\varepsilon = 1 & \text{in } B(x_{j_0}, \varepsilon) \\ \vartheta_\varepsilon = 0 & \text{in } \Omega \setminus B(x_{j_0}, 2\varepsilon). \end{cases}$$

Since the sequence $\{u_n \vartheta_\varepsilon\}_n$ is still bounded in $X_0^s(\Omega)$, we have that

$$\lim_{n\to\infty} \mathcal{I}_{a,b}(u_n)\,[u_n\vartheta_\varepsilon] = 0,$$

thus

$$\begin{aligned} o(1) = \mathcal{I}'_{a,b}(u_n)\,[u_n\vartheta_\varepsilon] &= \left(a + b\|u_n\|^2\right) \langle u_n, u_n\vartheta_\varepsilon\rangle_{X_0^s(\Omega)} - \int_\Omega |u_n|^{2_s^*} \vartheta_\varepsilon\, dx \\ &= \left[\left(a + b\|u_n\|^2\right) \int_{\mathcal{Q}} u_n(y) \frac{(u_n(x) - u_n(y))(\vartheta_\varepsilon(x) - \vartheta_\varepsilon(y))}{|x-y|^{N+2s}}\, dx\, dy \right. \\ &\quad \left. + \int_{\mathcal{Q}} \vartheta_\varepsilon(x) \frac{(u_n(x) - u_n(y))^2}{|x-y|^{N+2s}}\, dx\, dy \right] - \int_\Omega |u_n|^{2_s^*} \vartheta_\varepsilon\, dx. \end{aligned} \tag{20}$$

as $n \to \infty$. By using the Hölder inequality, we estimate the first term of (20)

$$\begin{aligned} &\left(a + b\|u_n\|^2\right) \int_{\mathcal{Q}} u_n(y) \frac{(u_n(x) - u_n(y))(\vartheta_\varepsilon(x) - \vartheta_\varepsilon(y))}{|x-y|^{N+2s}}\, dx\, dy \\ &\leq \int_{\mathcal{Q}} \frac{(u_n(x) - u_n(y))^2}{|x-y|^{N+2s}}\, dx\, dy \int_{\mathcal{Q}} u_n^2(y) \frac{(\vartheta_\varepsilon(x) - \vartheta_\varepsilon(y))^2}{|x-y|^{N+2s}}\, dx\, dy \\ &C \int_{\mathcal{Q}} u_n^2(y) \frac{(\vartheta_\varepsilon(x) - \vartheta_\varepsilon(y))^2}{|x-y|^{N+2s}}\, dx\, dy \end{aligned}$$

for some $C > 0$. As in [3, Lemma 2.1], we have that

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \int_{\mathcal{Q}} |u_n(y)|^2 \frac{|\vartheta_\varepsilon(x) - \vartheta_\varepsilon(y)|^2}{|x-y|^{N+2s}}\, dx\, dy = 0. \tag{21}$$

Regarding the second term of (20), recalling (18), we get

$$\begin{aligned} &\lim_{n\to\infty} \left(a + b\|u_n\|^2\right) \int_{\mathcal{Q}} \vartheta_\varepsilon(x) \frac{(u_n(x) - u_n(y))^2}{|x-y|^{N+2s}}\, dx\, dy \\ &\geq \lim_{n\to\infty} \left[a \int_{\mathbb{R}^{2N} \setminus B(x_{j_0}, 2\varepsilon)^c \times \Omega^c} \vartheta_\varepsilon(x) \frac{(u_n(x) - u_n(y))^2}{|x-y|^{N+2s}}\, dx\, dy \right. \\ &\quad \left. + b \left(\int_{\mathcal{Q}} \vartheta_\varepsilon(x) \frac{(u_n(x) - u_n(y))^2}{|x-y|^{N+2s}}\, dx\, dy \right)^2 \right] \\ &\geq a \int_{\mathbb{R}^{2N} \setminus B(x_{j_0}, 2\varepsilon)^c \times \Omega^c} \vartheta_\varepsilon(x) \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}}\, dx\, dy + a\mu_{j_0} \end{aligned}$$

$$+b\left(\int_{Q}\vartheta_{\varepsilon}(x)\frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\,dx\,dy\right)^2+b\mu_{j_0}^2.$$

Hence

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\left(a+b\|u_n\|^2\right)\int_{Q}\vartheta_{\varepsilon}(x)\frac{(u_n(x)-u_n(y))^2}{|x-y|^{N+2s}}\,dx\,dy\geq a\mu_{j_0}+b\mu_{j_0}^2. \tag{22}$$

Finally, exploiting (17) we have

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_{\Omega}|u_n|^{2_s^*}\vartheta_{\varepsilon}\,dx=\lim_{\varepsilon\to 0}\int_{\Omega}|u|^{2_s^*}\vartheta_{\varepsilon}\,dx+\nu_{j_0}=\nu_{j_0}. \tag{23}$$

Putting together (21), (22) and (23), and using (19), we obtain

$$0\geq a\mu_{j_0}+b\mu_{j_0}^2-\nu_{j_0}\geq a\mu_{j_0}+b\mu_{j_0}^2-S_{N,s}^{-\frac{2_s^*}{2}}\mu_{j_0}^{\frac{2_s^*}{2}}=\mu_{j_0}\left(a+b\mu_{j_0}-S_{N,s}^{-\frac{2_s^*}{2}}\mu_{j_0}^{\frac{2_s^*}{2}-1}\right).$$

We define

$$\tilde{f}_{N,s}(\zeta)=a+b\zeta-S_{N,s}^{-\frac{2_s^*}{2}}\zeta^{\frac{2_s^*}{2}-1}\quad\text{for }\zeta\geq 0.$$

At this point, noting that the condition $a^{(N-4s)/2s}b>\mathsf{P}S_{N,s}$ implies $\tilde{f}_{N,s}(x)>0$, we deduce

$$a+b\mu_{j_0}-S_{N,s}^{-\frac{2_s^*}{2}}\mu_{j_0}^{\frac{2_s^*}{2}-1}>0.$$

Hence $\mu_{j_0}=0$, and recalling (19) $\nu_{j_0}=0$ as well.

So the set $J=\emptyset$, and using the Brezis-Lieb lemma (see [5, Theorem 1]) we can rewrite (17) as

$$\lim_{n\to\infty}\int_{\Omega}|u_n|^{2_s^*}\,dx=\int_{\Omega}|u|^{2_s^*}\,dx.$$

Hence $u_n\to u$ in $L^{2_s^*}(\Omega)$ and

$$\lim_{n\to\infty}\int_{\Omega}|u_n|^{2_s^*-2}u_n(u-u_n)\,dx=0. \tag{24}$$

Coupling (24) and the fact that $\mathcal{I}'_{a,b}(u_n)\to 0$ as $n\to\infty$ we get

$$\begin{aligned}0&=\lim_{n\to\infty}\mathcal{I}'_{a,b}(u_n)[u_n-u]=\lim_{n\to\infty}\Big[\left(a+b\|u_n\|^2\right)\langle u_n,u_n-u\rangle_{X_0^s(\Omega)}\\&\quad-\int_{\Omega}|u_n|^{2_s^*-2}u_n(u_n-u)\,dx\Big]\\&=\lim_{n\to\infty}\left(a+b\|u_n\|^2\right)\langle u_n,u_n-u\rangle_{X_0^s(\Omega)}.\end{aligned}$$

From the last chain of equalities, recalling that $\{u_n\}_n\subset X_0^s(\Omega)$ is bounded, we obtain

$$\lim_{n\to\infty}\langle u_n,u_n-u\rangle_{X_0^s(\Omega)}=0. \tag{25}$$

To conclude the proof it suffices to notice that thanks to (25) and $u_n\rightharpoonup u$ we have

$$\|u_n-u\|^2=\langle u_n,u_n-u\rangle_{X_0^s(\Omega)}-\langle u,u_n-u\rangle_{X_0^s(\Omega)}\to 0$$

as $n\to\infty$. $\square$

Proof of Theorem 1 (iii) In order to establish the convexity we will show that

$$\mathcal{I}''_{a,b}(u)\,[v,v] \geq 0 \quad \text{for all } u, v \in X_0^s(\Omega).$$

Differentiating (7) we notice that

$$\mathcal{I}''_{a,b}(u)\,[v,v] = a\|v\|^2 + b\|u\|^2\|v\|^2 - (2_s^* - 1)\int_\Omega |u|^{2_s^*-2}v^2\,dx. \tag{26}$$

Using the Hölder and the Sobolev inequalities we get

$$\int_\Omega |u|^{2_s^*-2}v^2\,dx \leq \|u\|_{2_s^*}^{2_s^*-2}\|v\|_{2_s^*}^2 \leq S_{N,s}^{-\frac{2_s^*}{2}}\|u\|^{2_s^*-2}\|v\|^2. \tag{27}$$

Putting together (26) and (27) we obtain

$$\mathcal{I}''_{a,b}(u)\,[v,v] \geq \|v\|^2\left[a + b\|u\|^2 - (2_s^* - 1)S_{N,s}^{-\frac{2_s^*}{2}}\|u\|^{2_s^*-2}\right].$$

At this point we set

$$\hat{f}_{N,s}(\zeta) = a + b\zeta^2 - (2_s^* - 1)S_{N,s}^{\frac{2_s^*}{2}}\zeta^{2_s^*-2} \quad \text{for all } \zeta \geq 0,$$

and we want to prove that it is positive on $[0,\infty)$. Indeed, with a simple computation it is possible to show that $\hat{f}_{N,s}$ attains its global minimum at

$$\hat{m}_{N,s} = \left(\frac{2bS_{N,s}^{\frac{2_s^*}{2}}}{(2_s^* - 1)(2_s^* - 2)}\right)^{\frac{1}{2_s^*-4}}$$

and that

$$\hat{f}_{N,s}(\zeta) \geq 0 \Leftrightarrow a^{\frac{N-4s}{2s}}b \geq C_{N,s}$$

for all $\zeta \geq 0$. □

Remark 1 It is clear from the proof that the functional $\mathcal{I}_{a,b}$ is strictly convex provided that $a^{(N-4s)/2s}b > C_{N,s}$.

3 Application to a perturbed Kirchhoff problem

This section is devoted to study an application of Theorem 1. More precisely we want to study the set of solutions of the perturbed problem

$$\begin{cases} \left(a + b\displaystyle\int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\right)(-\Delta)^s u = |u|^{2_s^*-2}u + \lambda g(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{$P_{a,b}^\lambda$}$$

where as before a, b are real positive parameter, Ω is a bounded domain and $\lambda > 0$. As for g, we generalize to the fractional case the assumptions present in [13]. Namely, we make the following assumptions:

(H_1) $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $g(x,0) = 0$ a.e. in Ω;

(H_2) $g(x,t) > 0$ for every $t > 0$ and $g(x,t) < 0$ for every $t < 0$ a.e. in Ω. In addition, we require that there is a $\mu > 0$ such that $g(x,t) \geq \mu > 0$ a.e in Ω and for every $t \in I$, where I is some open interval of $(0,\infty)$;

(H_3) there is a constant $c > 0$ and $p \in (2, 2_s^*)$ such that $g(x,t) \leq c(1+|t|^{p-1})$ a.e. in Ω;

(H_4) $\lim_{t\to 0} g(x,t)/|t| = 0$ uniformly with respect to $x \in \Omega$.

Using a variational approach, we investigate the existence of critical points of the functional defined on the space $X_0^s(\Omega)$

$$\mathcal{I}_{a,b}^{\lambda}(u) := \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{2_s^*}\|u\|_{2_s^*}^{2_s^*} - \lambda\int_\Omega G(x,u)\,dx$$

where we denote with $G(x,t) = \displaystyle\int_0^t g(x,\omega)d\omega$.

We begin the treatment of our problem by proving a series of technical results that will be useful throughout this section.

Remark 2 Before starting, let us recall the functions

$$f_{N,s}(\zeta) := \frac{a}{2} + \frac{b}{4}\zeta^2 - \frac{S_{N,s}^{-\frac{2_s^*}{2}}}{2_s^*}\zeta^{2_s^*-2}$$

and

$$\tilde{f}_{N,s}(\zeta) = a + b\zeta - S_{N,s}^{-\frac{2_s^*}{2}}\zeta^{\frac{2_s^*}{2}-1}$$

defined in the proofs of Theorems 1 (i) and 1 (ii). As we have already seen these functions have a unique local minimizer attained respectively at

$$m_{N,s} = \left[\frac{b}{2}\frac{2_s^*}{2_s^*-2}S_{N,s}^{\frac{2_s^*}{2}}\right]^{\frac{1}{2_s^*-4}},$$

and

$$\tilde{m}_{N,s} = \left[\frac{2b}{2_s^*-2}S_{N,s}^{\frac{2_s^*}{2}}\right]^{\frac{1}{2_s^*-4}}.$$

Furthermore, $f_{N,s}(m_{N,s}) > 0$ if and only if $a^{\frac{N-4s}{2s}}b > L_{N,s}$ and $f_{N,s}(m_{N,s}) = 0$ when $a^{\frac{N-4s}{2s}}b = L_{N,s}$. Analogously $\tilde{f}_{N,s}(\tilde{m}_{N,s}) > 0$ if and only if $a^{(N-4s)/2s}b > \mathsf{PS}_{N,s}$ and $\tilde{f}_{N,s}(\tilde{m}_{N,s}) = 0$ when $a^{(N-4s)/2s}b = \mathsf{PS}_{N,s}$.

Proposition 1 *Let $u \in X_0^s(\Omega)\setminus\{0\}$. We have that:*

(i) *for every $\zeta > 0$ it holds*

$$\frac{a}{2}\|u\|^2 + \frac{b}{4}\zeta^2\|u\|^4 - \frac{1}{2_s^*}\zeta^{2_s^*-2} > f_{N,s}(\zeta\|u\|)\|u\|^2;$$

(ii) *for every $\zeta > 0$ it holds*

$$a\|u\|^2 + b\zeta^2\|u\|^4 - \|u\|_{2_s^*}^{2_s^*}\zeta^{2_s^*-2} > \tilde{f}_{N,s}(\zeta\|u\|)\|u\|^2.$$

Proof From the boundedness of Ω it follows as in [10] that

$$\zeta^2\left[\frac{a}{2}\|u\|^2+\frac{b}{4}\zeta^2\|u\|^4-\frac{1}{2_s^*}\zeta^{2_s^*-2}\|u\|_{2_s^*}^{2_s^*}\right]=\frac{a}{2}(\zeta\|u\|)^2+\frac{b}{4}(\zeta\|u\|)^4-\frac{\|u\|_{2_s^*}^{2_s^*}}{\|u\|^{2_s^*}}\frac{(\zeta\|u\|)^{2_s^*}}{2_s^*}$$
$$>\frac{a}{2}(\zeta\|u\|)^2+\frac{b}{4}(\zeta\|u\|)^4-S_{N,s}^{-\frac{2_s^*}{2}}\frac{(\zeta\|u\|)^{2_s^*}}{2_s^*} \tag{28}$$

where in the last expression we used the Sobolev inequality. Dividing by ζ^2 we get the first statement. (ii) follows similarly. □

As we did in the previous section, we show in the following lemma that the functional $\mathcal{I}_{a,b}^{\lambda}$ is sequentially lower semicontinuous and satisfies the Palais–Smale condition for a and b sufficiently large.

Lemma 1 *Let $a, b \in \mathbb{R}^+$, $(u_k)_k \subset X_0^s(\Omega)$ and $\lambda_k \to \lambda \geq 0$ as $k \to \infty$:*

(1) *if $a^{(N-4s)/2s}b \geq L_{N,s}$ and $u_k \rightharpoonup u$ in $X_0^s(\Omega)$ then*

$$\mathcal{I}_{a,b}^{\lambda}(u) \leq \liminf_{k\to\infty} \mathcal{I}_{a,b}^{\lambda}(u_k);$$

(2) *if $a^{(N-4s)/2s}b > PS_{N,s}$, $\mathcal{I}_{a,b}^{\lambda}(u_k) \to c$ and $\left(\mathcal{I}_{a,b}^{\lambda}\right)'(u_k) \to 0$ then $(u_k)_k$ is convergent to some u in $X_0^s(\Omega)$ up to subsequence.*

Proof The proof follows closely the arguments of Theorem 1 (i) and (ii) with minor changes. □

Now choose $\lambda \geq 0$ and $u \in X_0^s(\Omega)$. For every $\zeta > 0$ we introduce the fiber map

$$\mathcal{J}_{a,b}^{\lambda,u}(\zeta) := \mathcal{I}_{a,b}^{\lambda}(\zeta u) = \frac{a}{2}\zeta^2\|u\|^2+\frac{b}{4}\zeta^4\|u\|^4-\frac{\zeta^{2_s^*}}{2_s^*}\|u\|_{2_s^*}^{2_s^*}-\int_{\Omega}G(x,\zeta u)\,dx.$$

Proposition 2 *Let $\lambda \in \mathbb{R}$ be nonnegative and $u \in X_0^s(\Omega \setminus \{0\}$. Then there exists a neighbourhood V_λ of 0 such that $\mathcal{J}_{a,b}^{\lambda,u}(\zeta) > 0$ for every $\zeta \in V_\lambda \cap (0,\infty)$. We also have that $\mathcal{J}_{a,b}^{\lambda,u}(\zeta) \to \infty$ as $\zeta \to \infty$.*

Remark 3 The previous proposition shows indirectly that the map $\mathcal{J}_{a,b}^{\lambda,u}(\zeta)$ is bounded from below

Proof Fix $\varepsilon > 0$. Exploiting (H_4), for ζ small enough, we get

$$\mathcal{J}_{a,b}^{\lambda,u}(\zeta)=\zeta^2\left(\frac{a}{2}\|u\|^2+\frac{b}{4}\zeta^2\|u\|^4-\frac{\zeta^{2_s^*-2}}{2_s^*}\|u\|_{2_s^*}^{2_s^*}-\lambda\int_{\Omega}\frac{G(x,\zeta u)}{\zeta^2}dx\right)$$
$$\geq \zeta^2\left(\frac{a}{2}\|u\|^2+\frac{b}{4}\zeta^2\|u\|^4-\frac{\zeta^{2_s^*-2}}{2_s^*}\|u\|_{2_s^*}^{2_s^*}-\lambda\frac{\varepsilon}{2}\|u\|_2^2\right).$$

Using the Sobolev inequality, taking ε appropriately and choosing ζ even smaller if necessary we obtain the first part of the statement. In order to complete the proof, it is sufficient to remember that G has subcritical growth and to notice that $2 < p < 2_s^* < 4$. □

Now we choose $u \in X_0^s(\Omega)$ and we consider the system

$$\begin{cases} \mathcal{J}_{a,b}^{\lambda,u}(\zeta) = 0 \\ (\mathcal{J}_{a,b}^{\lambda,u})'(\zeta) = 0 \\ \mathcal{J}_{a,b}^{\lambda,u}(\zeta) = \inf_{\varrho>0} \mathcal{J}_{a,b}^{\lambda,u}(\varrho) \end{cases} \tag{29}$$

in the unknowns λ and ζ.

Proposition 3 *Let T and Z be two topological space, and assume that Z is compact. Let $h\colon T \times Z \to \mathbb{R}$ be a continuous function. Then the function $\hat{h}(t) := \inf_{z\in Z} h(t,z)$ is continuous on T.*

Proof We first observe that for any $t \in T$ the function $\hat{h}$ is well defined since Z is compact and the infimum is always attained at some point $z(t) \in Z$. Recalling that the sets $(-\infty, a)$ and (b, ∞) for some $a, b \in \mathbb{R}$ form a subbase of $\mathbb{R}$, our proof is reduced to the following:
Claim: $\hat{h}^{-1}(-\infty, a)$ and $\hat{h}^{-1}(b, \infty)$ are open in T.
We start showing the truthfulness of the claim for $(-\infty, a)$. Denote with $\pi_T : T \times Z \to T$ the usual projection and remember that is a map continuous and open. Noticing that $\hat{h}^{-1}(-\infty, a) = \left(\pi_T \circ h^{-1}\right)(-\infty, a)$ it is straightforward to conclude. On the other hand, consider an half line (b, ∞) for some $b \in \mathbb{R}$. If $t \in T$ is such that $\hat{h}(t) > b$ then $h(t,z) > b$ for any $z \in Z$. In other words if $t \in \hat{h}^{-1}(b, \infty)$ then $(t, z) \in h^{-1}(b, \infty)$ for any $z \in Z$. Since h is continuous and (b, ∞) is open, for any $(t, z) \in \hat{h}^{-1}(b, \infty) \times Z$ it is possible to find a neighbourhood $U_{t,z} \times V_{t,z}$ such that

$$(t, z) \in U_{t,z} \times V_{t,z} \subset h^{-1}(b, \infty).$$

Hence $\left\{V_{t,z}\right\}_{z\in Z}$ is an open covering of Z. Exploiting the compactness of Z, we can extract a finite subcovering indexed by a finite set $\mathcal{K}(t)$ with the property

$$\{t\} \times Z \subset \bigcap_{i\in\mathcal{K}(t)} U_{t,z_i} \times Z \subset h^{-1}(b, \infty).$$

Thus, we can conclude observing that

$$\hat{h}^{-1}(b, \infty) = \bigcup_{t\in\hat{h}^{-1}(b,\infty)} \bigcap_{i\in\mathcal{K}(t)} U_{(t,z_i)}. \tag{30}$$

□

Remark 4 We can even strengthen the result above for functions defined on non compact spaces requiring divergence at infinity. For instance, suppose $h\colon (\mathbb{R}_+)^2 \to \mathbb{R}$ is continuous and such that $\lim_{z\to\infty} h(t,z) = \infty$ for any $t \in \mathbb{R}_+$. The proof for sets as $(-\infty, a)$ is the same. As regard sets of the type (b, ∞) we observe that $\hat{h}^{-1}(b, \infty)$ can be written as in (30) plus an half line due to the divergence of the function at infinity.

Proposition 4 *Let $a, b \in \mathbb{R}^+$ such that $a^{(N-4s)/2s} b \geq L_{N,s}$. For any $u \in X_0^s(\Omega) \setminus \{0\}$ there is a unique $\lambda = \lambda_0^s(u)$ that solves (29).*

Proof Define the continuous function $h(\lambda, \zeta) := \mathcal{J}_{a,b}^{\lambda,u}(\zeta)$. We start pointing out that $h(0, \zeta)$ is positive on $(0, \infty)$ (see Remark 2) and goes to $+\infty$ as $\zeta \to \infty$. By continuity we have that for λ small $h(\lambda, \zeta)$ is nonnegative for all $\zeta \in \mathbb{R}^+$. Moreover, from Proposition 2 it follows that for any $\lambda \geq 0$ there is a neighbourhood V_λ such that $h(\lambda, \zeta) > 0$ for all $\zeta \in V_\lambda \cap (0, \infty)$. We also have that $h(\lambda, \zeta) \to -\infty$ as $\lambda \to \infty$ for any $\zeta > 0$. At this point we define the continuous function (refer to Proposition 3) $i(\lambda) = \inf_{\zeta\in[0,\infty)} h(\lambda,\zeta)$. From the previous considerations, we can deduce that for λ sufficiently large the function i is negative, while if we restrict λ it is equal to zero. This is due to the fact that the function $h(\lambda,\zeta)$ for λ big enough has a global minimizer in the variable ζ at a negative level. Shrinking λ, and remembering that all continuous functions are homotopically equivalent, this minimizer becomes local and attained at a positive level. All these arguments ensure us the existence of the desired $\lambda_0^s(u)$ that solves (29). □

Corollary 1 *Let $u \in X_0^s(\Omega)\setminus\{0\}$. The number $\lambda_0^s(u)$ is the only parameter such that $\inf_{\zeta\in(0,\infty)} \mathcal{J}_{a,b}^{\lambda_0^s(u),u}(\zeta) = 0$. In addition,*

$$\inf_{\zeta\in(0,\infty)} \mathcal{J}_{a,b}^{\lambda,u}(\zeta) \begin{cases} < 0 & \text{if } \lambda > \lambda_0^s(u) \\ = 0 & \text{if } 0 \le \lambda \le \lambda_0^s(u). \end{cases}$$

Proof The statement follows immediately from the proof of Proposition 4. □

Now we define a suitable parameter independent from u that will play a crucial role in the following. More precisely, we set

$$\overline{\lambda}_0^s := \inf_{u\in X_0^s(\Omega)\setminus\{0\}} \lambda_0^s(u).$$

The next Proposition shows how the parameter $\overline{\lambda}_0^s$ could vary depending on the choice made on a and b.

Proposition 5 *The following statements hold:*

- (i) *if $a^{(N-4s)/2s}b > L_{N,s}$ then $\overline{\lambda}_0^s > 0$;*
- (ii) *if $a^{(N-4s)/2s}b = L_{N,s}$ then $\overline{\lambda}_0^s = 0$. Furthermore, if $(u_k)_k \subset X_0^s(\Omega)\setminus\{0\}$ is a sequence such that $\lambda_0^s(u_k) \to \overline{\lambda}_0^s$ as $k\to\infty$, we have that $u_k \rightharpoonup 0$ and $\frac{\|u_k\|_2^2}{\|u_k\|_{2_s^*}^2} \to S_{N,s}$.*

Proof (i) As a first step we observe that the function $u \to \lambda_0^s(u)$ is well defined and homogeneous of degree zero. In fact, taking a couple $(\zeta, \lambda_0^s(u))$ that solves (29) and $\mu > 0$, since $\mathcal{J}_{a,b}^{\lambda,\mu u}(\zeta) = \mathcal{J}_{a,b}^{\lambda,u}(\mu\zeta)$ and $\left(\mathcal{J}_{a,b}^{\lambda,\mu u}\right)'(\zeta) = \left(\mathcal{J}_{a,b}^{\lambda,u}\right)'(\mu\zeta)$ we have that also $(\frac{\zeta}{\mu}, \lambda_0^s)$ is a solution of (29). From the uniqueness of the parameter $\lambda_0^s(\mu u)$ it follows that $\lambda_0^s(\mu u) = \lambda_0^s(u)$. Now, assume by contradiction that $\overline{\lambda}_0^s = 0$. If that, there is a sequence $(u_k)_k \subset X_0^s(\Omega)\setminus\{0\}$ such that $\lambda_k := \lambda_0^s(u_k) \to 0$. By homogeneity, we may assume that $\|u_k\| = 1$. From Proposition 4 it follows that there exists $\zeta_k > 0$ such that $\mathcal{J}_{a,b}^{\lambda_k,u_k}(\zeta_k) = 0$, that is

$$\frac{a}{2} + \frac{b}{2}\zeta_k^2 - \frac{1}{2_s^*}\|u_k\|_{2_s^*}^{2_s^*}\zeta_k^{2_s^*-2} - \lambda_k \int_\Omega \frac{G(x,\zeta_k u)}{\zeta_k^2}\,dx = 0.$$

Recalling Remark 2, we get that

$$f_{N,s}(\zeta_k) < \frac{a}{2} + \frac{b}{2}\zeta_k^2 - \frac{1}{2_s^*}\|u_k\|_{2_s^*}^{2_s^*}\zeta_k^{2_s^*-2} = \lambda_k \int_\Omega \frac{G(x,\zeta_k u)}{\zeta_k^2}\,dx. \tag{31}$$

Hypotheses H_3 and H_4 implies that for any $\varepsilon > 0$ there exists a positive constant $c > 0$ such that $|G(x,t)| < \frac{\varepsilon}{2}t^2 + \frac{c}{p}|t|^p$ for all $x \in \Omega$ and all $t \in \mathbb{R}$. So, the sequence $(\zeta_k)_k$ must be bounded, and up to subsequence converges to some $\overline{\zeta} > 0$. At this point, letting $k \to \infty$, (31) becomes

$$0 < f_{N,s}(\overline{\zeta}) = \lim_{k\to\infty} \lambda_k \int_\Omega \frac{G(x, \zeta_k u_k)}{\zeta_k^2}\, dx = 0$$

which is clearly a contradiction.

(ii) Up to a translation, we can suppose that $0 \in \Omega$. Take a nonnegative cut-off function such that $\varphi(x) = 1$ in $B_R(0)$ for some $R > 0$. Fix $\varepsilon > 0$ and consider

$$v_\varepsilon(x) := \frac{\varphi(x)}{\left(\varepsilon + |x|^2\right)^{\frac{N-2s}{2}}}.$$

We set $u_\varepsilon := v_\varepsilon / \|v_\varepsilon\|$ and we notice that from [29, Propositions 21 and 22] it follows that

$$\|u_\varepsilon\| = 1, \quad \|u_\varepsilon\|_{2_s^*}^{2_s^*} \geq S_{N,s}^{-\frac{2_s^*}{2}} + O(\varepsilon^{\frac{N-2s}{2}}), \quad \|v_\varepsilon\| \leq \varepsilon^{-\frac{N-2s}{4}} C_1 + O(1)$$

as $\varepsilon \to 0$ for some $C_1 > 0$. In virtue of the previous estimates, we get

$$\begin{aligned} \mathcal{J}_{a,b}^{\lambda,u_\varepsilon}(\zeta) &= \frac{a}{2}\zeta^2 + \frac{b}{4}\zeta^4 - \frac{\zeta^{2_s^*}}{2_s^*}\|u_\varepsilon\|_{2_s^*}^{2_s^*} - \lambda \int_\Omega G(x, \zeta u_\varepsilon)\, dx \\ &\leq \zeta^2 f_{N,s}(\zeta) - \frac{\zeta^{2_s^*}}{2_s^*} O(\varepsilon^{\frac{N-2s}{2}}) - \lambda \int_\Omega G(x, \zeta u_\varepsilon)\, dx. \end{aligned}$$

Choosing as $\zeta = m_{N,s}$ we obtain

$$\mathcal{J}_{a,b}^{\lambda,u_\varepsilon}(m_{N,s}) = -\frac{m_{N,s}^{2_s^*}}{2_s^*} O(\varepsilon^{\frac{N-2s}{2}}) - \lambda \int_\Omega G(x, m_{N,s} u_\varepsilon)\, dx. \tag{32}$$

Claim: There exists a constant $C_2 > 0$ such that $\int_\Omega G(x, m_{N,s} u_\varepsilon)\, dx \geq C_2 \varepsilon^{\frac{N}{2}}$ as $\varepsilon \to 0$.

assumptions H_2 implies the existence of $\mu > 0$ such that $g(x,t) \geq \chi_I$ where I is an open interval of $(0, \infty)$ and χ_I is its characteristic function. So, there exists $\beta > 0$ such that $G(x,t) \geq \tilde{G}(t) := \mu \int_0^t \chi_I(\omega) d\omega \geq \beta$ for any $t \geq \alpha$ where $\alpha := \inf I$ is positive. At this point, we have

$$\begin{aligned} \int_\Omega G(x, m_{N,s} u_\varepsilon)\, dx \geq \int_{|x|\leq R} G(x, m_{N,s} u_\varepsilon)\, dx &= \int_{|x|\leq R} G\left(x, \frac{m_{N,s}}{\|v_\varepsilon\|(\varepsilon + |x|^2)^{\frac{N-2s}{2}}}\right) dx \\ &\geq \int_{|x|\leq R} \tilde{G}\left(\frac{m_{N,s}}{\|v_\varepsilon\|(\varepsilon + |x|^2)^{\frac{N-2s}{2}}}\right) dx \qquad (33) \\ &= \int_0^R \tilde{G}\left(\frac{m_{N,s}}{\|v_\varepsilon\|(\varepsilon + w^2)^{\frac{N-2s}{2}}}\right) w^{N-1}\, dw \\ &\geq \int_0^{\sqrt{\varepsilon}R} \tilde{G}\left(\frac{m_{N,s}}{\|v_\varepsilon\|(\varepsilon + w^2)^{\frac{N-2s}{2}}}\right) w^{N-1}\, dw. \qquad (34) \end{aligned}$$

We emphasize that if

$$\frac{m_{N,s}}{\|v_\varepsilon\|(\varepsilon + w^2)^{\frac{N-2s}{2}}} \geq \alpha$$

then

$$\int_0^{\sqrt{\varepsilon}R} \tilde{G}\left(\frac{m_{N,s}}{\|v_\varepsilon\|(\varepsilon+w^2)^{\frac{N-2s}{2}}}\right) w^{N-1}\,dw \geq \beta \int_0^{\sqrt{\varepsilon}R} w^{N-1}\,dw = C_2\varepsilon^{\frac{N}{2}}.$$

Since $w \in \left[0, \sqrt{\varepsilon}R\right]$, we have

$$\begin{aligned}\frac{m_{N,s}}{\|v_\varepsilon\|(\varepsilon+w^2)^{\frac{N-2s}{2}}} &\geq \frac{m_{N,s}}{\varepsilon^{-\frac{N-2s}{4}}\left(C_1+O(\varepsilon^{\frac{N-2s}{4}})\right)\left(\varepsilon+R^2\varepsilon\right)^{\frac{N-2s}{2}}} \\ &= \frac{m_{N,s}}{\varepsilon^{\frac{N-2s}{4}}\left(C_1+O(\varepsilon^{\frac{N-2s}{4}})\right)\left(1+R^2\right)^{\frac{N-2s}{2}}} \geq \alpha\end{aligned}$$

as $\varepsilon \to 0$ proving the claim.

As a consequence of the claim and (33) we obtain

$$\mathcal{J}_{a,b}^{\lambda,u_\varepsilon}(m_{N,s}) \leq \varepsilon^{\frac{N-2s}{2}}\left(-\frac{1}{2_s^*}O(1) - \lambda C_2\varepsilon^s\right) < 0.$$

Hence, $\lambda_0^s(u_\varepsilon) < \lambda$. We can now let $\lambda \to 0$ and we get $\overline{\lambda}_0^s = 0$ as desired. In order to see the last part, let $(u_k)_k \subset X_0^s(\Omega)\setminus\{0\}$ be a sequence such that $\lambda_k := \lambda_0^s(u_k) \to \overline{\lambda}_0^s = 0$. As we did in part i), we suppose $\|u_k\| = 1$, $u_k \rightharpoonup u$ and that there exists $\zeta_k > 0$ such that

$$\frac{a}{2} + \frac{b}{4}\zeta_k^2 + \frac{\zeta^{2_s^*-2}}{2_s^*}\|u_k\|_{2_s^*}^{2_s^*} - \lambda_k \int_\Omega \frac{G(x,\zeta_k u_k)}{\zeta_k^2}\,dx = 0. \tag{35}$$

Combining assumptions H_3, H_4 and (35), we can deduce that, up to subsequence, $\zeta_k \to \overline{\zeta}$ and $\|u_k\|_{2_s^*}^{2_s^*} \to \gamma$ as $k \to \infty$. Passing to the limit in (35), we get

$$\frac{a}{2} + \frac{b}{4}\overline{\zeta}^2 - \frac{\overline{\zeta}^{2_s^*-2}}{2_s^*}\gamma = 0.$$

From $a^{(N-4s)/2s}b = L_{N,s}$ it follows that $\gamma = S_{N,s}^{-\frac{2_s^*}{2}}$, thus $(u_k)_k$ is a minimizing sequence for $S_{N,s}$. Now, by contradiction assume $u \neq 0$. We point out that by the lower semicontinuity of the norm we have $\|u\| \leq 1$. Coupling this fact with Remark 2, we obtain

$$\begin{aligned}0 &\leq \frac{a}{2} + \frac{b}{4}\overline{\zeta}^2 - \frac{S_{N,s}^{-\frac{2_s^*}{2}}}{2_s^*}\overline{\zeta}^{2_s^*-2}\|u\|^{2_s^*} \leq \frac{a}{2} + \frac{b}{4}\overline{\zeta}^2 - \frac{\overline{\zeta}^{2_s^*-2}}{2_s^*}\|u\|_{2_s^*}^{2_s^*} \\ &\leq \limsup_{k\to\infty}\left(\frac{a}{2} + \frac{b}{2}\zeta_k^2 - \frac{\zeta_k^{2_s^*}}{2_s^*}\|u_k\|_{2_s^*}^{2_s^*} - \lambda_k\int_\Omega \frac{G(x,\zeta_k u_k)}{\zeta_k^2}\,dx\right) = 0,\end{aligned}$$

which cannot happen since Ω is bounded, see [10]. □

Next proposition summarize the situation of the infimum depending on the choice of the parameter λ for the functional $\mathcal{J}_{a,b}^{\lambda,u}(\zeta)$.

Proposition 6 *If $\lambda \leq \overline{\lambda}_0^s$ then $\inf_{\zeta>0}\mathcal{J}_{a,b}^{\lambda,u}(\zeta) = 0$ for any $u \in X_0^s(\Omega)\setminus\{0\}$. On the other hand, if $\lambda > \overline{\lambda}_0^s$ there exists $u \in X_0^s(\Omega)\setminus\{0\}$ such that $\inf_{\zeta>0}\mathcal{J}_{a,b}^{\lambda,u}(\zeta) < 0$.*

Proof Take $\lambda \leq \overline{\lambda}_0^s$. We have that $\lambda \leq \overline{\lambda}_0^s \leq \lambda_0^s(u)$ for any $u \in X_0^s(\Omega) \setminus \{0\}$, then the conclusion comes from Corollary 1. Instead, let us consider $\lambda \in \mathbb{R}^*$ such that $\lambda > \overline{\lambda}_0^s$. By the definition of infimum, it is possible to find $u \in X_0^s(\Omega) \setminus \{0\}$ such that $\lambda \geq \lambda_0^s(u) > \overline{\lambda}_0^s$. Again, the assertion it is a consequence of Corollary 1. □

After some preliminary results we are ready to study the set of solutions of problem $(P_{a,b}^\lambda)$. The first step will consists in giving the proof for Theorems 2 and 3 providing the existence of global minimizers for $\lambda \geq \overline{\lambda}_0^s$.

Proof of Theorem 2 By the use of assumptions (H_3) and (H_4) it is easy to verify that $\mathcal{I}_{a,b}^\lambda$ is coercive. Furthermore, from 1 we also have the lower semicontinuity. At this point, as a consequence of the well known Weiestrass Theorem, we have that the infimum is attained. To conclude, we recall that Proposition 6 implies the existence of a function in which the functional turns out to be negative. □

Proof of Theorem 3 (i) Let $(\lambda_k)_k \subset \mathbb{R}^+$ a sequence such that $\lambda_k \searrow \overline{\lambda}_0^s$. Theorem 2 implies the existence of a sequence $(u_k)_k \subset X_0^s(\Omega) \setminus \{0\}$ such that $\iota_{\lambda_k}^s = \mathcal{I}_{a,b}^{\lambda_k}(u_k) < 0$. As we did in Proposition 5, after fixing $\varepsilon > 0$ we have

$$|G(x,t)| \leq \frac{\varepsilon}{2}t^2 + \frac{c}{p}|t|^p \tag{36}$$

for all $(x,t) \in \Omega \times \mathbb{R}$. Hence

$$\begin{aligned} \frac{a}{2}\|u_k\|^2 + \frac{b}{4}\|u_k\|^4 - \frac{1}{2_s^*}\|u_k\|_{2_s^*}^{2_s^*} &< \lambda_k \int_\Omega G(x,u_k)\,dx \\ &\leq \lambda_k \left(\frac{\varepsilon}{2}\|u_k\|_2^2 + \frac{c}{p}\|u_k\|_p^p\right) \leq \tilde{C}\left(\|u_k\|^2 + \|u_k\|^p\right) \end{aligned} \tag{37}$$

for some $\tilde{C} > 0$ since $X_0^s(\Omega) \hookrightarrow L^q(\Omega)$ continuously for any $q \in [2, 2_s^*]$. From $4 > 2_s^*$ it follows that $(\|u_k\|)_k$ must be bounded and it is not restrictive to assume $u_k \rightharpoonup u$ in $X_0^s(\Omega)$. Applying Lemma 1[(1)] we obtain

$$\mathcal{I}_{a,b}^{\overline{\lambda}_0^s}(u) \leq \liminf_{k\to\infty} \mathcal{I}_{a,b}^{\overline{\lambda}_0^s}(u_k) \leq 0.$$

On the other hand , Proposition 6 states that $\mathcal{I}_{a,b}^{\overline{\lambda}_0^s}(v) \geq 0$ for any $v \in X_0^s(\Omega)$, and so

$$\iota_{\overline{\lambda}_0^s}^s = \mathcal{I}_{a,b}^{\overline{\lambda}_0^s}(u) = 0. \tag{38}$$

It remains only to prove that u is a non trivial minimizer. To see that, observe that

$$\begin{aligned} &\frac{a}{2}\|u_k\|^2 + \frac{b}{4}\|u_k\|^4 - \frac{S_{N,s}^{-\frac{2_s^*}{2}}}{2_s^*}\|u_k\|^{2_s^*} \leq \frac{a}{2}\|u_k\|^2 + \frac{b}{4}\|u_k\|^4 - \frac{1}{2_s^*}\|u_k\|_{2_s^*}^{2_s^*} < \lambda_k \\ &< \lambda_k \int_\Omega G(x,u_k)\,dx \end{aligned}$$

where we used the fractional Sobolev inequality. Dividing by $\|u_k\|^2$ and exploiting (36), we get

$$f_{N,s}(\|u_k\|) \leq \lambda_k \left(\frac{\varepsilon}{2} + \frac{c}{p}\|u_k\|_p^p\right).$$

Were $u=0$, recalling that $X_0^s(\Omega)\hookrightarrow L^q(\Omega)$ for any $q\in\left[2,2_s^*\right)$, we would have

$$f_{N,s}\left(\|u_k\|\right)\to 0$$

as $k\to\infty$ since $\varepsilon>0$ is arbitrary. This fact is in contradiction with

$$f_{N,s}\left(\|u_k\|\right)\geq f_{N,s}(m_{N,s})>0$$

since $a^{(N-2s)/2s}b>L_{N,s}$. So u must be different from zero.
(ii) From Proposition 5[(ii)] we have $\overline{\lambda}_0^s$, and so

$$\mathcal{I}_{a,b}^{\overline{\lambda}_0^s}(u)=\frac{a}{2}\|u\|^2+\frac{b}{4}\|u\|^4-\frac{1}{2_s^*}\|u\|_{2_s^*}^{2_s^*}.$$

In virtue of Remark 2, we have

$$\mathcal{I}_{a,b}^{\overline{\lambda}_0^s}(u)=\|u\|^2 f_{N,s}(\|u\|)>0$$

for any $u\in X_0^s\setminus\{0\}$. Since (38) is still valid, we have that the infimum can be attained only in the case in which $u=0$. □

Corollary 2 *If $a^{(N-4s)/2s}b>L_{N,s}$ and $u\in X_0^s(\Omega)\setminus\{0\}$ is such that $\iota_{\overline{\lambda}^s}=\mathcal{I}_{a,b}^{\overline{\lambda}_0^s}(u)$ then $\overline{\lambda}_0^s=\lambda_0^s(u)$.*

Proof The pair $(\overline{\lambda}_0^s,u)$ solve the system (29). The conclusion follows by uniqueness. □

Proof of Theorem 4 Fix $\varepsilon>0$ and recall the Aubin-Talenti functions u_ε defined in Proposition 5. Choose $\zeta>0$ and keep $\lambda>0$ free. We have

$$\begin{aligned}\mathcal{J}_{a_k,b_k}^{\lambda,u_\varepsilon}(\zeta)&=\frac{a_k}{2}\zeta^2+\frac{b_k}{4}\zeta^4-\frac{\zeta^{2_s^*}}{2_s^*}\|u_k\|_{2_s^*}^{2_s^*}-\lambda\int_\Omega G(x,\zeta u_\varepsilon)\,dx\\&=\zeta^2 f_{N,s}^k(\zeta)-\frac{\zeta^{2_s^*}}{2_s^*}O\left(\varepsilon^{\frac{N-2s}{2}}\right)-\lambda\int_\Omega G(x,\zeta u_\varepsilon)\,dx\end{aligned}$$

where we defined with $f_{N,s}^k$ the map $f_{N,s}$ depending on the parameters a_k, b_k. We select $\zeta=m_{N,s}^k$ (here $m_{N,s}^k$ is the point point in which $f_{N,s}^k$ attains its minimum), and since $m_{N,s}^k\to m_{N,s}$ as $k\to\infty$, we get

$$\lim_{k\to\infty}\mathcal{J}_{a_k,b_k}^{\lambda,u_\varepsilon}(m_{N,s}^k)=-\frac{m_{N,s}^{2_s^*}}{2_s^*}O\left(\varepsilon^{\frac{N-2s}{2}}\right)-\lambda\int_\Omega G(x,m_{N,s}u_\varepsilon)\,dx. \tag{39}$$

Recalling that in Proposition 5 we obtained the estimate

$$\int_\Omega G(x,m_{N,s}u_\varepsilon)\,dx\geq C_2\varepsilon^{\frac{N}{2}},$$

from (39) we deduce

$$\lim_{k\to\infty}\mathcal{J}_{a_k,b_k}^{\lambda,u_\varepsilon}(m_{N,s}^k)\leq\varepsilon^{\frac{N-2s}{2}}\left(-\frac{1}{2_s^*}O(1)-\lambda C_2\varepsilon^s\right).$$

Hence for k sufficiently large and small ε

$$\mathcal{J}_{a_k,b_k}^{\lambda,u_\varepsilon}(m_{N,s}^k)<0.$$

As a consequence of that, we have that $\lambda_k \leq \lambda_0^s(u_\varepsilon) \leq \lambda$. Letting $\lambda \to 0$ we obtain that $\lambda_k \to 0$ as $k \to \infty$. Now, exploiting the homogeneity of degree zero of the function $\lambda_0^s(\cdot)$ proved in Proposition 5, we suppose $\|u_k\| = 1$ and $u_k \rightharpoonup u$. Arguing similarly as we did to get (35) we are able to deduce the existence of $\zeta_k > 0$ such that

$$\frac{a_k}{2} + \frac{b_k}{4}\zeta_k^2 - \frac{\zeta^{2_s^*-2}}{2_s^*}\|u_k\|_{2_s^*}^{2_s^*} - \lambda_k \int_\Omega \frac{G(x, \zeta_k u_k)}{\zeta_k^2}\, dx = 0 \tag{40}$$

Moreover, from H_3, H_4 and (40) it follows that $\zeta_k \to \overline{\zeta} > 0$ and that $\|u_k\|_{2_s^*}^{2^*s} to \gamma$ up to a subsequence as $k \to \infty$. Thus, passing to the limit in (40) we get

$$\frac{a}{2} + \frac{b}{4}\overline{\zeta}^2 - \frac{1}{2_s^*}\gamma\overline{\zeta}^{2_s^*-2} = 0.$$

Since $a^{(N-4s)/2s} b = L_{N,s}$ it must be $\gamma = S_{N,s}$ and that means $(u_k)_k$ is a minimizing sequence for the optimal Sobolev constant. We also have that $u = 0$. Indeed, if $u \neq 0$, combining Remark 2, the fact that by the sequentially lower semicontinuity of the norm $\|u\| \leq 1$ and Lemma 1[(1)], we obtain

$$\begin{aligned} 0 &\leq \frac{a}{2} + \frac{b}{4}\overline{\zeta}^2 - \frac{S_{N,s}^{-\frac{2_s^*}{2}}}{2_s^*}\overline{\zeta}^{2_s^*-2}\|u\|^{2_s^*} \leq \frac{a}{2} + \frac{b}{4}\overline{\zeta}^2 - \frac{\overline{\zeta}^{2_s^*-2}}{2_s^*}\|u\|_{2_s^*}^{2_s^*} \\ &\leq \liminf_{k\to\infty}\left(\frac{a_k}{2} + \frac{b_k}{4}\zeta_k^2 - \frac{\zeta_k^{2_s^*-2}}{2_s^*}\|u_k\|_{2_s^*}^{2_s^*} - \lambda_k \int_\Omega \frac{G(x, \zeta_k u_k)}{\zeta_k^2}\, dx\right) = 0 \end{aligned}$$

The conclusion comes from the nonexistence of minimizers for $S_{N,s}$ in bounded sets as shown in [10]. □

Now, we begin to investigate solutions of mountain pass type. As we will see, the situation changes if $\lambda \geq \overline{\lambda}_0^s$ or $\lambda < \overline{\lambda}_0^s$. The reader should keep in mind that from now to the end of the section we will consider positive parameters $a, b \in \mathbb{R}$ such that $a^{(N-4s)/2s} b > L_{N,s}$.

Proof of theorem 5 Take $\varepsilon > 0$. Recalling (36) and that $X_0^s(\Omega) \hookrightarrow L^q(\Omega)$ continuously for any $q \in [2, 2_s^*]$ we obtain

$$\mathcal{I}_{a,b}^\lambda \geq \left(\frac{a}{2} - \lambda C\varepsilon\right)\|u\|^2 + \frac{b}{4}\|u\|^4 - C\|u\|^{2_s^*} - \lambda C\|u\|^p \tag{41}$$

where $C > 0$ is a constant chosen adequately. By selecting $\varepsilon < a/(2\lambda C)$ there exists R_λ^s such that

$$\inf_{\|u\|=R_\lambda^s} \mathcal{I}_{a,b}^\lambda > 0.$$

Now, observe that $\mathcal{I}_{a,b}^\lambda(0) = 0$ and $\mathcal{I}_{a,b}^\lambda(u_{\overline{\lambda}_0^s}^s) \leq 0$. Indeed, $\mathcal{I}_{a,b}^\lambda(u_{\overline{\lambda}_0^s}^s) = 0$ if $\lambda = \overline{\lambda}_0^s$ while $\mathcal{I}_{a,b}^\lambda(u_{\overline{\lambda}_0^s}^s) < 0$ for $\lambda > \overline{\lambda}_0^s$ by Proposition 6. As a consequence of that, the functional possesses a mountain pass geometry. Furthermore, recalling Lemma 1[(2)] we have that $\mathcal{I}_{a,b}^\lambda$ satisfies the Palais–Smale condition. At this point the conclusion is obtained by applying the classic mountain pass theorem. □

Having analysed the situation for $\lambda \geq \overline{\lambda}_0^s$, now we draw our attention to the case $\lambda < \overline{\lambda}_0^s$. Namely, we will show the existence of non trivial solutions that are local minimizer or of mountain pass type.

Proposition 7 *If* $\lambda \leq \overline{\lambda}_0^s$ *then it is possible to find* $r = r(s)$, $M = M(s) > 0$ *such that*

$$\inf\left\{\mathcal{I}_{a,b}^{\lambda}(u) : u \in X_0^s(\Omega),\ \|u\| = r\right\} \geq M. \tag{42}$$

Proof Given $\varepsilon > 0$, because $\lambda \leq \overline{\lambda}_0^s$ and recalling (41), we get

$$\mathcal{I}_{a,b}^{\lambda} \geq \left(\frac{a}{2} - \overline{\lambda}_0^s C\varepsilon\right)\|u\|^2 + \frac{b}{4}\|u\|^4 - C\|u\|^{2_s^*} - \overline{\lambda}_0^s C\|u\|^p$$

for any $u \in X_0^s(\Omega)$. The statement follows by taking ε so that $a/2 - \overline{\lambda}_0^s C\varepsilon > 0$. □

Now, we are able to characterize the infimum in Theorem 6. More precisely, considering the $r > 0$ given by the previous Proposition, we can set

$$\hat{\iota}_{\lambda}^s := \inf\{\mathcal{I}_{a,b}^{\lambda}(u) : u \in X_0^s(\Omega),\ \|u\| \geq r\}.$$

Remark 5 It is straightforward to see that $\hat{\iota}_0^s \to 0$ as $\lambda \to \overline{\lambda}_0^s$. Indeed, it suffices to consider a function $u \in X_0^s(\Omega)$ such that $\overline{\lambda}_0^s = \lambda_0^s(u)$ (the existence of a such u is guaranteed by Theorem 3) and to observe that

$$0 \leq \hat{\iota}_{\lambda}^s \leq \mathcal{I}_{a,b}^{\lambda}(u) \to 0 \quad \text{as} \quad \lambda \to \overline{\lambda}_0^s.$$

Remark 6 The function w_{λ}^s obtained in the previous Theorem represents a critical point for the functional $\mathcal{I}_{a,b}^{\lambda}$ and it is a local minimizer.

Proof of Theorem 6 Consider the $r, M > 0$ given by Proposition 7 and notice that if $\lambda \in (\overline{\lambda}_0^s - \delta, \overline{\lambda}_0^s)$ we have that $\hat{\iota}_{\lambda}^s < M$ for an appropriate $\delta > 0$. As a consequence of that, if $(u_k)_k$ is a minimizing sequence there must be a $\upsilon > 0$ such that $\|u_k\| \geq M + \upsilon$ for k sufficiently large. At this point, invoking the Ekeland's variational principle (see [12]) we have the existence of a minimizing sequence and the convergence to a local minimizer $w_{\lambda}^s \in X_0^s(\Omega)$ such that $\|w_{\lambda}^s\| > M$ and $\hat{\iota}_{\lambda}^s = \mathcal{I}_{a,b}^{\lambda}(w_{\lambda}^s)$ is established remembering the validity of the Palais–Smale condition as showed in Lemma 1[(2)]. □

Finally, we prove Theorem 7 that ensure the existence of mountain pass solutions for $\lambda < \overline{\lambda}_0^s$ close enough to $\overline{\lambda}_0^s$. In the following we will denote with $\delta > 0$ the number obtained in Theorem 6.

Proof of Theorem 7 Notice $\min\{\mathcal{I}_{a,b}^{\lambda}(0), \mathcal{I}_{a,b}^{\lambda}(w_{\lambda}^s)\} < M$, recall $\|w_{\lambda}^s\| > M$ and (42). So, we have a mountain pass geometry. Since the Palais–Smale condition is satisfied, we exploit the the Mountain Pass Theorem (see [2]) to get the conclusion. □

Remark 7 If in addition to assumptions $(H_1) - (H_4)$ we require

(H_5) for any $u \in X_0^s(\Omega)$, the function $\zeta \mapsto \int_{\Omega} g(x, \zeta u(x))\,dx$ is C^1 on $(0, \infty)$

we are able to state a non-existence result for problem $(P_{a,b}^{\lambda})$. Namely, we claim there is $\overline{\lambda}^s := \overline{\lambda}^s(a,b) \in (0, \overline{\lambda}_0^s)$ such that if $\lambda \in (0, \overline{\lambda}^s)$ then $(P_{a,b}^{\lambda})$ does not admit non trivial solutions. Consider the system

$$\begin{cases} \left(\mathcal{J}_{a,b}^{\lambda,u}\right)'(\zeta) = 0 \\ \left(\mathcal{J}_{a,b}^{\lambda,u}\right)''(\zeta) = 0 \\ \left(\mathcal{J}_{a,b}^{\lambda,u}\right)'(\zeta) = \inf_{\varrho > 0}\left(\mathcal{J}_{a,b}^{\lambda,u}\right)'(\varrho). \end{cases} \tag{43}$$

After fixing $u \in X_0^s(\Omega)$, similarly to proposition 4 it is possible to find a unique $\lambda^s(u) > 0$ that solves (43). We point out that the parameter $\lambda^s(u)$ is the unique $\lambda > 0$ for which the fiber map $\mathcal{J}_{a,b}^{\lambda,u}(\zeta)$ has a critical point with null second derivative. Furthermore, observe that if $0 < \lambda < \lambda^s(u)$ then

$$\mathcal{J}_{a,b}^{\lambda,u}(\zeta) > \mathcal{J}_{a,b}^{\lambda^s(u),u}(\zeta) > 0.$$

So, $\mathcal{J}_{a,b}^{\lambda,u}(\zeta)$ has no critical points. As a consequence of that, it is immediate to prove that

$$\lambda^s(u) < \lambda_0^s(u). \tag{44}$$

If not, we would have that $\mathcal{J}_{a,b}^{\lambda_0^s(u),u}(\zeta)$ is increasing contradicting the existence of solutions for system (29). At this point we set

$$\overline{\lambda}^s := \inf_{u \in X_0^s(\Omega)\setminus\{0\}} \lambda^s(u).$$

Now, we observe that if $a^{(N-4s)/2s} b > PS_{N,s}$ then $0 < \overline{\lambda}^s < \overline{\lambda}_0^s$. In fact, we know from corollary 2 that there is $u \in X_0^s(\Omega) \setminus \{0\}$ such that $\overline{\lambda}_0^s = \lambda_0^s(u)$. From (44) it follows that

$$\overline{\lambda}^s \leq \lambda^s(u) < \lambda_0^s(u) = \overline{\lambda}_0^s.$$

To conclude, we observe that for any $\lambda \in (0, \overline{\lambda}^s)$ the map $\mathcal{J}_{a,b}^{\lambda,u}(\zeta)$ is increasing and that $\left(\mathcal{J}_{a,b}^{\lambda,u}\right)'(\zeta) > 0$ for all $\zeta > 0$. Hence $u = 0$ is the only admissible critical point.

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